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Output feedback exponential stabilization of uncertain chained systems

Zairong Xi[a,*], Gang Feng[b], Z.P. Jiang[c], Daizhan Cheng[a]

[a]*Laboratory of Systems and Control, Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100080, China*
[b]*Department of Manufacture Engineering and Engineering Management, The City University of Hong Kong, Tat Chee Avenue, Kowloon Tong, Hong Kong*
[c]*Department of Electrical and Computer Engineering, Polytechnic University, Six Metrotech Center, Brooklyn, NY 11201, USA*

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Abstract

This paper deals with chained form systems with strongly nonlinear disturbances and drift terms. The objective is to design robust nonlinear output feedback laws such that the closed-loop systems are globally exponentially stable. The systematic strategy combines the input-state-scaling technique with the so-called backstepping procedure. A dynamic output feedback controller for general case of uncertain chained system is developed with a filter of observer gain. Furthermore, two special cases are considered which do not use the observer gain filter. In particular, a switching control strategy is employed to get around the smooth stabilization issue (difficulty) associated with nonholonomic systems when the initial state of system is known.

Keywords: Chained form systems; Input-state scaling; Backstepping; Observer gain filter; Output feedback

1. Introduction

The control and stabilization of nonholonomic dynamic systems has been studied by many researchers within the nonlinear control community since Brockett's work [1], for example, see the recent survey paper by Kolmanovsky and McClamroch [2] and references

*Corresponding author.
E-mail address: zrxi@iss.ac.cn (Z. Xi).

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cited therein. In 1983 Brockett [1] showed that the origin of a simple nonholonomic integrator would not be made asymptotically stable in the sense of Lyapunov by any continuous state feedback control law, and at the same time he obtained a necessary condition for asymptotic stabilization of general nonlinear systems. This condition implies that a nonholonomic system is not stabilizable by stationary continuous state-feedback, although it is (open-loop) controllable. In the past decade, many researchers have been attracted to the search for new control strategies. So far, several interesting solutions, such as open-loop periodic steering control, smooth or continuous time-varying feedback control, and discontinuous feedback control, have been found to overcome the above-described obstruction in stabilizing a nonholonomic system, for example, [3–21].

It is noted that the majority of these constructive methods have been developed for an important class of driftless nonholonomic systems in chained form, which was first introduced by Murray and Sastry [13]. It has been shown in [2,13] and references therein that many nonlinear mechanical systems with nonholonomic constraints on velocities can be transformed, either locally or globally, to the chained form system via coordinates and state feedback transformation. The typical examples include tricycle-type mobile robots, cars towing several trailers, the knife edge, a vertical rolling wheel, and a rigid spacecraft with two torque actuators.

In many practical applications, both asymptotic stabilization and exponential convergence of regulation are often demanded. However, it has been known that for nonholonomic system a smooth time-varying state feedback law can be applied to achieve asymptotic stabilization but fails to meet the requirement of exponential convergence, while a continuous time-varying and/or discontinuous feedback law guarantees the exponential regulation of nonholonomic systems in chained form but fails to achieve asymptotic stabilization [3,4,6–8,11,12,15,17,18,22]. More recently, Marchand and Alamir [19] obtained Lyapunov stability and exponential rate of convergence in the absence of disturbances. Unfortunately, their method could not easily be extended to the case with occurrence of uncertain disturbances if not impossible. We recently proposed a switching scheme to achieve Lyapunov stability and exponential convergence for uncertain chained form systems using state feedback in [23]. It is noted that most control laws proposed in the past literatures are based on state feedback, and hence can be used only if the whole state vector is measurable. If this is not the case, a dynamic output feedback control law needs to be designed. The output feedback issue has been considered for chained systems without uncertainties in [24] and with a special class of uncertainties in [17,25], where the x_0-subsystem is assumed to be linear when considering output feedback.

The objective of this paper is to obtain both robust global exponential regulation and Lyapunov stability with output feedback for a class of disturbed nonlinear chained systems. We make use of a particular input-state scaling, the backstepping technique, the switching scheme and observer gain filter to design a dynamic output feedback controller such that the closed-loop system is exponentially convergent and Lyapunov stable. The contribution of the paper is twofold. First, we propose a systematic control design procedure to construct a robust nonlinear output feedback control law which solves global exponential regulation problem for all plants in the considered class, including the ideal chained systems, the systems with linear x_0-subsystem which was considered in [17,25], and in particular, the systems which have uncertainties in the x_0-subsystem. Second, a switching scheme is proposed to achieve Lyapunov stability when the initial states are in a specified manifold. For the Lyapunov stability with global exponential regulation using output feedback, to the best of our knowledge, there is still no robustification design tool for nonholonomic systems.

The remainder of this paper is organized as follows. In Section 2, the class of nonholonomic systems with strongly nonlinear disturbances is introduced, and the problem of global exponential stabilization is formulated, and some techniques are introduced. Section 3 presents the backstepping design procedure and switching control strategy. Then, in Section 4, we illustrate our novel output feedback control design methodology via a simple practical nonholonomic system with disturbances. The numerical simulations testify to the effectiveness and robustness aspects of the proposed robustification tool. Finally, some conclusions are given in Section 5.

2. Problem formulation and preliminaries

Consider the following perturbed version of nonlinear system in chained form [17]

$$\begin{cases} \dot{x}_0 = u_0 + x_0\phi_0(t, x_0), \\ \dot{x}_1 = x_2 u_0 + \phi_1^d(t, x_0, x_1, u_0), \\ \vdots \\ \dot{x}_{n-2} = x_{n-1}u_0 + \phi_{n-2}^d(t, x_0, x_1, u_0), \\ \dot{x}_{n-1} = u + \phi_{n-1}^d(t, x_0, x_1, u_0), \\ y = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}, \end{cases} \tag{1}$$

where $x = (x_1, \ldots, x_{n-1}) \in R^{n-1}$, the functions ϕ_i^d's denote the possible modeling error and neglected dynamics, and $\phi_0(t, x_0)$ is a known smooth function.

The following assumption is usually used in output feedback design [17,26].

Assumption 1. For every $1 \leqslant i \leqslant n-1$, there are (known) smooth nonnegative functions ϕ_i such that

$$|\phi_i^d(t, x_0, x_1, u_0)| \leqslant |x_1|\phi_i(t, y, u_0)$$

for all $(t, x_0, x_1, u_0) \in R_+ \times R \times R \times R$.

First of all, we introduce a notion of exponential stability, which is usually called $\mathscr{K}$-exponential stability in literature. It should be noted that the function $\gamma(\cdot)$ is of class $\mathscr{K}$.

Definition 1. Consider a nonlinear system $\sum: \dot{x} = f(t, x)$ with $x \in R^n$. Let $\gamma: R^+ \to R^+$ be of class $\mathscr{K}$ which means γ is continuous, monotonically increasing and $\gamma(0) = 0$. The system is said to be *globally $\mathscr{K}$-exponentially stable* (*GES*) if there exist a strictly positive constant λ and a function γ of class $\mathscr{K}$ such that $\forall x(0) \in R^n$, $\forall t \geqslant 0$

$$|x(t)| \leqslant \gamma(|x(0)|)\mathrm{e}^{-\lambda t}. \tag{2}$$

For a subset $A \subseteq R^n$, if $\forall x(0) \in A$, there are a strictly positive constant λ, $T(x(0)) > 0$ and a function $\gamma': R^+ \to R^+$ such that $\forall t \geqslant T$, $|x(t)| \leqslant \gamma'(|x(0)|)\mathrm{e}^{-\lambda t}$, then the system $\sum$ is said to be *$\mathscr{K}$-exponentially regulated with respect to A*.

Remark 1. *$\mathscr{K}$-exponential stability* is different from the *uniform global asymptotical stability* (*UGAS*). If there is a $\mathscr{K}\mathscr{L}$ function $\beta(\cdot, \cdot)$ such that $x(t) \leqslant \beta(\|x(0)\|, t)$, then the system $\sum$ is said to be *UGAS*. From [27, Proposition 7] there exist $\mathscr{K}_\infty$ functions θ_1, θ_2 so that $\beta(s,t) \leqslant \theta_1(\theta_2(s)\mathrm{e}^{-t})$, $s>0 \quad t>0$.

In this paper, we are interested in finding a dynamic output feedback

$$\begin{cases} \dot{\eta} = \Xi(\eta, y), \\ u_0 = \mu_0(\eta, y), \\ u = \mu(\eta, y), \end{cases} \tag{3}$$

such that the closed-loop systems consisting of Eqs. (1) and (3) is $\mathscr{K}$-exponentially stable in the sense of Definition 1.

In order to apply backstepping technique we introduce the following input-state scaling discontinuous transformation defined by [18,19]

$$\xi_i = \frac{x_i}{u_0^{n-(i+1)}}, \quad 1 \leqslant i \leqslant n-1. \tag{4}$$

Under the new ξ-coordinates, the x-subsystem is transformed into

$$\begin{cases} \dot{\xi}_1 = \xi_2 - (n-2)\xi_1 \dfrac{\dot{u}_0}{u_0} + \dfrac{\phi_1^d(t, x_0, x_1, u_0)}{u_0^{n-2}}, \\ \dot{\xi}_2 = \xi_3 - (n-3)\xi_2 \dfrac{\dot{u}_0}{u_0} + \dfrac{\phi_2^d(t, x_0, x_1, u_0)}{u_0^{n-3}}, \\ \vdots \\ \dot{\xi}_{n-2} = \xi_{n-1} - \xi_{n-2} \dfrac{\dot{u}_0}{u_0} + \dfrac{\phi_{n-2}^d(t, x_0, x_1, u_0)}{u_0}, \\ \dot{\xi}_{n-1} = u + \phi_{n-1}^d(t, x_0, x_1, u_0). \end{cases} \tag{5}$$

It should be noted that the measurement of state ξ_1 can be obtained if the to-be-designed control u_0 is only dependent on output y.

If $u_0(t) \neq 0$ for every $t \geqslant 0$, the discontinuous state transformation (4) is applicable. Then the system (5) has the following form:

$$\dot{\xi} = \left(A - \frac{\dot{u}_0}{u_0} L\right)\xi + bu + \Psi^d(t, x_0, x_1, u_0), \tag{6}$$

where

$$A = \begin{pmatrix} 0_{(n-2)\times 1} & I_{(n-2)\times(n-2)} \\ 0 & 0_{1\times(n-2)} \end{pmatrix}, \quad b = \begin{pmatrix} 0_{(n-2)\times 1} \\ 1 \end{pmatrix}, \quad L = diag((n-2) \ \cdots \ 1 \ \ 0)$$

and

$$\begin{aligned} \Psi^d(t, x_0, x_1, u_0) &:= (\Psi_1^d(t, x_0, x_1, u_0) \ \cdots \ \Psi_{n-2}^d(t, x_0, x_1, u_0) \ \ \Psi_{n-1}^d(t, x_0, x_1, u_0))^{\mathrm{T}} \\ &:= \left(\frac{\phi_1^d(t, x_0, x_1, u_0)}{u_0^{n-2}} \ \cdots \ \frac{\phi_{n-2}^d(t, x_0, x_1, u_0)}{u_0} \ \ \phi_{n-1}^d(t, x_0, x_1, u_0) \right)^{\mathrm{T}}. \end{aligned}$$

From Assumption 1 it is easy to obtain the following lemmas.

Lemma 1. *For each* $1 \leqslant i \leqslant n-1$, *there exists a smooth nonnegative functions* Ψ_i *such that*

$$|\Psi_i^d(t, x_0, x_1, u_0)| \leqslant |\xi_1| \Psi_i(t, x_0, \xi_1, u_0).$$

Proof. In view of Eq. (4) and Assumption 1, we have

$$\begin{aligned} |\Psi_i^d(t, x_0, x_1, u_0)| &\leqslant \frac{|x_1|}{|u_0^{n-(i+1)}|} \phi_i(t, y, u_0) \\ &= \frac{|x_1|}{|u_0^{n-(i+1)}|} \phi_i(t, x_0, x_1, u_0) \\ &= |\xi_1||u_0|^{(i-1)} \phi_i(t, x_0, x_1, u_0) \\ &\leqslant |\xi_1|((u_0)^{2(i-1)} + 1)\phi_i(t, x_0, (u_0)^{n-2}\xi_1, u_0). \end{aligned}$$

Therefore, the proof of Lemma 1 is completed. □

Denote $C = (1 \; 0_{1\times(n-2)})$ and then $Y = C\xi$ is a measurable variable. It is easily verified that (A, b) is controllable and (C, A) observable.

Lemma 2. *For any continuous function* $\varkappa_0(t)$ *there exist two strictly positive real numbers* $p_{\min}$ *and* $p_{\max}$ *such that the unique solution* $P(t)$ *of the following matrix differential equation*:

$$\begin{cases} \dot{P} = P(A - \varkappa_0(t)L)^{\mathrm{T}} + (A - \varkappa_0(t)L)P - PC^{\mathrm{T}}CP + I, \\ P(0) = P_0 > 0, \end{cases}$$

satisfies

$$p_{\min} I \leqslant P(t) \leqslant p_{\max} I, \quad t \geqslant 0.$$

Proof. Since

$$A = \begin{pmatrix} 0_{(n-2)\times 1} & I_{(n-2)\times(n-2)} \\ 0 & 0_{1\times(n-2)} \end{pmatrix}, \quad C = (1 \;\; 0_{1\times(n-2)}), \quad L = diag((n-2) \;\; \cdots \;\; 1 \;\; 0)$$

it is directly verified that for $y = Cx$ along $\dot{x} = (A - \varkappa_0(t)L)x$ we have

$$y^{(i-1)}(t) = [* \;\; \cdots \;\; * \;\; \underset{i}{1} \;\; 0 \;\; \cdots \;\; 0]x$$

$$i = 1, \ldots, n.$$

Then $(C, A - \varkappa_0(t)L)$ is observable. From duality of controllability and observability and [28,29] the proof is completed. □

Then the following full-order observer can be designed:

$$\begin{cases} \dot{\hat{\xi}} = \left(A - \dfrac{\dot{u}_0}{u_0}L\right)\hat{\xi} + bu + PC^{\mathrm{T}}(Y - C\hat{\xi}), \\ \dot{P} = P\left(A - \dfrac{\dot{u}_0}{u_0}L\right)^{\mathrm{T}} + \left(A - \dfrac{\dot{u}_0}{u_0}L\right)P - PC^{\mathrm{T}}CP + I, \\ P(0) = P_0 > 0. \end{cases} \tag{7}$$

Remark 2. It is noted that the observer gain P is determined by a filter, which is time-varying and dependent on the nonlinearity of x_0-subsystem. The observer is reminiscent of high-gain observer, for example, see the more recent paper [30].

3. Dynamic output feedback

In this section we focus on dynamic output feedback design for system (1). In order to apply the input-state scaling discontinuous transformation (4) $u_0(t)$ should not be equal to 0. The inherently triangular structure of system (5) suggests that we should design the control inputs u_0 and u in two separate stages.

3.1. General case

It is noted that there exists a smooth nonnegative function $\omega_0(t, x_0)$ such that

$$|\phi_0(t, x_0)| \leqslant \omega_0(t, x_0),$$

for example, $\omega_0(t, x_0) = 1 + \phi_0^2(t, x_0)$. Then for systems (1) we have

Lemma 3. *The x_0-subsystem of the uncertain system* (1) *can be globally $\mathscr{K}$-exponentially regulated at the origin by the following switching control scheme*:

$$u_0(t) = \begin{cases} -(\lambda_0 + \omega_0)x_0, \quad \lambda_0 > 0 & \text{if} \quad x_0(0) \neq 0, \\ \begin{cases} \beta - x_0\omega_0 & \text{if} \quad t < t_s \\ -(\lambda_0 + \omega_0)x_0 & \text{if} \quad t \geqslant t_s, \end{cases} & \text{if} \quad x_0(0) = 0, \end{cases}$$

where $t_s(T) = \min\{\delta, 1/2\Delta, T\}$ is a positive constant, $\delta > 0$, $\varepsilon > 0$, $\beta > 0$ and $T > 0$ are strictly positive constants, and $\Delta = \max_{\substack{0 \leqslant v \leqslant T \\ |s| \leqslant \beta T}}\{\varepsilon + \omega_0(v, s)\}$. At the same time, the designed control u_0 is an almost continuous and differentiable function such that

(1) *for all $t \geqslant 0$, $u_0(t) \neq 0$,*
(2) *for almost all $t \geqslant 0$,*

$$\left|\frac{\mathrm{d}u_0}{\mathrm{d}t}\right| \leqslant \varphi_0(t, x_0)|u_0(t)|, \tag{8}$$

where $\varphi_0(t, x_0)$ is a known smooth nonnegative function.

Proof. The proof is divided into two cases.

Case 1: $x_0(0) \neq 0$. First, it can be easily seen that the x_0-subsystem is exponentially stable since $\dot{V}_0 \leqslant -2\lambda_0 V_0$ where $V_0 = \frac{1}{2}x_0^2$.

Second, it follows from

$$\dot{x}_0 = -(\lambda_0 + \omega_0 - \phi_0)x_0$$

that

$$x_0(t) = x_0(0)\exp\left(-\int_0^t (\lambda_0 + \omega_0(s, x_0(s)) - \phi_0(s, x_0(s)))\,\mathrm{d}s\right),$$

and thus $x_0(t) \neq 0$. Then $u_0(t) \neq 0$.

Third, we need to verify that there exists a smooth nonnegative function φ_0 such that

$$\left|\frac{\dot{u}_0}{u_0}\right| \leqslant \varphi_0.$$

In fact, since $u_0 = -\lambda_0 x_0 - x_0\omega_0$, then

$$\dot{u}_0 = (\lambda_0 + \omega_0 - \phi_0)\left(\lambda_0 + \omega_0 + \frac{\partial \omega_0}{\partial x_0}x_0\right)x_0 - \frac{\partial \omega_0}{\partial t}x_0.$$

So

$$\begin{aligned}\frac{\dot{u}_0}{u_0} &= -\frac{(\lambda_0 + \omega_0 - \phi_0)(\lambda_0 + \omega_0 + ((\partial\omega_0/\partial x_0)x_0) - \partial\omega_0/\partial t}{\lambda_0 + \omega_0},\\ &\coloneqq -\bar{\varphi}_0(t, x_0)\end{aligned}$$

which is smooth, and

$$\left|\frac{\dot{u}_0}{u_0}\right| \leqslant 1 + \bar{\varphi}_0^2(t, x_0) \coloneqq \varphi_0(t, x_0).$$

Case 2: $x_0(0) = 0$. Firstly, when $t \leqslant t_s$ it is easy to see that $0 \leqslant x_0(t) \leqslant \beta t$. At the same time,

$$\begin{aligned}\frac{\mathrm{d}V_0}{\mathrm{d}t} &= 2x_0(\beta - (\omega_0 - \phi_0)x_0)\\ &= 2\beta\sqrt{V_0} - 2(\omega_0 - \phi_0)V_0,\end{aligned}$$

where $V_0 = x_0^2$. Then using the variable coefficient method we have

$$\begin{aligned}V_0(t) &= \beta^2 \exp(-2\sigma(t))\left(\int_0^t \exp(\sigma(s))\,\mathrm{d}s\right)^2\\ &\geqslant \beta^2 \exp(-2\sigma(t))t^2,\end{aligned}$$

where $\sigma(t) = \int_0^t(\omega_0(s, x_0(s)) - \phi_0(s, x_0(s)))\,\mathrm{d}s$. So $x_0(t) > 0$ for all $0 < t \leqslant t_s$.

Secondly, the following inequality is satisfied.

$$u_0(t) \geqslant \frac{\beta}{2}, \quad t < t_s.$$

It is easy to know that when $t < t_s$

$$\begin{aligned}u_0(t) &= \beta - \omega_0 x_0(t)\\ &\geqslant \beta - \Delta\beta t\\ &\geqslant \frac{\beta}{2}, \quad t < t_s.\end{aligned}$$

Thirdly, from above discussion we know that $x_0(t_s) \neq 0$. So we can switch from $u_0(t) = \beta - \omega_0 x_0$ when $t < t_s$ to $u_0(t) = -(\lambda_0 + \omega_0)x_0$ when $t \geqslant t_s$. It follows from the discussion of Case 1 that the x_0-subsystem is exponentially regulated.

Fourthly, it is easy to know that there exists smooth nonnegative function φ_0 such that for almost $t \geqslant 0$,

$$\left|\frac{\dot{u}_0}{u_0}\right| \leqslant \varphi_0,$$

and it is easy to see that $\dot{u}_0/u_0$ is smooth except at $t = t_s$.

Remark 3. Notice that $t_s(T) = \min\{\delta, 1/2\Delta, T\}$, where $\delta > 0$, $\varepsilon > 0$, $\beta > 0$ are arbitrarily chosen constants and $T > 0$. In particular, $\Delta = \max_{\substack{0 \leqslant v \leqslant T\\ |s| \leqslant \beta T}}\{\varepsilon + \omega_0(v, s)\}$ is a continuous function of T. Then it is easy to know that the switching time $t_s(T)$ is continuous with respect to T and its computation is dependent on T explicitly. If T is given and known $t_s(T)$ can be calculated in advance.

From previous discussion the observer (7) can be designed. Then let

$$e = \xi - \hat{\xi} := (e_1, \ldots, e_{n-1}),$$

and C_i $(i = 1, \ldots, n-1)$ are $n-1$-dimensional row vectors with zero elements except the ith element being 1. So the overall system to be controlled can be expressed as

$$\begin{cases} \dot{P} = P\left(A - \dfrac{\dot{u}_0}{u_0}L\right)^{\mathrm{T}} + \left(A - \dfrac{\dot{u}_0}{u_0}L\right)P - PC^{\mathrm{T}}CP + I, \\ \dot{e} = \left(A - PC^{\mathrm{T}}C - \dfrac{\dot{u}_0}{u_0}L\right)e + \Psi^d(t, x_0, x_1, u_0), \\ \dot{\xi}_1 = \hat{\xi}_2 - (n-2)\xi_1\dfrac{\dot{u}_0}{u_0} + \Psi_1^d(t, x_0, x_1, u_0) + e_2, \\ \dot{\hat{\xi}}_2 = \hat{\xi}_3 - (n-3)\hat{\xi}_2\dfrac{\dot{u}_0}{u_0} + C_2PC^{\mathrm{T}}Ce, \\ \vdots \\ \dot{\hat{\xi}}_{n-2} = \hat{\xi}_{n-1} - \hat{\xi}_{n-2}\dfrac{\dot{u}_0}{u_0} + C_{n-2}PC^{\mathrm{T}}Ce, \\ \dot{\hat{\xi}}_{n-1} = u + C_{n-1}PC^{\mathrm{T}}Ce. \end{cases} \tag{9}$$

Then the design of the control input u will be obtained using the standard backstepping method shown in [17,18,23,26] to the transformed system (9), i.e.,

3.1.1. *Design procedure*

Step 1: Let us begin with the (P, e, ξ_1)-subsystem of Eq. (9),

$$\begin{cases} \dot{P} = P\left(A - \dfrac{\dot{u}_0}{u_0}L\right)^{\mathrm{T}} + \left(A - \dfrac{\dot{u}_0}{u_0}L\right)P - PC^{\mathrm{T}}CP + I, \\ \dot{e} = \left(A - PC^{\mathrm{T}}C - \dfrac{\dot{u}_0}{u_0}L\right)e + \Psi^d(t, x_0, x_1, u_0), \\ \dot{\xi}_1 = \hat{\xi}_2 - (n-2)\xi_1\dfrac{\dot{u}_0}{u_0} + \Psi_1^d(t, x_0, x_1, u_0) + e_2, \end{cases}$$

where $\hat{\xi}_2$ is regarded as the virtual control input. Denote $z_1 = \xi_1$. From Lemma 2, $P(t)$ is bounded. Then the positive definite function $V_1 = e^{\mathrm{T}}P^{-1}e + \frac{1}{2}z_1^2$ of (e, z_1) can be chosen as Lyapunov function for (e, z_1)-subsystem. Using Lemma 1, the time derivative of V_1 along the solutions of Eq. (9) satisfies

$$\begin{aligned} \dot{V}_1 = & -e^{\mathrm{T}}P^{-2}e - e^{\mathrm{T}}C^{\mathrm{T}}Ce + z_1\hat{\xi}_2 - (n-2)z_1^2\frac{\dot{u}_0}{u_0} + 2e^{\mathrm{T}}P^{-1}\Psi^d + z_1\Psi_1^d + z_1e_2 \\ \leqslant & -\frac{1}{2}e^{\mathrm{T}}P^{-2}e - e_1^2 + z_1\hat{\xi}_2 + (n-2)z_1^2\varphi_0 + 4z_1^2\Psi^{\mathrm{T}}\Psi + z_1^2\Psi_1 + z_1^2C_2PPC_2^{\mathrm{T}} \end{aligned} \tag{10}$$

since $z_1e_2\leqslant z_1^2C_2PPC_2^{\mathrm{T}}+\frac{1}{4}e^{\mathrm{T}}P^{-2}e$. Then we are led to introduce a virtual control function α_1 and a new variable z_2 as

$$\alpha_1(t,P,x_0,z_1,u_0)=-\lambda_1z_1-\varphi_1(t,P,x_0,z_1,u_0)z_1,$$

$$z_2=\hat{\xi}_2-\alpha_1(t,P,x_0,z_1,u_0),$$

where λ_1 is a positive design parameter and $\varphi_1(t,P,x_0,z_1,u_0)=[(n-2)\varphi_0+4\Psi^{\mathrm{T}}\Psi+\Psi_1+C_2PPC_2^{\mathrm{T}}]z_1$. Consequently, Eq. (10) implies

$$\dot{V}_1\leqslant-\tfrac{1}{2}e^{\mathrm{T}}P^{-2}e-e_1^2-\lambda_1z_1^2+z_1z_2.$$

Note that α_1 is a smooth function satisfying

$$\alpha_1(t,P,x_0,0,u_0)=0.$$

Step i ($2\leqslant i\leqslant n-2$): As in [17,23,26], consider the Lyapunov function candidate $V_i=V_{i-1}(t,P,e,z_1,\ldots,z_{i-1})+\frac{1}{2}z_i^2$. Therefore, we can choose a virtual control function α_i and a new variable z_{i+1} as follows:

$$\alpha_i(t,P,x_0,z_1,\ldots,z_i,u_0)=-\lambda_iz_i-\sum_{j=1}^{i}\varphi_{ij}(t,P,x_0,z_1,\ldots,z_i,u_0)z_j,$$

$$z_{i+1}=\hat{\xi}_{i+1}-\alpha_i,$$

where $\varphi_{ij}(t,P,x_0,z_1,\ldots,z_i,u_0)$ ($j=1,\ldots,n-1$) are some suitable smooth functions, such that

$$\dot{V}_i\leqslant-\frac{1}{2^i}e^{\mathrm{T}}P^{-2}e-e_1^2-\sum_{j=1}^{i}(\lambda_j-i+j)z_j^2+z_iz_{i+1}.$$

Step $n-1$: At this last step, consider the whole $(P,e,\hat{\xi})$-system (9) where the true input u is to be designed on the basis of the virtual control functions α_{n-1}'s. To this end, consider a positive definite and radially unbounded Lyapunov function

$$V_{n-1}=V_{n-2}(t,P,e,z_1,\ldots,z_{n-2})+\tfrac{1}{2}z_{n-1}^2.$$

As in [26,17,18,23], it is easy to know that some suitable smooth functions $\varphi_{(n-1)j}(t,P,x_0,z_1,\ldots,z_{n-1},u_0)$ ($j=1,\ldots,n-1$) can be found such that along the solutions of Eq. (9)

$$\dot{V}_{n-1}\leqslant-\frac{1}{2^{n-1}}e^{\mathrm{T}}P^{-2}e-e_1^2-\sum_{j=1}^{n-1}(\lambda_j-n+1+j)z_j^2 \tag{11}$$

when choosing control law u as

$$u=-\lambda_{n-1}z_{n-1}-\sum_{j=1}^{n-1}\varphi_{(n-1)j}(t,P,x_0,z_1,\ldots,z_{n-1},u_0)z_j. \tag{12}$$

So the following theorem can be obtained.

Theorem 4 (*Main theorem*). *Let*

- β, δ, ε, λ_0 *and* λ_i ($1\leqslant i\leqslant n-1$) *be strictly positive real constants such that*

$$\lambda^*=\min\{\lambda_j-n+1+j|j=1,\ldots,n-1\}>0,$$

- $\Gamma = \{(0, x) : |x| \neq 0\}$.

Then, the following dy!namic discontinuous output feedback law globally $\mathscr{K}$*-exponentially stabilizes uncertain chained system* (1). *Moreover, the feedback law is bounded.*

(i) $(x_0(0), x(0)) = (0, x(0)) \in \Gamma$

$$u_0(t) = \begin{cases} \beta - x_0\omega_0 & \text{if } t < t_s(|x(0)|), \\ -(\lambda_0 + \omega_0)x_0 & \text{if } t \geqslant t_s(|x(0)|), \end{cases} \tag{13}$$

$$\begin{cases} \dot{\hat{\xi}} = \left(A - \dfrac{\dot{u}_0}{u_0}L\right)\hat{\xi} + bu + PC^{\mathrm{T}}(Y - C\hat{\xi}), \\ \dot{P} = P\left(A - \dfrac{\dot{u}_0}{u_0}L\right)^{\mathrm{T}} + \left(A - \dfrac{\dot{u}_0}{u_0}L\right)P - PC^{\mathrm{T}}CP + I, \\ P(0) = P_0 > 0, \\ u(t) = \alpha_{n-1}(t, P, x_{0,}\xi_1, \hat{\xi}_2, \ldots, \hat{\xi}_{n-1}, u_0), \end{cases} \tag{14}$$

where $\Delta = \max_{\substack{0 \leqslant v \leqslant |x(0)| \\ |s| \leqslant \beta|x(0)|}}\{\varepsilon + \omega_0(v, s)\}$ and $t_s(|x(0)|) = \min\{\delta, 1/2\Delta, |x(0)|\}$.

(ii) $(x_0(0), x(0)) = (0, 0)$,

$$u_0 = 0, \tag{15}$$

$$u = 0, \tag{16}$$

(iii) $(x_0(0), x(0)) \notin \Gamma \cup \{(0, 0)\}$,

$$u_0 = -(\lambda_0 + \omega_0)x_0, \tag{17}$$

$$\begin{cases} \dot{\hat{\xi}} = \left(A - \dfrac{\dot{u}_0}{u_0}L\right)\hat{\xi} + bu + PC^{\mathrm{T}}(Y - C\hat{\xi}), \\ \dot{P} = P\left(A - \dfrac{\dot{u}_0}{u_0}L\right)^{\mathrm{T}} + \left(A - \dfrac{\dot{u}_0}{u_0}L\right)P - PC^{\mathrm{T}}CP + I, \\ P(0) = P_0 > 0, \\ u(t) = \alpha_{n-1}(t, P, x_{0,}\xi_1, \hat{\xi}_2, \ldots, \hat{\xi}_{n-1}, u_0). \end{cases} \tag{18}$$

Proof. Choose $T = |x(0)|$. Hence it follows from Lemma 3 that the x_0-subsystem can be globally $\mathscr{K}$-exponentially regulated at the origin. Moreover, at the same time from the proof of Lemma 3 we have

$$\begin{cases} |x_0(t)| \leqslant \beta t, & 0 \leqslant t \leqslant t_s(|x(0)|), \\ |x_0(t)| \leqslant \beta|x(0)|\mathrm{e}^{-\lambda_0(t - t_s)}, & t > t_s(|x(0)|). \end{cases} \tag{19}$$

It is easy to see that the x_0-subsystem is globally $\mathscr{K}$-exponentially stable at the origin from the definition of t_s.

If parameters λ_i satisfy

$$\lambda^* = \min\{\lambda_j - n + 1 + j | j = 1, \ldots, n - 1\} > 0,$$

and $\lambda = \min\{\lambda^*, 1/2^{n-1}p_{\max}\} > 0$, then the above backstepping control strategy (7)–(12) yields that the x-subsystem of uncertain system (1) with the observer $\hat{\xi}$ is well defined and globally stabilized at the origin from backstepping design.

Let $z = (z_1, \ldots, z_{n-1})$. According to Eq. (11), we have

$$\dot{V}_{n-1} \leqslant -\lambda V_{n-1},$$

which implies

$$V_{n-1}(t) \leqslant V_{n-1}(0)\mathrm{e}^{-\lambda t}, \quad t \geqslant 0.$$

Then it follows from [17] that

$$|(e(t), \xi(t))| \leqslant \gamma(|(e(0), x_0(0), \xi(0), u_0(0), P_0)|)\mathrm{e}^{-\varepsilon t}, \quad t \geqslant 0, \tag{20}$$

where $\varepsilon > 0$, $\xi = (\xi_1, \ldots, \xi_{n-1})$ and γ is a class-$\mathscr{K}$ function.

In the following we are to prove that x is $\mathscr{K}$-exponentially stable if ξ is $\mathscr{K}$-exponentially stable. From Eqs. (4), (19) and (20) it is easy to know that

(a) $(x_0(0), x(0)) = (0, x(0)) \in \Gamma$,

$$\begin{aligned}|x_i(t)| &\leqslant |u_0(t)|^{n-(i+1)}|\xi_i(t)| \\ &\leqslant \begin{cases} (\beta + \beta|x(0)|\Delta)^{n-(i+1)}\gamma\mathrm{e}^{-\varepsilon t}, & 0 \leqslant t \leqslant t_s(|x(0)|), \\ ((\lambda_0 + \Delta)\beta|x(0)|)^{n-(i+1)}\gamma\mathrm{e}^{-[\lambda_0(n-(i+1))+\varepsilon](t-t_s)}, & t > t_s(|x(0)|). \end{cases}\end{aligned}$$

(b) $(x_0(0), x(0)) = (0, 0)$, $|x_i(t)| = 0$.

(c) $(x_0(0), x(0)) \notin \Gamma \cup \{(0, 0)\}$,

$$\begin{aligned}|x_i(t)| &\leqslant |u_0(t)|^{n-(i+1)}|\xi_i(t)| \\ &\leqslant ((\lambda_0 + \Delta)\beta|x(0)|)^{n-(i+1)}\gamma(|(e(0), x_0(0), \xi(0), u_0(0), P_0)|)\mathrm{e}^{-[\lambda_0(n-(i+1))+\varepsilon]t}.\end{aligned}$$

Hence, the claim of Theorem 4 is true. □

Remark 4. It is known from the proof of Theorem 4 that the switching time relies on the knowledge of the initial state of the system, which is the penalty in order to obtain the Lyapunov stability. It is this dependence on the initial state that makes the closed-loop system Lyapunov stable. If the initial state is unknown the switching time $t_s(T)$ could be set to be dependent on any known positive constant T and independent on the initial state of the system. From the design procedure it is easy to see that $x(t) \to 0$ as $t \to \infty$. However, the closed-loop system cannot be made Lyapunov stable in this case.

Remark 5. From the previous discussion, it is known that under Assumption 1, there is a smooth row vector $\chi(t, P, x_0, z, u_0)$ such that $u = \chi z$. Thus, it follows that the boundedness of $x_0(t)$ and $z(t)$ would imply the boundedness of $u_0(t)$ and $u(t)$. It is easily seen that the boundedness of $x_0(t)$ and $z(t)$ can be deduced from the backstepping procedure, and thus the boundedness of $u_0(t)$ and $u(t)$ can be concluded.

Remark 6. From Lemma 2, Theorem 4 and Remark 5, the signals of closed-loop are bounded in interval of existence of solution. Then the interval of existence of solution is $[0, \infty)$.

3.2. Two special cases

3.2.1. Case one

In this case in addition to Assumption 1 we further assume that $\phi_0(t, x_0)$ is a known constant, i.e.,

$$\phi_0(t, x_0) = c_0.$$

Then it is easily seen from the proof of Lemma 3 that there is a constant c_1 such that $\varphi_0(t, x_0) = c_1$. Similar to the last subsection, the following full-order observer can be designed.

$$\dot{\hat{\xi}} = \left(A - \frac{\dot{u}_0}{u_0} L\right)\hat{\xi} + bu + K(Y - C\hat{\xi}), \tag{21}$$

where $K = (k_1, \ldots, k_{n-1})$ is chosen such that $A - KC$ is Hurwitz. So there is a symmetric positive definite matrix P satisfying

$$P(A - KC) + (A - KC)^{\mathrm{T}} P + (c_1 + 2)P + c_1 LPL \leqslant 0.$$

Remark 7. It is noted that the observer gain K is static which can be determined in advance such that $A - KC$ is Hurwitz. This is different from the last subsection since the x_0-dynamics is linear in this case.

Let

$$e = \xi - \hat{\xi} := (e_1, \ldots, e_{n-1}).$$

So the overall system to be controlled can be expressed as

$$\begin{cases} \dot{e} = \left(A - KC - \dfrac{\dot{u}_0}{u_0} L\right) e + \Psi^d(t, x_0, x_1, u_0), \\ \dot{\xi}_1 = \hat{\xi}_2 - (n-2)\xi_1 \dfrac{\dot{u}_0}{u_0} + \Psi_1^d(t, x_0, x_1, u_0) + e_2, \\ \dot{\hat{\xi}}_2 = \hat{\xi}_3 - (n-3)\hat{\xi}_2 \dfrac{\dot{u}_0}{u_0} + k_2 e_1, \\ \vdots \\ \dot{\hat{\xi}}_{n-2} = \hat{\xi}_{n-1} - \hat{\xi}_{n-2} \dfrac{\dot{u}_0}{u_0} + k_{n-2} e_1, \\ \dot{\hat{\xi}}_{n-1} = u + k_{n-1} e_1. \end{cases} \tag{22}$$

The design of the control input u can be obtained by following the similar but simpler *Design Procedure* described in the last subsection to the transformed system (22). The major difference lies in Step 1. In this case we should choose the candidate Lyapunov function $V_1 = e^{\mathrm{T}}Pe + \frac{1}{2}\xi_1^2$ for the (e, ξ_1)-subsystem of Eq. (22), i.e.,

$$\begin{cases} \dot{e} = \left(A - KC - \dfrac{\dot{u}_0}{u_0}L\right)e + \Psi^d(t, x_0, x_1, u_0), \\ \dot{\xi}_1 = \hat{\xi}_2 - (n-2)\xi_1\dfrac{\dot{u}_0}{u_0} + \Psi_1^d(t, x_0, x_1, u_0) + e_2. \end{cases}$$

Then the following theorem is in order.

Theorem 5. *Under Assumption* 1 *and* $\phi_0(t, x_0) = c_0$, *the uncertain system* (1) *can be globally* $\mathscr{K}$*-exponentially stabilized at the origin by*

$$u_0(t) = \begin{cases} -\lambda_0 x_0 - x_0 c_0 & \text{if } x_0(0) \neq 0, \\ \begin{cases} \beta - x_0 c_0 & \text{if } t < t_s(|x(0)|) \\ -(\lambda_0 + c_0)x_0 & \text{if } t \geqslant t_s(|x(0)|), \end{cases} & \text{if } x_0(0) = 0, \end{cases} \tag{23}$$

and the corresponding output feedback control strategy $u(y, \hat{\xi}, u_0)$ *with the observer* (21), *which is designed based on the similar but simpler* Design Procedure *described in the last subsection, where* β, δ, ε *and* λ_0 $(\lambda_0 + c_0 \neq 0)$ *be strictly positive real constants, and* $t_s(|x(0)|) = \min\{\delta, 1/2(|c_0| + \varepsilon), |x(0)|\}$.

In the discussion up to here, the x_0-dynamics, whether or not it is linear, must be completely known because of the observer dynamics (7) or (21). In the next subsection the case of x_0-dynamics with uncertainties will be considered.

3.2.2. Case two

If only variables x_0 and x_1 are measurable it is known from the previous discussion that the dynamics of x_0 should be completely known in general. If there are some uncertainties in the x_0-dynamics the output feedback control design discussed in the previous subsections cannot be used directly. In this subsection we will show that the following system with unknown x_0-dynamics

$$\begin{cases} \dot{x}_0 = d_0(t)u_0 + x_0\phi_0^d(t, x_0), \\ \dot{x}_1 = x_2 u_0 + \phi_1^d(t, x_0, x_1, u_0), \\ \dot{x}_2 = u + \phi_2^d(t, x_0, x_1, u_0), \end{cases} \tag{24}$$

which can be transformed into

$$\begin{cases} \dot{x}_0 = d_0(t)u_0 + x_0\phi_0^d(t, x_0), \\ \dot{\xi}_1 = \xi_2 - \xi_1\dfrac{\dot{u}_0}{u_0} + \dfrac{\phi_1^d(t, x_0, x_1, u_0)}{u_0}, \\ \dot{\xi}_2 = u + \phi_2^d(t, x_0, x_1, u_0) \end{cases} \tag{25}$$

by input-state scaling $\xi_1 = x_1/u_0$, $\xi_2 = x_2$, can be globally $\mathscr{K}$-exponentially stabilized based on the backstepping technique and switching scheme. The following assumptions are supposed to be satisfied.

Assumption 2. $0 < c_{01} \leqslant d_0(t) \leqslant c_{02}$.

Assumption 3. For every $0 \leqslant i \leqslant 2$, there are smooth nonnegative functions ϕ_i such that

$$|\phi_0^d(t, x_0)| \leqslant \phi_0(x_0),$$
$$|\phi_i^d(t, x_0, x_1, u_0)| \leqslant |x_1|\phi_i(y, u_0), \quad i = 1, 2$$

for all $(t, x_0, x, u_0) \in R_+ \times R \times R^2 \times R$.

Similar to the previous subsections, the following reduced-order observer can be constructed

$$\begin{cases} \dot{\chi} = -l\chi - l^2\xi_1 + u, \\ \hat{\xi}_2 = \chi + l\xi_1. \end{cases} \tag{26}$$

Then we have the following dynamics

$$\begin{cases} \dot{e} = -le + l\xi_1\dfrac{\dot{u}_0}{u_0} - l\dfrac{\phi_1^d(t, x_0, x_1, u_0)}{u_0} + \phi_2^d(t, x_0, x_1, u_0), \\ \dot{\xi}_1 = \chi + l\xi_1 - \xi_1\dfrac{\dot{u}_0}{u_0} + \dfrac{\phi_1^d(t, x_0, x_1, u_0)}{u_0} + e, \\ \dot{\chi} = -l\chi - l^2\xi_1 + u, \end{cases} \tag{27}$$

where l is a positive design parameter and $e = \xi_2 - \hat{\xi}_2$.

Similar to the previous subsections, based on Lemma 3, the control u can be designed using the backstepping technique to system (27). Let

$$\alpha_1 = \frac{3}{2}(l^2\varphi_0^2 + l^2\phi_1^2 + u_0^2\phi_2^2) + \left(l + \frac{1}{l} + \lambda_1 + \varphi_0 + \phi_1\right),$$

$$\chi = z - \alpha_1\xi_1,$$

and choose the candidate Lyapunov function $V_1 = \frac{1}{2}e^2 + \frac{1}{2}\xi_1^2$ for the (e, ξ_1)-subsystem of Eq. (27). Then

$$\dot{V}_1 \leqslant -\frac{l}{4}e^2 - \lambda_1\xi_1^2 + \xi_1 z.$$

Thus, letting

$$\begin{aligned} \alpha_2 = {} & \lambda_2 z + \xi_1 + \left(\frac{\partial\alpha_1}{\partial\xi_1}\xi_1 + \alpha_1 - l\right)(\chi + l\xi_1) + \left(\frac{2}{l} + \frac{5}{2\lambda_1}\phi_1^2 + \frac{5}{2\lambda_1}\varphi_0^2\right)\left(\frac{\partial\alpha_1}{\partial\xi_1}\xi_1 + \alpha_1\right)^2 z \\ & + \frac{5}{2\lambda_1}\left[\left(\frac{\partial\alpha_1}{\partial x_0}x_0\phi_0\right)^2 + \left(\frac{\partial\alpha_1}{\partial x_0}c_{02}u_0\right)^2 + \left(\frac{\partial\alpha_1}{\partial u_0}\varphi_0 u_0\right)^2\right]z, \end{aligned} \tag{28}$$

$$u = -\alpha_2,$$

and choosing the candidate Lyapunov function $V = V_1 + \frac{1}{2}z^2$ for the whole system (27) we have

$$\dot{V} \leqslant -\frac{l}{8}e^2 - \frac{\lambda_1}{2}\xi_1^2 - \lambda_2 z^2.$$

Hence the following result is obtained.

Theorem 6. *Under Assumptions* 2 *and* 3, *the uncertain system* (24) *can be globally* $\mathscr{K}$*-exponentially stabilized at the origin by*

$$u_0(t) = \begin{cases} -\lambda_0 x_0 - \dfrac{\phi_0(x_0)}{c_{01}} x_0 & \text{if } x_0(0) \neq 0, \\ \begin{cases} \beta - \dfrac{\phi_0(x_0)}{c_{01}} x_0 & \text{if } t < t_s(|x(0)|) \\ -\left(\lambda_0 + \dfrac{\phi_0(x_0)}{c_{01}}\right) x_0 & \text{if } t \geqslant t_s(|x(0)|), \end{cases} & \text{if } x_0(0) = 0, \end{cases} \tag{29}$$

and (28) *with the observer* (26), *where* β, δ, ε *and* λ_0 *be strictly positive real constants, and* $T_0(s) = \max_{\tau \in [-s,s]}\{\varepsilon + \phi_0(\tau)\}$ *and*

$$t_s(|x(0)|) = \min\left\{\delta, \frac{1}{2c_{02}T_0(c_{02}\beta|x(0)|)}, |x(0)|\right\}.$$

Remark 8. In fact, if $(x_0, x_1, \ldots, x_{n-2})$ is measurable then a dynamic output feedback can also be designed by the input-state scaling, backstepping technique and reduced-order observer for the chained system (1).

4. Example

Consider the following bilinear model, which is an approximation of a mobile robot with small angle measurement error [17,31],

$$\begin{cases} \dot{x}_l = \left(1 - \dfrac{\varepsilon^2}{2}\right) v, \\ \dot{y}_l = \theta_l v + \varepsilon v, \\ \dot{\theta}_l = \omega, \end{cases} \tag{30}$$

where x_l and y_l can be measured. As indicated in [17] system (30) can be transformed into

$$\begin{cases} \dot{x}_0 = \left(1 - \dfrac{\varepsilon^2}{2}\right) u_0, \\ \dot{x}_1 = x_2 u_0, \\ \dot{x}_2 = u \end{cases}$$

by the following transformation:

$$x_0 = x_l, \quad x_1 = y_l, \quad x_2 = \theta_l + \varepsilon, \quad u_0 = v, \quad u = \omega.$$

Introducing

$$\xi_1 = \frac{x_1}{u_0},$$
$$\xi_2 = x_2,$$

then

$$\begin{cases} \dot{x}_0 = \left(1 - \frac{\varepsilon^2}{2}\right)u_0, \\ \dot{\xi}_1 = \xi_2 - \xi_1\frac{\dot{u}_0}{u_0}, \\ \dot{\xi}_2 = u. \end{cases}$$

At first assume that the designed control u_0 satisfies $|\dot{u}_0/u_0| \leqslant \alpha$ where α is a positive constant. So the following controller can be designed.

Step 0: Recall that ξ_1 is measured and but not ξ_2. The following reduced-order observer is built up to reconstruct ξ_2 and thus x_2:

$$\dot{\chi} = u - l\chi - l^2\xi_1, \tag{31}$$

where $l > 0$ is a design parameter. Define $\hat{\xi}_2 = \chi + l\xi_1$ as an estimate of ξ_2 and $e = \xi_2 - \hat{\xi}_2$ as the estimation error. We have

$$\dot{e} = -le + l\xi_1\frac{\dot{u}_0}{u_0}.$$

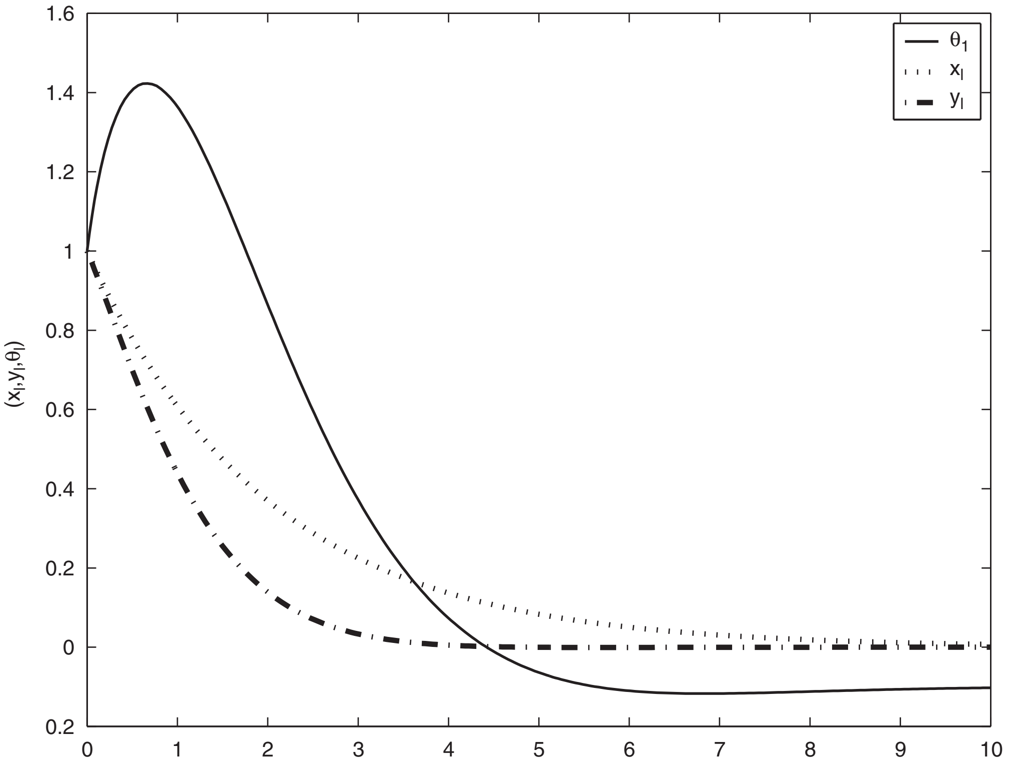

Fig. 1. The systems behaviour at initial condition $(x_l(0), y_l(0), \theta_l(0), \chi(0)) = (1, 1, 1, 3.5)$.

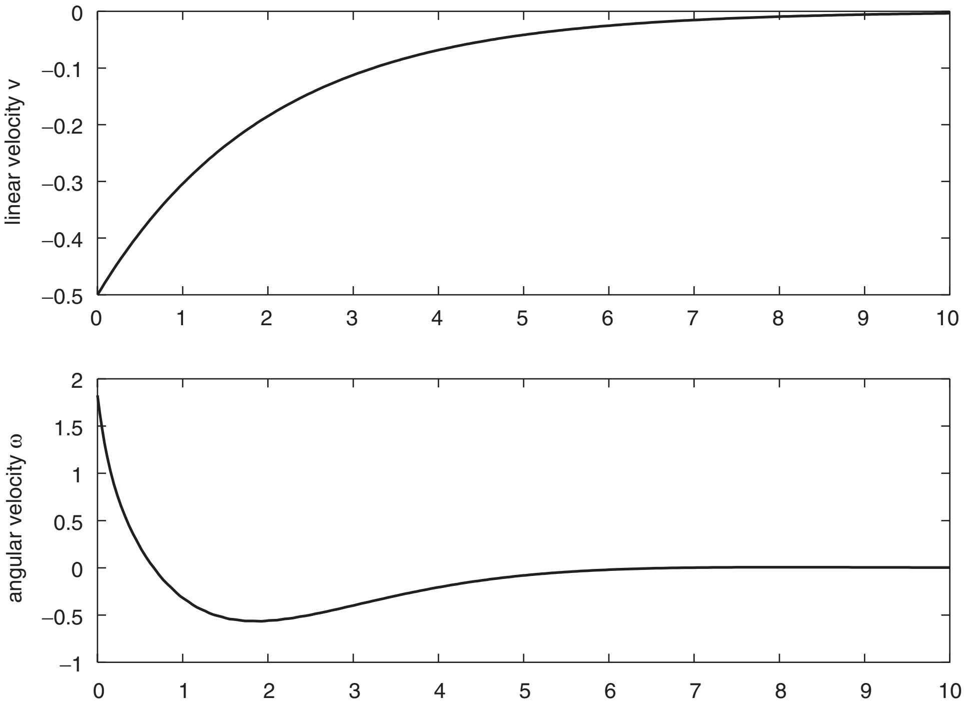

Fig. 2. The systems behaviour at initial condition $(x_l(0), y_l(0), \theta_l(0), \chi(0)) = (1, 1, 1, 3.5)$.

Then, noticing that $\xi_2 = \chi + l\xi_1 + e$, the output feedback design of u is based on the following controlled system:

$$\begin{cases} \dot{e} = -le + l\xi_1 \dfrac{\partial u_0}{\partial x_0}\left(1 - \dfrac{\varepsilon^2}{2}\right), \\ \dot{\xi}_1 = \chi + l\xi_1 + e - \xi_1 \dfrac{\dot{u}_0}{u_0}, \\ \dot{\chi} = u - l\chi - l^2\xi_1. \end{cases} \tag{32}$$

Step 1: Consider the (e, ξ_1)-subsystem of Eq. (32). Differentiating the quadratic function $W_1 = \frac{1}{2}e^2 + \frac{1}{2}\xi_1^2$ yields

$$\begin{aligned} \dot{W}_1 = & -le^2 + el\xi_1 \frac{\dot{u}_0}{u_0} + \xi_1\left(\chi + l\xi_1 + e - \xi_1 \frac{\dot{u}_0}{u_0}\right) \\ & \leqslant -\frac{l}{2}e^2 - \lambda_1\xi_1^2 + \xi_1 z_2, \end{aligned}$$

where

$$\begin{aligned} &\beta_1 = A\xi_1, \\ &A = \lambda_1 + \alpha + l + \frac{l}{2}(l\alpha + 1)^2, \\ &\chi = z_2 - \beta_1(\xi_1). \end{aligned}$$

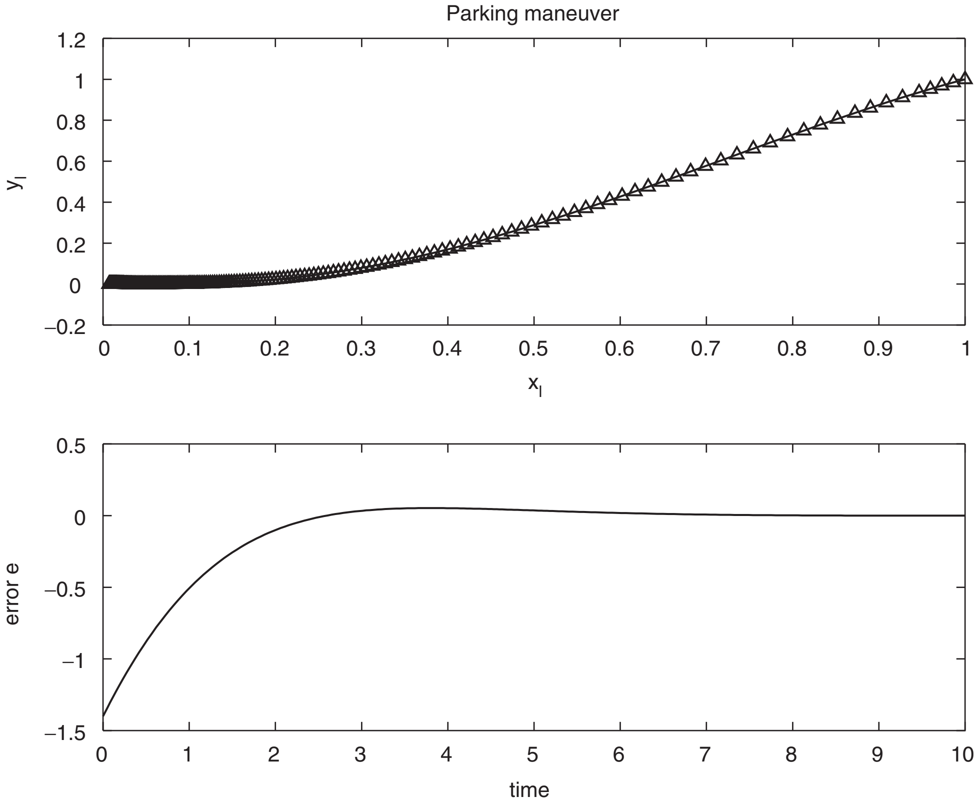

Fig. 3. The systems behaviour at initial condition $(x_l(0), y_l(0), \theta_l(0), \chi(0)) = (1, 1, 1, 3.5)$.

Step 2: Consider the (e, ξ_1, χ)-system of Eq. (32). The time derivative of the Lyapunov function $W_2 = W_1 + \frac{1}{2}z_2^2$ satisfies

$$
\begin{aligned}
\dot{W}_2 &\leqslant -\tfrac{l}{2}e^2 - \lambda_1\xi_1^2 + \xi_1 z_2 + z_2\left(u - l\chi - l^2\xi_1 + A\left(\chi + l\xi_1 + e - \xi_1\frac{\dot{u}_0}{u_0}\right)\right) \\
&\leqslant -\tfrac{l}{2}e^2 - \lambda_1\xi_1^2 + z_2\left(u + \xi_1 - l\chi - l^2\xi_1 + A\left(\chi + l\xi_1 + e - \xi_1\frac{\dot{u}_0}{u_0}\right)\right) \\
&\leqslant -\tfrac{l}{4}e^2 - (\lambda_1 - \tfrac{1}{2})\xi_1^2 - \lambda_2 z_2^2,
\end{aligned}
$$

where

$$
u = -(A - l)\chi - (Al + 1 - l^2)\xi_1 - \lambda_2 z_2 - A^2\left(\frac{\alpha^2}{2} + \frac{1}{l}\right)z_2,
$$

which is our controller expression of u in all cases.

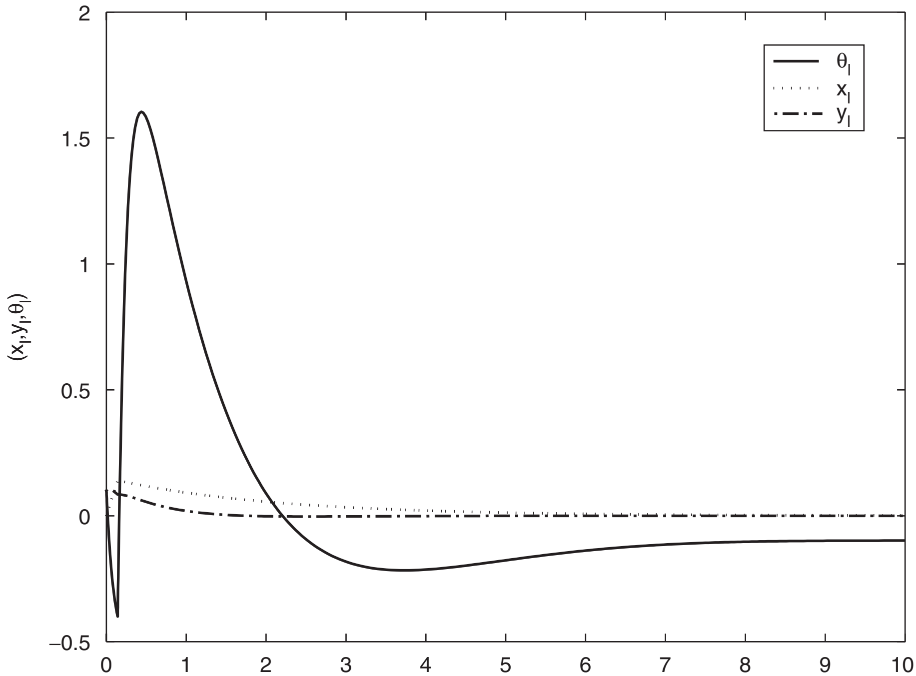

Fig. 4. The systems behaviour at initial condition $(x_l(0), y_l(0), \theta_l(0), \chi(0)) = (0, 0.1, 0.1, 0.5)$.

In the following a controller of u_0 can be designed according to the initial condition $x(0)$. Assume that $\varepsilon \in [-\varepsilon_{\max}, \varepsilon_{\max}]$ and $0 < \varepsilon_{\max} < \sqrt{2}$. Then $1 - \varepsilon_{\max}^2/2 \leqslant (1 - \varepsilon^2/2) \leqslant 1$.

- $x_0(0) = 0,\ x(0) \neq 0,$

$$u_0(t) = \begin{cases} \beta & \text{if } t < t_s(|x(0)|), \\ -\lambda_0 x_0 & \text{if } t \geqslant t_s(|x(0)|), \end{cases}$$

where $t_s = \min\{\delta, |x(0)|\},\ \delta > 0,\ \lambda_0 > 0,\ \beta > 0$. So

$$\dot{u}_0(t) = \begin{cases} 0 & \text{if } t < t_s(|x(0)|), \\ -\lambda_0\left(1 - \dfrac{\varepsilon^2}{2}\right)u_0 & \text{if } t > t_s(|x(0)|), \end{cases}$$

i.e.,

$$\left|\frac{\dot{u}_0}{u_0}\right| \leqslant \lambda_0 \triangleq \alpha.$$

- $(x_0(0), x(0)) = (0, 0),$

$$u_0 = 0,$$

$$u = 0.$$

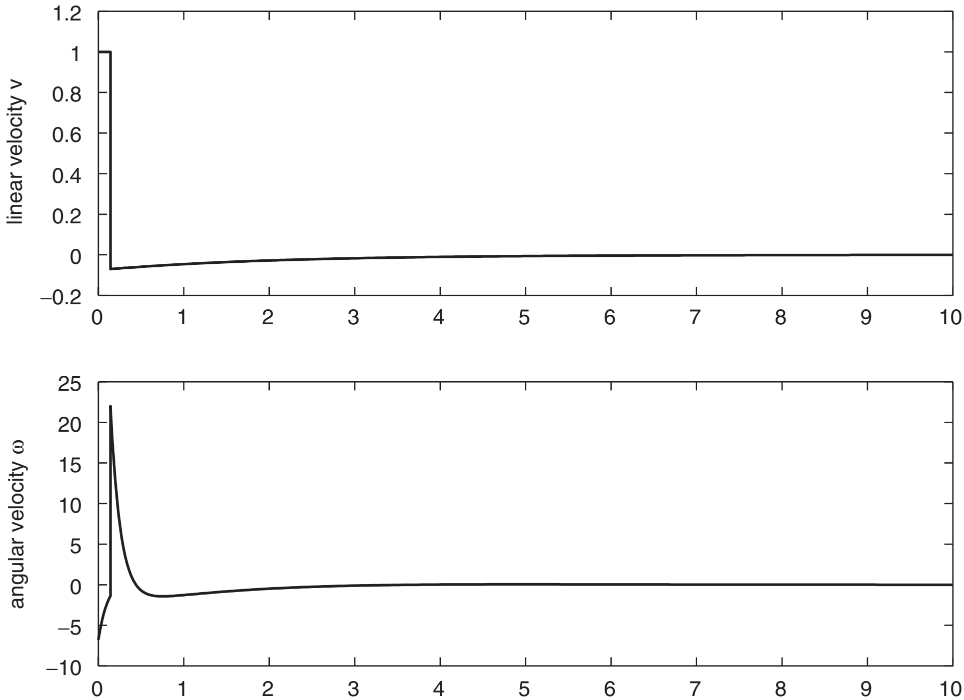

Fig. 5. The systems behaviour at initial condition $(x_l(0), y_l(0), \theta_l(0), \chi(0)) = (0, 0.1, 0.1, 0.5)$.

- $x_0(0) \neq 0$,

$$u_0 = -\lambda_0 x_0.$$

Remark 9. It is noted that in [17] the controller was dependent on the value of $\varepsilon_{\max}$, but it is not the case in our design.

Our simulations as shown in Figs. 1–6 are based on the following choice of design and system parameters [17]:

$$\lambda_0 = l = 0.5, \quad \lambda_1 = 0.6, \quad \lambda_2 = 1, \quad \varepsilon = 0.1, \quad \varepsilon_{\max} = 0.5, \quad \beta = 1, \quad \delta = 0.2.$$

In particular, suppose that the initial state of the system is unknown. Then the switching time is chosen as $t_s(0.1)$. As seen from the simulations, exponential rates of convergence are obtained for all signals and the control inputs. Due to the discontinuous state transformation, the initial transient value of the control input is relatively large.

5. Conclusions

In this paper a class of nonholonomic uncertain systems has been considered. Using input-state scaling and backstepping technique an output feedback controller has been proposed. Using a switching scheme dependent on the initial condition of the system the closed-loop system can be $\mathscr{K}$-exponentially stabilized. The properties of the discontinuous

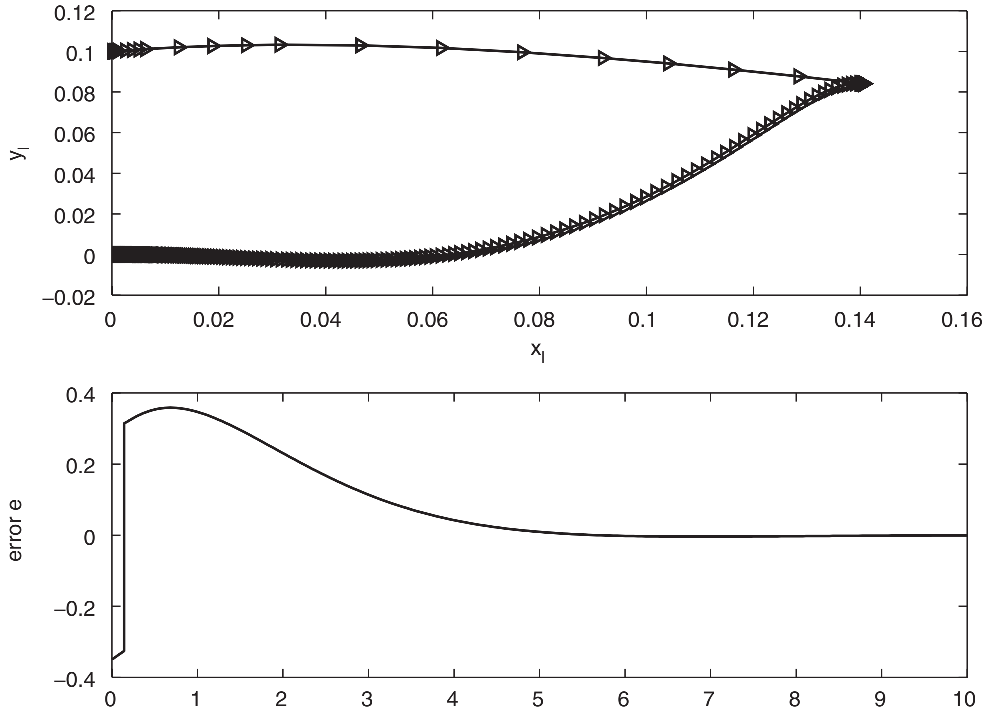

Fig. 6. The systems behaviour at initial condition $(x_l(0), y_l(0), \theta_l(0), \chi(0)) = (0, 0.1, 0.1, 0.5)$.

closed-loop system, in particular the existence of a unique solution, and the features of the control signals, in particular the boundedness, have been studied. The simulations results have demonstrated the effectiveness of the proposed control design tool.

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References

[1] W. Brockett, Asymptotic stability and feedback stabilization, in: R.W. Brockett, R.S. Millman, H.J. Sussman (Eds.), Differential Geometric Control Theory, 1983, pp. 181–191.

[2] I. Kolmanovsky, N.H. McClamroch, Developments in nonholonomic control problems, IEEE Control Systems Magazine 15 (6) (1995) 20–36.

[3] A. Astolfi, Discontinuous control of nonholonomic systems, Systems Control Lett. 27 (1996) 37–45.

[4] A. Astolfi, W. Schaufelberger, State and output feedback stabilization of multiple chained systems with discontinuous control, in: Proceedings of 35th IEEE Conference on Decision and Control, Kobe, Japan, December 1996, pp. 1443–1447.

[5] A. Bloch, M. Reyhanoglu, N.H. McClamroch, Control and stabilization of nonholonomic dynamic systems, IEEE Trans. Automat. Control 37 (1992) 1746–1757.
[6] C. Canudas de Wit, O.J. Sordalen, Exponential stabilization of mobile robots with nonholonomic constraints, IEEE Trans. Automat. Control 37 (1992) 1791–1797.
[7] M. Reyhanoglu, S. Cho, N.H. McClamroch, I. Kolmanovsky, Discontinuous feedback control of a planar rigid body with an underactuated degree of freedom, in: Proceedings of 37th IEEE Conference on Decision and Control, Tampa, FL, USA, December 1998, pp. 433–438.
[8] C. Rui, M. Reyhanoglu, I. Kolmanovsky, S. Cho, N.H. McClamroch, Nonsmooth stabilization of an underactuated unstable two degrees of freedom mechanical system, in: Proceedings of the 36th IEEE Conference on Decision and Control, San Diego, CA, USA, December 1997, pp. 3998–4003.
[9] M. Fliess, J. Levine, P. Martin, P. Rouchon, Flatness and defect of nonlinear systems: introductory theory and examples, Internat. J. Control 61 (1995) 1327–1361.
[10] Z.P. Jiang, Iterative design of time-varying stabilizers for multi-input systems in chained form, Systems Control Lett. 28 (1996) 255–262.
[11] I. Kolmanovsky, N.H. McClamroch, Hybrid feedback laws for a class of cascaded nonlinear control systems, IEEE Trans. Automat. Control 41 (1996) 1271–1282.
[12] R. M'Closkey, R. Murray, Exponential stabilization of driftless nonlinear control systems using homogeneous feedback, IEEE Trans. Automat. Control 42 (5) (1997) 614–628.
[13] R.M. Murray, S. Sastry, Nonholonomic motion planning: steering using sinusoids, IEEE Trans. Automat. Control 38 (5) (1993) 700–716.
[14] J.B. Pomet, Explicit design of time-varying stabilizing control laws for a class of controllable systems without drift, Systems Control Lett. 18 (1992) 147–158.
[15] O.J. Sordalen, O. Egeland, Exponential stabilization of nonholonomic chained systems, IEEE Trans. Automat. Control 40 (1995) 35–49.
[16] D. Tilbury, R.M. Murray, S. Sastry, Trajectory generation for the n-trailer problem using goursat normal form, IEEE Trans. Automat. Control 40 (1995) 802–819.
[17] Z.P. Jiang, Robust exponential regulation of nonholonomic systems with uncertainties, Automatica 36 (2000) 189–209.
[18] K.D. Do, J. Pan, Adaptive global stabilization of nonholonomic systems with strong nonlinear drifts, Systems Control Lett. 46 (2002) 195–205.
[19] N. Marchand, M. Alamir, Discontinuous exponential stabilization of chained form systems, Automatica 39 (2003) 343–348.
[20] W. Huo, S.S. Ge, Exponential stabilization of non-holonomic systems: an ENI approach, Internat. J. Control 74 (15) (2001) 1492–1500.
[21] S.S. Ge, Z. Sun, T.H. Lee, M.W. Spong, Feedback linearization and stabilization of second-order non-holonomic chained systems, Internat. J. Control 74 (14) (2001) 1383–1392.
[22] J. Luo, P. Tsiotras, Exponentially convergent control laws for nonholonomic systems in power form, Systems Control Lett. 35 (1998) 87–95.
[23] Z. Xi, G. Feng, Z. Jiang, D. Cheng, A switching algorithm for global exponential stabilization of uncertain chained systems, IEEE Trans. Automat. Control 48 (10) (2003) 1793–1798.
[24] A. Astolfi, W. Schaufelberger, State and output feedback stabilization of multiple chained systems with discontinuous control, Systems Control Lett. 32 (1997) 49–56.
[25] S.S. Ge, Z. Wang, T.H. Lee, Adaptive stabilization of uncertain nonholonomic systems by state and output feedback, Automatica 39 (2003) 1451–1460.
[26] M. Krstić, I. Kanellakopoulos, P.V. Kokotović, Nonlinear and Adaptive Control Design, Wiley, New York, 1995.
[27] E.D. Sontag, Comments on integral variants of ISS, Systems Control Lett. 34 (1998) 93–100.
[28] L. Praly, I. Kanellakopoulos, Output feedback asymptotic stabilization for triangular systems linear in the unmeasured state components, in: Proceedings of the 39th IEEE Conference on Decision and Control, Sydney, Australia, December 2000, pp. 2466–2471.
[29] M. Ikeda, H. Maeda, S. Kodama, Stabilization of linear systems, SIAM J. Control 10 (4) (1972) 716–729.
[30] G. Besancon, J. De Leon-Morales, O. Huerta-Guevara, On adaptive observers for state affine systems and application to synchronous machines, in: Proceedings of the 42nd IEEE Conference on Decision and Control, Maui, Hawaii USA, December 2003, pp. 2192–2197.
[31] P. Morin, J.B. Pomet, C. Samson, Developments in time-varying feedback stabilization of nonlinear systems, Preprints of Nonlinear control systems design symposium (NOLCOS'98), vol. 1/2, 1998, pp. 587–594.