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Nonlinear systems possessing linear symmetry

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SUMMARY

This paper tackles linear symmetries of control systems. Precisely, the symmetry of affine nonlinear systems under the action of a sub-group of general linear group $GL(n,\mathbb{R})$. First of all, the structure of state space (briefly, ss) symmetry group and its Lie algebra for a given system is investigated. Secondly, the structure of systems, which are ss-symmetric under rotations, is revealed. Thirdly, a complete classification of ss-symmetric planar systems is presented. It is shown that for planar systems there are only four classes of systems which are ss-symmetric with respect to four linear groups. Fourthly, a set of algebraic equations are presented, whose solutions provide the Lie algebra of the largest connected ss-symmetry group. Finally, some controllability properties of systems with ss-symmetry group are studied. As an auxiliary tool for computation, the concept and some properties of semi-tensor product of matrices are included. Copyright © 2006 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Symmetry of dynamic systems under a group action is an important topic in both physics and mathematics [1–4], because many systems in the nature do possess symmetry, and because taking symmetry into consideration may simplify the system investigation tremendously. Symmetry of control systems has also been investigated by many authors. For instance, symmetric structure of control systems has been proposed and studied by Grizzle and Marcus [5] and Xie *et al*. [6], controllability of symmetric control systems was investigated by Zhao and Zhang [7], Respondek and Tall [8, 9] gave a complete description of symmetries around equilibria of single input systems, the application of symmetry in optimal control problems has

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been studied by Jurdjevic [10] and Koon and Marsden [11], the symmetry of feedback linearizable systems has been investigated by Gardner and Shadwick [12], etc.

The symmetry of dynamic systems considered in the paper is related to the action of a Lie group on $\mathbb{R}^n$. Let G be a Lie group. G is an action on $\mathbb{R}^n$ (or an open subset of $\mathbb{R}^n$), if there exists a mapping $\theta : G \times \mathbb{R}^n \to \mathbb{R}^n$ such that (i) $\theta(e)x = x$, $\forall x \in \mathbb{R}^n$; (ii) for any $\alpha_1, \alpha_2 \in G$ we have $\theta(\alpha_1\alpha_2)x = \theta(\alpha_1)(\theta(\alpha_2)x)$.

For a control system we define two kinds of symmetries as follows:

Definition 1.1
Given an analytic control system

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x)u_i, \quad x \in \mathbb{R}^n \tag{1}$$

where $f_i(x)$, $i = 0, \ldots, m$ are analytic vector fields. Let G be a Lie group acting on $\mathbb{R}^n$ (or an open subset $M \subset \mathbb{R}^n$).

(i) System (1) is said to be ss-symmetric with respect to G (or has an state space (ss)-symmetry group G) if for each $\alpha \in G$

$$\theta(\alpha)_* f_i(x) = f_i(\theta(\alpha)x), \quad i = 0, \ldots, m$$

where $\theta(\alpha)_*$ is the induced mapping of $\theta(\alpha)$, which is a diffeomorphism on $\mathbb{R}^n$. If $f_i(x)$ satisfies the above equation (for a given α), $f_i(x)$ is said to be $\theta(\alpha)$ invariant.

(ii) System (1) is said to be symmetric with respect to G (or has a symmetry group G) if for each $\alpha \in G$

$$\theta(\alpha)_* \mathscr{A} = \mathscr{A}$$

where

$$\mathscr{A} = \left\{ f(x) + \sum_{i=1}^{m} g_i(x)u_i \,\middle|\, u \in \mathbb{R}^m \right\}$$

If G is a sub-group of the general linear group, i.e. $G < GL(n, \mathbb{R})$, then system (1) is said to be linearly (ss-) symmetric with respect to G (or has a linear (ss-) symmetry group G.)

Remark 1.2

1. Definition (i) is proposed and used by Grizzle and Marcus [5] and Zhao and Zhang [7], (ii) is from Respondek and Tall [9]. It is easy to see that ss-symmetry is a special case of symmetry.
2. In this paper we consider linear symmetry(except Section 6), so the word 'linear' is omitted (except Section 6).

In this paper linear ss-symmetries of nonlinear systems are investigated. The rest of the paper is organized as follows. Section 2 investigates the general structure of ss-symmetric group and its Lie algebra for a given system. In Section 3, we consider the ss-symmetry under rotations. General structure of such symmetric systems is revealed. Section 4 studied the ss-symmetry of planar systems. Four classes of symmetric systems with their corresponding symmetry groups are obtained, which cover all possible planar ss-symmetric systems. Section 5 considers the

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linear ss-symmetry group for a given system. A system of linear algebraic equations are constructed. Its solutions provide the Lie algebra of largest connected linear ss-symmetry group. As an application, some controllability properties of ss-symmetric systems are studied in Section 6.

2. STRUCTURE OF SYMMETRY GROUP AND ITS LIE ALGEBRA

In this section we consider ss-symmetry of system (1). For ss-symmetry the control is not essential. So we may start with a free analytic system as

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \tag{2}$$

Using Taylor series expansion and denoting by $M_{p\times q}$ the set of real $p \times q$ matrices, we can express f as

$$f(x) = f_0 + f_1 x + f_2 x^2 + \cdots \tag{3}$$

where $f_i \in M_{n\times n^i}$, $x = (x_1, \ldots, x_n)^{\mathrm{T}}$, and all the products are left semi-tensor product, which is introduced in Appendix A.1.

Let $\alpha \in GL(n, \mathbb{R})$. $\theta_\alpha : x \mapsto y = \alpha x$. Then for f to be invariant under θ_α we need

$$(\theta_\alpha)_*(f(x)) = \alpha f(x) = \alpha f(\alpha^{-1} y) = f(y) \quad \forall y \in \mathbb{R}^n \tag{4}$$

It is equivalent to

$$\alpha f(x) = f(\alpha x) \quad \forall x \in \mathbb{R}^n \tag{5}$$

Now since $(\theta_\alpha)_*$ does not change the degree of a homogeneous vector field, if (4) holds for f, it should also hold for each homogeneous component of f. That is,

$$\alpha f_k x^k = f_k(\alpha x)^k, \quad \forall x \in \mathbb{R}^n; \quad k = 0, 1, \ldots \tag{6}$$

Using the definition of semi-tensor products and formula (A6), we have

$$\begin{aligned}
(\alpha x)^k &= \underbrace{\alpha x \ltimes \alpha x \ltimes \cdots \ltimes \alpha x}_{k} \\
&= \alpha(I_n \otimes \alpha)x^2 \underbrace{\alpha x \ltimes \alpha x \ltimes \cdots \ltimes \alpha x}_{k-2} \\
&= \cdots \\
&= (\alpha)(I_n \otimes \alpha)(I_{n^2} \otimes \alpha)\ldots(I_{n^{k-1}} \otimes \alpha)x^k \\
&= (\alpha \otimes I_n)(I_n \otimes \alpha)[(I_{n^2} \otimes \alpha)\ldots(I_{n^{k-1}} \otimes \alpha)]x^k \\
&= (\alpha \otimes \alpha)[(I_{n^2} \otimes \alpha)\ldots(I_{n^{k-1}} \otimes \alpha)]x^k \\
&= \cdots \\
&= [\underbrace{\alpha \otimes \alpha \otimes \cdots \otimes \alpha}_{k}]x^k := \alpha^{\otimes k} x^k
\end{aligned}$$

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It is clear that system (2) is α-invariant iff

$$\alpha f_k x^k = f_k \alpha^{\otimes k} x^k, \quad k = 0, 1, \ldots \tag{7}$$

Since x^k is a generator of kth degree homogeneous polynomials, to avoid redundancy we use a (conventional) basis. The basis, denoted by $x_{(k)}$, is the set as

$$x_{(k)} = \left\{ \prod_{i=1}^{n} x_i^{t_i} \,\middle|\, \sum_{i=1}^{n} t_i = k \right\}$$

$x_{(k)}$ is also used as a matrix. Then the elements in $x_{(k)}$ are arranged in alphabetic order. That is, let $b_1 = x_1^{p_1} \cdots x_n^{p_n}, b_2 = x_1^{q_1} \cdots x_n^{q_n}$. Define $b_1 \prec b_2$ if $p_s = q_s, s = 1, \ldots, t$ and $p_{t+1} > q_{t+1}$ for some $0 \leqslant t < n$. So when $x_{(k)}$ is considered as a matrix, it is expressed as $x_{(k)} = (b_1, \ldots, b_d)^{\mathrm{T}}$.

It is easy to verify that for $x \in \mathbb{R}^n$ the dimension of the vector space of kth homogeneous polynomials is

$$d := r_n^k = \frac{(n+k-1)!}{(n-1)!k!} \tag{8}$$

We use a simple example to describe the generator x^k and basis $x_{(k)}$.

Example 2.1
Assume $n = 2$ and $k = 3$. Then $x = (x_1, x_2)^{\mathrm{T}}$. Moreover, $d = (2+3-1)!/3! = 4$.

$$x^3 = (x_1^3 \quad x_1^2 x_2 \quad x_1 x_2 x_1 \quad x_1 x_2^2 \quad x_2 x_1^2 \quad x_2 x_1 x_2 \quad x_2^2 x_1 \quad x_2^3)^{\mathrm{T}}$$

and

$$x_{(3)} = (x_1^3 \quad x_1^2 x_2 \quad x_1 x_2^2 \quad x_2^3)^{\mathrm{T}}$$

Then we can construct a matrix $T_N(n,k) \in M_{n^k \times r_n^k}$ such that [13]

$$x^k = T_N(n,k) x_{(k)} \tag{9}$$

Since the coefficients for a basis are unique, from (7) we have

Proposition 2.2
System (2) is α-invariant, iff

$$\alpha f_k T_N(n,k) = f_k \alpha^{\otimes k} T_N(n,k), \quad k = 0, 1, 2, \ldots \tag{10}$$

Clearly, a sufficient condition for f to be α-invariant is that

$$\alpha f_k = f_k \alpha^{\otimes k}, \quad k = 0, 1, 2, \ldots \tag{11}$$

Using Proposition 2.2, we can reach the following result immediately.

Proposition 2.3
Let H be a subset of $GL(n, \mathbb{R})$, which consists of all α satisfying (10). Then H is a group.

Equation (10) provides a formula for solving α. But it is, in general, very difficult to solve such an infinite set of algebraic equations. We have to find an alternative easy way to solve the problem. We turn to Lie algebra approach.

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Denote by $g(G)$ the Lie algebra of G, which is a Lie sub-algebra of $gl(n, \mathbb{R})$. We refer to [14, 15] for some other related concepts, notations and terminologies used in the sequel.

We prove following lemma, which is fundamental.

Lemma 2.4

Let $G < GL(n, \mathbb{R})$ be a connected sub-group. System (2) (or briefly, vector field $f(x)$) has symmetry group G, iff

$$\mathrm{ad}_{Vx} f(x) = 0 \quad \forall V \in g(G) \tag{12}$$

where Vx is a linear vector field.

Proof

Let M be a given manifold. For a vector field $X \in V(M)$, we denote its integral curve with initial condition $x(0) = x$ by $\phi_X^t(x)$. Then it is well known that for any $Y \in V(M)$

$$(\phi_X^t)_* Y(x) = Y(\phi_X^t(x))$$

iff $[X, Y] = 0$ [16]. Now the integral curve of $Vx \in V(\mathbb{R}^n)$ is $\mathrm{e}^{Vt}x$. Hence

$$(\mathrm{e}^{Vt})_* f(x) = \mathrm{e}^{Vt} f(x(z)) = \mathrm{e}^{Vt} f(\mathrm{e}^{-Vt}z) = f(z)$$

where $z = \mathrm{e}^{Vt}x$. Equivalently

$$\mathrm{e}^{Vt} f(x) = f(\mathrm{e}^{Vt}x)$$

iff $\mathrm{ad}_{Vx} f(x) = 0$. □

Note that in Lemma 2.4 and thereafter discrete groups have been excluded.

Now consider system (1). Using Taylor series expansion to each f_j, $j = 0, 1, \ldots, m$, we denote

$$f_j = \sum_{k=0}^{\infty} f_k^j x^k, \quad i = 0, \ldots, m$$

Since $deg(\mathrm{ad}_{Vx} f_k^j x^k) = k$, that is, ad_{Vx} doesn't change the degree of each term, we can define

$$\mathscr{V}_k^j = \{V \in gl(n, \mathbb{R}) | \mathrm{ad}_{Vx} f_k^j x^k = 0\}$$

Using Jacobi identity, it is easy to see that $\mathscr{V}_k^j$ is a Lie algebra. According to Lemma 2.4, if $G < GL(n, \mathbb{R})$ is the largest ss-symmetry group of system (1), then its Lie algebra is

$$g(G) := \mathscr{V} = \bigcap_{j=0}^{m} \bigcap_{k=0}^{\infty} \mathscr{V}_k^j$$

Then the corresponding connected group $G(g)$, which has g as its Lie algebra, can be constructed as

$$G(g) = \left\{ \prod_{i=1}^{k} \exp(t_i V_i) \middle| V_i \in \mathscr{V}, k < \infty \right\} \tag{13}$$

Summarizing them yields the following result.

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Theorem 2.5
System (1) has a unique largest connected ss-symmetry group $G < GL(n, \mathbb{R})$, which has its Lie algebra as

$$g(G) = \bigcap_{j=0}^{m} \bigcap_{k=0}^{\infty} \mathscr{V}_k^j \tag{14}$$

Finally, assume a Lie algebra, $g \subset gl(n, \mathbb{R})$ is given, we give an algebraic condition for the set of vector fields, $f(x)$, which have $G(g)$ as their ss-symmetry group.

Denote by $\mathscr{H}_n^k$ the set of vector fields with components of kth degree homogeneous polynomials. It is easy to see that $\mathscr{H}_n^k$ is a linear space over $\mathbb{R}$ and for any $V \in gl(n, \mathbb{R})$ the mapping $\mathrm{ad}_{Vx} : \mathscr{H}_n^k \to \mathscr{H}_n^k$ is a linear mapping. We refer to [14] for some details of $\mathscr{H}_n^k$. Using (8), dimension of $\mathscr{H}_n^k$, denoted by d_n^k, is $d_n^k = n(n+k-1)!/(n-1)!k!$. Then a basis of $\mathscr{H}_n^k$, denoted by the columns of matrix H_n^k, can be obtained as

$$H_n^k = I_n \otimes x_{(k)}^{\mathrm{T}} \tag{15}$$

It will be called the conventional basis of $\mathscr{H}_n^k$. In the sequel, the adjoint representation of ad_{Vx} means the representation with respect to this conventional basis.

Now we can define a mapping from $gl(n, \mathbb{R})$ to the adjoint representations of the Lie derivative, called the adjoint mapping, as

Definition 2.6
The adjoint mapping is defined as the following:

$$\Phi_n^k : gl(n, \mathbb{R}) \ni V \to \Phi_n^k(V) \in gl(d_n^k, \mathbb{R})$$

where $\Phi_n^k(V)$ is the adjoint representation of $\mathrm{ad}_{Vx} : \mathscr{H}_n^k \to \mathscr{H}_n^k$ (with respect to the conventional basis).

We give an example to illustrate it:

Example 2.7
Let $n = 2$ and

$$V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{16}$$

Then a straightforward computation shows that

$$\Phi_2^k(V) = \begin{pmatrix} A & -I \\ 0 & A \end{pmatrix} \tag{17}$$

where

$$A = \begin{pmatrix} 0 & & & & \\ k & 0 & & & \\ & k-1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix} \tag{18}$$

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The following lemma is an immediate consequence of Lemma 2.4 and the definition of Φ_n^k.

Proposition 2.8
Let $G \in GL(n, \mathbb{R})$ be a one-dimensional connected sub-group, and $V \in g(G)$. A vector field $f(x)$ with components of kth degree homogeneous polynomials is G-invariant, if and only if, $f(x) \in (\Phi_n^k(V))$.

Example 2.9
Recall Example 2.7 again. Let's consider the $(\Phi_n^k(V))$ where V is given in (16). Using (17), we have to solve the following:

$$\begin{pmatrix} A & -I \\ 0 & A \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Then we have

$$Y = AX$$
$$A^2X = 0$$

Using (18)

$$A^2 = \begin{pmatrix} 0 & & & & & \\ 0 & 0 & & & & \\ k(k-1) & 0 & & & & \\ & \ddots & \ddots & & & \\ 0 & & 2\times 1 & 0 & 0 \end{pmatrix}$$

So $X = (0, \ldots, a, b)^{\mathrm{T}}$, $Y = AX = (0, \ldots, 0, a)^{\mathrm{T}}$, where a and b are any two real numbers. Recall that $(\mathrm{col}(X), \mathrm{col}(Y))^{\mathrm{T}}$ is the coefficient with respect to conventional basis of $\mathscr{H}_2^k$, it follows that

$$f(x) = (ax_1x_2^{k-1} + bx_2^k, ax_2^k)^{\mathrm{T}}, \quad k \geqslant 1 \tag{19}$$

According to Proposition 2.8, we conclude that such a vector field $f(x)$ has a one-dimensional symmetry group G as

$$G = \exp\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t\right] \tag{20}$$

3. SYMMETRY UNDER ROTATION

This section considers ss-symmetry under rotations. The motivation is from the following result.

Consider system (1) and assume $n = 2$. Then the following result answers when it has ss-symmetry group $SO(2, \mathbb{R})$.

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Theorem 3.1 (Xie et al. [6])
When $n = 2$ system (1) has ss-symmetry group $SO(2, \mathbb{R})$, iff

$$f_j(x) = \sum_{i=0}^{\infty} (x_1^2 + x_2^2)^i \begin{pmatrix} a_i^j & b_i^j \\ -b_i^j & a_i^j \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad a_i^j, b_i^j \in \mathbb{R}, \ j = 0, \ldots, m \tag{21}$$

We consider when system (1) has an ss-symmetry group $SO(n, \mathbb{R})$. The problem discussed is a generalization of [6]. Our necessary and sufficient condition is as follows.

Theorem 3.2
System (1) with $n \geqslant 3$ has an ss-symmetry group $G = SO(n, \mathbb{R})$, iff

$$f_j(x) = \sum_{i=0}^{\infty} a_i^j \|x\|^{2i} x, \quad a_i^j \in \mathbb{R}, \ j = 0, 1, \ldots, m \tag{22}$$

(The proof is in Appendix A.2.)

Remark 3.3
Comparing Theorem 3.2 with Theorem 3.1, one sees that for $n = 2$ and $n \geqslant 3$, the corresponding $f(x)$ are quite different. An intuitive reason may be found from the structures of their Lie algebras. The centre of $o(2, \mathbb{R})$ is

$$Z(o(2, \mathbb{R})) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$$

while the centre of $o(n, \mathbb{R})$, $n \geqslant 3$ is

$$Z(o(n, \mathbb{R})) = \{ r I_n | r \in \mathbb{R} \}$$

They are quite different. When $n \geqslant 3$ the $o(n, \mathbb{R})$ does not have non-trivial centre. Roughly speaking, there is no freedom for 'swap'. For reader's convenience, recall that the centre Z of a Lie algebra L is [15]

$$Z = \{ z \in L | \, [z, l] = 0, \forall l \in L \}$$

4. SYMMETRY OF PLANAR SYSTEMS

This section considers the ss-symmetry of planar systems. The following main result characterizes all the possible ss-symmetries.

Theorem 4.1
1. Assume system (1) with $n = 2$ has a connected ss-symmetry group $G < GL(2, \mathbb{R})$. Then G is conjugated to one of the following four groups:

$$G_1 = \left\{ \exp \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} t \middle| t \in \mathbb{R} \right\} \tag{23}$$

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with

$$\frac{\lambda_2}{\lambda_1} = \frac{p}{q}, \quad q > 0, \;\; p \leqslant 0$$

where p and q are two integers, and if $p = 0$, we set $\lambda_2 = 0$.

$$G_2 = \left\{ \exp\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t \,\middle|\, t \in \mathbb{R} \right\} \tag{24}$$

$$G_3 = \left\{ \exp\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} t \,\middle|\, t \in \mathbb{R} \right\} = SO(2, \mathbb{R}) \tag{25}$$

$$G_4 = \left\{ \prod_{i<\infty} \exp(A_i t_i) \,\middle|\, A_i = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \text{or } A_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \;\; t_i \in \mathbb{R} \right\} \tag{26}$$

2. Assume system (1) with $n = 2$ is ss-symmetric with respect to $G = TG_iT^{-1}$, for some $T \in GL(2, \mathbb{R})$, then (1) satisfies that

$$f_j = \sum_{n=0}^{\infty} a_n^j p_n^i (T^{-1}y) T B_n^i T^{-1} y, \quad j = 0, \ldots, m, \quad i = 1, 2, 3, 4 \tag{27}$$

where

$$p_n^1(x) = x_1^{-np} x_2^{nq}, \quad B_n^1 = \begin{pmatrix} \alpha_n & 0 \\ 0 & \beta_n \end{pmatrix}$$

$$p_n^2(x) = x_2^n, \quad B_n^2 = \begin{pmatrix} \alpha_n & \beta_n \\ 0 & \alpha_n \end{pmatrix}$$

$$p_n^3(x) = (x_1^2 + x_2^2)^n, \quad B_n^3 = \begin{pmatrix} \alpha_n & \beta_n \\ -\beta_n & \alpha_n \end{pmatrix}$$

$$p_n^4(x) = x_2^n, \quad B_n^4 = I$$

(The proof is in Appendix A.3.)

Remark 4.2
If system (1) satisfies (27), then it is a straightforward verification to show that it has the corresponding ss-symmetry group TG_iT^{-1}. So Theorem 4.1 gives a complete description for all planar ss-symmetric systems and their ss-symmetry groups.

5. LARGEST SS-SYMMETRY GROUP

In this section we will find the largest connected ss-symmetry group for a given system. We need some preparations.

Given a matrix $A(x) \in M_{p\times q}$ with smooth function entries $a_{i,j}(x)$ $(x \in \mathbb{R}^n)$. We define the differential of $A(x)$, denoted by $DA(x)$, as a $p \times nq$ matrix, obtained by replacing $a_{i,j}$ by its differential $(\partial a_{i,j}/\partial x_1, \ldots, \partial a_{i,j}/\partial x_n)$. The higher-order differentials can be defined recursively as

$$D^{k+1}A(x) = D(D^k A(x)), \quad k \geqslant 1$$

The advantage of this notation can be seen from the following observation: for Taylor series expression (3), the coefficients can be obtained as

$$f_k = \frac{1}{k!} D^k f(0), \quad k = 0, 1, \ldots$$

Given a matrix $A = (a_{ij}) \in M_{m\times n}$, its row staking form, $V_r(A)$, and column staking form, $V_c(A)$, are defined as

$$V_r(A) = (a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, \ldots, a_{mn})^T$$

$$V_c(A) = (a_{11}, a_{21}, \ldots, a_{m1}, a_{12}, \ldots, a_{mn})^T$$

Using swap matrix $W_{[m,n]}$, (see Appendix A.4 for swap matrix) we define two matrices as

$$\Psi_k = \sum_{s=0}^{k} I_{n^s} \otimes W_{[n^{k-s},n]}$$

$$E_k^n := I_{n^{k-1}} \otimes W_{[n^{k-1},n]} \ltimes V_c(I_{n^{k-1}})$$

Then we have

Theorem 5.1
Assume system (1) has an ss-symmetry group G with its Lie algebra $g(G)$. Then $\alpha \in g(G)$, iff $\xi = V_c(\alpha)$ is the solution of the following linear algebraic equations.

$$\begin{aligned} &([T_N^T(n,k) \otimes (f_k^j \Phi_{k-1})]E_k^n - [T_N^T(n,k)(f_k^j)^T] \otimes I_n)\xi = 0 \\ &k = 0, 1, 2, \ldots, \quad j = 0, 1, \ldots, m \end{aligned} \tag{28}$$

We refer to (9) for the matrix $T_N(n,k)$.

(See Appendix A.5 for the proof of Theorem 5.1.)

Theorem 5.1 provides a numerical method to calculate the largest connected ss-symmetry group for system (1).

Example 5.2
Consider the following system:

$$\dot{x} = f(x) = f_3 x^3, \quad x \in \mathbb{R}^3 \tag{29}$$

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where $f_3 = (f_{ij}) \in M_{3\times 27}$. Let $C = (c_{ij}) \in M_{4\times 2}$ be a parameter set. We set the coefficient matrix as

$$f_{1,2} = f_{1,4} = f_{1,10} = c_{1,1} \quad f_{1,3} = f_{1,7} = f_{1,19} = c_{1,2}$$

$$f_{2,6} = f_{2,8} = f_{2,12} = f_{2,16} = f_{2,20} = f_{2,22} = c_{2,1} \quad f_{3,6} = f_{3,8} = f_{3,12} = f_{3,16} = f_{3,20} = f_{3,22} = c_{2,2}$$

$$f_{2,5} = f_{2,11} = f_{2,13} = c_{3,1} \quad f_{3,5} = f_{3,11} = f_{3,13} = c_{3,2}$$

$$f_{2,9} = f_{2,21} = f_{2,25} = c_{4,1} \quad f_{3,9} = f_{3,21} = f_{3,25} = c_{4,2}$$

$$f_{i,j} = 0 \quad \text{for other} \quad (i,j)$$

A careful computation shows that such a group of parameters assure the existence of non-trivial symmetric group.

According to Theorem 5.1, we can construct the matrix

$$S_3^3 = T_N^{\mathrm{T}}(3,3) \otimes (f_3\Psi_2)E_2^3 - (T_N^{\mathrm{T}}(3,3)f_3^{\mathrm{T}}) \otimes I_3$$

and we have only to solve ξ for $S_3^3\xi = 0$.

Case 1: Let C be a set of randomly chosen parameters. Particularly, if we choose $c_{12} = 2$ and $c_{21} = c_{22} = c_{31} = c_{32} = 3$, and the other $c_{ij} = 1$, then a computation via computer shows that S_3^3 is an 30×9 matrix. To save space, we listed its non-zero entries only

$s_{1,2} = 3$	$s_{1,3} = 6$	$s_{4,1} = 3$	$s_{4,5} = 3$	$s_{4,6} = 6$	$s_{5,2} = 3$	$s_{5,3} = 18$	$s_{6,2} = 6$
$s_{6,3} = 15$	$s_{7,1} = 6$	$s_{7,8} = 3$	$s_{7,9} = 6$	$s_{8,2} = 12$	$s_{8,3} = 6$	$s_{9,2} = 18$	$s_{10,4} = 3$
$s_{10,7} = -3$	$s_{11,1} = 3$	$s_{11,5} = 3$	$s_{11,6} = 18$	$s_{11,8} = -3$	$s_{12,1} = 3$	$s_{12,5} = 6$	$s_{12,6} = 15$
$s_{12,9} = -3$	$s_{13,4} = -6$	$s_{13,7} = -12$	$s_{14,1} = 18$	$s_{14,6} = 6$	$s_{14,8} = -12$	$s_{14,9} = 18$	$s_{15,1} = 18$
$s_{15,5} = 18$	$s_{15,6} = -12$	$s_{15,8} = 6$	$s_{16,4} = -3$	$s_{16,7} = 9$	$s_{17,1} = 3$	$s_{17,5} = -3$	$s_{17,8} = 15$
$s_{17,9} = 6$	$s_{18,1} = 3$	$s_{18,6} = -3$	$s_{18,8} = 18$	$s_{18,9} = 3$	$s_{20,4} = 3$	$s_{21,4} = 3$	$s_{23,4} = 18$
$s_{23,7} = 3$	$s_{24,4} = 18$	$s_{24,7} = 3$	$s_{26,4} = 3$	$s_{26,7} = 18$	$s_{27,4} = 3$	$s_{27,7} = 18$	$s_{29,7} = 3$
$s_{30,7} = 3$							

The non-trivial solution is

$$\xi = (1\ 0\ 0\ 0\ -1\ 0\ 0\ 0\ -1)^{\mathrm{T}}$$

A program shows for random C this ξ is always the solution.

That is, the largest connected invariant linear group of the system (28) with above parameters is

$$G_{\mathrm{r}} = \left\{ \exp\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} t \,\middle|\, t \in \mathbb{R} \right\}$$

Case 2: Set $c_{ij} = 1$, $\forall i, j$. Then the 30×9 matrix S_3^3 has non-zero entries as

$s_{1,2} = 3$ $s_{1,3} = 3$ $s_{4,1} = 3$ $s_{4,5} = 3$ $s_{4,6} = 3$ $s_{5,2} = 3$ $s_{5,3} = 6$ $s_{6,2} = 6$
$s_{6,3} = 3$ $s_{7,1} = 3$ $s_{7,8} = 3$ $s_{7,9} = 3$ $s_{8,2} = 3$ $s_{8,3} = 6$ $s_{9,2} = 6$ $s_{9,3} = 3$
$s_{10,4} = 3$ $s_{10,7} = -3$ $s_{11,1} = 3$ $s_{11,5} = 3$ $s_{11,6} = 6$ $s_{11,8} = -3$ $s_{12,1} = 3$ $s_{12,5} = 6$
$s_{12,6} = 3$ $s_{12,9} = -3$ $s_{14,1} = 6$ $s_{14,6} = 6$ $s_{14,9} = 6$ $s_{15,1} = 6$ $s_{15,5} = 6$ $s_{15,8} = 6$
$s_{16,4} = -3$ $s_{16,7} = 3$ $s_{17,1} = 3$ $s_{17,5} = -3$ $s_{17,8} = 3$ $s_{17,9} = 6$ $s_{18,1} = 3$ $s_{18,6} = -3$
$s_{18,8} = 6$ $s_{18,9} = 3$ $s_{20,4} = 3$ $s_{21,4} = 3$ $s_{23,4} = 6$ $s_{23,7} = 3$ $s_{24,4} = 6$ $s_{24,7} = 3$
$s_{26,4} = 3$ $s_{26,7} = 6$ $s_{27,4} = 3$ $s_{27,7} = 6$ $s_{29,7} = 3$ $s_{30,7} = 3$

The solution is

$$\xi_1 = (0\ 0\ 0\ 0\ \ -1\ 1\ 0\ 1\ \ -1)^{\mathrm{T}}$$
$$\xi_2 = (-2\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 1)^{\mathrm{T}}$$

When we convert ξ_1 and ξ_2 back to matrices, still denote them by $\xi_1, \xi_2 \in gl(n, \mathbb{R})$, then $[\xi_1, \xi_2] = 0$. That is they are commutative, which means $g = \mathrm{Span}\{\xi_1, \xi_2\}$ is a Lie algebra. Then it is ready to show that the largest connected invariant linear group of the system (28) for this set of coefficients is

$$G_1 = \left\{ \exp\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} t_1 \exp\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} t_2 \,\middle|\, t_1, t_2 \in \mathbb{R} \right\}$$

We may explore Case 2 in a little bit more. A careful computation shows that

$$f(x) = f_3 x^3 = \begin{pmatrix} 3x_1^2(x_2 + x_3) \\ 3x_1x_2^2 + 6x_1x_2x_3 + 3x_1x_3^2 \\ 3x_1x_2^2 + 6x_1x_2x_3 + 3x_1x_3^2 \end{pmatrix} = 3x_1(x_2 + x_3)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} := p(x)Kx$$

Then it is easy to verify that the system satisfies (A22)–(A23). That is, $p(x)$ is $g(G_1)$ invariant and

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \in Z(g(G_1))$$

6. SYMMETRY *VS* CONTROLLABILITY

In this section we briefly discuss some controllability properties of the affine nonlinear systems possessing symmetry.

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First we consider a linear system

$$\dot{x} = Ax + \sum_{i=1}^{m} b_i u_i := Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \tag{30}$$

Its linear symmetry can be described by the following proposition.

Proposition 6.1
System (30) has a connected ss-symmetry group $G < GL(n, \mathbb{R})$, iff for any $\alpha \in g(G)$

$$\begin{aligned} \alpha A - A\alpha &= 0 \\ \alpha B &= 0 \end{aligned} \tag{31}$$

Proof
(Necessity) Since $\alpha \in g(G)$, $e^{\alpha t} \in G$, $\forall t \in \mathbb{R}$. Since system is symmetric with respect to G, by definition we have

$$(e^{\alpha t})_* (Ax + Bu) = e^{\alpha t} A e^{-\alpha t} x + e^{\alpha t} B = Ax + Bu \quad \forall x \in \mathbb{R}^n, \quad u, v \in \mathbb{R}^m$$

Hence,

$$\begin{aligned} e^{\alpha t} A e^{-\alpha t} &= A \\ e^{\alpha t} B &= B \end{aligned} \tag{32}$$

Differentiating both sides of the first equation in (32) with respect to t, we have

$$\alpha e^{\alpha t} A e^{-\alpha t} - e^{\alpha t} A \alpha e^{-\alpha t} = 0$$

Set $t = 0$ yields the first equation of (31). Similarly, we can get the second one.
(Sufficiency) Using Taylor series expansion on e^{ht}, one sees easily that (31) implies (32). The conclusion follows from the structure (13) of G. □

Expressing (31) into matrix form, we have

Corollary 6.2
System (30) has a non-trivial ss-symmetry group ($G \neq \{I_n\}$), iff the equation

$$\begin{pmatrix} A^T \otimes I_n - I_n \otimes A \\ B^T \otimes I_n \end{pmatrix} \xi = 0 \tag{33}$$

has a non-zero solution.

Corollary 6.3
If system (30) is completely controllable, it doesn't allow a non-trivial linear ss-symmetry $G < GL(n, \mathbb{R})$.

Proof
Assume $\Phi \in G$. Then

$$\Phi A \Phi^{-1} = A, \quad \Phi B = B$$

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Hence

$$\Phi(A^{n-1}B, \ldots, B) = (A^{n-1}B, \ldots, B)$$

which leads to: $\Phi = I_n$. □

Now assume system (1) has an ss-symmetry group G (G may not be a linear group.) For any $\alpha \in G$, the mapping $\theta_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism. Denote

$$J_\alpha = \frac{\partial \theta_\alpha(x)}{\partial x}(0), \quad \alpha \in G$$

Using Taylor series expansion on θ_α and the system and verifying the linear terms, one can easily prove the following result:

Proposition 6.4
Assume system (1) with $f_0(0) = 0$ has an ss-symmetry group G. Then

1. $G_L := \{J_\alpha | \alpha \in G\} < GL(n, \mathbb{R})$ is a Lie sub-group.
2. Let $A = \partial f_0 / \partial x(0)$ and $b_i = f_i(0)$, $i = 1, \ldots, m$. Then the linear approximate system

$$\dot{z} = Az + \sum_{i=1}^{m} b_i u_i$$

has G_L as its ss-symmetry group.

The following result may be considered as a necessary condition for general symmetry. (Where the G_L, A, B are as in Proposition 6.4.)

Corollary 6.5
Assume system (1) with $f_0(0) = 0$ has an ss-symmetry group G and (A, B) is controllable. Then

$$G_L = \{I_n\}$$

The following result adds some new (but related in certain sense) observation to [9]:

Proposition 6.6
Assume system (1) has a non-trivial ss-symmetry group $G < GL(n, \mathbb{R})$. Then it does not satisfy accessibility rank condition [17] at the origin.

Proof
Since G is non-trivial, which means there exists $0 \neq V \in g(G)$. Using Lemma 2.4, we have

$$[Vx, f_i(x)] = 0, \quad i = 0, 1, \ldots, m$$

Using Jacobi identity, for accessibility Lie algebra

$$\mathscr{L} = \{f_0, f_1, \ldots, f_m\}_{\mathrm{LA}}$$

we also have

$$[Vx, \mathscr{L}] = 0$$

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Now if $\dim \mathrm{Span}\{\mathscr{L}\}(0) = n$, we can find $\eta_1(x), \ldots, \eta_n(x) \in \mathscr{L}$ which are linearly independent at $x = 0$. Express

$$\eta_i(x) = \eta_i(0) + O(\|x\|), \quad i = 1, \ldots, n$$

Then

$$0 = [Vx, \eta_i(x)] = -V\eta_i(0) + O(\|x\|), \quad i = 1, \ldots, n$$

which implies that

$$V\eta_i(0) = 0, \quad i = 1, \ldots, n$$

Therefore, $V = 0$, which leads to a contradiction. □

Note that in fact the above Proposition says that system (1) does not satisfy accessibility rank condition at any $x_0 \in \mathbb{R}^n$ if it is ss-symmetric with respect to a non-trivial $G < GL(n, \mathbb{R})$ about any point x_0. The statement 'symmetric about point x_0' means for any $\alpha \in G$, the system is ss- invariant under the action $\theta_\alpha : x - x_0 \mapsto \alpha(x - x_0)$.

7. CONCLUSION

This paper considered linear symmetries of nonlinear control systems. First of all, the state space (ss) symmetry was investigated from two aspects: Lie group and its Lie algebra. Certain necessary and sufficient conditions were obtained. Secondly, some special cases were considered: (1) Assume the Lie group consisted of the rotations ($SO(n, \mathbb{R})$). Then the only possible form of symmetric systems was obtained for $n \geqslant 3$. (2) The classification of ss-symmetries of planar systems was obtained. It was shown that planar systems have only four classes of linear ss-symmetries. Any symmetric planar dynamic systems should be conjugate to one of them. Then a set of algebraic equations were given to calculate the Lie algebra of the largest ss-symmetry group for a given system. From this Lie algebra the largest connected ss-symmetry group of the system is easily constructible. Finally, certain controllability properties of symmetric control systems were revealed.

Linear symmetry is co-ordinate dependent. Converting the results of linear symmetry to co-ordinate free symmetries remains for further study.

APPENDIX A

A.1. Semi-tensor product of matrices

Here we briefly introduce the semi-tensor product of matrices. It can be considered as a notation, and will be used as an auxiliary tool in our computations.

Definition A.1 (Cheng [13])
Let $A \in M_{m\times n}$ and $B \in M_{p\times q}$. If $n = pt$, i.e. p is a divisor of n, the left semi-tensor product (right semi-tensor product) of M and N, denoted by $M \ltimes N$ ($M \rtimes N$), is defined as

$$A \ltimes B = A(B \otimes I_t), \quad (A \rtimes B = A(I_t \otimes B)) \tag{A1}$$

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If $nt = p$, then

$$A \ltimes B = (A \otimes I_t)B, \quad (A \rtimes B = (I_t \otimes A)B) \tag{A2}$$

In either (A1) or (A2) when $n = p$, $A \ltimes B$ becomes conventional matrix product. Hence, the left semi-tensor product is obviously a generalization of the conventional matrix product. So in the following we assume the default matrix product is the left semi-tensor product and use AB for $A \ltimes B$. (In fact, the right semi-tensor product is also a generalization of conventional product. But the left semi-tensor product has more nice properties [13]. It is, therefore, more useful.)

We cite some fundamental properties of the semi-tensor product, which will be used in the sequel.

Proposition A.2 (Cheng [13])

1. If $A \in M_{m\times n}$ and either m is a divisor of n or n is a divisor of m, then A^k ($A^{\rtimes k}$) is defined as

$$A^1 = A, \quad (A^{\rtimes 1} = A,)$$

$$A^{k+1} = A^k \ltimes A, \quad (A^{\rtimes (k+1)} = A^{\rtimes k} \rtimes A)$$

Particularly if V is a row or column vector, then V^k is always well defined.

2. Denote by $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$. Then x^k is a redundant pseudo-basis of the kth homogeneous polynomials. (A set is called a pseudo-basis if it contains a basis.) Therefore, a kth homogeneous polynomial $p_k(x)$ can be expressed as

$$p_k(x) = fx^k \quad \text{where} \quad f^{\mathrm{T}} \in \mathbb{R}^{n^k}$$

But f is not unique because x^k contains linearly dependent components.

3. Let A, B, C be three matrices with proper dimensions such that the involved left (right) semi-tensor products are well defined, then

$$A \ltimes (B \ltimes C) = (A \ltimes B) \ltimes C \quad (A \rtimes (B \rtimes C) = (A \rtimes B) \rtimes C) \tag{A3}$$

$$(A + B) \ltimes C = A \ltimes C + B \ltimes C \quad ((A + B) \rtimes C = A \rtimes C + B \rtimes C) \tag{A4}$$

$$A \ltimes (B + C) = A \ltimes B + A \ltimes C \quad (A \rtimes (B + C) = A \rtimes B + A \rtimes C) \tag{A5}$$

That is, the left (right) semi-tensor product is associative and distributive.

4. If $x \in \mathbb{R}^t$ and $A \in M_{m\times n}$, then

$$x \ltimes A = (I_t \otimes A) \ltimes x \tag{A6}$$

Remark A.3

From the definition one sees that the semi-tensor product can be expressed directly by tensor product and conventional product. One significant advantage of semi-tensor product is that the associative rule holds between semi-tensor product and conventional product because the conventional product can be considered as a particular case of the

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semi-tensor product, while the associativity doesn't hold between tensor product and conventional product.

A.2. Proof of Theorem 3.2

Definition A.4

A smooth function $p(x) \in V(\mathbb{R}^n)$ is $A \in M_{n\times n}$ invariant if

$$L_{Ax}p(x) = 0 \tag{A7}$$

The meaning of 'invariant' is from the following observation: since $L_{Ax}p(x) \equiv 0$, then $L^k_{Ax}p(x) = 0$, $k > 1$. Using the Taylor series expansion, we have

$$(\phi^t_{Ax})^* p(x) = p(\mathrm{e}^{At}x) = \sum_{k=0}^{\infty} L^k_{Ax}p(x)\frac{t^k}{k!} = p(x)$$

That is, $p(x)$ is invariant with respect to the integral curve of Ax.

In some literatures, $p(x)$ is also called a first integral of the linear vector field Ax.

Lemma A.5

Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then A has no invariant polynomial of odd degrees.

Proof

We have only to prove the claim with respect to homogeneous polynomials. Assume $g(y_1, y_2) = \sum_{i=0}^{2l+1} a_i y_1^i y_2^{2l+1-i}$ and $L_{Ay}g(y_1, y_2) = 0$. Then

$$0 = L_{Ay}g(y_1, y_2) = \frac{\partial g}{\partial y}\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix} = -\sum_{i=0}^{2l+1} \mathrm{i} a_i y_1^{i-1} y_2^{2l-i+2} + \sum_{j=0}^{2l+1} (2l+1-j) a_j y_1^{j+1} y_2^{2l-j}$$

$$= -\sum_{j=0}^{2l-1} (j+2) a_{j+2} y_1^{j+1} y_2^{2l-j} + \sum_{j=0}^{2l-1} (2l+1-j) a_j y_1^{j+1} y_2^{2l-j} - a_1 y_2^{2l+1} + a_{2l} y_1^{2l+1}$$

Comparing the coefficients on both sides yields

$$a_1 = 0, \quad a_{2l} = 0 \quad (j+2)a_{j+2} = (2l+1-j)a_j, \ \ j = 1, 2, \ldots, 2l-1$$

Hence,

$$a_i = 0, \quad 0 \leqslant i \leqslant 2l+1$$

□

Lemma A.6

Let A be as in Lemma A.5. A has no invariant polynomial of even degree with odd powers on both two variables.

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Proof
Set $g(y_1, y_2) = \sum_{i=0}^{m-1} a_{2i+1} y_1^{2i+1} y_2^{2m-2i-1}$, then

$$
\begin{aligned}
0 = L_{Ay} g(y_1, y_2) &= \frac{\partial g}{\partial y} \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix} \\
&= -\sum_{i=0}^{m-1} (2i+1) a_{2i+1} y_1^{2i} y_2^{2m-2i} + \sum_{j=0}^{m-1} (2m-2j-1) a_{2j+1} y_1^{2j+2} y_2^{2m-2j-2} \\
&= -\sum_{i=1}^{m-1} (2i+1) a_{2i+1} y_1^{2i} y_2^{2m-2i} + \sum_{i=1}^{m-1} (2m-2i+1) a_{2i-1} y_1^{2i} y_2^{2m-2i} \\
&\quad - a_1 y_2^{2m} + a_{2m-1} y_1^{2m}
\end{aligned}
$$

Comparing the coefficients yields

$$a_1 = 0, \quad a_{2m-1} = 0, \quad (2i+1) a_{2i+1} = (2m-2i+1) a_{2i-1}, \ \ i = 1, 2, \ldots, m-1$$

which implies

$$a_{2i+1} = 0, \quad 0 \leqslant i \leqslant m-1$$

□

Lemma A.7
Consider $o(3, \mathbb{R})$, and a polynomial

$$g(x_1, x_2, x_3) = \sum_{i_1+i_2+i_3=2k} a_{i_1 i_2 i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3}$$

If $g(x)$ is $o(3, \mathbb{R})$ invariant, then for the terms with at least one of i_1, i_2, or i_3 is odd, we have

$$a_{i_1 i_2 i_3} = 0$$

Proof
Let v_1, v_2, v_3 be a set of canonical basis of $o(3, \mathbb{R})$ as

$$v_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then $L_{v_i x} g = 0$, $i = 1, 2, 3$.

Assume i_1 is odd. From $L_{v_1 x} g = 0$, we have

$$\frac{\partial g}{\partial x} \begin{pmatrix} 0 \\ -x_3 \\ x_2 \end{pmatrix} = 0$$

Using Lemma A.5, we have $a_{i_1 i_2 i_3} = 0$.

Similarly, when i_2 or i_3 is odd, we also have $a_{i_1 i_2 i_3} = 0$. □

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Lemma A.8
Let $f \in \mathscr{H}_3^{2k}$ be expressed as

$$f(x) = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}, \quad f_i = \sum_{i_1+i_2+i_3=2k} a^i_{i_1 i_2 i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3}, \quad i = 1, 2, 3$$

Assume it is $o(3, \mathbb{R})$ invariant, i.e.

$$[v_i x, f] = \frac{\partial f}{\partial x} v_i x - v_i f = 0, \quad i = 1, 2, 3$$

Then $f(x) \equiv 0$.

Proof
Since

$$\frac{\partial f}{\partial x} v_i x = v_i f, \quad i = 1, 2, 3$$

a straightforward computation yields the following;

$$\frac{\partial f_1}{\partial x}\begin{pmatrix} 0 \\ -x_3 \\ x_2 \end{pmatrix} = \frac{\partial f_2}{\partial x}\begin{pmatrix} x_3 \\ 0 \\ -x_1 \end{pmatrix} = \frac{\partial f_3}{\partial x}\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} = 0 \tag{A8}$$

$$\frac{\partial f_2}{\partial x}\begin{pmatrix} 0 \\ -x_3 \\ x_2 \end{pmatrix} = -f_3, \quad \frac{\partial f_3}{\partial x}\begin{pmatrix} 0 \\ -x_3 \\ x_2 \end{pmatrix} = f_2, \quad \frac{\partial f_3}{\partial x}\begin{pmatrix} x_3 \\ 0 \\ -x_1 \end{pmatrix} = -f_1 \tag{A9}$$

Consider (A8). According to Lemma A.7, every variable in each non-zero term of f_i should have even degree.

Observe (A9). On the left-hand side of the equation, each term has at least one variable with odd degree, while on the right-hand side the degrees of all variables are even. It follows that

$$f_1 = f_2 = f_3 \equiv 0, \quad \Rightarrow f \equiv 0$$

□

Lemma A.9
Given a polynomial

$$g(x_1, x_2, x_3) = \sum_{i_1+i_2+i_3=2k+1} a_{i_1 i_2 i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3}$$

and assume $L_{v_1 x} g = 0$. Then

$$g = \left(\sum_{j_1+j_2+j_3=k} b_{j_1 j_2 j_3} x_1^{2j_1} x_2^{2j_2} x_3^{2j_3} \right) x_1 \tag{A10}$$

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Proof
Since $L_{v_1x}g = 0$, we have

$$\frac{\partial g}{\partial x}\begin{pmatrix} 0 \\ -x_3 \\ x_2 \end{pmatrix} = 0$$

If i_1 is even, then $i_2 + i_3$ is odd. From Lemma A.7, we have $a_{i_1i_2i_3} = 0$.

If i_1 is odd, then either both i_2 and i_3 are odd, or both i_2 and i_3 are even. In the first case, according to Lemma A.6, $a_{i_1i_2i_3} = 0$. In the second case assume $i_1 = 2j_1 + 1, i_2 = 2j_2, i_3 = 2j_3$, it follows that

$$b_{j_1j_2j_3} = a_{(2j_1+1)(2j_2)(2j_3)}, \quad j_1 + j_2 + j_3 = k$$

The conclusion follows. □

Remark A.10
If $L_{v_ix}g = 0$, $i = 2$ or $i = 3$, similar argument shows that

$$g = \left(\sum_{j_1+j_2+j_3=k} b_{j_1j_2j_3} x_1^{2j_1} x_2^{2j_2} x_3^{2j_3}\right) x_i$$

Lemma A.11
Let $f(x) \in \mathscr{H}_3^{2k+1}$ be expressed as

$$f(x) = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

where f_1, f_2, f_3 are $2k + 1$ homogeneous polynomials. If

$$\frac{\partial f}{\partial x} v_i x = v_i f, \quad i = 1, 2, 3$$

then

$$\begin{aligned} f_1 &= \left(\sum_{j_1+j_2+j_3=k} a_{j_1j_2j_3} x_1^{2j_1} x_2^{2j_2} x_3^{2j_3}\right) x_1 \\ f_2 &= \left(\sum_{j_1+j_2+j_3=k} b_{j_1j_2j_3} x_1^{2j_1} x_2^{2j_2} x_3^{2j_3}\right) x_2 \\ f_3 &= \left(\sum_{j_1+j_2+j_3=k} c_{j_1j_2j_3} x_1^{2j_1} x_2^{2j_2} x_3^{2j_3}\right) x_3 \end{aligned} \tag{A11}$$

Moreover, if $a_{k00} = 0$, then

$$a_{j_1j_2j_3} = 0, \quad b_{j_1j_2j_3} = 0, \quad c_{j_1j_2j_3} = 0$$

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Proof

From Equation (A8) and Lemma A.9 we have (A11). Then using

$$\frac{\partial f(x)}{\partial x} v_3 x = v_3 f(x)$$

the following can be deduced directly.

$$\frac{\partial f_2}{\partial x} \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} = f_1 \tag{A12}$$

$$\frac{\partial f_1}{\partial x} \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} = -f_2 \tag{A13}$$

It follows from (A12) and (A13), respectively, that

$$\begin{aligned} &\sum_{j_1+j_2+j_3=k} -b_{j_1 j_2 j_3}(2j_1) x_1^{2j_1-1} x_2^{2j_2+2} x_3^{2j_3} + \sum_{j_1+j_2+j_3=k} b_{j_1 j_2 j_3}(2j_2+1) x_1^{2j_1+1} x_2^{2j_2} x_3^{2j_3} \\ &\quad = \sum_{i_1+i_2+i_3=k} a_{i_1 i_2 i_3} x_1^{2i_1+1} x_2^{2i_2} x_3^{2i_3} \\ &\quad = \sum_{\substack{i_1+i_2+i_3=k \\ i_1 \leqslant k-1,\ i_2 \geqslant 1}} [-2(i_1+1) b_{(i_1+1)(i_2-1)i_3} + (2i_2+1) b_{i_1 i_2 i_3}] x_1^{2i_1+1} x_2^{2i_2} x_3^{2i_3} + b_{k00} x_1^{2k+1} \end{aligned} \tag{A14}$$

and

$$\begin{aligned} &- \sum_{j_1+j_2+j_3=k} (2j_1+1) a_{j_1 j_2 j_3} x_1^{2j_1} x_2^{2j_2+1} x_3^{2j_3} + \sum_{j_1+j_2+j_3=k} (2j_2) a_{j_1 j_2 j_3} x_1^{2j_1+2} x_2^{2j_2-1} x_3^{2j_3} \\ &\quad = - \sum_{i_1+i_2+i_3=k} b_{i_1 i_2 i_3} x_1^{2i_1} x_2^{2i_2+1} x_3^{2i_3} \\ &\quad = \sum_{\substack{i_1+i_2+i_3=k \\ i_1 \geqslant 1,\ i_2 \leqslant k-1}} [-(2i_1+1) a_{i_1 i_2 i_3} + 2(i_2+1) a_{(i_1-1)(i_2+1)i_3}] x_1^{2i_1} x_2^{2i_2+1} x_3^{2i_3} + a_{0k0} x_2^{2k+1} \end{aligned} \tag{A15}$$

Comparing coefficients on both sides of (A14), we have

$$\begin{aligned} (2i_2+1) b_{i_1 i_2 i_3} - 2(i_1+1) b_{(i_1+1)(i_2-1)i_3} &= a_{i_1 i_2 i_3} \\ i_1 + i_2 + i_3 &= k, \quad i_1 \leqslant k-1, \quad i_2 \geqslant 1 \\ b_{k00} = a_{k00} &= 0 \end{aligned} \tag{A16}$$

Similarly, (A15) provides

$$\begin{aligned} (2i_1+1) a_{i_1 i_2 i_3} - 2(i_2+1) a_{(i_1-1)(i_2+1)i_3} &= b_{i_1 i_2 i_3} \\ i_1 + i_2 + i_3 &= k, \quad i_1 \geqslant 1, \quad i_2 \leqslant k-1 \\ b_{0k0} &= a_{0k0} \end{aligned} \tag{A17}$$

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Therefore,

$$\begin{aligned} 2(i_2+1)a_{(i_1-1)(i_2+1)i_3} &= (2i_1+1)a_{i_1i_2i_3} - b_{i_1i_2i_3} \\ (2i_2+1)b_{i_1i_2i_3} &= a_{i_1i_2i_3} + 2(i_1+1)b_{(i_1+1)(i_2-1)i_3} \\ a_{k00} = b_{k00} &= 0 \end{aligned} \tag{A18}$$

It follows that $a_{i_1i_20} = b_{i_1i_20} = 0, \quad i_1 + i_2 = k$.

Similarly, we have

$$2(i_3+1)a_{(i_1-1)i_2(i_3+1)} = (2i_1+1)a_{i_1i_2i_3} - c_{i_1i_2i_3}$$

and

$$(2i_3+1)c_{i_1i_2i_3} = a_{i_1i_2i_3} + (2i_1+1)c_{(i_1+1)i_2(i_3-1)}$$

and hence

$$a_{i_1i_2i_3} = c_{i_1i_2i_3} = 0, \quad i_1 + i_3 = k - i_2 \quad \text{or} \quad i_1 + i_2 + i_3 = k$$

Similarly, we also have

$$b_{i_1i_2i_3} = 0$$

□

The following lemma is motivated by the main result of [6].

Lemma A.12

Consider the following system

$$\dot{x} = f(x) = \sum_{i=1}^{t} p_i(x)K_ix, \quad x \in \mathbb{R}^n, \quad t \in Z_+ \tag{A19}$$

where $p_i(x)$ is a polynomial and $K_i \in M_{n\times n}$. System (A19) is ss-symmetric with respect to $G < GL(n, \mathbb{R})$ if

1. $p_i(x)$, $i = 1, \ldots, t$ are $g(G)$ invariant;
2. K_i, $i = 1, \ldots, t$ are in the centre of $g(G)$ [15], where $g(G)$ is the Lie algebra of G.

Proof

Let $V \in g(G)$.

$$\begin{aligned} \mathrm{ad}_{Vx} f(x) &= \sum_{i=1}^{t} (L_{Vx}p_i(x)K_ix + p_i(x)\mathrm{ad}_{Vx}K_ix) \\ &= \sum_{i=1}^{t} (L_{Vx}p_i(x)K_ix - p_i(x)[V, K_i]x) = 0 \end{aligned}$$

The conclusion follows from Lemma 2.4. □

Lemma A.13

System (1) with $n = 3$ has an ss-symmetry group $G = SO(3, \mathbb{R})$, iff

$$f_j(x) = \sum_{i=0}^{\infty} a_k^j \|x\|^{2i}x, \quad a_k^j \in \mathbb{R}, \quad j = 0, 1, \ldots, m \tag{A20}$$

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Proof
(Sufficiency) The sufficiency follows from Lemma A.12.

(Necessity) Consider system (1) with $n = 3$. Assume it is state space symmetric with respect to $G = SO(3, \mathbb{R})$, and

$$f_j(x) = \sum_{r=0}^{\infty} f_r^j(x)$$

where $f_r^j(x) \in \mathscr{H}_n^r$. Now if r is even, according to Lemma A.8, $f_r^j = 0$. So we assume $r = 2k + 1$. Denote the coefficient of x_1^{2k+1} in $f_{2k+1}^j(x)$ by $a_k = a_{k00}$. Set

$$g_{2k+1}^j(x) = f_{2k+1}^j(x) - a_k \|x\|^{2k} x$$

According to Lemma 2.4, a straightforward computation shows

$$\frac{\partial g_{2k+1}}{\partial x} v_i x = v_i g_{2k+1}, \quad i = 1, 2, 3$$

Now Lemma A.9 assures

$$g_{2k+1}(x) \equiv 0$$

It follows that

$$f_{2k+1}(x) = a_k \|x\|^{2k} x$$

□

Proof of Theorem 3.2
From the proof of Lemma A.13 one sees easily that the basic trick used in the proof is comparing a pair of variables. It is obvious that this method can be extended to the case of $n > 3$. Theorem 3.2 follows. □

A.3. Proof of Theorem 4.1

First, we want to show that if system (1) with $n = 2$ is ss-symmetric, then it can be expressed in a particular form, satisfying certain conditions. To get a motivation for this form we recall (21). It is easy to see that (21) has the form as

$$f_j(x) = \sum_{n=0}^{\infty} p_n^j(x) B_n^j x, \quad x \in \mathbb{R}^2, \quad j = 0, \ldots, m \tag{A21}$$

(Since the following argument is independent of j, for notational ease, j is omitted in the rest of this proof.) Moreover, for any $S \in so(2, \mathbb{R})$, or, equivalently, simply choose a basis as

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

we have

$$L_{Sx} p_n(x) = 0 \tag{A22}$$

$$[S, B_n] = 0 \tag{A23}$$

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According to Lemma A.12, (1) has ss-symmetry group $so(2,\mathbb{R})$ if it has the form (A21), satisfying (A22)–(A23).

In the following lemma we claim that the aforementioned form and conditions are universal for all planar ss-symmetric systems.

Lemma A.14

Let $V \in gl(2,\mathbb{R})$ and $G = \{e^{Vt} | t \in \mathbb{R}\}$. A planar system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^2 \tag{A24}$$

is symmetric with respect to G, iff

(i) $f(x)$ can be expressed as (A21);
(ii) p_n and B_n satisfy (A22) and (A23), respectively.

Proof

Note that ad_{Vx} does not change the degree of each homogeneous component in $f(x)$, so we can simply assume $f(x)$ is a homogeneous vector field. That is, set

$$f(x) = \begin{pmatrix} \sum_{i=1}^{n} a_i x_1^{n-i} x_2^i \\ \sum_{j=1}^{n} b_j x_1^{n-j} x_2^j \end{pmatrix} \tag{A25}$$

To begin with, we assume V is in a Jordan canonical form.

Case 1: Assume

$$V = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Using Lemma 2.4, $[Vx, f] = 0$ yields

$$\begin{aligned} &((n-i-1)\lambda_1 + \mathrm{i}\lambda_2)a_i = 0 \\ &((n-j)\lambda_1 + (j-1)\lambda_2)b_j = 0, \quad i,j = 0,\ldots,n \end{aligned} \tag{A26}$$

To get non-zero a_i, b_j, we need

$$det\begin{pmatrix} n-i-1 & i \\ n-j & j-1 \end{pmatrix} = (j-i-1)(n-1) = 0 \tag{A27}$$

If $n = 1$, $f(x)$ is linear, and the conclusion comes from a straightforward computation. We consider $n > 1$ case. From (A27) we have

$$j - i - 1 = 0 \tag{A28}$$

From (A26) we also have

$$(n-j)\lambda_1 + (j-1)\lambda_2 = 0 \tag{A29}$$

Since λ_1 and λ_2 cannot be zero simultaneously, we may assume $\lambda_1 \neq 0$, and set $\mu = \lambda_2/\lambda_1$. According to (A29), μ is a rational number. First, we assume $\lambda_2 \neq 0$. Then there exist two

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co-prime integers p, q, such that

$$\mu = \frac{p}{q} \tag{A30}$$

Then (A29) yields that

$$\begin{aligned} i &= j - 1 = tq, \quad q > 0, \quad p < 0 \\ n &= t(q - p) + 1, \quad t = 1, 2, \ldots \end{aligned}$$

The form of $f(x)$ follows as

$$f_{t(q-p)+1}(x) = x_1^{-tp} x_2^{tq} \begin{pmatrix} a_t & 0 \\ 0 & b_t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad t = 1, 2, \ldots \tag{A31}$$

On the other hand, consider V-invariant polynomial. Assume

$$p_{n-1}(x) = \sum_{k=0}^{n-1} c_k x_1^{n-k-1} x_2^k$$

From $L_{Vx} p_{n-1}(x) = 0$ we have that

$$(n - k - 1)\lambda_1 + k\lambda_2 = 0 \tag{A32}$$

Comparing (A32) with (A29), one sees easily that $x_1^{-tp} x_2^{tq}$ is the set of solutions of (A22) under this pair of (λ_1, λ_2). Moreover assume $\lambda_1 \neq \lambda_2$. Then (A31) presents all the solutions satisfying (A22)–(A23).

Now assume $\lambda_2 = 0$. It is easy to see that the vector fields, satisfying (A26), have the form as

$$f_t(x) = x_2^{t-1} \begin{pmatrix} a_t & 0 \\ 0 & b_t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{A33}$$

which is the set of solutions of (A22)–(A23) with respect to $\lambda_2 = 0$.

Finally, assume λ_1, λ_2 are complex numbers. We may allow $f(x)$ to have complex coefficients. Then the above argument remains available. Say, $\lambda_{1,2} = \alpha \pm \beta J$, where $J = \sqrt{-1}$. Then from (A29)–(A30) we have $\alpha = 0$, $\mu = -1$. It implies that

$$V = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \tag{A34}$$

Case 2: Assume

$$V = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Lemma 2.4 yields

$$\begin{aligned} \lambda(n-1)a_i + (n - i + 1)a_{i-1} - b_i &= 0 \\ \lambda(n-1)b_i + (n - i + 1)b_{i-1} &= 0, \quad i = 0, \ldots, n+1 \end{aligned} \tag{A35}$$

where for notational ease, we use $a_{-1} = b_{-1} = a_{n+1} = b_{n+1} = 0$.

First, we assume $\lambda \neq 0$. Using the second equation of (A35) and setting $i = 0$, we get $b_0 = 0$. Then we can show recursively that all $b_i = 0$. Then the first equation implies all $a_i = 0$. So there is no non-trivial solution. Next, let $\lambda = 0$. The second equation provides non-zero solution as

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$b_n \neq 0$ and $b_i = 0,\ i \neq n$. Plugging them into the first equation yields: $a_n \neq 0$ and $a_{n-1} = b_n \neq 0$, $a_i = 0,\ i \leqslant n-2$. Then the non-trivial solution f_n becomes

$$f_n = \begin{pmatrix} b_n x_1 x_2^{n-1} + a_n x_2^n \\ b_n x_2^n \end{pmatrix} = x_2^{n-1} \begin{pmatrix} b_n & a_n \\ 0 & b_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{A36}$$

Similarly, we can prove that it consists of all the solutions of (A22)–(A23).

Finally, we consider the case if V is not in the Jordan canonical form. Taking a linear transformation $y = Tx$, equation (12) becomes

$$[T_*(Vx), T_*(f(x))] = 0$$

Now assume $f(x)$ has the form as in (A25). Then

$$T_*(Vx) = TVT^{-1}y$$

$$T_*(f(x)) = \begin{pmatrix} \sum_{i=1}^n \tilde{a}_i y_1^{n-i} y_2^i \\ \sum_{j=1}^n \tilde{b}_j y_1^{n-j} y_2^j \end{pmatrix}$$

We, therefore, can assume TVT^{-1} has a Jordan canonical form. Assume it is symmetric with respect to a one-dimensional group

$$G = \{e^{TVT^{-1}t} | t \in \mathbb{R}\}$$

then the original system is obviously symmetric with respect to

$$G = \{e^{Vt} | t \in \mathbb{R}\}$$

because (12) is co-ordinate independent. Moreover, since under y the system has the form of (A21), then

$$T_*^{-1}(f_j(y)) = T_*^{-1}(p_n(y)B_n y) = p_n(Tx)T^{-1}B_n T x$$

That is, the original system also has the form of (A19). Since (A22) and (A23) are co-ordinate independent, they hold for the original system too. The proof is completed. □

The following generalization is an immediate consequence of the proof of Lemma A.14.

Lemma A.15
A planar system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^2 \tag{A37}$$

has a symmetry group $G < GL(2, \mathbb{R})$, iff

(i) $f(x)$ can be expressed as (A21);
(ii) the p_n and B_n satisfy (A22) and (A23) with respect to any $S \in g(G)$.

Next, we consider a possible symmetry group, G, of dimension greater than one. Let $0 \neq A \in g(G)$. $g(G)$ is the Lie algebra of G.

Case 1: Assume

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

From Lemma A.14 we have

$$f = \sum_{i=0}^{\infty} c_i p_i(x) \begin{pmatrix} \alpha_i & 0 \\ 0 & b_i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where $p_i(x) = x_1^{-tp} x_2^{tq}$. Let

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in g(G)$$

Then

$$L_{Bx} p_i(x) = -tp x_1^{-tp-1} x_2^{tq}(b_{11}x_1 + b_{12}x_2) + tq x_1^{-tp} x_2^{tq-1}(b_{21}x_1 + b_{22}x_2) = 0 \tag{A38}$$

If $p \neq 0$, it follows that

$$-b_{11}p + b_{22}q = 0$$
$$b_{12} = b_{21} = 0$$

which implies that

$$\frac{b_{22}}{b_{11}} = \frac{p}{q} = \frac{\lambda_2}{\lambda_1}$$

That is A and B are linearly dependent, and $\dim(G) = 1$. We have to assume $p = 0$ for exploring new elements. It implies that

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{equivalently} \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Then from (A38), we have $b_{21} = b_{22} = 0$. That is,

$$B = \begin{pmatrix} b_{11} & b_{12} \\ 0 & 0 \end{pmatrix}$$

To make

$$\left[B, \begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix} \right] = 0$$

it is obvious that if $b_{12} \neq 0$ then $a_i = b_i$. We conclude that

$$g = \mathrm{Span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} \tag{A39}$$

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and the corresponding system is

$$\dot{x} = \sum_{i=0}^{\infty} a_i x_2^i \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{A40}$$

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1
In Case 1, if $\lambda_2 \neq 0$, we can exchange x_1, x_2 to get the required form. In fact, Cases 1, 2, and 4 are discussed in above. The only new thing is Case 3. Previously, it was treated as a special case of Case 1 with complex eigenvalues. Starting from Case 1 with V as in (A34), we can do the following transformation: Set

$$x = Ty = \begin{pmatrix} 1 & 1 \\ J & -J \end{pmatrix}$$

Then

$$T_*(Vy) = TVT^{-1}x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x$$

and

$$(T^{-1})^* p_{2n}(y) = p_{2n}(T^{-1}x) = \frac{a_n}{2^n}(x_1^2 + x_2^2)^n$$

which is the required form. □

A.4. Swap matrix

Definition A.16 (Cheng [13], Magnus and Neudecker [18])
A swap matrix, $W_{[m,n]} \in M_{mn \times mn}$, is constructed in the following way: index its columns by $(11, 12, \ldots, 1n, \ldots, m1, m2, \ldots, mn)$ and its rows by $(11, 21, \ldots, m1, \ldots, 1n, 2n, \ldots, mn)$. Then the elements of $W_{[m,n]}$ are defined as

$$w_{(IJ),(ij)} = \delta_{i,j}^{I,J} = \begin{cases} 1 & I = i \quad \text{and} \quad J = j \\ 0 & \text{otherwise} \end{cases} \tag{A41}$$

(In [18] it is called the permutation matrix. But we reserve this name for general permutation case.)

We cite some basic properties of the swap matrix here.

Proposition A.17

1.
$$W_{[m,n]}^{\mathrm{T}} = W_{[m,n]}^{-1} = W_{[n,m]} \tag{A42}$$

2. Given a matrix $A \in M_{m\times n}$ with its row staking form $V_{\mathrm{r}}(A)$ and column staking form $V_{\mathrm{c}}(A)$. Then
$$V_{\mathrm{c}}(A) = W_{[m,n]} V_{\mathrm{r}}(A), \quad V_{\mathrm{r}}(A) = W_{[n,m]} V_{\mathrm{c}}(A) \tag{A43}$$

3. Let $V \in \mathbb{R}^t$ and $A \in M_{m\times n}$. Then

$$VA = (I_t \otimes A)V \tag{A44}$$

A.5. Proof of Theorem 5.1

A straightforward computation shows the following lemma:

Lemma A.18

The differential of a product of two matrices of function entries satisfies the following:

$$D(A(x)B(x)) = DA(x) \ltimes B(x) + A(x)DB(x) \tag{A45}$$

Using (A45), we can prove the following differential formula inductively:

Lemma A.19

$$D(x^{k+1}) = \Psi_k(x^k \otimes I_n) = \Psi_k \ltimes x^k \tag{A46}$$

Combining (A46) with (A44), it is easy to prove the following formula:

$$L_{Vx}f_k x^k = f_k\Psi_{k-1}x^{k-1}Vx - Vf_k x^k = f_k\Psi_{k-1}(I_{n^{k-1}} \otimes V)x^k - Vf_k x^k, \quad k = 1, 2, \ldots \tag{A47}$$

Now to get unique solution, we convert it back to the conventional basis as

$$L_{Vx}f_k x^k = [f_k\Psi_{k-1}(I_{n^{k-1}} \otimes V) - Vf_k]T_N(n,k)x_k, \quad k = 1, 2, \ldots \tag{A48}$$

Therefore, the derivative is zero, iff

$$[f_k\Psi_{k-1}(I_{n^{k-1}} \otimes V) - Vf_k]T_N(n,k) = 0, \quad k = 1, 2, \ldots \tag{A49}$$

To simplify (A49) we need the following formula [13], which can be proved *via* direct computation.

Lemma A.20

Let $A \in M_{m\times n}$, $B \in M_{q\times p}$, and $Z \in M_{n\times q}$. Then the column stacking form of the product is

$$V_c(AZB) = (B^T \otimes A)V_c(Z) \tag{A50}$$

Using (A44) again, (A49) can be converted as

$$L_{Vx}f_k x^k = (T_N^T(n,k) \otimes (f_k\Psi_{k-1}))V_c(I_{n^{k-1}} \otimes V) - ((T_N^T(n,k)f_k^T) \otimes I_n)V_c(V) = 0$$
$$k = 1, 2, \ldots \tag{A51}$$

To convert (A51) to a standard linear equation, we need one more formula, which itself is important.

Proposition A.21

Let $A \in M_{m\times n}$ and $B \in M_{p\times q}$. Then

$$V_c(A \otimes B) = (I_n \otimes W_{[m,q]}) \ltimes V_c(A) \ltimes V_c(B)$$
$$= (I_n \otimes W_{[m,q]}) \ltimes W_{[pq,mn]} \ltimes V_c(B) \ltimes V_c(A) \tag{A52}$$

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$$V_r(A \otimes B) = W_{[mp,nq]}(I_n \otimes W_{[m,q]}) \ltimes W_{[m,n]} \ltimes (I_{mn} \otimes W_{p,q}) \ltimes V_r(A) \ltimes V_r(B) = W_{[mp,nq]}(I_n \otimes W_{[m,q]}) \ltimes W_{[m,n]} \ltimes (I_{mn} \otimes W_{p,q}) \ltimes W_{[pq,mn]} \ltimes V_r(B) \ltimes V_r(A) \quad \text{(A53)}$$

Proof

We prove the first formula of (A52) only. The others are the immediate consequences of it.

To begin with, we assume $n = 1$. Then it is obvious that

$$V_c(A) \ltimes V_c(B) = \mathrm{col}(a_{11}B_1, \ldots, a_{11}B_q, \ldots, a_{m1}B_1, \ldots, a_{m1}B_q)$$

and

$$V_c(A \otimes B) = \mathrm{col}(a_{11}B_1, \ldots, a_{m1}B_1, \ldots, a_{11}B_q, \ldots, a_{m1}B_q)$$

Note that they consist of the same set of p-dimensional vectors but with different order of double indexes. A straightforward computation shows that

$$V_c(A \otimes B) = W_{[m,q]} \ltimes V_c(A) \ltimes V_c(B)$$

Now for general case, we have only to do the swap for n blocks. The first formula of (A52) follows immediately.

Denote by

$$E_k^n := I_{n^{k-1}} \otimes W_{[n^{k-1},n]} \ltimes V_c(I_{n^{k-1}})$$

Then using (A52), we have

$$V_c(I_{n^{n-1}}) \otimes V = E \ltimes V_c(V)$$

Plugging it into (A51) yields Theorem 5.1. □

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