Accessibility of Switched Linear Systems

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***Abstract*—This note considers the controllability of switched linear systems. The structure of accessibility Lie algebra is revealed. Some accessibility properties are proved. Certain necessary and sufficient conditions for (local or global, weak or normal) controllability of a large class of switched linear systems are obtained.**

***Index Terms*—Accessibility, controllability, Lie algebra, switched linear system.**

I. Introduction

Consider a switched linear system

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \qquad x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m \tag{1}$$

where the switching functions $\sigma(t) : [0, \infty) \to \Lambda$ are piecewise constant, right continuous mappings with $\Lambda = \{1, 2, \ldots, N\}$, and where controls $u(t) : [0, \infty) \to \mathbb{R}^m$ are piecewise constant functions. We use $x(\cdot; t_0, x_0, u, \sigma)$ to denote the solution of the system satisfying the initial value $x(t_0) = x_0$ with the switching function σ and the input function u. The controllability property of such switched linear systems was investigated by many authors, e.g., [3], [13], [8], [9], and [14]. The following notion of controllability was adopted in [9] and [14].

Definition 1.1: Consider system (1). A state $p \in \mathbb{R}^n$ is controllable at time t_0, if there exist a time instant $t_f > t_0$, a switching path $\sigma : [t_0, t_f] \to \Lambda$, and an input function $u(t)$, such that $x(t_f; t_0, p, u, \sigma) = 0$. The set of controllable points is a vector space. The largest subspace V of $\mathbb{R}^n$ in which every point is controllable is called the controllable subspace. The system is controllable if $V = \mathbb{R}^n$.

Let $\mathcal{L} = \langle A_1 \cdots A_N \mid B_1, \ldots, B_N \rangle$, the smallest space containing the column vectors of $B_1, \ldots, B_N$ that is invariant under the transformations $A_1, \ldots, A_N$. The following result, a significant contribution of [9], reveals the structure of the controllable subspace of the system.

Theorem 1.2: [9] The controllable subspace of system (1) is $\mathcal{L}$. Hence, the system is controllable if and only if $\dim(\mathcal{L}) = n$.

It can be seen that the controllability notion given in Definition 1.1 is an analogue of the linear case. It deals only with controllability at the origin. We would like to point out that a switched linear system is essentially a nonlinear system with the switching functions acting as controls. For a nonlinear system, controllability at the origin is in general not sufficient to describe reachability or controllability at other points. The following example shows how the controllable subspace at (or the reachable subspace from) 0 and the reachable sets from other points may be unrelated.

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Example 1.3: Consider the following system:

$$\dot{x} = A_{\sigma(t)}x, \qquad x \in \mathbb{R}^2 \tag{2}$$

for which $\Lambda = \{1, 2, 3\}$ and

$$A_{1,2} = \pm I_2, \quad A_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

By Theorem 1.2, the controllable subspace is $\{0\}$. However, it is easy to see that the submanifold $M_c = \mathbb{R}^2 \backslash \{0\}$ is a controllable sub-manifold in the sense that every point q on M_c can be reached from any other point p on M_c: With A_1 and A_2, a trajectory can go in the radius direction (either increasing or decreasing), and with A_3 it can go along a circular path. So, for any two points $p, q \in M_c$, there is a switching law that drives a trajectory from p to q. □

The previous example shows that even though the controllable subspace is $\{0\}$, the reachable set of a point $p \neq 0$ still consists of almost every point in the state–space.

For general nonlinear control systems, a main tool for investigating their controllability is the accessibility Lie algebras generated by the vector fields of the systems (cf., [10] and [12]). The purpose of this note is to apply the Lie algebra approach for controllability of general nonlinear control systems to switched linear systems. We are particularly interested in the case when the accessiblity rank condition fails.

The rest of the note is organized as follows. In Section II, we discuss some preliminaries for the Lie algebras associated with the controllability of nonlinear systems and the integrability of Lie algebras. In Section III, we study the structure of accessibility Lie algebras for switched linear systems. In Section IV, we investigate the controllability, including global, local, weak and normal types, of switched linear systems. A topological structure of the controllable sub-manifolds and some necessary and sufficient conditions for certain controllability are obtained there. In Section V, we provide an illustrating example. In Section VI, we summarize the main conclusions.

II. Lie Algebra and Its Integrability

Consider a nonlinear control system

$$\dot{x} = f(x, u), \qquad x \in \mathbb{R}^n \tag{3}$$

where f is assumed to be analytic and defined on $\mathbb{R}^{n+m}$. The controls are piecewise constant, right continuous functions from $[0, \infty)$ to $\mathcal{U} \subseteq \mathbb{R}^m$. If f is affine in u, (3) becomes an affine nonlinear system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i := f(x) + g(x)u. \tag{4}$$

A switched linear system as in (1) can be treated as a nonlinear system as follows:

$$\dot{x}(t) = \sum_{i=1}^{N} \theta_i(t)(A_i x(t) + B_i u(t))$$

where the controls of the system are (θ, u) with $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$, taking values in the set $\mathcal{S} := \{(\sigma_1, \sigma_2, \ldots, \sigma_N) : \sigma_k = 1, \sigma_i = 0 \text{ if } i \neq k, 1 \leq k \leq N\}$.

Definition 2.1: Consider system (3).

1) A point q is reachable from p at a given time $T > 0$, denoted by $q \in R_T(p)$, if for some control function the corresponding trajectory satisfies
$$x(0) = p \quad \text{and} \quad x(T) = q.$$
A point q is reachable from p, denoted by $q \in R(p)$, if q is reachable from p at some time T.
2) The system is said to be controllable at p if $R(p) = \mathbb{R}^n$. The system is said to be controllable if $R(p) = \mathbb{R}^n$, for all $p \in \mathbb{R}^n$.
A control system is said to be T-controllable at p if $R_T(p) = \mathbb{R}^n$. The system is said to be T-controllable if $R_T(p) = \mathbb{R}^n$ for all $p \in \mathbb{R}^n$.
3) A point q is weakly reachable from p, denoted by $q \in \mathrm{WR}(p)$, if there exist $p_0 = p, p_1, \ldots, p_s = q$ such that either $p_i \in R(p_{i-1})$ or $p_{i-1} \in R(p_i), i = 1, \ldots, s$.
A point q is weakly T-reachable from p, denoted by $q \in \mathrm{WR}_T(p)$ if there exist $p_0 = p, p_1, \ldots, p_s = q$ such that either $p_i \in R_{T_i}(p_{i-1})$ or $p_{i-1} \in R_{T_i}(p_i), i = 1, \ldots, s$, and $\sum_{i=1}^s T_i = T$.
4) The system is said to be weakly controllable at p if $\mathrm{WR}(p) = \mathbb{R}^n$. The system is said to be weakly controllable if $\mathrm{WR}(p) = \mathbb{R}^n$ for all $p \in \mathbb{R}^n$.
The system is said to be weakly T-controllable at p if $\mathrm{WR}_T(p) = \mathbb{R}^n$. The system is said to be weakly T-controllable if $\mathrm{WR}_T(p) = \mathbb{R}^n$ for all $p \in \mathbb{R}^n$. □

Observe that WR defines an equivalent relation. Consequently, if a system is weakly controllable at a point p, then it is weakly controllable.

It was proved in [9] that for system (1) global weak T-controllability is equivalent to controllability.

The controllability of systems (3) and (4) via the Lie algebra approach has been discussed thoroughly in the 1970s and 1980s (cf., [10]–[12] and [5]).

We briefly review the construction of Lie algebras related to the controllability of nonlinear systems. For system (3) (assuming that controls are piecewise constant functions taking vaules in an open subset $\mathcal{U} \subseteq \mathbb{R}^m$), let
$$F = \{f(x, a) | a \in \mathcal{U}\}.$$
Then F is a set of analytic vector fields. The accessibility Lie algebra of system (3) is defined to be the Lie algebra generated by F:
$$\mathcal{L}_a = \{F\}_{\mathrm{LA}}. \tag{5}$$
The strong accessibility Lie algebra of the system is the Lie subalgebra of $\mathcal{L}_a$ defined by
$$\mathcal{L}_{\mathrm{sa}} = \left\{ \sum_{i=1}^k l_i X_i + Y \,\middle|\, X_i \in F, \sum_{i=1}^k l_i = 0, k < \infty, Y \in [\mathcal{L}_a, \mathcal{L}_a] \right\}. \tag{6}$$
For an affine nonlinear system (4), a straightforward computation shows the following.

Proposition 2.2: For (4), the accessibility Lie algebra is
$$\mathcal{L}_a = \{f(x), g(x)\}_{\mathrm{LA}}; \tag{7}$$
and the strong accessibility Lie algebra is
$$\mathcal{L}_{\mathrm{sa}} = \left\{ \mathrm{ad}_f^k g(x) | k \geq 0 \right\}_{\mathrm{LA}}. \tag{8}$$
For (3), if the rank of the (strong) accessibility Lie algebra is n at a point p it is said that the system satisfies the (strong, respectively) accessibility rank condition at p. A basic result about accessibility is as follows.

Theorem 2.3: [10] If (3) satisfies the accessibility rank condition at p, the reachable set $R(p)$ contains a nonempty set of interior points. If system (3) satisfies strong accessibility rank condition at p, then for any $T > 0$, the reachable set $R_T(p)$ contains a nonempty set of interior points.

The key point in the relationship between the controllability properties and the Lie algebras is the integrability of Lie algebras. Chow's Theorem plays a fundamental role in this context (cf. [5] and [12]).

Theorem 2.4: (Chow's Theorem) [4] Let M be an n-dimensional C^∞ manifold, $L = \{X_1, \ldots, X_k\} \subseteq V^\infty(M)$ a set of C^∞ vector fields, and $\mathcal{L} = \{X_1, \ldots, X_k\}_{\mathrm{LA}}$ the Lie algebra generated by L. Assume $\dim(\mathcal{L}(p)) = \text{constant} \leq n$ on M, and for any $p_0 \in M$, denote by $\mathcal{I}(\mathcal{L})(p_0)$ the largest integral submanifold of $\mathcal{L}$. Then, for any $p \in \mathcal{I}(\mathcal{L})(p_0)$ there exist $X_{i_1}, \ldots, X_{i_s} \in L$ and $t_1, \ldots, t_s \in \mathbb{R}$, such that
$$p = e_{t_1}^{X_{i_1}} \cdots e_{t_s}^{X_{i_s}}(p_0). \tag{9}$$
Remark 2.5: [2] (Generalized Chow's Theorem) When the manifold and the distribution are analytic, the regularity assumption $(\dim(\mathcal{L}(p)) = \text{constant})$ can be removed.

From Chow's Theorem one sees that for (3) and a given point p, its weakly reachable set is the largest integral manifold of the accessibility Lie algebra passing through p, assuming either $\mathcal{L}_a$ is regular or the system is analytic.

Lemma 2.6: [6] (Generalized Frobinius' Theorem) Let M be an n-dimensional C^ω (analytic) manifold, and Δ an analytic involutive distribution. Then, for any $p \in M$ there exists a largest integral manifold of Δ, passing through p. The following lemma is an immediate consequence of the Campbell–Baker–Hausdorff formula [7].

Lemma 2.7: Let Δ be an analytic involutive distribution, and $X \in \Delta$ an analytic vector field. Assume p_1 and p_2 are two points connected by an integral curve of X, i.e., there exists $t > 0$ such that the integral curve $e_t^X(p)$ of X satisfies $e_t^X(p_1) = p_2$, then $\dim(\Delta(p_1)) = \dim(\Delta(p_2))$.

III. Accessibility Versus Its Lie Algebra

When applying the definition of the accessibility algebra to system (1), one sees that the accessibility Lie algebra for system (1) is given by
$$\mathcal{L}_a := \{A_i x + B_i u \,|\, i = 1, \ldots, N, u = \text{constant}\}_{\mathrm{LA}}. \tag{10}$$
In analogue to $\mathcal{L}_{\mathrm{sa}}$ defined for affine systems, we adopt the following natural definition of strong accessibility algebra for a switched system as in (1).

Definition 3.1: For (1), the strong accessibility Lie algebra is the set of constant vector fields given by
$$\mathcal{L}_{\mathrm{sa}}^\sigma := \{\xi \,|\, \xi \in \langle A_1, \ldots, A_N \,|\, B_1, \ldots, B_N \rangle\}. \tag{11}$$
Indeed (11) may be considered as a generalization of (8), which can be expressed alteratively as $\mathcal{L}_{\mathrm{sa}} := \langle f(x) \,|\, g(x) \rangle_{\mathrm{LA}}$.

Denote $\mathbf{A} = \{A_1, A_2, \ldots, A_N\}$. The collection $\mathbf{A}$ is said to be symmetric if $-A_i \in \mathbf{A}$ for all i.

We consider $A_i, i = 1, \ldots, N$ as the elements of the Lie algebra $gl(n, \mathbb{R})$, and denote by $\mathcal{A}$ the Lie sub-algebra generated by $\mathbf{A}$. That is, $\mathcal{A} = \{A_1, \ldots, A_N\}_{\mathrm{LA}} \subset gl(n, \mathbb{R})$. Then, we define the set of linear homogeneous vector fields, denoted by

$$LH(\mathbb{R}^n) = \{g | g(x) = Ex \qquad \forall x, \text{ some } E \in M_n\}$$

where M_n is the set of $n \times n$ matrices. A straightforward computation shows that $LH(\mathbb{R}^n) \subset V^{\omega}(\mathbb{R}^n)$ is a Lie sub-algebra of $V^{\omega}(\mathbb{R}^n)$ (the Lie algebra of the analytic vector fields on $\mathbb{R}^n$). Define a mapping, $\Phi : LH(\mathbb{R}^n) \to gl(n, \mathbb{R})$ by $\Phi(Ex) := -E$. It is easy to prove the following lemma.

Lemma 3.2: For the map Φ, the following statements hold.

1) The map $\Phi : LH(\mathbb{R}^n) \to gl(n, \mathbb{R})$ is a Lie algebra isomorphism.
2) Let $\mathcal{L} \subset gl(n, \mathbb{R})$ be a Lie sub-algebra, then its inverse image $\Phi^{-1}(\mathcal{L})$ is a Lie sub-algebra of $V^{\omega}(\mathbb{R}^n)$, which can be expressed as $\Phi^{-1}(\mathcal{L}) = \mathcal{L}x$. □

Theorem 3.3: For system (1) the accessibility Lie algebra can be expressed as

$$\mathcal{L}_a = \{Px + Q \mid P \in \mathcal{A}, Q \in \mathcal{L}_{\mathrm{sa}}^{\sigma}\}. \tag{12}$$

Proof: Observe that the "$\supseteq$" relation is obvious because $Px \in \mathcal{L}_a$ for any $P \in \mathcal{A}$, and consequently

$$\mathcal{L}_a \supseteq \{Px + Q \,| P \in \mathcal{A}, Q \in \mathcal{L}_{\mathrm{sa}}^{\sigma}\}.$$

Next, we consider the structure of vector fields in $\mathcal{L}_a$. It is easy to see that $A_i x \in \mathcal{L}_a, B_i \in \mathcal{L}_a, i = 1, \ldots, N$. Then since $A_i x + B_i u$ is a linear combination of $A_i x$ and B_i, we have that

$$\mathcal{L}_a = \{A_i x, B_i \mid i = 1, \ldots, N\}_{\mathrm{LA}}. \tag{13}$$

Consider a bracket-product of $A_i x$ and B_i. A straightforward computation shows that: (a) a product containing more than one B_i is zero; b) a product containing one B_i, using Jacobi identity, can be expressed as the sum of terms of the form

$$A_{i_1} \cdots A_{i_k} B_i \tag{14}$$

which is obviously contained in $\mathcal{L}_{\mathrm{sa}}^{\sigma}$; and c) a product containing no B_i is in $\mathcal{A}$, because Lemma 3.2 says that

$$\{A_1 x, \ldots, A_N x\}_{\mathrm{LA}} = \mathcal{A}x. \tag{15}$$

From (14) and (15), any $\xi \in \mathcal{L}_a$ can be expressed as $\xi = Px + Q$, where $P \in \mathcal{A}$ and $Q \in \mathcal{L}_{\mathrm{sa}}^{\sigma}$, which proves "$\subseteq$" relation. □

Next, we prove a result which is the counterpart of Theorem 2.3 of nonlinear systems.

Proposition 3.4: If (1) satisfies accessibility rank condition at p_0, then the reachable set $R(p_0)$ contains a nonempty set of interior points. If system (1) satisfies strong accessibility rank condition at p_0, then for any $T > 0$ the reachable set $R_T(p_0)$ contains a nonempty set of interior points.

Proof: Note that each switching model is an analytic system. Denote $L = \{A_i x + B_i u | u = \mathrm{constant} \in \mathbb{R}^m\}$. Then, every vector field in L is a vector field of (1) with a suitable choice of constant controls and switching laws.

Assume that (1) satisfies accessibility rank condition at p_0, then there exists a neighborhood U of p_0 such that $\dim(\mathcal{L}_a(p)) = n$ for all $p \in U$. Choose any nonzero vector field $X_1 \in L$ and denote its integral curve as $\pi_1(t_1, x(0)) = e_{t_1}^{X_1}(p_0)$. If all $X \in L$ are linearly dependent with X_1 in U, then $\dim(\mathcal{L}_a(p)) = 1, p \in U$, which is a contradiction. So, there exists a t_1^0, a vector field $X_2 \in L$, such that at $p_1 = e_{t_1^0}^{X_1}(p_0)$ the vectors $X_1(p_1)$ and $X_2(p_1)$ are linearly independent. Therefore, we can construct a mapping as $\pi_2(t_1, t_2, x(0)) = e_{t_2}^{X_2} e_{t_1}^{X_1}(p_0)$, such that its Jacobian matrix $J_{\pi_2}(t_1^0, 0)$ has full rank. So it is a local diffeomorphism. Same argument shows that we can find (t_1^0, t_2^0) and $X_3 \in L$ (t_1^0 may not be the same as before), such that $\pi_3(t_1, t_2, t_3, x(0)) = e_{t_3}^{X_3} e_{t_2}^{X_2} e_{t_1}^{X_1}(p_0)$ is a local diffeomorphism at $(t_1^0, t_2^0, 0)$. Repeating the same procedure, one can construct a mapping

$$\pi_n(t_1, t_2, \ldots, t_n, x(0)) = e_{t_n}^{X_n} \cdots e_{t_2}^{X_2} e_{t_1}^{X_1}(p_0) \tag{16}$$

which is a local diffeomorphism at $(t_1^0, \ldots, t_n^0) \in \mathbb{R}^n$. Then there exists an open neighborhood V of $(t_1^0, \ldots, t_n^0)$, such that $\pi_n^{-1}(V) \subset R(p_0)$ is an open set. The conclusion follows.

Next, we prove the second part of the proposition. Assume (1) satisfies strong accessibility rank condition at x_0. Similar to the discussion for nonlinear systems [12], an additional variable $x_{n+1} = t$ can be added to the original system to get an extended system as

$$\begin{cases} \dot{x} = A_{\sigma(t)} x + B_{\sigma(t)} u \\ \dot{x}_{n+1} = 1. \end{cases} \tag{17}$$

It is easy to show that the extended system satisfies the accessibility rank condition at (p_0, T) for any $T > 0$. Recall the aformentioned proof for the first part of this proposition. It is easy to see that in the mapping (16), $t_1 + t_2 + \cdots + t_n > 0$ can be arbitrarily small.

Now, consider a moment $T_0 > 0$ and $T_0 < T$. Since (17) satisfies accessibility rank condition at (p_0, T_0), (16) can be constructed for $t_1 + t_2 + \cdots + t_n < T - T_0$. (Precisely, for (17) the mapping should be from $(t_1, \ldots, t_{n+1})$ to $\mathbb{R}^n$, and where $t_{n+1} = T_0 + t_1 + \cdots + t_n$.) So, (17) has a nonempty interior at $T_0 + t_1 + \cdots + t_n := T_1 < T$, which implies that for (1) $R_{T_1}(p_0)$ has a nonempty interior. Since $T_1 < T$, choosing $u(t)$ and $\sigma(t)$ such that

$$A_{\sigma(t)} + B_{\sigma(t)} u(t) \neq 0, \qquad T_1 \leq t \leq T$$

then $R_{T_1}(p_0)$ is diffeomorphic to $R_T(p_0)$. The conclusion follows. □

An alternative (more direct) proof of the second part of Proposition 3.4 can be found in [9].

Using generalized Chow's theorem, we have the following result.

Proposition 3.5: Consider system (1). The weakly reachable set of any $p \in \mathbb{R}^n$, denoted by $\mathrm{WR}(p)$, is the largest integral submanifold of $\mathcal{L}_a$ passing through p.

IV. Controllability and Weak Controllability

In this section, we will first reveal a topological structure of (weak) controllable submanifolds for a switched linear system with $\dim(\mathcal{L}_a) < n$.

Let V be the controllable subspace of system (1) (defined as in Definition 1.1). The following proposition is an immediate consequence of Theorem 1.2.

Proposition 4.1: Consider (1). The controllable subspace is $V = \mathcal{I}(\mathcal{L}_{\mathrm{sa}}^{\sigma})(0)$, i.e.,the integral submanifold of the strong accessibility Lie algebra passing through 0.

Now, the state–space can be split as $x = (x^1, x^2)$, where x^1 is the coordinates of V. Since V contains B_i and is A_i invariant for all i, system (1) can be expressed as

$$\begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix} = \begin{pmatrix} A^{11}_{\sigma(t)} & A^{12}_{\sigma(t)} \\ 0 & A^{22}_{\sigma(t)} \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} + \begin{pmatrix} B_{\sigma(t)} \\ 0 \end{pmatrix} u. \tag{18}$$

The following proposition gives a partition for possible (weakly) controllable submanifolds.

Proposition 4.2: Consider (1). For any $p_0 \in \mathbb{R}^n$, if $p_0 \notin V$, then $\mathrm{WR}(p_0) \cap V = \emptyset$.

Proof: Case 1) Assume $Ap_0 \in \mathcal{L}^\sigma_{\mathrm{sa}}(p_0)$ for all $A \in \mathbf{A}$. By Lemma 2.7, at any point $p \in \mathrm{WR}(p_0)$

$$\dim(\mathcal{L}_a(p)) = \dim(\mathcal{L}_a(p_0)) = \dim\left(\mathcal{L}^\sigma_{\mathrm{sa}}(p_0)\right).$$

Since $\mathcal{L}^\sigma_{\mathrm{sa}}$ consists of constant vector fields, one sees that $\dim(\mathcal{L}_a(p)) = \dim(\mathcal{L}^\sigma_{\mathrm{sa}}(p))$ for all $p \in \mathrm{WR}(p_0)$, that is

$$\dim(\mathcal{L}_a(p)) = \dim\left(\mathcal{L}^\sigma_{\mathrm{sa}}(p)\right) \quad \forall p \in \mathcal{I}(\mathcal{L}_a)(p_0).$$

Consequently, $\mathcal{I}(\mathcal{L}_a)(p_0) = \mathcal{I}(\mathcal{L}^\sigma_{\mathrm{sa}})(p_0)$. Again, since $\mathcal{L}^\sigma_{\mathrm{sa}}$ consists of constant vector fields, it follows that $\mathcal{I}(\mathcal{L}^\sigma_{\mathrm{sa}})(p_0) = p_0 + V$. Hence, $WP(p_0) \subseteq \mathcal{I}(\mathcal{L}_a)(p_0) = p_0 + V$. Since $p_0 \notin V$, $(p_0 + V) \cap V = \emptyset$.

Case 2) Assume $\mathcal{A}p_0 \not\subset \mathcal{L}^\sigma_{\mathrm{sa}}(p_0)$, then

$$\dim(\mathcal{L}_a(p_0)) > \dim(\mathcal{L}^\sigma_{\mathrm{sa}}(p_0)). \tag{19}$$

If $\mathrm{WR}(p_0) \cap V \neq \emptyset$, then $p_0 \in \mathrm{WR}(0)$. By Lemma 2.7 and the fact that $\mathcal{L}_a(0) = \mathcal{L}^\sigma_{\mathrm{sa}}(0)$

$$\dim(\mathcal{L}_a(p_0) = \dim(\mathcal{L}_a(0)) = \dim\left(\mathcal{L}^\sigma_{\mathrm{sa}}(0)\right)$$

which contradicts (19). □

Remark 4.3: From Proposition 4.2, it is clear that if the initial point $x(0) = p_0 \notin V$ there is no piecewise constant control which can drive it to zero.

From the decomposed form of (18), we denote

$$\mathcal{A}_0 := \left\{ A^{22}_i \middle| i = 1, \ldots, N \right\}_{\mathrm{LA}}. \tag{20}$$

Then, we can define a projection $\pi : \mathcal{A} \to \mathcal{A}_0$ in a natural way by letting $\pi(A_i) = A^{22}_i, i = 1, \ldots, N$. Note that the projection π is defined under a chosen coordinate frame. However, it is easy to prove that this projection is coordinate independent, because the quotient space $\mathbb{R}^n/V$ is A_i invariant. Let $\mathcal{L}^o_a = \{(0, w)^T | w = Ex^2, E \in \mathcal{A}_0\}$. Then, one can see the following.

Proposition 4.4: The mapping π is a Lie algebra homomorphism. Moreover, $\mathcal{L}_a = \mathcal{L}^o_a + \mathcal{L}^\sigma_{\mathrm{sa}}$. □

It is obvious that $\mathcal{A}_0$ is uniquely determined by

$$\mathbf{A_0} := \{A_i \,|\, 1 \leq i \leq N; A_i \notin \ker(\pi)\}.$$

Remark 4.5: From Proposition 4.2 and the structure of $\mathcal{L}_a$, the topological structure of the (weak) controllability sub-manifolds is characterized as follows.

- System (1) is globally controllable if and only if $\dim(\mathcal{L}_a) = n$.
- As $\dim(\mathcal{L}_a) < n$, (1) cannot even be globally weakly controllable. In this case, the state space is split into two disjoint parts: Subspace V and its complement $V^c = \mathbb{R}^n \backslash V$.
- With (18), let $V = \{x \in \mathbb{R}^n | x^2 = 0\}$, so $V^c = \{x \in \mathbb{R}^n | x^2 \neq 0\}$. Assume $\dim(V) = n - k$. If $k > 1, V^c$ is a pathwise connected open set, and if $k = 1$, V^c is composed of two path-wise connected open sets: $\{x | x^2 < 0\}$ and $\{x | x^2 > 0\}$.

Based on the previous argument, it is clear that the (weak) controllability of (1) is determined by its (weak) controllability on V^c. In a later discussion, we assume V^c is a pathwise connected open set. As for $k = 1$, it will be replaced by two pathwise connected open sets: $\{x | x^2 < 0\}$ and $\{x | x^2 > 0\}$.

Definition 4.6: For a subspace $W \subset \mathbb{R}^n$, a transformation P is called W-invariant if $PW \subset W$.

Definition 4.7: System (18) is V-invariant, feedback block diagonalizable, if there exist a V-invariant transformation P, feedback controls $u_i = K_i x + v_i, i = 1, \ldots, N$ such that the feedback models have block-diagonal form. That is, with $z = Px$, (18) becomes

$$\begin{pmatrix} \dot{z}^1 \\ \dot{z}^2 \end{pmatrix} = \begin{pmatrix} A^{11}_{\sigma(t)} & 0 \\ 0 & A^{22}_{\sigma(t)} \end{pmatrix} \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} + \begin{pmatrix} B_{\sigma(t)} \\ 0 \end{pmatrix} v \tag{21}$$

where the matrices A^{11}_θ, A^{22}_θ and B_θ, using same notations for the sake of simplicity, are different from those in (18).

The following lemma is necessary for proving the main result in this section.

Lemma 4.8: Consider system (3). Let U be a pathwise connected open set, and $\dim(\mathcal{L}_a(p)) = n$ for all $p \in U$. Then, for any $p \in U$, the largest integral submanifold of the Lie algebra satisfies $\mathcal{I}(\mathcal{L}_a)(p) \supseteq U$.

Proof: If it is not true, there exists a point $q \in U$, which is also on the boundary of $\mathcal{I}(p)$, i.e., $q \in \overline{\mathcal{I}(p)} \backslash \mathcal{I}(p)$, where we have used $\mathcal{I}(p)$ to denote $\mathcal{I}(\mathcal{L}_a)(p)$ for simplicity of notations. Since $\dim(\mathcal{L}_a(q)) = n$ on $U, \mathcal{I}(q)$ contains a neighborhood of q, and consequently, there exists $q_0 \in \mathcal{I}(p) \cap \mathcal{I}(q)$. It then follows that $\mathcal{I}(p) \cup \mathcal{I}(q)$ is a connected integral manifold of $\mathcal{L}_a$ passing through p, and it is larger than $\mathcal{I}(p)$, which is a contradiction. □

By slightly abusing notations, we say that a subset $S \subseteq \mathbb{R}^n$ is controllable (weakly controllable) if $R(p) = S$ ($\mathrm{WR}(p) = S$ respectively) for all $p \in S$. The following is the main result about controllability.

Theorem 4.9: For (21), assume that $\dim(V) = n - k$, and V^c is pathwise connected. Then, the following hold.

i) V^c is weakly controllable if and only if

$$\dim(\mathcal{A}_0 p) = k, \qquad p \in V^c. \tag{22}$$

ii) Assume $\mathbf{A_0}$ is symmetric, then V^c is controllable if and only if (22) holds.

Proof: i) (Necessity) If there exists a $p_0 \in V^c$ with $\dim \mathcal{A}_0(p_0) = s < k$, then, $\mathrm{WR}(p_0) = \mathcal{I}(\mathcal{L}_a)(p_0)$, and according to the generalized Chow's theorem, the right-hand side is a submanifold of dimension $n - k + s < n$. Thus, V^c can not be weakly controllable. (Sufficiency) Let $p, q \in V^c$. We have to find a $T > 0$, a switching law $\sigma(t)$ with switching moments

$$0 = t_0 < t_1 < \cdots < t_s = T$$

and a switched vector field

$$X(t) = A_{\sigma(t)} + B_{\sigma(t)} u_{\sigma(t)}, \qquad 0 < t < T$$

such that $q = x(T) = e^{X_s}_{d_s} \cdots e^{X_1}_{d_1}(p)$, where $X_i = X(t), \quad t_{i-1} \le t < t_i, d_i = t_i - t_{i-1}$.

Denote the starting point and the destination as

$$p = \begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix} \quad q = \begin{pmatrix} x_1^1 \\ x_1^2 \end{pmatrix}.$$

We design the control in three steps. First, since V is controllable, we can find switching law and controls such that at a moment T_0 we have $x^1(T_0) = 0$. Note that $x^2(T_0)$ is determined accordingly.

Now, by controllability [9], there are $X_k^1 = A_{i_k}^{11} x + B_{i_k} u_{i_k}, k = 1, \ldots, s$, such that

$$x_1^1 = x^1(T_2) = e^{X_s^1}_{d_s} \cdots e^{X_1^1}_{d_1}(0) \tag{23}$$

for some T_2. Let $\{s_1, \ldots, s_l\} \subset \{1, 2, \ldots, s\}$ be such that

$$A_{s_j} \in A_0, \qquad j = 1, \ldots, l.$$

Define

$$x_2^2 := e^{X_{s_1}^2}_{-d_{s_1}} \cdots e^{X_{s_l}^2}_{-d_{s_l}} \left(x_1^2\right) \tag{24}$$

where $X_{s_i}^2 = A_{s_i}^{22} x^2$.

Then, in step 2), $x^2(T_0)$ can be steered to x_2^2 (in the weak sense) in time T_1 for some T_1. This can be done because of Lemma 4.8. Note that in step 2), the first component of states x^1 can be kept in the origin by setting all controls $v_i = 0$. Now, in step 3), we use the controls in (23) to move x^1 to the destination x_1^1. At the same time, according to (24), x^2 moves from x_2^2 to x_1^2.

ii) Same argument for weak controllability can be used for controllability. The only difference is, now since $\mathbf{A_0}$ is symmetric, by Chow's theorem, x_2^2 is reachable from $x^2(T_0)$. □

Next, we define the local controllability.

Definition 4.10: System (1) is locally controllable at p_0, if there exists a neighborhood U of p_0 such that

$$p \in R(p_0) \qquad \forall p \in U.$$

The system is locally weakly controllable at p_0, if there exists a neighborhood U of p_0 such that

$$p \in \mathrm{WR}(p_0) \qquad \forall p \in U.$$

Using the similar argument as in Theorem 4.9, we can easily prove the following local controllability result.

Corollary 4.11: For (21), assume $\dim(V) = n - k$.

i) For a point $p_0 \in V^c$, the system is weakly locally controllable at p_0 if and only if

$$\dim(\mathcal{A}_0 p_0) = k. \tag{25}$$

ii) Assume $\mathbf{A_0}$ is symmetric. Then, the system is locally controllable at x_0 if and only if (25) holds.

It was remarked by an anonymous referee for the original version of this note that "how to verify (25) in finite steps is a major question. For second-order systems, [15] has addressed this in detail." We give the following proposition for verifying it.

Since $\mathcal{A}_0 \subset gl(k, \mathbb{R})$ is a finite dimensional Lie algebra, it is easy to find its basis. Let $E_1, \ldots, E_t$ be a basis of $\mathcal{A}_0$. Then, we have the following.

Proposition 4.12: Define $M(x_0) := \sum_{i=1}^t E_i x_0 x_0^T E_i^T$. Then, (25) holds if and only if $det(M(x_0)) > 0$.

Proof: It is obvious that (25) holds iff

$$\xi^T (E_1 x_0, \ldots, E_t x_0) = 0$$

implies $\xi = 0$. Since

$$M(x_0) = (E_1 x_0, \ldots, E_t x_0)(E_1 x_0, \ldots, E_t x_0)^T$$

the conclusion follows. □

Finally, we consider a stabilization problem for (21).

Definition 4.13: A controllable switched linear system is said to be proper if there exist two $\mathcal{K}$-functions φ_1 and φ_2 such that for any $x_0 \in \mathbb{R}^n$, there is a reachable time $T(x_0)$ (under suitable control and switching for x_0 to reach zero) satisfies the following:

$$\begin{aligned} &T(x_0) \le \varphi_1(\|x_0\|) \\ &\|x(t)\| \le \varphi_2(\|x_0\|), \qquad 0 \le t \le T(x_0) \end{aligned} \tag{26}$$

(and $x(T(x_0)) = 0$.)

For the z^2-subsystem in (21):

$$\dot{z}^2(t) = A_{\theta(t)}^{22} z^2(t), \qquad z^2(t) \in \mathbb{R}^n \tag{27}$$

we have the following result which by itself is interesting.

Proposition 4.14: For (27), assume that: i) (22) holds; and ii) $\mathbf{A_0}$ is symmetric. Then the system is exponentially stabilizable.

Proof: Let $y(t; p, \theta)$ denote the solution of the z^2-subsystem with the initial state $z^2(0) = p$ corresponding to the switching law θ. Let $S = \{p \in \mathbb{R}^{n-k} : |p| = 1\}$. Then, by Theorem 4.9, for each $q \in S$, there exists some switching function θ_q that steers q to a point inside the neighborhood $B(0, 1/2) := \{p : |p| < 1/2\}$ of 0 in time T_q for some $T_q > 0$. By continuity, one sees that for each q, there exists a neighborhood $\mathcal{N}_q$ such that θ_q steers every point $p \in \mathcal{N}_q$ into $B(0, 1/2)$ in time T_q. Let $\{\mathcal{N}_{q_i}\}_{i=1}^l$ be a finite cover of S. Denote θ_{q_i} by θ_i, T_{q_i} by T_i, and $\mathcal{N}_{q_i}$ by $\mathcal{N}_i$. For each $p \in S$, $p \in \mathcal{N}_i$ for some i, and $|y(\tau; p, \theta_i)| = 1/2$ for some $\tau \le T_i$. If $p \in \mathcal{N}_i \cap \mathcal{N}_j$, choose either θ_i or θ_j for p.

Assume now that $p \in \mathbb{R}^{n-k}, p \ne 0$. Choose $\tilde{\theta}_1 \in \{\theta_1, \ldots, \theta_l\}$ such that $|y(t_1; p/|p|, \tilde{\theta}_1)| = 1/2$ for some $t_1 \le T := \max\{T_1, \ldots, T_l\}$. By linearity, $|y(t_1; p, \tilde{\theta}_1)| = |p|/2$. Let $p_1 = x(t_1; p, \theta)$. Repeat the process with p_{k-1} replaced by p_k for $k \ge 1$ inductively. Let $\theta(t) = \tilde{\theta}_i(t)$ on $[t_{i-1}, t_i)$. Let $M = \max_{1 \le \sigma \le N}\{|A_\sigma^{22}|\}$. Then $|y(t_k; p, \theta)| \le |p|/2^k$ for each k, and for $t \in [t_{k-1}, t_k)$, $|y(t; p, \theta)| \le |y(t_{k-1})| e^{M(t - t_{k-1})}$. Hence, working with $t \in [t_{k-1}, t_k)$, one has

$$|y(t; p, \theta)| \le \frac{|p|}{2^{k-1}} e^{MT} \le L|p|e^{-at}$$

where $L = 2e^{MT}$, $a = (\ln 2)/(T)$. □

Note that in the above discussions, the choices of switching functions are based on "event driven." It remains to be explored further to design switching laws in the form of a state feedback.

The following holds as a consequence of Proposition 4.14.

Corollary 4.15: Assume that: i) (22) holds; ii) $\mathbf{A_0}$ is symmetric; and iii) the controllable sub-system of (21) is proper. Then, (21) is stabilizable.

Proof: First, drive the z^1-component to 0 in a finite time, and then work with the switching functions to drive the z^2-component to neighborhoods of 0. So we have

$$\lim_{t\to\infty} z(t) = 0. \tag{28}$$

Then the assumption of "properness" assures that for any $\delta > 0$ there exists $\epsilon > 0$ such that $\|z(0)\| < \epsilon$ implies that $\|x(t)\| < \delta$ for all $t > 0$. □

Remark:

1) If one is only required to get the attraction property (28), then the "properness" assumption is not needed.
2) Our conjecture is that every controllable switched linear system is proper. (It is obviously true for no-switched linear system.) We leave this for future study.

V. An Illustrating Example

Example 5.1: Consider the following system with $n = 4, m = 1$ and $N = 2$. Two models $(A_i, B_i), i = 1, 2$ are

$$\left(\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right) \quad \left(\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right).$$

We skip some routine computation, and show the results directly. The controllable subspace is

$$\mathcal{L}_{\rm sa}^{\sigma} = \mathrm{span}\left\{\begin{pmatrix}I_2\\0\end{pmatrix}\right\}.$$

Let $x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$. Then

$$\mathcal{L}_a(x) = \begin{cases}\mathcal{L}_{\rm sa}^{\sigma}, & x_3 = x_4 = 0\\ T_x(\mathbb{R}^4), & \text{otherwise}\end{cases}$$

where $T_x(M)$ stands for the tangent space of a manifold M at x. Define a subspace

$$V = \{x \in \mathbb{R}^4 \mid x_3 = x_4 = 0\}.$$

Then, we know that the weakly reachable set of x is

$$\mathrm{WR}(x) = \begin{cases}\mathcal{I}(\mathcal{L}_{\rm sa}^{\sigma})(0) = V, & x \in V\\ \mathbb{R}^4\backslash V, & x \notin V\end{cases}$$

where $\mathcal{I}(D)$ is used for the integral manifold of a distribution D. Now assume we add two switching models $(A_i, B_i), i = 3, 4$ to i) as

$$\left(\begin{pmatrix}-1&2&0&2\\1&2&0&2\\0&0&-1&0\\0&0&0&-1\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\end{pmatrix}\right)$$
$$\left(\begin{pmatrix}1&0&-1&0\\3&1&1&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}, \begin{pmatrix}1\\-1\\0\\0\end{pmatrix}\right).$$

Then one sees easily that $\mathbf{A_0}$ is symmetric. So the weak controllability becomes controllability. Precisely,

$$R(x) = \begin{cases}\mathcal{I}(\mathcal{L}_{\rm sa}^{\sigma})(0) = V, & x \in V\\ \mathbb{R}^4\backslash V, & x \notin V\end{cases}$$

Note that we need pre-feedback controls to block diagonalize A_3 and A_4 first. □

VI. Conclusion

In this note the controllability of switched linear system was considered. The main contribution of the note consists of: 1) the (strong) accessibility Lie algebra for nonlinear systems has been extended to switched linear systems; 2) a clear topological structure for the possible controllable submanifolds is provided; and 3) some necessary and sufficient conditions for the (standard or weak, global or local) controllability of a large class of switched linear systems is obtained.

References

[1] C. Altafini, "The reachable set of a linear endogenous switching system," *Syst. Control Lett.*, vol. 47, no. 4, pp. 343–353, 2002.

[2] R. W. Brockett, "Nonlinear systems and differential geometry," *Proc. IEEE*, vol. 64, no. 1, pp. 61–71, Jan. 1976.

[3] J. Ezzine and A. H. Haddad, "Controllability and observability of hybrid systems," *Int. J. Control*, vol. 49, no. 6, pp. 2045–2055, 1989.

[4] R. Hermann, *Differential Geometry and the Calculus of Variations*. New York: Academic, 1968.

[5] A. J. Krener, "A generalization of Chow's theorem and the bang-bang theorem to nonlinear control problems," *SIAM J. Control*, vol. 12, no. 1, pp. 43–52, 1974.

[6] T. Nagano, "Linear differential sysems with singularities and an application to transitive Lie algebra," *J. Math. Soc.*, vol. 18, pp. 398–404, 1966.

[7] P. J. Olver, *Applications of Lie Groups to Differential Equations*, 2nd ed. New York: Springer-Verlag, 1993.

[8] Z. Sun and D. Zheng, "On reachability and stabilization of switched linear systems," *IEEE Trans. Autom. Control*, vol. 46, no. 2, pp. 291–295, Feb. 2001.

[9] Z. Sun, S. S. Ge, and T. H. Lee, "Controllability and reachability criteria for switched linear systems," *Automatica*, vol. 38, pp. 775–786, 2002.

[10] H. J. Susmann and V. Jurdjevic, "Controllability of nonlinear systems," *J. Diff. Equat.*, vol. 12, pp. 95–116, 1972.

[11] H. J. Susmann, "Lie brackets and local controllability: A sufficient condition," *SIAM J. Control*, vol. 21, pp. 686–713, 1983.

[12] H. J. Susmann, "A general theorem on local controllability," *SIAM J. Control Optim.*, vol. 25, pp. 158–194, 1987.

[13] F. Szigeti, "A differential-algebraic condition for controllability and observability of time varying linear systems," in *Proc. 31st Conf. Decision and Control*, 1992, pp. 3088–3090.

[14] G. Xie, D. Zheng, and L. Wang, "Controllability of switched Linear systems," *IEEE Trans. Autom. Control*, vol. 47, no. 8, pp. 1401–1405, Aug. 2002.

[15] X. Xu and P. J. Antsaklis, On the reachability of a class of second-order switched systems, Univ. Notre Dame, Notre Dame, IN, Tech. Rep. ISIS-99-0003, Feb. 1999.