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Brief paper

Problems on time-varying port-controlled Hamiltonian systems: geometric structure and dissipative realization ☆

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Abstract

To apply time-varying port-controlled Hamiltonian (PCH) systems to practical control designs, two basic problems should be dealt with: one is how to provide such time-varying systems a geometric structure to guarantee the completeness of representations in mathematics; and the other is how to express the practical system under consideration as a time-varying PCH system, which is called the dissipative Hamiltonian realization problem. The paper investigates the two basic problems. A suitable geometric structure for time-varying PCH systems is proposed first. Then the dissipative realization problem of time-varying nonlinear systems is investigated, and serval new methods and sufficient conditions are presented for the realization.

Keywords: Time-varying PCH system; Geometric structure; Dissipative Hamiltonian realization

1. Introduction

In recent years, time-invariant port-controlled Hamiltonian (PCH) systems have been well investigated (see, e.g., van der Schaft, 1999; Nijmeijer & van der Schaft, 1990; Maschke, Ortega, & van der Schaft, 2000; Ortega, van der Schaft, Maschke, & Escobar, 2002; Escobar, van der Schaft, & Ortega, 1999). The Hamiltonian function in a PCH system is considered as the total energy, which is the sum of potential and kinetic energies in mechanical systems, and it can play the role of Lyapunov function for the system. Because of this, based on time-invariant PCH systems, various effective controllers have been designed for many control problems (see, e.g., Shen, Ortega, Lu, Mei, & Tamura, 2000; Wang, Cheng, Li, & Ge, 2003; Xi & Cheng, 2000). However, for some practical systems the time-invariant PCH structure does not easily apply and its time-varying form is really needed. Please see the following example.

Example 1. Consider a single-machine infinite-bus power system (Lu & Sun, 1993):

$$\begin{aligned}
\dot{\delta} &= \omega - \omega_0,\\
\dot{\omega} &= \frac{\omega_0}{M}P_m - \frac{D}{M}(\omega-\omega_0) - \frac{\omega_0 E_q' V_s}{M x_{d\Sigma}'}\sin\delta + w_1,\\
\dot{E}_q' &= -\frac{1}{T_d'}E_q' + \frac{1}{T_{do}}\frac{x_d - x_d'}{x_{d\Sigma}'}V_s\cos\delta + \frac{1}{T_{d0}}u_f + w_2,
\end{aligned}$$

where w_1 and w_2 are disturbances, δ is the power angle, ω the rotor speed, E_q' the q-axis internal transient voltage, u_f the control input, and V_s the infinite-bus voltage. As for other parameters, please refer to Lu and Sun (1993). In the case that all the parameters are constant, we can use the time-invariant PCH structure to design an effective controller to attenuate the disturbances w_1 and w_2 (Xi & Cheng, 2000). But as well known, in power systems there are always uncertainties caused by load-level variations, faults, or

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changes of network structure, etc. When a parameter of the above system is affected by a time-varying signal, say, V_s is affected by a sine signal $\sin t$, the time-invariant structure is no longer valid for the system. In this case, to design an effective energy-based controller, the time-varying PCH structure is really needed.

Therefore, it is necessary to develop the theory of time-varying PCH systems for some practical control problems. Recently, time-varying PCH systems have been studied by Fujimoto and Sugie (2001a,b), Fujimoto, Sakurama, and Sugie (2003) and Cheng (2002). It is worth noticing that Fujimoto et al. (2003) set up a very important way to the trajectory tracking control of time-varying PCH systems via generalized canonical transformations, whose key idea was to preserve the structure of PCH systems under both coordinate and feedback transformations. At present, in order to apply time-varying PCH systems to practical control designs, two basic problems should be dealt with: one is how to define a geometric structure on a manifold for such systems to guarantee the completeness of representations in mathematics; and the other is how to express the practical system under consideration into a time-varying PCH system. The latter is the so-called dissipative Hamiltonian realization problem.

This paper investigates the above-mentioned two problems. First, by defining a time-varying generalized Poisson bracket, we provide a geometric structure for time-varying PCH systems. Then, we deal with the dissipative Hamiltonian realization of time-varying nonlinear systems, and propose some new methods and sufficient conditions for the realization.

The rest of the paper is organized as follows. Section 2 briefly reviews the classical Poisson structure, and Section 3 provides the geometric structure for time-varying PCH systems. In Section 4, we deal with the dissipative Hamiltonian realization problem, which is followed by the conclusion in Section 5.

2. A brief review of Poisson structure

This section briefly reviews the classical Poisson structure with Lie algebraic properties, which will motivate the next section of the paper.

In order to define a Hamiltonian system on a manifold, one should equip the manifold with a suitable geometric structure first. Let $\mathscr{M}$ be a smooth manifold and $C^\infty(\mathscr{M})$ be the set of smooth functions on $\mathscr{M}$. A Poisson bracket on $\mathscr{M}$, denoted by $\{\cdot,\cdot\}$, is a map: $C^\infty(\mathscr{M}) \times C^\infty(\mathscr{M}) \longmapsto C^\infty(\mathscr{M})$, satisfying (Ortega & Planas-Bielsa, 2004; Olver, 1993):

(i) Bilinearity:

$$\{aF + bG, H\} = a\{F, H\} + b\{G, H\},$$

$$\{F, aG + bH\} = a\{F, G\} + b\{F, H\};$$

(ii) skew-symmetry: $\{F, H\} = -\{H, F\}$;

(iii) Jacobian identity:

$$\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0; \quad \text{and}$$

(iv) Leibniz' rule: $\{F, HG\} = \{F, H\}G + H\{F, G\}$,

where $\forall F, G, H \in C^\infty(\mathscr{M})$, $\forall a, b \in \mathbb{R}^1$. Obviously, the Poisson bracket defines a Lie algebra structure on the algebra $C^\infty(\mathscr{M})$ (Ortega & Planas-Bielsa, 2004). The pair $(\mathscr{M}, \{\cdot,\cdot\})$ is called a Poisson manifold, and the bracket defines a Poisson structure on $\mathscr{M}$.

Assume $H \in C^\infty(\mathscr{M})$ is an arbitrary smooth function. We define $X_H := \{\cdot, H\}$, which is called a Hamiltonian vector field. System $\dot{x} = X_H$ is called a Hamiltonian system defined on $\mathscr{M}$, and H is its Hamiltonian function.

It should be pointed out that the manifold $\mathscr{M}$ used here does not need to be an even-dimensional one, for the Poisson bracket defined above has dropped the property of non-degeneracy (Libermann & Marle, 1986).

In recent years, it has been well noticed that a weakening of the defining conditions of the Poisson bracket is sometimes a necessary and useful way to accommodate the description of more general dynamical systems (Ortega & Planas-Bielsa, 2004; van der Schaft, 1999; Olver, 1993). Motivated by this, in the next section we will provide a geometric structure for time-varying PCH systems.

3. Geometric structure for time-varying PCH systems

This section is to provide a geometric structure for time-varying PCH systems. First, we give the concept of time-varying generalized Poisson brackets, and then, we present the geometric structure for time-varying PCH systems.

Definition 1. Let $\mathscr{M}$ be an n-dimensional manifold and time $t \in \mathbb{R}^+ := [0, \infty)$. A time-varying generalized Poisson bracket (GPB), denoted by $\{\cdot,\cdot\}_t$, is a map: $C^\infty(\mathscr{M} \times \mathbb{R}^+) \times C^\infty(\mathscr{M} \times \mathbb{R}^+) \longmapsto C^\infty(\mathscr{M} \times \mathbb{R}^+)$, satisfying

(i) Bilinearity:

$$\begin{aligned}&\{aF(x,t) + bG(x,t), H(x,t)\}_t\\&\quad = a\{F(x,t), H(x,t)\}_t + b\{G(x,t), H(x,t)\}_t,\\&\{F(x,t), aG(x,t) + bH(x,t)\}_t\\&\quad = a\{F(x,t), G(x,t)\}_t + b\{F(x,t), H(x,t)\}_t;\end{aligned} \tag{1}$$

(ii) Leibniz' rule:

$$\begin{aligned}&\{F(x,t), G(x,t)H(x,t)\}_t\\&\quad = \{F(x,t), G(x,t)\}_t H(x,t)\\&\qquad + G(x,t)\{F(x,t), H(x,t)\}_t,\\&\{F(x,t)G(x,t), H(x,t)\}_t\\&\quad = \{F(x,t), H(x,t)\}_t G(x,t)\\&\qquad + F(x,t)\{G(x,t), H(x,t)\}_t,\end{aligned} \tag{2}$$

where $\forall F(x,t), G(x,t), H(x,t) \in C^\infty(\mathscr{M} \times \mathbb{R}^+)$, and $\forall a, b \in \mathbb{R}^1$.

Remark 1. The time-varying GPB defined above has dropped the following: (a) nondegeneracy; (b) integrability (Jacobian identity); and (c) skew-symmetry. As for the detailed development of Poisson brackets, please see Libermann and Marle (1986), Ortega and Planas-Bielsa, 2004, van der Schaft (1999) and references therein.

Definition 2. A time-varying GPB is called to be symmetric if $\{F(x,t), G(x,t)\}_t = \{G(x,t), F(x,t)\}_t$, and it is called to be skew-symmetric if $\{F(x,t), G(x,t)\}_t = -\{G(x,t), F(x,t)\}_t$, for $\forall F(x,t), G(x,t) \in C^\infty(\mathscr{M} \times \mathbb{R}^+)$.

With Definitions 1 and 2, we obtain the following result.

Proposition 1. (i) *Assume that* $\{\cdot,\cdot\}_t^{(1)}, \{\cdot,\cdot\}_t^{(2)}$ *are two time-varying GPBs. Then,*

$$a\{\cdot,\cdot\}_t^{(1)} + b\{\cdot,\cdot\}_t^{(2)} \tag{3}$$

is still a time-varying GPB.

(ii). *An arbitrary time-varying GPB* $\{\cdot,\cdot\}_t$ *can be uniquely decomposed as*

$$\{\cdot,\cdot\}_t = \{\cdot,\cdot\}_t^{\mathrm{J}} + \{\cdot,\cdot\}_t^{\mathrm{S}}, \tag{4}$$

where $\{\cdot,\cdot\}_t^{\mathrm{J}}$ *is skew-symmetric and* $\{\cdot,\cdot\}_t^{\mathrm{S}}$ *is symmetric.*

Proof. (i). Obviously.

(ii). Assume that $\{\cdot,\cdot\}_t$ is an arbitrary time-varying GPB. Set

$$\{F, G\}_t^{\mathrm{S}} = \tfrac{1}{2}(\{F, G\}_t + \{G, F\}_t),$$

$$\{F, G\}_t^{\mathrm{J}} = \tfrac{1}{2}(\{F, G\}_t - \{G, F\}_t). \tag{5}$$

From (i), we know that $\{\cdot,\cdot\}_t^{\mathrm{S}}, \{\cdot,\cdot\}_t^{\mathrm{J}}$ are time-varying GPBs. It is easy to show that $\{\cdot,\cdot\}_t^{\mathrm{S}}$ is symmetric, $\{\cdot,\cdot\}_t^{\mathrm{J}}$ is skew-symmetric and

$$\{\cdot,\cdot\}_t = \{\cdot,\cdot\}_t^{\mathrm{J}} + \{\cdot,\cdot\}_t^{\mathrm{S}}. \tag{6}$$

Assume that $\{\cdot,\cdot\}_t$ has another decomposition as follows:

$$\{\cdot,\cdot\}_t = \{\cdot,\cdot\}_t^{(1)} + \{\cdot,\cdot\}_t^{(2)}, \tag{7}$$

where $\{\cdot,\cdot\}_t^{(1)}$ is skew-symmetric and $\{\cdot,\cdot\}_t^{(2)}$ is symmetric. (6) minus (7) yields

$$\{\cdot,\cdot\}_t^{\mathrm{J}} - \{\cdot,\cdot\}_t^{(1)} = \{\cdot,\cdot\}_t^{(2)} - \{\cdot,\cdot\}_t^{\mathrm{S}}. \tag{8}$$

The left-hand side of (8) is skew-symmetric and the right-hand side is symmetric, thus $\{\cdot,\cdot\}_t^{\mathrm{J}} - \{\cdot,\cdot\}_t^{(1)} \equiv 0$, $\{\cdot,\cdot\}_t^{(2)} - \{\cdot,\cdot\}_t^{\mathrm{S}} \equiv 0$. Therefore, the decomposition (4) is unique. □

Assume that $\{\cdot,\cdot\}_t$ is a time-varying GPB. For $\forall H(x,t) \in C^\infty(\mathscr{M} \times \mathbb{R}^+)$, let $X_H := \{\cdot, H(x,t)\}_t$. Then, X_H is a map: $C^\infty(\mathscr{M} \times \mathbb{R}^+) \longmapsto C^\infty(\mathscr{M} \times \mathbb{R}^+)$. Obviously, it is a time-varying vector field on $\mathscr{M}$. We call X_H a time-varying generalized Hamiltonian vector field.

Definition 3. (i) A time-varying generalized Poisson manifold is a manifold equipped with a time-varying GPB.

(ii). A time-varying generalized Hamiltonian system is a triple $(\mathscr{M}, \{\cdot,\cdot\}_t, H(x,t))$. Its dynamic expression is $\dot{x} = X_H$.

We define the structure matrix of the time-varying GPB $\{\cdot,\cdot\}_t$ as follows:

$$M(x,t) := \begin{bmatrix} \{x_1, x_1\}_t & \{x_1, x_2\}_t & \cdots & \{x_1, x_n\}_t \\ \{x_2, x_1\}_t & \{x_2, x_2\}_t & \cdots & \{x_2, x_n\}_t \\ & \cdots & & \cdots \\ \{x_n, x_1\}_t & \{x_n, x_2\}_t & \cdots & \{x_n, x_n\}_t \end{bmatrix}, \tag{9}$$

which is expressed in a set of local coordinates $x_1, \ldots, x_n$.

Since X_H is a vector field, X_H can be expressed as $X_H = \sum_{i=1}^n \xi_i(x,t)\partial/\partial x_i$. From Proposition 1, $X_H = \{\cdot, H\}_t = \{\cdot, H\}_t^{\mathrm{J}} + \{\cdot, H\}_t^{\mathrm{S}} := X_H^{\mathrm{J}} + X_H^{\mathrm{S}}$. It is easy to see that X_H^{J} and X_H^{S} are also vector fields and $X_H^{\mathrm{J}} = \sum_{i=1}^n \xi_i^{\mathrm{J}}(x,t)\partial/\partial x_i$, $X_H^{\mathrm{S}} = \sum_{i=1}^n \xi_i^{\mathrm{S}}(x,t)\partial/\partial x_i$. Thus,

$$X_H = \sum_{i=1}^n \xi_i(x,t)\frac{\partial}{\partial x_i} = \sum_{i=1}^n [\xi_i^{\mathrm{J}}(x,t) + \xi_i^{\mathrm{S}}(x,t)]\frac{\partial}{\partial x_i}. \tag{10}$$

With Eq. (10), Leibniz' rule and the bilinearity, we can prove the following result.

Theorem 1. (i) *For* $\forall F(x,t), H(x,t) \in C^\infty(\mathscr{M} \times \mathbb{R}^+)$,

$$\{F(x,t), H(x,t)\}_t = (\nabla F)^{\mathrm{T}} M(x,t) \nabla H; \tag{11}$$

(ii) *The time-varying generalized Hamiltonian system defined in Definition* 3 *can be expressed as*

$$\dot{x} = M(x,t)\nabla H(x,t). \tag{12}$$

From Theorem 1, we can see that, as in the standard Poisson bracket case, the time-varying GPB is also determined uniquely by its structure matrix.

Assume that $y = \Phi(x)$ is a coordinate transformation. We can prove that under the new coordinates, the structure matrix $M(x,t)$ becomes

$$\bar{M}(y,t) = J_\Phi M(x,t) J_\Phi^{\mathrm{T}}|_{x=\Phi^{-1}(y)}, \tag{13}$$

where J_Φ is the Jacobian matrix of $\Phi(x)$; (13) indicates that the structure matrix $M(x,t)$ is consistent with the changing law of structure matrices under coordinate transformations.

From the above discussion, we know that the time-varying generalized Poisson manifold $(\mathscr{M}, \{\cdot,\cdot\}_t)$ can serve as a suitable geometric structure for time-varying generalized Hamiltonian systems and, of course, for time-varying PCH systems.

4. Dissipative Hamiltonian realization

This section investigates the dissipative Hamiltonian realization of time-varying nonlinear systems, and proposes several new results. First, we give some concepts and properties.

4.1. Concepts and properties

Definition 4. A time-varying dynamic system

$$\dot{x} = f(x,t), \quad x \in \mathscr{M}, \quad t \in \mathbb{R}^+ \tag{14}$$

is said to have a generalized Hamiltonian realization (GHR) if there exists a suitable coordinate chart and a Hamiltonian function $H(x,t)$ such that (14) can be expressed as

$$\dot{x} = M(x,t)\nabla H(x,t), \tag{15}$$

where $\mathscr{M}$ is an n-dimensional manifold and $M(x,t)$ is the structure matrix of some time-varying GPB defined on $\mathscr{M}$. Furthermore, if $M(x,t)$ can be decomposed as $M(x,t) = J(x,t) - R(x,t)$, with $J(x,t)$ skew-symmetric and $R(x,t)\geqslant 0$ symmetric, then (15) is called a dissipative Hamiltonian realization.

Definition 5. A controlled dynamic system

$$\dot{x} = f(x,t) + \sum_{i=1}^{m} g_i(x,t)u_i \tag{16}$$

is said to have a feedback GHR if there exists a feedback law $u=\alpha(x,t)+v$ such that the closed-loop system can be expressed as

$$\dot{x} = M(x,t)\nabla H(x,t) + g(x,t)v, \tag{17}$$

where $g(x,t) = (g_1(x,t),\ldots,g_m(x,t))$ and $u=(u_1,\ldots,u_m)^{\mathrm{T}}$.

From Definition 4, we know that system (15) is a dissipative realization iff $M(x,t)+M(x,t)^{\mathrm{T}}\leqslant 0$.

Recall the concept of $\mathscr{K}$-functions (Slotine & Li, 1991). A continuous function $\alpha : \mathbb{R}^+ \longmapsto \mathbb{R}^+$ is called a $\mathscr{K}$-function if (i) $\alpha(0)=0$; (ii) $\alpha(p)>0, \ \forall p>0$; (iii) $\alpha \uparrow$ strictly.

Assume that system (16) has a dissipative realization as follows:

$$\dot{x} = (J(x,t) - R(x,t))\nabla H(x,t) + g(x,t)v. \tag{18}$$

We can prove the following proposition.

Proposition 2. *If* $\partial H/\partial t \leqslant 0$ *and there exists a* $\mathscr{K}$*-function* α *such that* $H(x,t)\geqslant\alpha(||x||)>0, \forall x\neq 0$, *then system* (18) *with* $v=0$ *is Lyapunov stable*.

The following lemma is equivalent to Part 1 of Theorem 1 in Fujimoto et al. (2003).

Lemma 1 (Fujimoto et al., 2003). *Assume that* $(\mathscr{M}, \{\cdot,\cdot\}_t)$ *is the geometric structure of system* (18). *Then, under the structure, the dissipative Hamiltonian system* (18) *is changed into another dissipative Hamiltonian system by a coordinate transformation* $z=\Phi(x,t)$ *if and only if there exists a scalar function* $\bar{H}$ *such that* $X_{\bar{H}} = J_{\Phi}^{-1}(\partial\Phi/\partial t)$ *holds on* $(\mathscr{M},\{\cdot,\cdot\}_t)$.

Corollary 1. *The dissipativeness of system* (18) *is invariant under time-invariant coordinate transformations.*

4.2. New results on dissipative Hamiltonian realization

A function $V(x)$ is called a regular positive definite function (Wang, Li, & Cheng, 2003) if $V(x)>0$ $(x\neq 0)$, $V(0)=0$, $\partial V/\partial x|_{x=0}=0$ and $\partial V/\partial x|_{x\neq 0}\neq 0$. For example, $H(x)=\frac{1}{2}\sum_{i=1}^n x_i^2$ is a regular positive definite function on $\mathbb{R}^n$.

Consider the following time-varying nonlinear system

$$\dot{x} = f(x,t) + g(x,t)u, \quad f(0,t)=0, \tag{19}$$

where $x\in\mathscr{M}$, $t\in\mathbb{R}^+$, $u\in\mathbb{R}^m$. Motivated by Ortega et al. (2002), we obtain the following result.

Proposition 3. *For arbitrary regular positive definite function* $H(x)$, *system* (19) *can be expressed as*

$$\dot{x} = (J(x,t)+P(x,t))\frac{\partial H(x)}{\partial x} + g(x,t)u, \tag{20}$$

where

$$P(x,t) = \begin{cases} \frac{\langle f(x,t),\nabla H(x)\rangle}{||\nabla H(x)||^2} I_n, & x\neq 0,\\ 0, & x=0 \end{cases} \tag{21}$$

is symmetric,

$$J(x,t) = \begin{cases} \frac{1}{||\nabla H(x)||^2}\left[f_{\mathrm{td}}(x,t)\frac{\partial H^{\mathrm{T}}(x)}{\partial x} - \frac{\partial H(x)}{\partial x} f_{\mathrm{td}}^{\mathrm{T}}(x,t)\right], & x\neq 0,\\ 0, & x=0 \end{cases} \tag{22}$$

is skew-symmetric, $\langle\cdot,\cdot\rangle$ *denotes the inner product,* I_n *is the* $n\times n$ *identity matrix, and*

$$\begin{aligned} f_{\mathrm{td}}(x,t) &= f(x,t) - f_{\mathrm{gd}}(x,t),\\ f_{\mathrm{gd}}(x,t) &= \frac{\langle f(x,t),\nabla H(x)\rangle}{||\nabla H(x)||^2}\nabla H(x), \quad x\neq 0. \end{aligned} \tag{23}$$

Proof. From (23), we get

$$\begin{aligned} L_{f_{\mathrm{td}}}H(x) &= \langle f_{\mathrm{td}}(x,t),\nabla H(x)\rangle\\ &= \langle f(x,t),\nabla H(x)\rangle - \frac{\langle f(x,t),\nabla H(x)\rangle}{||\nabla H(x)||^2}\langle\nabla H(x),\nabla H(x)\rangle\\ &= \langle f(x,t),\nabla H(x)\rangle - \langle f(x,t),\nabla H(x)\rangle = 0, \end{aligned}$$

from which we know that when $x \neq 0$,

$$\begin{aligned}
&J(x,t)\frac{\partial H(x)}{\partial x}\\
&\quad= \frac{1}{||\nabla H(x)||^2}\left[f_{\mathrm{td}}(x,t)\frac{\partial H^{\mathrm{T}}(x)}{\partial x}\right.\\
&\qquad\left.-\frac{\partial H(x)}{\partial x}f_{\mathrm{td}}^{\mathrm{T}}(x,t)\right]\frac{\partial H(x)}{\partial x}\\
&\quad= \frac{1}{||\nabla H(x)||^2}f_{\mathrm{td}}(x,t)\frac{\partial H^{\mathrm{T}}(x)}{\partial x}\frac{\partial H(x)}{\partial x}\\
&\qquad- \frac{1}{||\nabla H(x)||^2}\frac{\partial H(x)}{\partial x}f_{\mathrm{td}}^{\mathrm{T}}(x,t)\frac{\partial H(x)}{\partial x}\\
&\quad= \frac{1}{||\nabla H(x)||^2}f_{\mathrm{td}}(x,t)||\nabla H(x)||^2\\
&\qquad- \frac{1}{||\nabla H(x)||^2}\frac{\partial H(x)}{\partial x}L_{f_{\mathrm{td}}}H(x) = f_{\mathrm{td}}(x,t).
\end{aligned}$$

Thus, when $x \neq 0$,

$$\begin{aligned}
f(x,t) &= f_{\mathrm{td}}(x,t) + f_{\mathrm{gd}}(x,t)\\
&= J(x,t)\frac{\partial H(x)}{\partial x} + P(x,t)\frac{\partial H(x)}{\partial x}\\
&= (J(x,t)+P(x,t))\frac{\partial H(x)}{\partial x}.
\end{aligned}$$

Note that, when $x=0$, the above equation still holds. Then, the theorem follows. $\square$

Remark 2. (i) f_{gd} is the projection of $f(x,t)$ in the gradient direction ∇H and f_{td} is the projection in the tangential direction of equi-value surfaces of $H(x)$. Obviously, $f_{\mathrm{gd}} \perp f_{\mathrm{td}}$ and $f = f_{\mathrm{gd}} + f_{\mathrm{td}}$. Therefore, realization (20) has a clear physical meaning. (ii) Since there always exist regular positive definite functions, an arbitrary time-varying dynamic system always has the realization (20), which can be calculated by the formulas (21)–(23).

Remark 3. From the proof of Proposition 3, we know that $f(x,t) \equiv (J(x,t)+P(x,t))(\partial H(x)/\partial x)$. Thus, when system (19) is smooth, realization (20) is smooth, too. However, even if system (19) is smooth, Proposition 3 cannot ensure that matrices $P(x,t)$ and $J(x,t)$ are smooth around the origin. In general, the proposition can only guarantee the continuity of $P(x,t)$ and $J(x,t)$ if $H(x)$ is chosen properly.

Now, at regular points[1] of $P(x,t)$, decompose $P(x,t)$ as $P(x,t) = -R(x,t) + S(x,t)$, where $R(x,t)\geqslant 0$ and $S(x,t)\geqslant 0$ are symmetric. Then, system (20) can be expressed as

$$\dot{x} = (J(x,t) - R(x,t) + S(x,t))\nabla H(x) + g(x,t)u. \tag{24}$$

[1] We call x a regular point of matrix $P(x,t)$, if there exists a neighborhood, Ω, of x such that the number of positive eigenvalues and the number of negative eigenvalues are invariant for $x \in \Omega$ and $t \in \mathbb{R}^+$.

Proposition 4. *If*

$$S(x,t) \subset Span\{g(x,t)\} + Ker\{\mathrm{d}H(x)\}, \tag{25}$$

then there is a control law $u = \alpha(x,t) + v$ such that the closed-loop system consisted of (24) *and the control law can be expressed as a dissipative Hamiltonian system with $H(x)$ as its Hamiltonian function.*

Proof. If (25) holds, there exist matrices B and C with proper dimensions such that

$$S(x,t) = g(x,t)B + (\xi_1, \xi_2, \ldots, \xi_r)C, \tag{26}$$

where $\{\xi_1, \ldots, \xi_r\}$ is a basis of $Ker\{\mathrm{d}H(x)\}$. Choose $u = -B\nabla H(x) + v$ and substitute it into (24), then we get

$$\begin{aligned}
\dot{x} &= (J(x,t) - R(x,t))\nabla H(x)\\
&\quad + (\xi_1, \ldots, \xi_r)C\nabla H(x) + g(x,t)v.
\end{aligned} \tag{27}$$

Let $\bar{f}(x,t) = (\xi_1, \ldots, \xi_r)C\nabla H(x)$. Since $\nabla H^{\mathrm{T}}(x)\bar{f}(x,t) = 0$, from the proof of Proposition 3 we know

$$\bar{f}(x,t) = \bar{J}(x,t)\nabla H(x), \tag{28}$$

where

$$\bar{J}(x,t) = \begin{cases} \frac{1}{||\nabla H(x)||^2}[\bar{f}(x,t)\frac{\partial H^{\mathrm{T}}(x)}{\partial x} - \frac{\partial H(x)}{\partial x}\bar{f}^{\mathrm{T}}(x,t)], & x \neq 0,\\ 0, & x = 0. \end{cases}$$

With (27) and (28), we obtain

$$\dot{x} = (J(x,t) + \bar{J}(x,t) - R(x,t))\nabla H(x) + g(x,t)v, \tag{29}$$

which is a dissipative Hamiltonian realization. $\square$

In the following, we consider the single-input case of system (19). In this case, we have the following result.

Theorem 2. *Assume that $g(x,t) \in \mathbb{R}^{n\times 1}$. If there exists a regular positive definite function $H(x)$ such that $L_{g(x,t)}H(x) \neq 0$ ($x \neq 0$), then system* (19) *has a feedback dissipative Hamiltonian realization with $H(x)$ as its Hamiltonian function.*

Proof. Assume that there exists a regular positive definite function $H(x)$ such that $L_g H \neq 0$, for all $x \neq 0$. Similar to the above discussion, we can obtain system (24) with this $H(x)$.

When $x = 0$, it is obvious that the theorem holds. When $x \neq 0$, choose control law

$$u = \frac{1}{L_g H(x)}[L_g H(x)v - \nabla H^{\mathrm{T}}(x)S(x,t)\nabla H(x)] \tag{30}$$

and substitute it into (24), then we have

$$\begin{aligned}\dot{x} &= [J(x,t) - R(x,t) + S(x,t)]\nabla H(x) \\ &\quad + \frac{1}{L_g H(x)} g[L_g H(x) v - \nabla H^{\mathrm{T}}(x) S(x,t) \nabla H(x)] \\ &= (J(x,t) - R(x,t))\nabla H(x) + S(x,t)\nabla H(x) \\ &\quad - \frac{1}{L_g H(x)} g \nabla H^{\mathrm{T}}(x) S(x,t) \nabla H(x) + gv \\ &= (J(x,t) - R(x,t))\nabla H(x) \\ &\quad + \frac{1}{L_g H(x)} [S(x,t)\nabla H L_g H - g\nabla H^{\mathrm{T}} S(x,t)\nabla H] \\ &\quad + gv \\ &= (J(x,t) - R(x,t))\nabla H \\ &\quad + \frac{1}{L_g H(x)} [S(x,t)\nabla H g^{\mathrm{T}} - g\nabla H^{\mathrm{T}} S(x,t)]\nabla H(x) \\ &\quad + gv.\end{aligned}$$

Thus,

$$\dot{x} = (J(x,t) + \tilde{J}(x,t) - R(x,t))\nabla H(x) + g(x,t)v, \tag{31}$$

where

$$\begin{aligned}\tilde{J}(x,t) &= \frac{1}{L_g H(x)} [S(x,t)\nabla H(x) g^{\mathrm{T}} \\ &\quad - g\nabla H^{\mathrm{T}}(x) S(x,t)]\end{aligned} \tag{32}$$

is skew-symmetric. Therefore, (31) is a dissipative Hamiltonian realization. □

4.3. An illustrative example

In this subsection, we give an example to illustrate how to apply the results proposed in Section 4.2 to get Hamiltonian realizations.

Example 2. Find a feedback law such that the following system can be expressed as a dissipative Hamiltonian system:

$$\begin{aligned}\dot{x} &= \begin{bmatrix} tx_1 + t^2 x_1^2 x_2 \\ -t^2 x_1^3 + t x_2 \\ t x_3 \end{bmatrix} + \begin{bmatrix} x_1 + t(x_1 + x_2) \\ x_2 + t(x_2 - x_1) \\ x_3 + t x_3 \end{bmatrix} u \\ &:= f(x,t) + g(x,t)u, \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}^+.\end{aligned} \tag{33}$$

First, we apply Proposition 3 to express (33) as a time-varying generalized Hamiltonian system. Choose a regular positive definite function as: $H(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$. When $x \neq 0$, from (23) we obtain

$$\begin{aligned}f_{\mathrm{td}}(x,t) &= f(x,t) - \frac{\langle f(x,t), \nabla H(x)\rangle}{||\nabla H(x)||^2} \frac{\partial H(x)}{\partial x} \\ &= \begin{bmatrix} tx_1 + t^2 x_1^2 x_2 \\ -t^2 x_1^3 + t x_2 \\ t x_3 \end{bmatrix} - \frac{t x_1^2 + t x_2^2 + t x_3^2}{x_1^2 + x_2^2 + x_3^2} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= [t^2 x_1^2 x_2, \quad -t^2 x_1^3, \quad 0]^{\mathrm{T}}.\end{aligned}$$

Thus, when $x \neq 0$, from (22) and (21) we have

$$\begin{aligned}J(x,t) &= \frac{1}{||\nabla H(x)||^2} [f_{\mathrm{td}}(x,t)\nabla H^{\mathrm{T}}(x) \\ &\quad - \nabla H(x) f_{\mathrm{td}}^{\mathrm{T}}(x,t)] \\ &= \frac{t^2 x_1^2}{x_1^2 + x_2^2 + x_3^2} \begin{bmatrix} 0 & x_1^2 + x_2^2 & x_2 x_3 \\ -x_1^2 - x_2^2 & 0 & -x_1 x_3 \\ -x_2 x_3 & x_1 x_3 & 0 \end{bmatrix},\end{aligned}$$

$$P(x,t) = \frac{\langle f(x,t), \nabla H(x)\rangle}{||\nabla H(x)||^2} I_3 = \begin{bmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{bmatrix}.$$

Therefore, (33) can be expressed as

$$\dot{x} = \begin{cases} (J(x,t) + P(x,t)) \frac{\partial H(x)}{\partial x} + g(x,t)u & x \neq 0, \\ g(x,t)u, & x = 0. \end{cases} \tag{34}$$

Second, we design a control law to make the system be expressed as a dissipative form. Because $L_g H(x) = (1+t)(x_1^2 + x_2^2 + x_3^2) \neq 0$ ($x \neq 0$), it can be seen from Theorem 2 that system (34) has a feedback dissipative realization with $H(x)$ as its Hamiltonian function. Decompose $P(x,t)$ as follows: $P(x,t) = -R(x,t) + S(x,t)$, where $R(x,t) = Diag\{1,1,1\} > 0$, $S(x,t) = Diag\{1+t, 1+t, 1+t\} > 0$. When $x \neq 0$, according to (30) we choose the control law as

$$\begin{aligned}u &= \frac{1}{L_g H} [L_g H(x) v - \nabla H^{\mathrm{T}}(x) S(x,t) \nabla H(x)] \\ &= \frac{1}{(1+t)(x_1^2 + x_2^2 + x_3^2)} \left\{ (1+t)(x_1^2 + x_2^2 + x_3^2) v \right. \\ &\qquad \left. - [x_1, x_2, x_3] \begin{bmatrix} 1+t & 0 & 0 \\ 0 & 1+t & 0 \\ 0 & 0 & 1+t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right\} \\ &= -1 + v;\end{aligned}$$

and, when $x = 0$, we choose $u = v$. Under the control law, system (34) can be expressed as

$$\begin{aligned}\dot{x} &= \begin{cases} (J(x,t) + \tilde{J}(x,t) - R(x,t))\nabla H(x) & \\ \quad + g(x,t)v, & x \neq 0 \\ g(x,t)v, & x = 0, \end{cases} \\ &= \begin{cases} (\bar{J}(x,t) - R(x,t))\nabla H(x) + g(x,t)v, & x \neq 0 \\ g(x,t)v, & x = 0, \end{cases}\end{aligned} \tag{35}$$

where

$$\begin{aligned}\tilde{J}(x,t) &= \frac{1}{L_g H} [S(x,t)\nabla H g^{\mathrm{T}} - g\nabla H^{\mathrm{T}} S(x,t)] \\ &= \frac{-t}{x_1^2 + x_2^2 + x_3^2} \begin{bmatrix} 0 & x_1^2 + x_2^2 & x_2 x_3 \\ -x_1^2 - x_2^2 & 0 & -x_1 x_3 \\ -x_2 x_3 & x_1 x_3 & 0 \end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}\bar{J}(x,t) &= J(x,t) + \tilde{J}(x,t) \\ &= \frac{t^2 x_1^2 - t}{x_1^2 + x_2^2 + x_3^2} \begin{bmatrix} 0 & x_1^2 + x_2^2 & x_2 x_3 \\ -x_1^2 - x_2^2 & 0 & -x_1 x_3 \\ -x_2 x_3 & x_1 x_3 & 0 \end{bmatrix}\end{aligned}$$

are skew-symmetric. System (35) is the desired feedback dissipative Hamiltonian realization.

5. Conclusion

Through defining a time-varying generalized Poisson bracket, we have provided a suitable geometric structure for time-varying PCH systems, which can guarantee the mathematical completeness of representations of time-varying PCH systems. In order to apply time-varying PCH systems to practical control problems, we have also investigated the dissipative Hamiltonian realization problem of time-varying nonlinear systems, and proposed serval new methods and sufficient conditions for the realization.

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