show the stability of the network dominated by the TCP congestion control algorithm under the normal offered load condition.

An assumption of the bandwidth allocation in this note is that the *instant* traffic load at each link can not exceed the link capacity; see the constraint (4). Concerning the practical operation of data networks supporting elastic traffic (including in particular the Internet), one may question whether such assumption would be realistic. For the Internet, when some links are carrying multiple TCP connections, it is possible for there to be a steady-state packet drop rate on those links, and for the (instant) arrival rate at those links to equal or slightly exceed the link capacity. Future work will include investigating whether the results in this note provide a good approximation to this environment.

It is known that the exponential assumption on the document size of a connection is often violated in the real network. Can we relax this assumption in our model so that the stability result would be more robust? We believe that the relaxation would be a significant but challenging step toward a better understanding of the network dynamics. Under the exponential assumption, the network can be modeled as a continuous time Markov chain for which analytical tools are available. To extend the model to allow more general document size assumption, it is necessary to keep track of remaining untransmitted document sizes on all connections in order to capture the network dynamics. For this purpose, more sophisticated stochastic model is required and studying the stability for the network model with exponential document size assumption would be helpful.

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On p-Normal Forms of Nonlinear Systems

Daizhan Cheng and Wei Lin

***Abstract*—Using the differential-geometric control theory, we present in this note a necessary and sufficient condition under which an affine system is locally feedback equivalent to, via a change of coordinates and restricted smooth state feedback, a generalized normal form called p—*normal form*, which includes Brunovsky canonical form and feedback linearizable systems in a lower-triangular form as its special cases. We also give an algorithm for computing the appropriate coordinate transformations and feedback control laws.**

***Index Terms*—Differential geometric approach, feedback equivalence, local diffeomorphism, nonlinear systems, p-normal form, state feedback.**

I. INTRODUCTION

The past two decades have witnessed a rapid growth of research efforts aimed at the development of systematic analysis and design methodologies for nonlinear control systems. Many powerful analysis and synthesis techniques have been developed based on the use of differential geometric approach [5], [15].

The differential geometric approach was emerged in the 1970s and gained strong momentum around 1980s due to a series of original work [1], [4], [6]–[8], [10], [22], [23]. In [8], the problem of equivalence between an affine system and a linear system was first investigated and solved by a change of coordinates (local diffeomorphism) *without feedback*. Later, Brockett gave a necessary and sufficient condition for affine systems to be locally diffeomorphic to linear controllable systems by using not only coordinate transformations but also state feedback of the type $u = \alpha(\xi) + v$. This is the so-called exact feedback linearization problem which has been widely studied in the literature. For instance, the works by Jakubczyk and Respondek [7], Su [22], and Hunt *et al.* [4] were stimulated directly by [1] and [8]. These papers provided a complete solution to the feedback linearization problem. Subsequent contributions by Krener *et al.* [9], Marino [14], and Respondek [20] addressed the partial feedback linearization problem by identifying a class of systems that consists of a maximal linear subsystem cascaded by a lower-dimensional nonlinear subsystem. On the other hand, the discovery of “zero-dynamics” of a nonlinear control system [2], [5] and systematic use of this notion which leads to Byrnes–Isidori’s normal form (composed of a nonlinear zero dynamics driven by a chain of integrators), have led to a number of significant advances in the area of nonlinear feedback design, including asymptotic stabilization of minimum-phase systems by state feedback, output regulation of nonlinear systems, feedback equivalence to a passive system and robust and adaptive control of nonlinear systems.

When a control system is inherently nonlinear and is neither fully nor partially feedback linearizable (e.g., the linearized system is *null* or uncontrollable and the uncontrollable modes are associated with eigen-

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values on the right-half plane), the papers by Respondek [21] and by Celikovsky and Nijmeijer [3] studied the question whether there exists a higher order normal form that is locally diffeomorphic to an affine system. This important and fundamental issue will be further addressed in this note. In particular, we are interested in a class of nonlinear systems of the form

$$\begin{aligned}
\dot{x}_1 &= x_2^{p_1} + \sum_{i=0}^{p_1-1} x_2^i \phi_i^1(x_1)\\
&\vdots\\
\dot{x}_{n-1} &= x_n^{p_{n-1}} + \sum_{i=0}^{p_{n-1}-1} x_n^i \phi_i^{n-1}(x_1, x_2, \ldots, x_{n-1})\\
\dot{x}_n &= v
\end{aligned} \tag{1}$$

where $x = (x_1, \ldots, x_n)^T \in R^n$ and $v \in R$ are the system state and control input, respectively. p_i, $i = 1, \ldots, n-1$, are positive integers, and $\phi_i^l : R^l \to R, i = 1, \ldots, p_l - 1, l = 1, \ldots, n-1$, are smooth functions with $\phi_i^l(0, \ldots, 0) = 0$. For system (1) with p_i, $1 \le i \le n-1$, being *odd* positive integers, a series of exciting results have been obtained recently, including global strong stabilization by non-Lipschitz continuous feedback [16], [17], global practical output tracking [18], disturbance attenuation or decoupling [19], adaptive control of nonlinearly parameterized systems [12], [13].

When $p_i = 1$, $i = 1, \ldots, n-1$, (1) reduces to a feedback linearizable system in a lower-triangular form, which has been extensively investigated over the last decade. If $p_i \ge 1$, $i = 1, \ldots, n-1$, are positive integers and $\phi_i^l(x_1, \ldots, x_l) \equiv 0$ for all $i = 1, \ldots, p_l - 1$, $l = 1, \ldots, n-1$, (1) becomes a chain of power integrators perturbed by a lower triangular vector field—a class of nonlinear systems that received considerable attention recently [11], [16]. Finally, (1) also includes the class of systems in [21] and the well-known Brunovsky canonical form as its special cases. In view of the previous discussions, (1) can be naturally regarded as a generalized canonical form and we refer it as a p*-normal form* throughout this note.

The purpose of this note is to study the problem of when a single-input affine control system

$$\dot{\xi} = f(\xi) + g(\xi)u, \qquad f(0) = 0 \tag{2}$$

with f and g being smooth vector fields defined on an open set U in $\mathbb{R}^n$ containing $\xi = 0$, is locally diffeomorphic to the p-normal form (1) by a change of coordinates and restricted state feedback, i.e., by the following actions:

i) a local diffeomorphism $x = T(\xi)$ defined on U;
ii) a smooth state feedback $u = \alpha(\xi) + \beta v$, with $\alpha(0) = 0$, $0 \ne \beta = \text{constant}, \forall \xi \in U$.

For the sake of convenience, the problem is called the p*-normalization problem*. Accordingly, an affine control system that can be transformed into (1) is said to be p*-normalizable*.

In this note, we shall address the p-normalization problem and provide a partial answer in terms of differential geometric control theory. In particular, we present a necessary and sufficient condition for the p*-normalization problem* to be solvable for affine control systems. We also give an algorithm that enables one to find the transformation $x = T(\xi)$ and the state feedback $u = \alpha(\xi) + \beta v$ systematically. Finally, we use an example to demonstrate the theoretic results developed in this note. The example illustrates how an affine system with uncontrollable linearization can be transformed into the p-normal form (1) via a change of coordinates and state feedback, although it is *not feedback linearizable*.

II. Solvability Conditions of the p-Normalization Problem

Motivated by the study of exact feedback linearization, we investigate in this section the question of when an affine system is locally feedback equivalent to the p-normal form (1), under some appropriate assumptions. To begin with, we first introduce a number of basic concepts related to the p-normalization problem, which will be used in the rest of the note.

Definition 2.1: Given the vector fields $f(\xi)$ and $g(\xi)$ in (2), if there exists a sequence of m vector fields defined as

$$X_0 = g \quad \text{and} \quad X_{k+1} = ad_{X_k}^{q_{k+1}} f, \qquad k = 0, 1, \ldots, m-2 \tag{3}$$

where $q_{k+1} > 0$, $0 \le k \le m-2$, are the smallest positive integers such that $X_0, X_1, \ldots, X_{k+1}$, are linearly independent at $\xi = 0$, system (2) is said to have a *normalizable order* m and *minimum index* $(q_1, \ldots, q_{m-1})$.

Associated with the vector fields $X_0, \ldots, X_{m-1}$, one can define a set of nested distributions

$$\Delta_k = \text{span}\{X_0, X_1, \ldots, X_k\}, \qquad k = 0, 1, \ldots, m-1. \tag{4}$$

The next concept is a natural generalization of the notion of the relative degree.

Definition 2.2: The single-input–single-output (SISO) nonlinear system

$$\begin{aligned}
\dot{\xi} &= f(\xi) + g(\xi)u\\
y &= h(\xi)
\end{aligned} \tag{5}$$

with well-defined normalizable order m and minimum index $(q_1, \ldots, q_{m-1})$ is said to have a *generalized relative degree* ρ at $\xi = 0$ if there is an open neighborhood U containing $\xi = 0$, such that

1) $L_{X_i} h(\xi) = 0, 0 \le i \le \rho - 2, \forall \xi \in U$;
2) $L_{X_{\rho-1}} h(0) \ne 0$.

Remark 2.3: According to Definitions 2.1 and 2.2, a system in the p-normal form (1) has a minimum index $(p_{n-1}, \ldots, p_2, p_1)$, normalizable order n, and generalized relative degree n when setting $y = x_1$. For a feedback linearizable system, it is clear from [5] that its generalized relative degree is identical to the relative degree n, its normalizable order is equal to n, and the minimum index is $(1, 1, \ldots, 1)$ with $X_i = (-1)^i ad_f^i g$, $\Delta_i = \text{span}\{g, ad_f g, \ldots, ad_f^i g\}$.

Lemma 2.4: Assume that a SISO nonlinear system

$$\begin{aligned}
\dot{\xi} &= f(\xi) + g(\xi)u\\
y &= h(\xi)
\end{aligned} \tag{6}$$

has well-defined normalizable order m, minimum index $(q_1, \ldots, q_{m-1})$ and generalized relative degree $\rho \le m$. Then, they are all invariant under the actions of a change of coordinates $x = T(\xi)$ and a nonsingular state feedback $u = \alpha(\xi) + \beta(\xi)v$, where $\beta(\xi) \ne 0, \xi \in U$, if the following conditions hold:

A1) Δ_k, $k = 1, \ldots, \rho - 1$, are involutive;
A2) $ad_{X_{k-1}}^j f(\xi) \in \Delta_k \;\; \forall \xi \in U$, whenever $j < q_k$.

The proof of this lemma is given in the Appendix. Notably, the class of feedback $u = \alpha(\xi) + \beta(\xi)v$ used in Lemma 2.4 is a general one rather than the restricted feedback $u = \alpha(\xi) + \beta v$.

Remark 2.5: In the case of feedback linearizable systems, i.e., $p_1 = \cdots = p_{n-1} = 1$ in (1), conditions A1) and A2) are automatically satisfied.

Now we are ready to present the main results of this note. The first result characterizes a necessary and sufficient condition for the p-normalization problem to be solvable via a change of coordinates and the state feedback

$$u = \alpha(\xi) + v, \qquad \xi \in U. \tag{7}$$

Note that under an additional linear nonsingular transformation, (7) is equivalent to

$$u = \alpha(\xi) + \beta v \quad \beta = \text{constant} \neq 0, \qquad \xi \in U.$$

Theorem 2.6: The analytic affine system (2) can be transformed into the p-normal form (1) via a local diffeomorphism $x = T(\xi)$ and state feedback (7) if and only if the following conditions hold:

- C1) (2) has a normalizable order n and minimum index $(p_{n-1}, p_{n-2}, \ldots, p_1)$;
- C2) distributions Δ_k defined by (4), $k = 0, 1, \ldots, n-2$, are involutive on U;
- C3) $ad^j_{X_{k-1}} f(\xi) \in \Delta_k \ \forall \xi \in U$ whenever $1 \leq j \leq p_{n-k}$, for $k = 1, \ldots, n-1$;
- C4) $ad^j_{X_{k-1}} f(\xi) \in \Delta_{k-1}, \ \forall \xi \in U$ whenever $j > p_{n-k}$, for $k = 1, \ldots, n-1$.

Remark 2.7: It is clear from Remark 2.3 that C1) and C2) are a natural generalization of the well-known conditions of [5, Th. 4.2.6]. Indeed, in the case of exact feedback linearization (i.e., the minimal index of (2) is $(p_{n-1}, p_{n-2}, \ldots, p_1) = (1, 1, \ldots, 1)$), C1) reduces to the controllability condition i) of [5, Th. 4.2.6], while C2) amounts to the statement that Δ_k defined by (4), with $q_{k+1} = 1$ and $k = 0, 1, \ldots, n-1$, are involutive near the origin. Although the latter appears to be redundant and less intuitive than ii) of [5, Th. 4.2.6], it turns out that they are identical in the feedback linearizable case. Conditions C3) and C4) come from the highest order restriction on each power integral channel.

Proof of Theorem 2.6 (Necessity): The proof is carried out in two steps. First, we show that C1)–C4) are invariant under the change of coordinates and state feedback $u = \alpha(\xi) + \beta v$. That is, if C1)–C4) hold for (1), they also hold for the system after the two actions and vice versa.

Similar to the proof of Lemma 2.4, setting $\beta = \text{constant}$ in (36), (37) results in $\tilde{g} = \beta g$, $\tilde{X}_0 = \beta X_0$ and

$$ad^j_{\tilde{X}_k} \tilde{f} = \beta^{\left(j \prod_{i=1}^{k} p_{n-i}\right)} ad^j_{X_k} f + \sum_{i=0}^{k} a_{ij}(\xi) X_i \tag{8}$$

and, hence

$$\begin{aligned} \tilde{X}_k &= \beta^{\left(\prod_{i=1}^{k} p_{n-i}\right)} X_k + \sum_{i=0}^{k-1} \alpha_i(\xi) X_i \\ \tilde{\Delta}_i &= \Delta_i, \qquad i = 1, 2, \ldots, n-1. \end{aligned} \tag{9}$$

Clearly, C1)–C4) are invariant under the state feedback $u = \alpha(\xi) + \beta v$.

On the other hand, same arguments as the ones in Lemma 2.4 indicate that C1)–C4) are invariant under the local diffeomorphism $x = T(\xi)$.

Next, we verify that C1)–C4) hold for (1). Let $X_0 = g = (0, \ldots, 0, 1)^T$ and $\Delta_0 = \text{span}\{(\partial)/(\partial x_n)\}$. To verify C1)–C4), observe that

$$\begin{aligned} ad^j_{X_0} f = \Bigg(0, \ldots, 0, &\frac{p_{n-1}!}{(p_{n-1} - j)!} x_n^{p_{n-1}-j} \\ &+ \Sigma_{i=j}^{p_{n-1}-1} \frac{i!}{(i-j)!} x_n^{i-j} \phi_i^{n-1}, \ \star \Bigg)^T \end{aligned} \tag{10}$$

where $\star$ represents the last component of $ad^j_{X_0} f$.

From (10), it follows that $q_1 = p_{n-1}$ and $X_1 = (0, \ldots, 0, p_{n-1}!, *)^T$. Hence

$$\Delta_1 = \text{span}\{X_0, X_1\} = \text{span}\left\{\frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-1}}\right\}.$$

Obviously, rank $\{X_0(0), X_1(0)\} = 2$ and Δ_1 is involutive. Moreover, using (10) it is concluded that $ad^j_{X_0} f \in \Delta_1$ when $j \leq q_1 = p_{n-1}$ and $ad^j_{X_0} f \in \Delta_0$ when $j > q_1 = p_{n-1}$.

Inductively, when $k = i$ one can assume that $q_i = p_{n-i}$

$$X_i = \Bigg(0, \ldots, 0, \underbrace{p_{n-i}! \ldots p_{n-1}!}_{(n-i)\text{th}}, \star, \cdots, \star\Bigg)^T$$

$$\text{and} \quad \Delta_i = \text{span}\left\{\frac{\partial}{\partial x_n}, \ldots, \frac{\partial}{\partial x_{n-i}}\right\}.$$

Then, a direct calculation rank $\{X_0(0), X_1(0), \ldots, X_i(0)\} = i + 1$ and the distributions Δ_k, $k = 0, \ldots, i$, are involutive. In addition, C3)–C4) also hold for $k = i + 1$ with $q_{i+1} = p_{n-i-1}$. Finally

$$X_{i+1} = \Bigg(0, \ldots, 0, \underbrace{p_{n-i-1}! \ldots p_{n-1}!}_{(n-i-1)\text{th}}, \star, \cdots, \star\Bigg)^T$$

$$\text{and} \quad \Delta_{i+1} = \text{span}\left\{\frac{\partial}{\partial x_n}, \ldots, \frac{\partial}{\partial x_{n-i-1}}\right\}.$$

Using the aforementioned inductive argument, we conclude that C1)–C4) hold for (1).

(Sufficiency): Using the involutivity of Δ_i, one can assume, without loss of generality (by the Frobenius Theorem), that in the new coordinate $z = \Phi(\xi)$

$$\Delta_i(\xi) = \tilde{\Delta}_i(z) = \text{span}\left\{\frac{\partial}{\partial z_n}, \ldots, \frac{\partial}{\partial z_{n-i}}\right\}, \qquad i = 0, 1, \ldots, n-1$$

and $g(\xi)$ can be represented as $\tilde{g}(z) = (0, \ldots, 0, 1)^T$.

Denote $\tilde{f}(z) = ((\partial\Phi)/(\partial\xi)) f(\xi)|_{\xi=\Phi^{-1}(z)} = (\tilde{f}_1(z), \ldots, \tilde{f}_n(z))^T$. Note that C1)–C4) are invariant under a change of coordinates and $\tilde{X}_0 = \tilde{g}$. By C3), we have

$$ad_{\tilde{X}_0} \tilde{f} \in \tilde{\Delta}_1$$

which implies that $(\partial \tilde{f}_i)/(\partial z_n) = 0$ for $1 \leq i \leq n-2$ and, therefore $\tilde{f}_1(z), \ldots, \tilde{f}_{n-2}(z)$ are independent of z_n.

By analyticity of $\tilde{f}_{n-1}(z)$ and the Taylor expansion formula, we have

$$\tilde{f}_{n-1}(z_1, \ldots, z_{n-1}, z_n) = \Sigma_{i=0}^{\infty} c_i(z_1, \ldots, z_{n-1}) z_n^i$$

where $c_i(\cdot), i = 1, 2, \ldots$, are smooth functions.

Using the previous relationship, the $(n-1)$th component of the vector field $ad^j_{\tilde{X}_0} \tilde{f}$ is

$$\begin{aligned} \left(ad^j_{\tilde{X}_0} \tilde{f}\right)_{n-1} &= j! c_j(z_1, \ldots, z_{n-1}) + \sum_{i=j+1}^{\infty} \frac{i!}{(i-j)!} \\ &\quad \times c_i(z_1, \ldots, z_{n-1}) z_n^{i-j}, \qquad j \geq 1. \end{aligned} \tag{11}$$

By condition C3), $ad^j_{\tilde{X}_0} \tilde{f}(0)$ is linearly dependent with $\tilde{X}_0$ when $j < p_{n-1}$ and $ad^{p_{n-1}}_{\tilde{X}_0} \tilde{f}$ is independent of $\tilde{X}_0$ in the neighborhood of the origin. Thus

$$\begin{cases} c_j(0, \ldots, 0) = 0 & \text{when} \quad 0 \leq j \leq p_{n-1} - 1 \\ c_{p_{n-1}}(z_1, \ldots, z_{n-1}) \neq 0 & \forall (z_1, \ldots, z_{n-1}) \in V \subset R^{n-1}. \end{cases}$$

For those terms whose orders satisfy $j > p_{n-1}$, using (11) and C4) (i.e. $ad^j_{\tilde{X}_0} \tilde{f} \in \tilde{\Delta}_0$ when $j > p_{n-1}$) yields $c_j(z_1, \ldots, z_{n-1}) = 0$, $\forall j > p_{n-1}$.

In view of the aforementioned arguments, we conclude that

$$\tilde{f}(z) = \Big(\tilde{f}_1(z_1, \ldots, z_{n-1}), \ldots, \tilde{f}_{n-2}(z_1, \ldots, z_{n-1}), \ \tilde{f}_{n-1}(z), \ \tilde{f}_n(z)\Big)^T$$

where $\tilde{f}_{n-1}(z) = c_{p_{n-1}}(z_1, \ldots, z_{n-1}) z_n^{p_{n-1}} + \sum_{i=0}^{p_{n-1}-1} c_i(z_1, \ldots, z_{n-1}) z_n^i$ with $c_{p_{n-1}}(0, \ldots, 0) \neq 0$ and $c_i(0, \ldots, 0) = 0$ for $0 \leq i \leq p_{n-1} - 1$.

Since $c_{p_{n-1}}(z_1,\ldots,z_{n-1}) \neq 0 \ \forall(z_1,\ldots,z_{n-1}) \in V \subset R^{n-1}$, one can introduce the following transformation:

$$\tilde{z}_i = z_i, \qquad i \neq n-1 \quad \text{and} \quad \tilde{z}_{n-1} = \int_0^{z_{n-1}} \frac{1}{c_{p_{n-1}(z_1,\ldots,z_{n-2},s)}} ds \tag{12}$$

which leads to

$$\begin{aligned}\dot{\tilde{z}}_{n-1} = &\sum_{i=1}^{n-2} \dot{z}_i \int_0^{z_{n-1}} \frac{\partial}{\partial z_i}\left[\frac{1}{c_{p_{n-1}(z_1,\ldots,z_{n-2},s)}}\right] ds \\ &+ z_n^{p_{n-1}} + \frac{1}{c_{p_{n-1}(z_1,\ldots,z_{n-2},z_{n-1})}} \\ &\times \sum_{i=0}^{p_{n-1}-1} c_i(z_1,\ldots,z_{n-1}) z_n^i.\end{aligned} \tag{13}$$

Since $\dot{z}_i$, $i = 1,\ldots,n-2$, are independent of z_n and $\tilde{z}_{n-1}$ is only a function of $z_1,\ldots,z_{n-1}$, it follows immediately from (13) that

$$\begin{aligned}\dot{\tilde{z}}_1 &= \tilde{f}_1(\tilde{z}_1,\ldots,\tilde{z}_{n-1}) \\ &\vdots \\ \dot{\tilde{z}}_{n-2} &= \tilde{f}_{n-2}(\tilde{z}_1,\ldots,\tilde{z}_{n-1}) \\ \dot{\tilde{z}}_{n-1} &= \tilde{z}_n^{p_{n-1}} + \Sigma_{i=0}^{p_{n-1}-1} \tilde{c}_i(\tilde{z}_1,\ldots,\tilde{z}_{n-1})\tilde{z}_n^i \\ \dot{\tilde{z}}_n &= u + \tilde{f}_n(\tilde{z}).\end{aligned} \tag{14}$$

With this in mind, it is clear that

$$\tilde{X}_1 = ad_{\tilde{X}_0}^{p_{n-1}-1}\tilde{f}(\tilde{z}) = (0,\ldots,0,p_{n-1}!,\ \star)^T.$$

Using exactly the same argument, one can prove that in the new coordinates (with a little abuse of notations, we still use $(\tilde{z}_1,\ldots,\tilde{z}_n)$ to represent a new coordinate)

$$\tilde{f}_{n-2}(\tilde{z}_1,\ldots,\tilde{z}_{n-1}) = \tilde{z}_{n-1}^{p_{n-2}} + \Sigma_{i=0}^{p_{n-2}-1} c_i(\tilde{z}_1,\ldots,\tilde{z}_{n-2})\tilde{z}_{n-1}^i.$$

Inductively, we have for $i = 2,3,\ldots,n-2$

$$\tilde{X}_i = \Big(0,\ldots,0,\ \underbrace{p_{n-i}\cdots p_{n-1}!}_{(n-i)\text{th}},\ \star,\cdots,\star\Big)^T$$

and $\tilde{f}_{n-i-1}(\tilde{z}_1,\ldots,\tilde{z}_{n-i}) = \tilde{z}_{n-i}^{p_{n-i-1}} + \Sigma_{k=0}^{p_{n-i-1}-1}\tilde{c}_k(\tilde{z}_1,\ldots,\tilde{z}_{n-i-1})\tilde{z}_{n-i}^k$.

Finally, using the state feedback $v = u + \tilde{f}_n(\tilde{z}) = u + \alpha(\xi)$ yields the last equation of (1). This completes the proof. ■

Observe that the condition C4) only plays a role in restricting the highest order in each integral channel of (1). Then, it is not difficult to deduce the following result from Theorem 2.6.

Corollary 2.8: The *analytic* affine system (2) is locally diffeomorphic to

$$\begin{aligned}\dot{x}_1 &= x_2^{p_1} + \Sigma_{i=0,i\neq p_1}^{\infty} x_2^i \phi_i^1(x_1) \\ &\vdots \\ \dot{x}_{n-1} &= x_n^{p_{n-1}} + \Sigma_{i=0,i\neq p_{n-1}}^{\infty} x_n^i \phi_i^{n-1}(x_1,x_2,\ldots,x_{n-1}) \\ \dot{x}_n &= v\end{aligned} \tag{15}$$

via the transformation $x = T(\xi)$ and state feedback (7) iff C1)–C3) of Theorem 2.6 hold. ■

III. p-Normalization Algorithm

In this section, we discuss how to find the change of coordinates $T(\xi)$ and the state feedback $u = \alpha(\xi) + \beta v$ when the conditions C1)–C4) of Theorem 2.6 are satisfied. Our goal is to develop an algorithm, similar to the one for the problem of exact feedback linearization (see, e.g., [5]), which provides a systematic way to compute $T(\xi)$ and $\alpha(\xi)$ yielding a solution to the p-normalization problem.

To this end, we define a set of vector fields Y_i's as follows:

$$Y_i = ad_{X_{i-1}}^{p_{n-i}-1} f, \qquad i = 1,\ldots,n-1. \tag{16}$$

Then, one can prove the following result.

Lemma 3.1: Suppose C1)–C4) of Theorem 2.6 hold. Let $h(\xi)$ be a smooth function such that

$$dh \in \Delta_{n-2}^{\perp} \quad \text{and} \quad L_{X_{n-1}}h = 1.$$

Then, the following coordinates transformation:

$$z_1 = h(\xi) \quad z_2 = L_{Y_{n-1}}h(\xi),\ldots,z_n = L_{Y_1}L_{Y_2}\ldots L_{Y_{n-1}}h(\xi) \tag{17}$$

denoted as $z = \Psi(\xi)$, is a local diffeomorphism.

Proof: It suffices to show that $dz_k(0)$, $k = 1,\ldots,n$, are linearly independent. Assume that there exist real constants $c_1,\ldots,c_n$, such that $\Sigma_{k=1}^n c_k dz_k(0) = 0$.

$$\Phi(\xi) = \Sigma_{k=1}^n c_k z_k. \tag{18}$$

Since C1)–C4) are true, by Theorem 2.6, there is a local transformation $x = T(\xi)$ transforming (2) into

$$\begin{aligned}\dot{x}_1 &= x_2^{p_1} + \Sigma_{i=0}^{p_1-1} x_2^i \phi_i^1(x_1) \\ &\vdots \\ \dot{x}_{n-1} &= x_n^{p_{n-1}} + \Sigma_{i=0}^{p_{n-1}-1} x_n^i \phi_i^{n-1}(x_1,x_2,\ldots,x_{n-1}) \\ \dot{x}_n &= -\alpha\left(T^{-1}(x)\right) + u.\end{aligned} \tag{19}$$

In this coordinate frame

$$\begin{aligned}\tilde{X}_k(x) &= \Big(0,\ldots,0,\ \underbrace{p_{n-k}\cdots p_{n-1}!}_{(n-k)\text{th}},\ \star,\cdots,\star\Big)^T \\ &= \frac{\partial T}{\partial \xi} X_k(\xi)|_{\xi=T^{-1}(x)} \\ \tilde{Y}_k(x) &= (0,\ldots,0,\underbrace{p_{n-k}!\ldots p_{n-1}!x_{n+1-k}}_{(n-k)\text{th}},\ \star,\cdots,\star)^T \\ &= \frac{\partial T}{\partial \xi} Y_k(\xi)|_{\xi=T^{-1}(x)}\end{aligned} \tag{20}$$

which implies

$$\Delta_k(\xi) = \tilde{\Delta}_k(x) = \text{span}\left\{\frac{\partial}{\partial x_{n-k}},\ldots,\frac{\partial}{\partial x_n}\right\}, \quad k = 0,1,\ldots,n-1. \tag{21}$$

Hence

$$\Delta_{n-2}^{\perp} \ni h(\xi) = h\left(T^{-1}(x)\right) = \tilde{h}(x_1). \tag{22}$$

According to the forms of $\tilde{Y}_k$ and $\tilde{X}_k$, it is easy to show that

$$dL_{\tilde{Y}_{n-1}}\tilde{h} \in \tilde{\Delta}_{n-3}^{\perp} \quad dL_{\tilde{Y}_{n-2}}L_{\tilde{Y}_{n-1}}\tilde{h} \in \tilde{\Delta}_{n-4}^{\perp},\ldots, \quad dL_{\tilde{Y}_2}L_{\tilde{Y}_3}\ldots L_{\tilde{Y}_{n-1}}\tilde{h} \in \tilde{\Delta}_0^{\perp}. \tag{23}$$

Using (20)–(23) and the nonsingularity of $(\partial T/\partial \xi)$, we have

$$dL_{Y_{n-1}}h \in \Delta_{n-3}^{\perp} \quad dL_{Y_{n-2}}L_{Y_{n-1}}h \in \Delta_{n-4}^{\perp},\ldots \quad dL_{Y_2}L_{Y_3}\ldots L_{Y_{n-1}}h \in \Delta_0^{\perp}. \tag{24}$$

Now, consider

$$\begin{aligned}L_g\Phi(\xi) &= \sum_{k=1}^n c_k L_g z_k \\ &= c_1 L_g h(\xi) + \cdots + c_{n-1}L_g L_{Y_2}L_{Y_3}\ldots L_{Y_{n-1}}h(\xi) \\ &\quad + c_n L_g L_{Y_1}L_{Y_2}\ldots L_{Y_{n-1}}h(\xi).\end{aligned}$$

According to (24), the first $n-1$ terms are identical to zero. For the last term, using (24) repeatedly, we have

$$\begin{aligned} L_gL_{Y_1}L_{Y_2}\ldots L_{Y_{n-1}}h(\xi) &= \left(L_{ad_gY_1}+L_{Y_1}L_g\right)\\ &\quad\times L_{Y_2}\ldots L_{Y_{n-1}}h(\xi)\\ &= L_{ad_gY_1}L_{Y_2}\ldots L_{Y_{n-1}}h(\xi)\\ &= L_{X_1}L_{Y_2}\ldots L_{Y_{n-1}}h(\xi)\\ &= \left(L_{ad_{X_1}Y_2}+L_{Y_2}L_{X_1}\right)\\ &\quad\times L_{Y_3}\ldots L_{Y_{n-1}}h(\xi)=\cdots\\ &= L_{X_{n-2}}L_{Y_{n-1}}h(\xi)\\ &= \left(L_{ad_{X_{n-2}}Y_{n-1}}\right.\\ &\qquad\left.+L_{Y_{n-1}}L_{X_{n-2}}\right)h(\xi)\\ &= L_{X_{n-1}}h(\xi)\neq 0. \end{aligned}$$

Hence, $c_n = 0$. Next, consider $L_{X_1}\Phi(\xi)$. A similar argument shows that $c_{n-1}=0$. Continuing this procedure, it is easy to prove that $c_i = 0$, for $i=0,\ldots,n$. ■

Lemma 3.2: Under C1)–C4), there exists a smooth function $h(\xi)$ such that

$$dh\in\Delta_{n-2}^{\perp} \quad\text{and}\quad L_{X_{n-1}}h=1.$$

Proof: Using Theorem 2.6, we have (19). Then

$$\tilde{X}_{n-1}(x)=(p_1\ldots p_{n-1}!,\ \star,\cdots,\star)^T.$$

Choose $\tilde{h}(x)=(x_1)/(p_1!\ldots p_{n-1}!)$. Clearly, $h(\xi)=\tilde{h}(x(\xi))$ meets the requirements. ■

Theorem 3.3: Assume that C1)–C4) hold and let $h(\xi)$ be the function obtained from Lemma 3.2. Then, the p-normalization problem can be solved by the state feedback $u=\alpha(\xi)+v$ with

$$\alpha(\xi)=-L_fL_{Y_1}L_{Y_2}\ldots L_{Y_{n-1}}h(\xi)$$

and the coordinates transformation $x=T(\xi)$ defined as

$$x_1=T_1(\xi)=a_1z_1,\ \ldots,\ x_{n-1}=T_{n-1}(\xi)=a_{n-1}z_{n-1},\quad x_n=T_n(\xi)=z_n \tag{25}$$

where z_i is given by (17) and

$$a_{n-1}=p_{n-1}!\quad\text{and}\quad a_{k-1}=p_{k-1}!a_k^{p_{k-1}},\qquad k=2,3,\ldots,n-1. \tag{26}$$

Proof: By Lemma 3.1, the coordinates transformation $z=\Psi(\xi)$ defined by (17) transforms the affine system (2) into

$$\dot{z}=\tilde{f}(z)+\tilde{g}(z)u$$

where

$$\tilde{f}(z)=\begin{bmatrix}\tilde{f}_1(z)\\ \tilde{f}_2(z)\\ \vdots\\ \tilde{f}_n(z)\end{bmatrix}=\begin{bmatrix}L_fh(\xi)\\ L_fL_{Y_{n-1}}h(\xi)\\ \vdots\\ L_fL_{Y_1}L_{Y_2}\ldots L_{Y_{n-1}}h(\xi)\end{bmatrix}_{\xi=\Psi^{-1}(z)}$$

$$\tilde{g}(z)=\begin{bmatrix}L_gh(\xi)\\ L_gL_{Y_{n-1}}h(\xi)\\ \vdots\\ L_gL_{Y_1}L_{Y_2}\ldots L_{Y_{n-1}}h(\xi)\end{bmatrix}_{\xi=\Psi^{-1}(z)}=\begin{bmatrix}0\\0\\ \vdots\\1\end{bmatrix}. \tag{27}$$

Now, we claim that for $1\le i\le n-1$

$$\tilde{f}_i(z)=\frac{1}{p_i!}z_{i+1}^{p_i}+\Sigma_{k=0}^{p_i-1}\tilde{c}_k(z_1,\ldots,z_i)z_{i+1}^k. \tag{28}$$

If the claim is true, choose $x_i=a_iz_i$ with a_i defined by (26), and $u=\alpha(\xi)+v$ with

$$\alpha(\xi)=-\tilde{f}_n(\Psi(\xi))=-L_fL_{Y_1}L_{Y_2}\ldots L_{Y_{n-1}}h(\xi).$$

Then, the resulted closed-loop system is in the p-normal form (1).

So, the only thing needed to be shown is the relation (28). To this end, we first calculate

$$ad_{\tilde{g}}\tilde{f}=\begin{pmatrix}\frac{\partial f_1}{\partial z_n}\\ \frac{\partial f_2}{\partial z_n}\\ \vdots\\ \frac{\partial f_{n-1}}{\partial z_n}\\ \star\end{pmatrix}=\begin{pmatrix}L_{\tilde{g}}\tilde{f}_1(z)\\ L_{\tilde{g}}\tilde{f}_2(z)\\ \vdots\\ L_{\tilde{g}}\tilde{f}_{n-1}(z)\\ \star\end{pmatrix}$$

$$=\begin{pmatrix}L_gL_fh(\xi)\\ L_gL_fL_{Y_{n-1}}h(\xi)\\ \vdots\\ L_gL_fL_{Y_2}\ldots L_{Y_{n-1}}h(\xi)\\ \star\end{pmatrix}_{\xi=\Psi^{-1}(z)}$$

$$=\begin{pmatrix}\left(L_{ad_gf}+L_fL_g\right)h(\xi)\\ \left(L_{ad_gf}+L_fL_g\right)L_{Y_{n-1}}h(\xi)\\ \vdots\\ \left(L_{ad_gf}+L_fL_g\right)L_{Y_2}\ldots L_{Y_{n-1}}h(\xi)\\ \star\end{pmatrix}_{\xi=\Psi^{-1}(z)}$$

$$=\begin{pmatrix}0\\0\\ \vdots\\ L_{ad_gf}L_{Y_2}\ldots L_{Y_{n-1}}h(\xi)\\ \star\end{pmatrix}_{\xi=\Psi^{-1}(z)}. \tag{29}$$

The last step is a consequence of (24).

Similarly, a direct computation shows that the $(n-1)$th component of $ad_{\tilde{g}}^k\tilde{f}(z)$ is

$$\left(ad_{\tilde{g}}^k\tilde{f}(z)\right)_{n-1}=\frac{\partial^k\tilde{f}_{n-1}}{\partial z_n^k}=\begin{cases}L_{ad_g^kf}L_{Y_2}L_{Y_{n-1}}\times h(\xi)|_{\xi=\Psi^{-1}(z)}, & k<p_{n-1}\\ L_{X_{n-1}}h(\xi)=1, & k=p_{n-1}\\ L_g^{k-p_{n-1}}(1)=0, & k>p_{n-1}\end{cases}$$

By the properties of the minimum index, $((\partial^k\tilde{f}_{n-1})/(\partial z_n^k))(0)=0$ for $k<p_{n-1}$. Therefore

$$\tilde{f}_{n-1}(z)=\frac{1}{p_{n-1}!}z_n^{p_{n-1}}+\Sigma_{k=0}^{p_{n-1}-1}\tilde{c}_k(z_1,\ldots,z_{n-1})z_n^k.$$

Note that $\tilde{f}_k(z)$, $k=1,\ldots,n-1$, are independent of z_n. Hence

$$\tilde{X}_1=ad_{\tilde{g}}^{p_{n-1}}\tilde{f}(z)=(0,\ldots,0,1,\star)^T.$$

Using a similar argument, one can calculate $ad_{\tilde{X}_1}\tilde{f}(z)$. From (24) and the relation $(ad_{\tilde{X}_1}^k\tilde{f}(z))_{n-2}=(\partial^k\tilde{f}_{n-2})/(\partial z_{n-1}^k)$, it is deduced that

$$\tilde{f}_{n-2}(z)=\frac{1}{p_{n-2}!}z_{n-1}^{p_{n-2}}+\Sigma_{k=0}^{p_{n-2}-1}\tilde{c}_k(z_1,\ldots,z_{n-2})z_{n-1}^k.$$

Repeating this procedure leads to (28). ■

On the basis of the previous discussions, we now are able to provide the following algorithm resulting in a design procedure for the p-normalization problem.

p-**Normalization Algorithm**: Consider an affine system $\dot{\xi}=f(\xi)+g(\xi)u$.

Step 1) Calculate X_i's, p_i's and Δ_i's using (3) and (4).

Step 2) Verify the conditions C1)–C4) of Theorem 2.6.

Step 3) If C1)–C4) are satisfied, solve $h(\xi)$ from the partial differential equations

$$L_{X_i}h(\xi)=0,\ i=0,1,\ldots,n-2,\ L_{X_{n-1}}h(\xi)=1,\ \forall\xi\in U \tag{30}$$

and calculate the vector fields Y_i 's from (16).

Step 4) Construct the change of coordinates $x = T(\xi)$ as follows:

$$x_1 = a_1 h(\xi), \qquad x_i = a_i L_{Y_{n-i+1}} L_{Y_{n-i+2}} \dots L_{Y_{n-1}} h(\xi)$$
$$2 \le i \le n-1 \quad x_n = L_{Y_1} L_{Y_2} \dots L_{Y_{n-1}} h(\xi) \tag{31}$$

where the coefficients a_i's are given by (26).

Step 5) Compute the state feedback $u(\xi) = \alpha(\xi) + v$, where $\alpha(\xi) = -L_f L_{Y_1} L_{Y_2} \dots L_{Y_{n-1}} h(\xi)$.

After the change of coordinates and state feedback, the closed-loop system is of the p-normal form (1) in the new coordinates $(x_1, x_2, \dots, x_n)$.

IV. Illustrative Example

We now present an example to illustrate the theoretic results developed so far. In particular, we show how an affine system that is not feedback linearizable can be transformed into the p-normal form via a systematic procedure given in the previous section.

Example 4.1: Consider the smooth affine system

$$\begin{cases} \dot{\xi}_1 = \left(\xi_2 - \xi_3^2\right)^3 + \xi_3 \\ \dot{\xi}_2 = \xi_3 + \left(\xi_1 - \xi_2 + \xi_3^2\right)\left(\xi_2 - \xi_3^2\right)^2 - 2\xi_3^2 + 2\xi_3 u \\ \dot{\xi}_3 = -\xi_3 + u \end{cases} \tag{32}$$

where $f(\xi) = \begin{bmatrix} (\xi_2 - \xi_3^2)^3 + \xi_3 \\ \xi_3 + (\xi_1 - \xi_2 + \xi_3^2)(\xi_2 - \xi_3^2)^2 - 2\xi_3^2 \\ -\xi_3 \end{bmatrix}$, $g(\xi) = \begin{bmatrix} 0 \\ 2\xi_3 \\ 1 \end{bmatrix}$. Clearly, (32) is not locally feedback equivalent to a linear controllable system because its Jacobian linearization is uncontrollable.

On the other hand, a simple calculation gives $ad_g f = (1, 1 - 2\xi_3, -1)^T$ which is independent of $X_0 = g(\xi)$ at $\xi = 0$. Thus, $X_1 = ad_g f(\xi)$ and $q_1 = p_2 = 1$.

Note that

$$ad_{X_1} f = \begin{bmatrix} 3\left(\xi_2 - \xi_3^2\right)^2 - 1 \\ 2\left(\xi_1 - \xi_2 + \xi_3^2\right)\left(\xi_2 - \xi_3^2\right) + 2\xi_3 - 1 \\ 1 \end{bmatrix}$$
$$ad_{X_1}^2 f = \begin{bmatrix} 6\left(\xi_2 - \xi_3^2\right) \\ 2\left(\xi_1 - \xi_2 + \xi_3^2\right) \\ 0 \end{bmatrix}$$
$$ad_{X_1}^3 f = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}.$$

Clearly, $X_2 = ad_{X_1}^2 f$ and $q_2 = p_1 = 3$.

Now we are ready to verify if the conditions C1)–C4) of Theorem 2.6 hold. According to the computations above, the minimum index of (32) is (1,3) and the normalizable order is 3. Moreover, it is easy to check that the distributions Δ_i, $i = 0, 1, 2$, are involutive. C3) holds because $X_0 \in \Delta_1$ and $dim\ \Delta_2 = 3$. C4) is ensured by the fact that $[X_0, X_1] = 0$ and $[X_1, X_2] = 0$. By Theorem 2.6, (32) is p normalizable.

In what follows, we apply Theorem 3.3 to explicitly construct a smooth state feedback control law and a change of coordinates.

First, find a real-valued function $h(\xi) \in \Delta_1^\perp$ satisfying $L_{X_2} h(\xi) = 1$, i.e. solve the following partial differential equations:

$$0 = L_{X_0} h(\xi) = \frac{\partial h}{\partial \xi_2} 2\xi_1 + \frac{\partial h}{\partial \xi_3}$$
$$0 = L_{X_1} h(\xi) = \frac{\partial h}{\partial \xi_1} + \frac{\partial h}{\partial \xi_2}(1 - 2\xi_1) - \frac{\partial h}{\partial \xi_3}$$
$$1 = L_{X_2} h(\xi) = 6\frac{\partial h}{\partial \xi_1}. \tag{33}$$

It is not difficult to see that $h(\xi) = (1/6)(\xi_1 - \xi_2 + \xi_3^2)$.

Next, we calculate the vector fields Y_i's: $Y_1 = ad_g^{1-1} f = f$, $Y_2 = ad_{X_1}^{3-1} f = [6(\xi_2 - \xi_3^2),\ 2(\xi_1 - \xi_2 + \xi_3^2),\ 0]^T$. Using Theorem 3.3, we introduce the following coordinates transformation:

$$\begin{cases} z_1 = h(\xi) = \frac{1}{6}\left(\xi_1 - \xi_2 + \xi_3^2\right) \\ z_2 = L_{Y_2} h(\xi) = \left(\xi_2 - \xi_3^2\right) - \frac{1}{3}\left(\xi_1 - \xi_2 + \xi_3^2\right) \\ z_3 = L_{Y_1} L_{Y_2} h(\xi) = \xi_3 - \frac{1}{3}\left(\xi_2 - \xi_3^2\right)^3 \\ \qquad + \frac{4}{3}\left(\xi_1 - \xi_2 + \xi_3^2\right)\left(\xi_2 - \xi_3^2\right)^2 \end{cases}$$

whose inverse mapping is given by $\xi_1 = z_2 + 4z_1$, $\xi_2 = z_2 - 2z_1 + [z_3 + (1/3)(z_2 - 2z_1)^2 + 8z_1(z_2 - 2z_3)^2]^2$, and $\xi_3 = z_3 + 2(z_2 - 2z_1)^2 + 8z_1(z_2 - 2z_3)^2$. In the z coordinate

$$\tilde{g}(z) = \frac{\partial z}{\partial \xi} g(\xi)|_{\xi = z^{-1}(z)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$\tilde{f}(z) = \frac{\partial z}{\partial \xi} f(\xi)|_{\xi = z^{-1}(z)} = \begin{bmatrix} \frac{1}{6}\left(z_2^3 - 12z_2^2 z_1 + 36 z_2 z_1^2 - 32 z_1^3\right) \\ z_3 \\ f_3(z_1, z_2, z_3) \end{bmatrix}.$$

By Theorem 3.3, the nonsingular transformation $x_3 = z_3$, $x_2 = z_2$, $x_1 = 6z_1$ and the state feedback controller $u = -f_3((1/6)x_1, x_2, x_3) + v$ transform the system $\dot{z} = \tilde{f}(z) + \tilde{g}(z)u$ into the the p-normal form (1)

$$\begin{cases} \dot{x}_1 = x_2^3 - 2x_2^2 x_1 + x_2 x_1^2 - \frac{4}{27} x_1^3 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = v. \end{cases} \tag{34}$$

Appendix

Proof of lemma 2.4: Under the change of coordinates $x = T(\xi)$, define

$$\tilde{f}(x) = \frac{\partial T}{\partial \xi} f(\xi)|_{\xi = T^{-1}(x)} \quad \tilde{g}(x) = \frac{\partial T}{\partial \xi} g(\xi)|_{\xi = T^{-1}(x)}.$$

Then, a direct computation gives

$$\left[\tilde{g}(x), \tilde{f}(x)\right] = \frac{\partial T}{\partial \xi}\left[g(\xi), f(\xi)\right]|_{\xi = T^{-1}(x)} := \widetilde{[g, f]}(x).$$

This, in turn, implies that for any k

$$ad_{\tilde{g}(x)}^k \tilde{f}(x) = \frac{\partial T}{\partial \xi} ad_{g(\xi)}^k f(\xi)|_{\xi = T^{-1}(x)} := \widetilde{ad_g^k} f(x). \tag{35}$$

Consequently, denote $\tilde{X}_i(x) := ((\partial T)/(\partial \xi)) X_i(\xi)|_{\xi = T^{-1}(x)}$. Since $(\partial T)/(\partial \xi)$ is invertible, normalizable order, and minimum index q_i's are clearly unchanged by a change of coordinates. Observe that $\tilde{h}(x) = h(\xi)|_{\xi = T^{-1}(x)}$. Then

$$L_{\tilde{X}_i} \tilde{h}(x) = L_{X_i} h(\xi)|_{\xi = T^{-1}(x)}, \qquad i = 0, 1, 2, \dots$$

which implies that the generalized relative degree is invariant under the coordinates transformation $x = T(\xi)$.

On the other hand, using the nonsingular smooth state feedback $u = \alpha(\xi) + \beta(\xi)v$ yields $\tilde{f}(\xi) = f(\xi) + g(\xi)\alpha(\xi)$ and $\tilde{g}(\xi) = g(\xi)\beta(\xi)$. With this in mind, a straightforward calculation shows that

$$\tilde{X}_0 = \beta(\xi) X_0$$
$$ad_{\tilde{X}_0}^j \tilde{f} = \beta^j(\xi) ad_g^j f + \sum_{t=1}^{j-1} b_{tj}^0(\xi) ad_g^t f + a_{0j}^0(\xi) X_0$$
$$j \ge 1 \tag{36}$$

Using A1) and A2), it is not difficult to conclude that

$$\tilde{X}_k = \left[\beta^{\left(\prod_{i=1}^k q_i\right)}(\xi) + \gamma_k(\xi)\right] X_k + \sum_{i=0}^{k-1} \alpha_{ik}(\xi) X_i$$
$$ad_{\tilde{X}_k}^j \tilde{f} = \left[\beta^{\left(\prod_{i=1}^k q_i\right)}(\xi) + \gamma_k(\xi)\right]^j ad_{X_k}^j f$$
$$+ \sum_{t=1}^{j-1} b_{tj}^k(\xi) ad_{X_k}^t f + \sum_{i=0}^k a_{ij}^k(\xi) X_i, \qquad j \ge 1 \tag{37}$$

where $\alpha_{ik}(\xi)$, $a_{ij}^k(\xi)$, and $b_{tj}^k(\xi)$ are real-valued smooth functions, and $\gamma_k(0) = 0$.

In fact, (37) can be proved inductively. When $k = 0$, it is obvious that (36) is a particular form of (37), with $\gamma_0(\xi) = 0$ and $b_{tj}^0(\xi) = 0$. Suppose (37) holds for k. From the second equation of (37), we have

$$\begin{aligned} ad_{\tilde{X}_k}^{q_{k+1}}\tilde{f} = &\beta^{\left(\prod_{i=1}^{k+1} q_i\right)}(\xi) ad_{X_k}^{q_{k+1}} f + \gamma_k(\xi) \\ &\times \left(\sum_{i=1}^{q_{k+1}} \binom{q_{k+1}}{i} \gamma_k^{i-1}(\xi) \left[\beta^{\left(\prod_{i=1}^{k} q_i\right)}(\xi) \right]^{q_{k+1}-i} \right). \\ &\times ad_{X_k}^{q_{k+1}} f + \sum_{t=1}^{q_{k+1}-1} b_{tj}^k(\xi) ad_{X_k}^t f + \sum_{i=0}^{k} a_{ij}^k(\xi) X_i \end{aligned}$$

By A2), the third term in the previous equality can be expressed as

$$\sum_{t=1}^{q_{k+1}-1} b_{tj}^k(\xi) ad_{X_k}^t f = \sum_{i=1}^{k+1} c_i(\xi) X_i$$

where $c_i(\xi)$ are smooth functions. Since q_{k+1} is a component of minimum index and $t < q_{k+1}$, then $c_{k+1}(0) = 0$.

In view of the previous discussions, we have

$$\tilde{X}_{k+1} = \left[\beta^{\left(\prod_{i=1}^{k+1} q_i\right)}(\xi) + \gamma_{k+1}(\xi) \right] X_{k+1} + \sum_{i=0}^{k} \alpha_{ik}(\xi) X_i \quad (38)$$

which proves the first equality of (37).

Next, we prove that the second equation of (37) also holds for $k+1$. First of all, a direct computation shows that the second equation of (37) with $k+1$ is true when $j = 0$. Assume that it holds for j. Define

$$\bar{\beta} = \beta^{\left(\prod_{i=1}^{k+1} q_i\right)}(\xi) \quad \text{and} \quad \bar{\gamma} = \gamma_{k+1}(\xi).$$

Then

$$\begin{aligned} ad_{\tilde{X}_{k+1}}^{j+1}\tilde{f} = &\Bigg[(\bar{\beta}+\bar{\gamma}) X_{k+1} + \sum_{i=1}^{k} \alpha_{ik}(\xi) X_i,\ (\bar{\beta}+\bar{\gamma})^j ad_{X_{k+1}}^j f \\ &+ \sum_{t=1}^{j-1} b_{tj}^{k+1}(\xi) ad_{X_{k+1}}^t f + \sum_{i=0}^{k+1} \alpha_{ij}(\xi) X_i \Bigg] \end{aligned}$$

from which it is not difficult to deduce that the second equation of (37) holds, as long as

$$\left[X_i,\ ad_{X_{k+1}}^j f \right] \in \Delta_{k+1} + \operatorname{span}\left\{ ad_{X_{k+1}}^s f,\ s \le j \right\},\ \ i \le k. \quad (39)$$

To prove (39), consider the case when $j = 1$. Using the Jacobi identity and A1)–A2) gives

$$\begin{aligned} \left[X_i,\ ad_{X_{k+1}} f \right] &= [X_{k+1},\ ad_{X_i} f] + [f,\ [X_{k+1}, X_i]] \\ &\in [X_{k+1}, \Delta_{k+1}] + [f, \Delta_{k+1}] \subset \Delta_{k+1} + \operatorname{span}\left\{ ad_{X_{k+1}} f \right\} \end{aligned}$$

Now, suppose (39) is true for j. Then

$$\begin{aligned} \left[X_i, ad_{X_{k+1}}^{j+1} f \right] &= \left[X_{k+1},\ \left[X_i, ad_{X_{k+1}}^j f \right] \right] \\ &\quad + \left[ad_{X_{k+1}}^j f,\ [X_{k+1}, X_i] \right] \\ &\in \left[X_{k+1}, \Delta_{k+1} + \operatorname{span}\left\{ ad_{X_{k+1}}^s f, s \le j \right\} \right] \\ &\quad + \left[ad_{X_{k+1}}^{j-1} f, \Delta_{k+1} \right] \\ &\subset \Delta_{k+1} + \operatorname{span}\left\{ ad_{X_{k+1}}^s f,\ s \le j+1 \right\} \end{aligned}$$

which leads to (39). As a consequence of (37), normalizable order and q_i, $i = 1, 2, \ldots, \rho - 1$, are unchanged. In view of (36) and (37), the generalized relative degree remains same. ∎

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