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Normal form representation of control systems

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SUMMARY

This paper is to investigate the normal form representation of control systems. First, as numerical tools we develop an algorithm for normal form expression and the matrix representation of the Lie derivative of a linear vector field over homogeneous vector fields. The concept of normal form is modified. Necessary and sufficient conditions for a linear transformation to maintain the Brunowsky canonical form are obtained. It is then shown that the shift term can always be linearized up to any degree. Based on this fact, linearization procedure is proposed and the related algorithms are presented. Least square linear approximations are proposed for non-linearizable systems. Finally, the method is applied to the ball and beam example.

The efforts are focused on the numerical and computer realization of linearization process.

KEY WORDS: normal form; matrix representation of ad_L; kth degree linearization; Brunowsky canonical form; least square approximation

1. PRELIMINARY

Consider a dynamic system

$$\dot{x} = f(x). \tag{1}$$

The normal form is a powerful tool in dynamic system analysis. (cf. References [1, 2]). In this paper we will use normal form to consider the feedback equivalence of nonlinear systems, particularly, the problem of approximate linearization.

A lot of work has been done in this field [2–14]. These works provided necessary and sufficient conditions and algorithms for the approximate linerization of different control systems. Particularly, a homological equation is derived in References [3, 4] (refer also to References [13, 14])

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to characterize a linearizable systems. Two sets of canonical forms were found in Reference [13]. The second canonical form is a system without shifting term. Furthermore, a canonical form is also found for $g(x)$.

Our goal in this paper is to provide an easily computable conditions for approximate linearization. We emphasize on the mechanical and computer realization.

We first recall the normal form of (1) and some related properties. Let H_n^k be the set of kth degree homogeneous polynomial vector fields in R^n. Then the following facts are obvious:

1. H_n^k is a linear vector space over R.
2. Let $L \in H_n^1$ be a given vector field. Then the Lie derivative

$$ad_L : H_n^k \to H_n^k$$

is a linear mapping.

Now fix $L \in H^1$. According to above fact the range of the mapping $ad_L : H^k \to H^k$ is a subspace of H^k. Then we can decompose H^k as

$$H^k = ad_L(H^k) \oplus G_k$$

where G_k is a complement of $ad_L(H^k)$. Note that G_k is not unique. The following theorem provides a normal form expression of system (1).

Theorem 1.1. (Guckenheimer and Holmes [1])

Consider system (1) with $f(0) = 0$. Let $L = J_f(0)x$, where $J_f(0)$ is the Jacobian matrix of f at zero. Then there exists a local diffeomorphism $x = x(z)$ around zero such that (8.1.1) can be locally expressed as

$$\dot{z} = g_1(z) + g_2(z) + \cdots + g_r(z) + R_r(z) \tag{2}$$

where

$$g_1(z) = J_f(0)z;\ g_i(z) \in G_i;\ i = 2, \ldots, r;\ R_r(z) = o(\|z\|^{r+1})$$

Equation (2) is called a *normal form* of (1).

For convenience in further discussion, we give a mild modification to the normal form expression (2) as follows:

Let $E_k \subset aa_L(H^k)$ be a subspace and

$$H^k = E_k \oplus G_k. \tag{3}$$

Then equation (2) is called a *modified normal form* if $g_i(z) \in G_i$; $i = 2, \ldots, r$ where G_i is defined by relationship (3). Since we enlarged the subspace G_k, a normal form is also a modified normal form for any modification. The following algorithm is suitable for both the normal and modified normal form expression. This formulation will be helpful in the sequel.

Algorithm 1.2

Step 1: Use the Taylor expansion to express system (1) as

$$\dot{x} = J_f(0)x + \xi_2(x) + R_3 = Ax + \xi_2(x) + R_3(x), \quad \xi_2(x) \in H_n^2. \tag{4}$$

Step 2: (From Step 2 on is a loop, starting with $i = 1$.) Assume g_i, $i = 1, \ldots, k-1$, are as required. That is, at step $i = k - 1$ (8.1.2) is obtained as

$$\dot{z} = Az + g_2(z) + \cdots + g_{k-1}(z) + \xi_k(z) + R_{k+1}(z). \tag{5}$$

Choose G_k and decompose $\xi_k(z)$ as

$$\xi_k = h_k + g_k \tag{6}$$

where $h_k \in E^k \subset ad_L(H^k)$, $g_k \in G_k$.

Step 3: Find $T(z) \in H^k$ such that

$$h_k(z) = ad_{\mathrm{L}}(T(z)) \tag{7}$$

where $L = Ax$.

Step 4: Modify the right hand side of (4) by replacing z by $z + T(z)$ and then multiply it by a proper approximation of $[I + J_T(z)]^{-1}$. Precisely, modify (4) as follows:

$$\begin{aligned}\dot{z} = Q_k(z)[&g_1(a + T(z)) + \cdots + g_{k-1}(z + T(z)) + \xi_k(z + T(z)) \\ &+ R_{k+1}(z + T(z))]\end{aligned} \tag{8}$$

where

$$Q_k(z) = I - J_T(z) + J_T^2(z) \pm \cdots + (-1)^j J_T^j(z) \tag{9}$$

and the order j in the above equation is

$$j = \min\left\{t \,|\, t \geqslant \frac{r-1}{k-1}\right\} \tag{10}$$

If $k < r$, replace (5) by (8) then go back to step 2. Else end the algorithm.

Theorem 1.3

After $r - 1$ recursive computations the above algorithm provides the required modified normal form.

Proof. First of all, it is easy to check that under the j defined in (10), we have

$$(I + J_T(z))^{-1} = Q_k(z) + o(\|z\|^{r+1}).$$

Secondly, a straightforward computation shows that

$$\begin{aligned}\dot{z} &= (I - J_T(z))(Az + AT(z)) + g_2(z) + \cdots + g_{k-1}(z) + \xi^{(k)}(z) + o(\|z\|^{k+1}) \\ &= Az - ad_L T(z) + \xi_k(z) + g_2 + \cdots + g_{k-1}(z) + o(\|z\|^{k+1}).\end{aligned}$$

Thus

$$\xi_k(z) - ad_L T(z) = \xi_k(z) - h_k(z) := g_k(z) \in G_k$$

and the conclusion follows. □

2. MATRIX REPRESENTATION OF ad_L

It can be seen in previous section that the vector space of kth homogeneous vector fields, H_n^k, plays an important role in normal form representation. We investigate some properties of it in this section.

Proposition 2.1

The dimension of H_n^k is

$$\dim(H_n^k) = \frac{n(n+k-1)!}{k!(n-1)!}, \quad k \geqslant 0, \quad n \geqslant 1. \tag{11}$$

Proof. Let B_n^k be the set of homogeneous polynomial of degree k in R^n. Then

$$H_n^k = \oplus_{i=1}^n h_i^k \quad \text{where } h_i^k = B_n^k \times \delta_k.$$

Throughout this paper δ_k is used for a vector with all zero elements except the kth component, which is 1. By fixing the degree of one variable, one sees easily that

$$\dim(B_n^k) = \sum_{i=0}^{k} \dim(B_{n-1}^i).$$

Using the equality

$$\binom{n-1}{0} + \binom{n}{1} + \cdots + \binom{n+k-1}{k} = \binom{n+k}{k} \tag{12}$$

the conclusion follows via mathematical induction. □

It is interesting that the dimension of B_n^k can be obtained quickly by Table I which is constructed as follows: achieving each number by adding the upper and left numbers. Then $\dim(H_n^k) = n \dim(B_n^k)$.

Later on, we denote $s = s_n^k = \dim(B_n^k)$.

Consider the Algorithm 1.2. To get a fixed matrix representation of ad_L, we have to unify the order of the elements in a natural basis of H_n^k.

Table I. The dimension of B_n^k.

$k\backslash\dim(n)$	1	2	3	4	5	6	⋯
0	1	1	1	1	1	1	
1	1	2	3	4	5		
2	1	3	6	10			
3	1	4	10				
4	1	5					
5	1						
⋯							

First, we order the monic monomial elements of degree k as follows: Let $b_1 = x_1^{k_1^1} \cdots x_n^{k_n^1}$, $b_2 = x_1^{k_1^2} \cdots x_n^{k_n^2}$. Define $b_1 \prec b_2$ if $k_s^1 = k_s^2$, $s = 1, \ldots, t$ and $k_{t+1}^1 > k_{t+1}^2$ for some $0 \leqslant t < n$. Denote the set of such ordered monomials by B_n^k, or simply B^k if there is no confusion: e.g.

$$B_3^2 = (x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2)$$

$$B_3^3 = (x_1^3, x_1^2x_2, x_1^2x_3, x_1x_2^2, x_1x_2x_3, x_1x_3^2, x_2^3, x_2^2x_3, x_2x_3^2, x_3^3).$$

The basis $\{B_n^k\delta_1, \ldots, B_n^k\delta_n\}$ of H_n^k is called the *natural basis.*

For $X \in H_n^k$, X can be expressed as

$$X = (r_1^1, \ldots, r_s^1, r_1^2, \ldots, r_s^2, \ldots, r_1^n, \ldots, r_s^n) \in R^{n\times s} \tag{13}$$

Precisely, let $\{e_j, j = 1, \ldots, s\}$ be the natural basis of B_n^k. Then

$$X = \sum_{i=1}^{n} \sum_{j=1}^{s} r_j^i \delta_i e_j = \left(\sum_{j=1}^{s} r_j^1 e_j, \ldots, \sum_{j=1}^{s} r_j^n e_j \right)^{\mathrm{T}}. \tag{14}$$

In later discussion we need these two forms of X. We will call (13) the *expanded form*, while (8.1.12) the *vector field form.*

A matrix expression of L_A can be explained as to find a matrix M_n^k such that

$$L_A X = M_n^k X, \quad X \in R^{n\times s}.$$

Now assume L has a canonical form as

$$L = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix} x. \tag{15}$$

We use ∂_i for $\partial/\partial x_i$. Then

$$ad_L X = \begin{pmatrix} \sum_{j=1}^{s} r_j^1 \partial_1 e_j, \ldots, \sum_{j=1}^{s} r_j^1 \partial_n e_j \\ \cdots \\ \sum_{j=1}^{s} r_j^n \partial_1 e_j, \ldots, \sum_{j=1}^{s} r_j^n \partial_n e_j \end{pmatrix} Lx - L \begin{pmatrix} \sum_{j}^{s} r_j^1 e_j \\ \cdots \\ \sum_{j}^{s} r_j^n e_j \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{k=1}^{n-1} x_{k+1} \sum_{j=1}^{s} r_j^1 \partial_k e_j + \sum_{i=1}^{n} a_i x_i \sum_{j=1}^{s} r_j^1 \partial_n e_j - \sum_{j=1}^{s} r_j^2 e_j \\ \cdots \\ \sum_{k=1}^{n-1} x_{k+1} \sum_{j=1}^{s} r_j^{n-1} \partial_k e_j + \sum_{i=1}^{n} a_i x_i \sum_{j=1}^{s} r_j^{n-1} \partial_n e_j - \sum_{j=1}^{s} r_j^n e_j \\ \sum_{k=1}^{n-1} x_{k+1} \sum_{j=1}^{s} r_j^n \partial_k e_j + \sum_{i=1}^{n} a_i x_i \sum_{j=1}^{s} r_j^n \partial_n e_j - \sum_{i=1}^{n} a_i \sum_{j=1}^{s} r_j^i e_j \end{pmatrix}. \tag{16}$$

From (13) it is clear that the matrix expression, M_m^k of ad_L can be expressed as

$$ad_L = \begin{pmatrix} D & -I & 0 & \cdots & 0 \\ 0 & D & -I & \cdots & 0 \\ & & & \cdots & \\ -a_1 I & -a_2 I & -a_3 I & \cdots & D - a_n I \end{pmatrix} \tag{17}$$

where D is determined by the following mapping:

$$\sum_{j=1}^{s} r_j e_j \mapsto \sum_{k=1}^{n-1} x_{k+1} \sum_{j=1}^{s} r_j \partial_k e_j + \sum_{i=1}^{n} a_i x_i \sum_{j=1}^{s} r_j \partial_n e_j. \tag{18}$$

Example 2.2

Consider the following system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} x_2 e^{x_3} \\ x_3 + x_1 \sin x_2 \\ ax_1 + bx_2 + cx_3 + x_1 x_3 \end{pmatrix}$$

$$= \begin{pmatrix} x_2 \\ x_3 \\ ax_1 + bx_2 + cx_3 \end{pmatrix} + \begin{pmatrix} x_2 x_3 \\ x_1 x_2 \\ x_1 x_3 \end{pmatrix} + o(\|x\|^3). \tag{19}$$

Then we have

$$L = J_f(0)x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{pmatrix} x.$$

Using (14), when $n = 3$ the representation of $ad_L : H_3^k \to H_3^k$ is

$$ad_L = \begin{pmatrix} D & -I & 0 \\ 0 & D & -I \\ -aI & -bI & D - cI \end{pmatrix} \tag{20}$$

where D can be calculated by comparing coefficients of (18). For $k = 2$, D is expressed as

$$D = \begin{pmatrix} 0 & 0 & a & 0 & 0 & 0 \\ 2 & 0 & b & 0 & a & 0 \\ 0 & 1 & c & 0 & 0 & 2a \\ 0 & 1 & 0 & 0 & b & 0 \\ 0 & 0 & 1 & 2 & c & 2b \\ 0 & 0 & 0 & 0 & 1 & 2c \end{pmatrix}. \tag{21}$$

To simplify the computation let $a = b = c = 0$. According to the form of D one can choose G_2 as

$$G_2 = \mathrm{Span}\{x_1^2\delta_3,\, x_1x_2\delta_3,\, x_1x_3\delta_3,\, x_2^2\delta_3\}.$$

Then $H_3^2 = ad_L(H_3^2) \oplus G_2$. Moreover, matrix D provides that

$$\xi_2 = -t_1 - t_2 + s$$

where $t_1 = ad_L(x_2x_3\delta_2 + x_3^2\delta_3) \in H^2$, $t_2 = ad_L(x_1x_2\delta_3) \in H^2$ and $s \in G_2$. Following Algorithm 1.2, set

$$x = z + \begin{pmatrix} 0 \\ -x_2x_3 \\ -x_1x_2 - x_3^2 \end{pmatrix}.$$

Approximating $(I + J_T)^{-1}$ by $I - J_T$ with

$$J_T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -x_3 & -x_2 \\ -x_2 & -x_1 & -2x_3 \end{pmatrix}$$

we obtain the following normal form:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} z_2 \\ z_3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ z_2^2 + 2z_1z_3 \end{pmatrix} + o(\|x\|^3). \tag{22}$$

In the above example if we had chosen $k = 3$, then we would have had to choose $I - J_T + J_T^2$ to approximate $(I + J_T)^{-1}$ and would have had to keep the third order terms at each step. Instead of equation (19), we would have then obtained

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} z_2 \\ z_3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ z_2^2 + 2z_1z_3 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}z_2z_3^2 - z_1z_2^2 \\ z_2^3 + z_1z_2z_3 \\ -z_1^2z_2 + 3z_1z_3^2 + 2z_2^2z_3 \end{pmatrix} + o(\|x\|^4)$$

and then would have had to use the table for $ad_L : H^3 \to H^3$ to find a new T.

For general n and k, the algorithm for D is essential for further discussion. We consider the matrix representation of $ad_L : H^k \to H^k$ for a fixed L as in the Brunowsky canonical form. In fact, this matrix representation is a Lie algebra representation: let $L = Ax$ and $A \in gl(n, R)$ be arbitrary, then $A \to ad_L$ is a Lie algebra homomorphism and when we identify ad_L with its matrix representation as a matrix, say $M_L \in gl(t, R)$, where $t = ns = \dim(H_n^k)$, then the natural mapping becomes a Lie algebra homomorphism from $gl(n, R)$ to $gl(t, R)$.

To construct the representation form, we have to find the position of a monic polynomial $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ in B_n^k, denoted by $p_n^k(k_1, \ldots, k_n)$. It is basic for calculating the matrix form of the representation of ad_L on H_n^k. We prove a formula as follows:

Theorem 2.3

The position p_n^k is given by

$$p_n^k(k_1, \ldots, k_n) = \frac{(k-k_1)(k-k_1+1)\cdots(k-k_1+(n-2))}{(n-1)!} + \frac{(k-k_1-k_2)(k-k_1-k_2+1)\cdots(k-k_1-k_2+(n-3))}{(n-2)!} + \cdots + \frac{(k-k_1-\cdots-k_{n-1})}{1} + 1. \tag{23}$$

Proof. Consider k_1 as a fixed number. The following recursive expression is obtained by assigning $k_1 = k, k-1, \ldots, k_1+1$ to get the size of the corresponding blocks.

$$p_n^k(k_1, \ldots, k_n) = p_{n-1}^0(0, \ldots, 0) + p_{n-1}^1(0, \ldots, 0, 1) + p_{n-1}^2(0, \ldots, 0, 2) + \cdots + p_{n-1}^{k-k_1-1}(0, \ldots, 0, k-k_1-1) + p_{n-1}^{k-k_1}(k_2, \ldots, k_n). \tag{24}$$

Using Equation (24), we can derive Equation (23) by mathematical induction with respect to n. Equation (23) is obviously true for $n = 2$. Assume it is true for n, then

$$\begin{aligned} p_{n+1}^k(k_1, \ldots, k_{n+1}) &= p_n^0(0, \ldots, 0) + p_n^1(0, \ldots, 0, 1) + p_n^2(0, \ldots, 0, 2) \\ &\quad + \cdots + p_n^{k-k_1-1}(0, \ldots, 0, k-k_1-1) + p_n^{k-k_1}(k_2, \ldots, k_{n+1}) \\ &:= P_1 + P_2 \end{aligned} \tag{25}$$

where P_1 contains all but last terms, which, by induction assumption, is

$$\begin{aligned} P_1 &= (1+0+0+\cdots+0) + \left(1+1+\frac{1\times 2}{2!}+\cdots+\frac{1\times\cdots\times(n-1)}{(n-1)!}\right) \\ &\quad + \left(1+2+\frac{2\times 3}{2!}+\cdots+\frac{2\times\cdots\times n}{(n-1)!}\right) + (\cdots) + \left(1+(k-k_1-1)\right. \\ &\quad \left. + \frac{(k-k_1-1)\times(k-k_1)}{2!} + \cdots + \frac{(k-k_1-1)\times\cdots\times(k-k_1+n-3)}{(n-1)!}\right) \end{aligned}$$

and P_2 is the last term in Equation (25), which is

$$P_2 = p_n^{k-k_1}(k_2, \ldots, k_{n+1}).$$

Comparing Equation (25) with the expression of $p_{n+1}^k(k_1, k_2, \ldots, k_{n+1})$, one sees easily that we must show

$$P_1 = \frac{(k-k_1)(k-k_1+1)\cdots(k-k_1+(n-1))}{n!}.$$

Observe that the sum of the first elements in each parenthesis of P_1 is

$$\binom{k-k_1}{k-k_1-1}.$$

Adding it to the sum of second elements in each parenthesis and applying the formula (12), one sees easily that the sum of the elements of the first two columns is

$$\binom{k-k_1+1}{k-k_1-1}$$

Repeat the procedure by adding to it the sum of the third elements, using formula (12) and continuing we finally have

$$P_1 = \binom{k-k_1+n-1}{k-k_1-1}$$

which complete the proof. □

For instance, consider $p_3^4(0, 2, 2)$. Using (8.2.4),

$$p_3^4(0, 2, 2) = \frac{k(k+1)}{2!} + \frac{k-1}{1!} + 1 = 13.$$

Hence $x_2^2 x_3^2$ is the 13th element in B_3^4.

Using Theorem 2.3, one can easily obtain the matrix representation of $ad_L : H_n^k \to H_n^k$. For linearization, we are particularly interested in the form when $L = (x_2, x_3, \dots, x_n, 0)^{\mathrm{T}}$. For such L we have the following application.

Proposition 2.4

If $L = (x_2, x_3, \dots, x_n, 0)^{\mathrm{T}}$, the $ad_L : H_n^k \to H_n^k$ can be expressed as in (17), where D is constructed as follows: For all $k_i \geqslant 0$, $i = 1, \dots, n$, $\sum_{i=1}^n k_i = k$, set

$$j = p_n^k(k_1, k_2, \dots, k_n)$$
$$i = p_n^k(k_1, \dots, k_r - 1, k_{r+1} + 1, \dots, k_n), \quad k_r > 0, \quad r = 1, \dots, n-1.$$

Then the elements D_{ij} of the matrix D of dimension $s \times s$, which is the matrix form of the representation of ad_L, are determined as

$$d_{ij} = \begin{cases} k_r, & k_r \neq 0, \text{ for above } (i, j) \\ 0 & \text{otherwise.} \end{cases} \tag{26}$$

Proof. First of all, by definition different $(k_1, \dots, k_n)$ corresponds different j. Then for same j different s corresponds different i. So (26) is well defined. Next, from (18) it is clear that for each term of $ax_1^{k_1}, \dots, x_n^{k_n}$ D maps it to $a(k_s)x_1^{k_1} \cdots x_r^{k_r - 1} x_{r+1}^{k_{r+1}+1} \cdots x_n^{k_n}$, for $r = 1, \dots, n-1$. It is not difficult to see that D, defined in (26), realizes this mapping. □

The following example is used to describe the constructing process.

Example 2.5

Let $n = 3$ and $k = 4$. To construct D, we figure out its entries first. Denote by $K_0 = (k_1, k_2, k_3)$, $K_1 = (k_1 - 1,\ k_2 + 1,\ k_3)$, $K_2 = (k_1,\ k_2 - 1,\ k_3 + 1)$ where K_1 corresponds to $r = 1$ and K_2 corresponds to $r = 2$. Then the entries d_{ij} can be calculated in Table II.

Then D follows as

$$
D = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.
$$

In fact, the constructing procedure is particularly suitable for computer realization. A program can be created easily to calculate it.

Table II. Matrix representation of ad_L on H_3^4.

K_0	j	K_1	i	d_{ij}	K_2	i	d_{ij}
400	1	310	2	4	—	—	—
310	2	220	4	3	301	3	1
301	3	211	5	3	—	—	—
220	4	130	7	2	211	5	2
211	5	121	8	2	202	6	1
202	6	112	9	2	—	—	—
130	7	040	11	1	121	8	3
121	8	031	12	1	112	9	2
112	9	022	13	1	103	10	1
103	10	013	14	1	—	—	—
040	11	—	—	—	031	12	4
031	12	—	—	—	022	13	3
022	13	—	—	—	013	14	2
013	14	—	—	—	004	15	1
004	15	—	—	—	—	—	—

Now assume $X=(x_1^2x_2^2,\ 2x_2x_3^3,\ x_1^3x_3)^{\mathrm{T}}:=(X_1, X_2, X_3)^{\mathrm{T}}$. Then in the expanded basis X_i can be expressed as

$$X_1=(0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)^{\mathrm{T}}$$
$$X_2=(0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 2\ 0)^{\mathrm{T}}$$
$$X_3=(0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)^{\mathrm{T}}.$$

Let $Y=ad_LX$. Then

$$\begin{pmatrix}Y_1\\Y_2\\Y_3\end{pmatrix}\begin{pmatrix}D & -I & 0\\ 0 & D & -I\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}X_1\\X_2\\X_3\end{pmatrix}=\begin{pmatrix}DX_1-X_2\\ DX_2-X_3\\ DX_3\end{pmatrix}.$$

It turns out easily that

$$Y_1=(0\ 0\ 0\ 0\ 2\ 0\ 2\ 0\ 0\ 0\ 0\ 0\ 0\ -2\ 0)^{\mathrm{T}}$$
$$Y_2=(0\ 0\ -1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 2)^{\mathrm{T}}$$
$$Y_3=(0\ 0\ 0\ 0\ 3\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)^{\mathrm{T}}.$$

That is

$$ad_LX=\begin{pmatrix}B_3^4Y_1\\B_3^4Y_2\\B_3^4Y_3\end{pmatrix}=\begin{pmatrix}2x_1^2x_2x_3+2x_1x_2^3-2x_2x_3^3\\ 2x_3^4-x_1^3x_3\\ 3x_1^2x_2x_3\end{pmatrix}. \qquad \square$$

Later on, to get a normal form expression of a control system, the expresentation (17) is not convenient. To get more convenient matrix expression of the linear mapping $ad_L: H_n^k\to H_n^k$, we prefer to use the following transformation, which itself is interesting:

$$T=\begin{pmatrix}I & 0 & 0 & \cdots & 0\\ D & I & 0 & \cdots & 0\\ D^2 & 2D & I & \cdots & 0\\ \vdots & & \ddots & & \\ D^{n-1} & \binom{n-1}{1}D^{n-2} & \binom{n-1}{2}D^{n-3} & \cdots & I\end{pmatrix}. \tag{27}$$

Its inverse is

$$T^{-1}=\begin{pmatrix}I & 0 & 0 & \cdots & 0\\ -D & I & 0 & \cdots & 0\\ D^2 & -2D & I & \cdots & 0\\ \vdots & & \ddots & & \\ (-D)^{n-1} & \binom{n-1}{1}(-D)^{n-2} & \binom{n-1}{2}(-D)^{n-3} & \cdots & I\end{pmatrix}. \tag{28}$$

Using the transformation T to the natural basis, we get a new basis as

$$(N_1\ N_2 \cdots N_n) = (B_n^k\delta_1\ B_n^k\delta_2 \cdots B_n^k\delta_n) \times T$$

which is called the *normal basis.*

Assume the expanded form of $X \in H_n^k$ under original natural basis is X_o and under new normal basis is X_n. Then

$$X_n = T^{-1}X_o. \tag{29}$$

Using (17), it is clear that under normal basis the matrix expression of ad_L becomes

$$T^{-1}L_AT = \begin{pmatrix} 0 & -I & 0 & \cdots & 0 \\ 0 & 0 & -I & \cdots & 0 \\ \vdots & & \ddots & & \\ D^n & \binom{n}{1}D^{n-1} & \binom{n}{2}D^{n-2} & \cdots & \binom{n}{n-1}D \end{pmatrix}. \tag{30}$$

The following proposition is obvious, but we will find it useful in the linearization problem.

Proposition 2.6

Let $ad_L: H_n^k \to H_n^k$, where L is defined as (15). Then

(i)

$$\operatorname{codim}(ad_L(H_n^k)) = \frac{n(n+k-1)!}{k!(n-1)!} - \dim(D^n).$$

(ii) Consider the normal form decomposition: $H_n^k = ad_L(H_n^k) \oplus G_n^k$. The complement of the image, G_n^k, may be chosen as a subspace of $\operatorname{Span}\{N_n\}$, which is $\operatorname{Span}\{B_n^k\delta_n\}$.

(iii) Let $W_n^k \subset H_n^k$ be the subspace of H_n^k generated by $W_n^k = \operatorname{Span}\{N_2, N_3, \ldots, N_n\}$. If we restrict ad_L to W_n^k, then $ad_L: W_n^k \to H_n^k$ is a one to one mapping. Moreover, if $E_n^k := ad_L(W_n^k)$ is used to replace $ad_L(H_n^k)$, for the modified normal form with respect to E_n^k, statement (ii) remains true.

Proof. The conclusion follows immediately by Equation (30) and the Brunowsky canonical form. Particularly, from the structure of T it is obvious that $\operatorname{Span}\{N_n\} = \operatorname{Span}\{B_n^k\delta_n\}$. □

Remark

For multi-input case we have

$$L = \operatorname{diag}(L_1, L_2, \ldots, L_m)$$

$$T = \operatorname{diag}(T_1, T_2, \ldots, T_m)$$

$$T^{-1}ad_LT = \operatorname{diag}(T_1^{-1}ad_{L_1}T_1, T_2^{-1}ad_{L_2}T_2, \ldots, T_m^{-1}ad_{L_m}T_m).$$

3. ON LINEAR EQUIVALENCE

Linearization is one of the most basic and useful topics in the geometric theory of nonlinear control systems. We refer to References [15, 16] for necessary and sufficient conditions for the local feedback linearization of affine nonlinear systems and to Reference [17] for other linearization problems. The approximate linearization problem has received considerable study. The reader may refer to References [6–15] for related works and many useful results on this problem.

Consider the following system:

$$\dot{x} = f(x) + g(x)u = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \quad x \in R^n \tag{31}$$

where $f(x)$ and $g_i(x)$, $i = 1, \ldots, m$, are C^ω vector fields over $R^n, f(0) = 0$. Taking into consideration a state feedback control with non-zero leading linear terms, we always assume $u = \mathcal{O}(\|x\|)$, that is u has the same order as x. For instance, both x^3 and x^2u are considered as elements of $\mathcal{O}(\|x, u\|^3)$.

The approximate linearization is defined as follows:

Definition 3.1

System (31) is said to be kth degree linearizable at zero, if there exists a neighbourhood $N \ni 0$, a local diffeomorphism $z = z(x)$ from N to $z(N)$, and a state feedback $u = \alpha(x) + \beta(x)v$, with non-singular $\beta(x)$, such that the feedback system has the following form:

$$\dot{z} = Az + Bv + \mathcal{O}(\|x, u\|^{k+1}) \tag{32}$$

with the pair (A, B) in Brunowsky canonical form.

Before constructing the transformation, we consider the following problem: How much freedom do we have for linear equivalence and in particular, for kth order linear equivalence with state feedback?

The main purpose of this argument is to restrict the searching class of diffeomorphisms.

Consider again system (32) and let the pair (A, B) be the linear approximation of the system, i.e.

$$A = J_f(0)$$
$$B = g(0) = (g_1(0), \ldots, g_m(0))$$

then an obvious necessary condition for linearizability is that (A, B) is a controllable pair. If this condition is satisfied, we can convert system (31) to

$$\dot{x} = Ax + Bu + \mathcal{O}(\|x\|^2) + \mathcal{O}(\|x, u\|^2) \tag{33}$$

We call system (33) the first order *linearized form*, where (A, B) is assumed to be in Brunowsky canonical form. If a system satisfies this necessary condition, it is said that the *linear rank condition* is satisfied.

If system (31) is kth degree linearizable for any $k > 1$, it should be convertible to first degree linearized form as in Equation (32). Thus, we may start from Equation (33). From now on, we will only consider systems in the form of Equation (33).

Next, we want to investigate the set of diffeomorphisms, which preserves the Brunowsky canonical form. To find the particular form of such diffeomorphisms we need some preliminary results.

Definition 3.2

A square matrix A is said to have Brunowsky A-form if

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix}.$$

A vector b is said to have Brunowsky b-form if

$$b = (0 \cdots 0\ b_0)^{\mathrm{T}}$$

A matrix $N_{m\times n}$ is called a Brunovsky null-form if all but the last row elements are zero. That is

$$N = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & 0 \\ n_1 & n_2 & \cdots & n_m \end{pmatrix}.$$

For multi-input case, Brunowsky-A-form, Brunowsky B-form and Brunowsky null-form are defined similarly. That is:

$$A = \mathrm{diag}(A_1, A_2, \ldots, A_m)$$
$$B = \mathrm{diag}(b_1, b_2, \ldots, b_m)$$

and N is the matrix of $n \times n$ with all zero elements except the last rows $n_1, n_1 + n_2, \ldots, n$ at each block.

Lemma 3.3

Let $A_{n\times n}$ and $A_{m\times m}$ be two matrices with Brunowsky A-form, b_1 and b_2 be two vectors of dimensions n and m, respectively, as of Brunowsky b-form and $N_{m\times n}$ be a Brunowsky null-form. $J_{m\times n}$ is any matrix.

J satisfies the following conditions:

$$\begin{aligned} J \times b_1 &= b_2 && \text{(i)} \\ J \times A_{n\times n} &= A_{m\times m} \times J + N && \text{(ii)} \end{aligned} \tag{34}$$

if and only if

$$J = \begin{pmatrix} j_1 & j_2 & \cdots & j_{n-m+1} & 0 & \cdots & 0 \\ 0 & j_1 & & j_{n-m} & j_{n-m+1} & & 0 \\ & & \ddots & & & \ddots & \\ 0 & 0 & \cdots & j_1 & j_2 & \cdots & j_{n-m+1} \end{pmatrix} \tag{35}$$

where $j_1, \ldots, j_{n-m+1}$ are $n - m + 1$ real numbers.

Proof. (*Necessary*): From (i) the elements in the last column of J are all zero except the last one. Then consider the equation in (ii). The result follows by equalizing the elements in the first $m-1$ rows and using mathematical induction.

(*Sufficiency*): A straightforward computation shows sufficiency. □

The following corollary is obvious but useful in the proof of the following theorem.

Corollary 3.4

Suppose J satisfies the conditions in Lemma 3.3.

(i) If $m = n$, then $J = \lambda I$ where $\lambda \in R$;

(ii) If $m > n$, $J = 0$.

In fact, Lemma 3.3 and Corollary 3.4 tell us how much freedom we have if we want to preserve the Brunovsky canonical form. A larger Brunovsky block can be multiplied by a matrix (33) and added to smaller one to keep the Brunovsky form of the block unchanged. Corollary 3.4. claims the following fact: Each set of sequential m rows are multiplied by same number. If two blocks have the same size the only operation allowed is adding a constant multiple of one block to another one. A larger block may not be added to a smaller one.

For instance, assume for system (1) there are two diffeomorphisms: $\psi : x \mapsto y$ and $\phi : x \mapsto z$, such that

$$\psi_*(f) = \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & & & \\ & & 0 & 1 & 0 \\ & & 0 & 0 & 1 \\ & & 0 & 0 & 0 \end{pmatrix} y + o(\|y\|^2)$$

$$\phi_*(f) = \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & & & \\ & & 0 & 1 & 0 \\ & & 0 & 0 & 1 \\ & & 0 & 0 & 0 \end{pmatrix} z + o(\|z\|^2)$$

Then the Jacobian matrix of $\psi\phi^{-1}$ at zero should be

$$J_{\psi\phi^{-1}}|_0 = \begin{pmatrix} \alpha I_2 & 0 \\ J & \beta I_3 \end{pmatrix}$$

where α, β are non-zero real numbers and J is as in (35).

We hereafter will, without loss of generality, assume in Brunowsky, A-form all a_i are zero and in Brunowsky B-form all $b_0 = 1$. A pre-state feedback can be used to realize this.

The next theorem plays a fundamental role in the following linearization argument.

Theorem 3.5

Consider system (33). If it is kth degree linearizable, then there exists a diffeomorphism $x = z + \phi(z)$, which realized the linearization.

Proof. Let $x = T(z)$ be a diffeomorphism, which realized the linearization, then it should keep the Brunowsky canonical form unchanged. For notational ease, we assume $m = 2$. The proof for $m > 2$ is basically the same but involves a messy set of indexes. As for $m = 1$, as a particular case (with one block disappeared in the following proof), it is much easier.

If T realizes the kth degree linearization, it should convert system (33) into the following form:

$$\dot{z} = \begin{pmatrix} A_1 & N_1 \\ N_2 & A_2 \end{pmatrix} z + \begin{pmatrix} b_1 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ b_2 \end{pmatrix} u_2 + \begin{pmatrix} \mathcal{O}(\|x\|^{k+1})\}n_1 - 1 \\ \mathcal{O}(\|x\|^2) \\ \mathcal{O}(\|x\|^{k+1})\}n_2 - 1 \\ \mathcal{O}(\|x\|^2) \end{pmatrix}$$

$$+ \begin{pmatrix} \mathcal{O}(\|x\|^k)\}n_1 - 1 \\ \mathcal{O}(\|x\|) \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ \mathcal{O}(\|x\|^k)\}n_2 - 1 \\ \mathcal{O}(\|x\|) \end{pmatrix} u_2 \tag{36}$$

where A_1 and A_2 are Brunowsky A-forms, b_1 and b_2 are Brunowsky b-forms, N_1 and N_2 are Brunowsky null-forms. Assume the original Brunowsky canonical form in Equation (33) is

$$A = \begin{pmatrix} A_1^o & N_1^o \\ N_2^o & A_2^o \end{pmatrix}, \quad B = \begin{pmatrix} b_1^o & 0 \\ 0 & b_2^o \end{pmatrix}$$

and the Jacobian matrix of T at zero is

$$J_T = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}.$$

Since $J_{11}b_1$, $J_{21}b_1$, $J_{12}b_2$ and $J_{22}b_2$ should be Brunowsky b-forms, the elements in the last columns of J_{ij}, $i = 1, 2; j = 1, 2$, are all zero except the last one. If follows that for any Brunowsky-null form N, both $J_{ij}N$ and NJ_{ij} (if the dimensions are proper for multiplication) remain as a Brunowsky null-form.

Comparing the linear terms of (33) and (36) yields

$$\begin{pmatrix} A_1^o & N_1^o \\ N_2^o & A_2^o \end{pmatrix} \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \begin{pmatrix} A_1 & N_1 \\ N_2 & A_2 \end{pmatrix}.$$

Multiplying and setting corresponding blocks equal yields

$$A_1^o J_{11} = J_{11} A_1 + N$$

$$A_1^o J_{12} = J_{11} A_2 + N$$

$$A_2^o J_{21} = J_{21} A_1 + N$$

$$A_2^o J_{22} = J_{22} A_2 + N$$

where N is some Brunowsky null-form with proper dimension. Now using Lemma 3.3 and Corollary 3.4,

$$J_{11} = aI; \quad J_{22} = dI.$$

If $n_1 = n_2$

$$J_{12} = bI; \quad J_{21} = cI.$$

If $n_1 > n_2$

$$J_{12} = 0; \quad J_{21} = \begin{pmatrix} j_1 & & j_{n_1-n_2+1} & & \\ & \ddots & & \ddots & \\ & & j_1 & & j_{n_1-n_2+1} \end{pmatrix}$$

where a, b, c, d, $j_1, \ldots, j_{n_1-n_2+1}$ are real numbers.

Case 1: $n_1 = n_2$. Since J_T is non-singular, the inverse

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

exists. Hence we can define a linear transformation as

$$S: z = \begin{pmatrix} \alpha I_n & \beta I_n \\ \gamma I_n & \delta I_n \end{pmatrix} y, \quad n = n_1 = n_2. \tag{37}$$

Now the Jacobian matrix of the composed mapping $S \circ T$ at zero is

$$J_{S \circ T}(0) = \frac{\partial x}{\partial z}(0) \times \frac{\partial z}{\partial y}(0) = J_T \times J_S = I_{n_1+n_2}. \tag{38}$$

Note that S is a linear transformation. Using Equation (37) and the sufficient part of Lemma 3.3, it is clear that S leaves Equation (33) unchanged. (Precisely, only the remaining higher degree part may have been changed.) Then Equation (38) implies that $S \circ T$, expressed as $x = y + \mathcal{O}(\|y\|^2)$ is the required diffeomorphism.

Case 2: $n_1 > n_2$. Define a linear transformation S as

$$S: z = \begin{pmatrix} \frac{1}{a} I & 0 \\ -\frac{1}{a} \begin{pmatrix} j_1 & & j_{n_1-n_2+1} & & \\ & \ddots & & \ddots & \\ & & j_1 & & j_{n_1-n_2+1} \end{pmatrix} & \frac{1}{d} I \end{pmatrix} y. \tag{39}$$

It is easy to check that Equation (39) satisfies Equation (38) and has the particular form required by Lemma 3.3. The arguments in Case 1 remain correct. □

Remark

The physical meaning of the transformation discussed above is: In Case 1 the two blocks have been changed by block non-singular linear combination; while Case 2 means adding a larger block to a smaller block in such a way that multiplying each set of sequential n_2 rows by the same numbers (first set by j_1, second set by j_2, etc.) and then adding them to the smaller block. It is obvious that the Brunowsky canonical form remains unchanged. Basically, Lemma 3.3 says that only these two kinds of linear transformations are allowed if the Brunowsky canonical form is to remain invariant.

4. LINEAR APPROXIMATION

Now we consider the linearization problem. For shifting term we have

Definition 4.1
System (33) is shifting term kth degree linearizable, if there exists a diffeomorphism and a state feedback such that (33) becomes

$$\dot{x} = Ax + o(\|x\|^{k+1}) + Bv + g^{(1)}(x)v \tag{40}$$

where $g^{(1)}(x) = o(\|x\|)$ and (A, B) has feedback Brunowsky canonical form.

Theorem 4.2
System (33) is always shifting term kth degree linearizable for any $k \geqslant 1$.

Proof. We prove it by induction. To begin with, we assume $m = 1$. For $k = 1$ the conclusion is trivial. Suppose (31) is already shifting term kth degree linearized. We rewrite (40) as

$$\dot{x} = Ax + f^{(k+1)}(x) + o(\|x\|^{k+2}) + Bv + g^{(1)}(x)v$$

where $f^{(k+1)}(x) \in H_n^{k+1}$. We express $f^{(k+1)}(x)$ under the normal basis as $f^{(k+1)} = \text{col}(\alpha_1, \ldots, \alpha_n)$.
Correspondingly, we can use a diffeomorphism as $x = z + \phi(z)$, where $\phi(z) \in H_n^{k+1}$ is chosen as

$$\phi = -(N_2\alpha_1 + N_3\alpha_2 + \cdots + N_n\alpha_{n-1}).$$

It follows from (30) that

$$ad_L\phi = \begin{pmatrix} \alpha_1 \\ \cdots \\ \alpha_{n-1} \\ \sum_{i=0}^{n-1} \binom{n}{i} D^{n-i}\alpha_{i+1} \end{pmatrix}. \tag{41}$$

Now we can choose

$$u = -f_n^{k+1} + \sum_{i=0}^{n-1} \binom{n}{i} D^{n-i}\alpha_{i+1}.$$

Then

$$ad_L\phi = f + Bu.$$

So the normal form transformation $x = z + \phi(z)$ converts (40) into a $(k+1)$th degree linearized form.
For multi-input systems, we can simply do the above process block-wise. □

For systems with a single input the same result as Theorem 4.2 for quadratic case was proved in Reference [13].

Next, we provide a detailed algorithm for approximate linearization. The basic result is an algorithm of the kth degree linearization for a $(k-1)$th degree linearized system, which leads to a system of linear algebraic equations. That is, the $(k-1)$th degree linearized system is kth degree linearizable, if and only if, the algebraic equations have solution.

Assume we have already obtained a $(k-1)$th degree linearized system in the form

$$\dot{x} = Ax + f^{(k)}(x) + Bu + g^{(k-1)}(x)u + o(\|x, u\|^{k+1}) \tag{42}$$

where $f^{(k)}(x) \in H_n^k$, $g_i^{(k-1)}(x) \in H_n^{k-1}$, $i = 1, \ldots, m$.

To state our main algorithm, we need a little preparation and some additional notation.

First we express $f^{(k)}$ in normal basis as

$$f^{(k)} = (N_n^k(\alpha_1, \ldots, \alpha_n))^{\mathrm{T}} \tag{43}$$

where $\alpha_j \in R^s$, $s = s_n^k$.

Denote by

$$C = \{n_1, n_1 + n_2, n_1 + n_2 + n_3, \ldots, n\}$$

which corresponds the rows with controls of the linearized system. Let its complement be denoted by U, i.e., $U = \{1, 2, \ldots, n\} \backslash C$. We use C_i for ith element in C.

It should be emphasized that whether natural basis or normal basis is used, the transformation T assures that the index set C always corresponds to the rows with linear inputs.

Since we started from system (42), where the pair (A, B) has canonical Brunowsky form, a diffeomorphism should be chosen to keep them unchanged. Note that feedback can only affect the rows in U. According to Theorem 3.5 and the proof of Theorem 4.2, a diffeomorphism, $x = z + \phi(z)$, where $\phi(z) \in H_n^k$ should be chosen as

$$\begin{aligned}\phi(z) = (&\beta_1, -\alpha_1, \ldots, -\alpha_{n_1-1}, \beta_2, -\alpha_{n_1}, \ldots, -\alpha_{n_1+n_2-1}, \ldots, \\ &\beta_m, -\alpha_{n_1+\cdots+n_{m-1}+1}, \ldots, -\alpha_{n-1})^{\mathrm{T}}\end{aligned} \tag{44}$$

The above form is expressed as an expanded form with respect to the normal basis. $\alpha_i \in R^s$ are known from (43), which are used to eliminate the kth degree terms of $f_j^{(k)}$, $j \in U$ rows. $\beta_i \in R^{s_n^{k-1}}$ will be chosen in the following to eliminate the $(k-1)$th degree terms of $(g^{(k-1)})_j^i$ $i \in U$.

Denote by

$$D = \{1, n_1 + 1, n_1 + n_2 + 1, \ldots, n_1 + n_2 + \cdots + n_{m-1} + 1\}$$

which corresponds the β_j blocks in $\phi(z)$. Let its complement be denoted by V, i.e., $V = \{1, 2, \ldots, n\} \backslash D$, which corresponds the α_j blocks in $\phi(z)$.

Using Equations (8) and (9), one sees that to obtain the Brunowsky B-form up to $o(\|z\|)^k)$, we need

$$\left(I - \frac{\partial \phi}{\partial x}\right)(B + g^{(k-1)}) = B + O_n \tag{45}$$

where B is a Brunowsky B-form and O_n is a Brunowsky null-form. Only $(k-1)$th degree terms should be considered. So it is required that

$$-\frac{\partial \phi_r}{\partial x_{C_i}} = (g_i^{(k-1)})_r = 0, \quad r \in U, \, i = 1, \ldots, m. \tag{46}$$

Now the problem becomes: Find β_j for $\phi(z)$ in (44) such that the diffeomorphism $x = z + \phi(z)$ satisfies (46). The next effort will be focused on express (46) into an easily solvable linear algebraic equations.

Let $J_q = \partial/\partial x_q$, then J_q may be considered as a mapping $J_q : B_n^k \to B_n^{k-1}$, and expressed as an $s_n^{k-1} \times s_n^k$ matrix.

Using Theorem 2.3, the following algorithmic formula for J_q is obtained.

Algorithm 4.3

The derivative J_q, or precisely $(J_n^k)_q$, can be constructed as follows: For all $k_i \geqslant 0$, $i = 1, \ldots, n$, $\sum_{i=1}^n k_i = k$, if $k_q \neq 0$, set

$$j = p_n^k(k_1, k_2, \ldots, k_n)$$

$$i = p_n^k(k_1, \ldots, k_q - 1, k_{q+1}, \ldots, k_n).$$

Then the elements J_{ij} of the matrix $(J_n^k)_q$ of dimension $\dim(B_n^{k-1}) \times \dim(B_n^k)$, are determined as

$$J_{ij} = \begin{cases} k_q, & k_q \neq 0, \text{ for above } (i, j) \\ 0, & \text{otherwise.} \end{cases} \tag{47}$$

The following example explains the matrix J_i.

Example 4.4

Assume $n = 3$ and $k = 4$. Now for instance if we consider a monic $p(x) = x_1 x_3^3 \in B_3^4\, p(x)$ is the 10th element in B_3^4. So $j = 10$. Now

$$\frac{\partial p}{\partial x_1} = x_3^3, \quad \frac{\partial p}{\partial x_2} = 0, \quad \frac{\partial p}{\partial x_3} = 3x_1 x_3^2.$$

Table III. Derivative operator.

$p(x)$	$J_1(p(x))$		$J_2(p(x))$		$J_3(p(x))$	
j	i	J_{ij}	i	J_{ij}	i	J_{ij}
1	1	4	—	—	—	—
2	2	3	1	1	—	—
3	3	3	—	—	1	1
4	4	2	2	2	—	—
5	5	2	3	1	2	1
6	6	2	—	—	3	2
7	7	1	4	3	—	—
8	8	1	5	2	4	1
9	9	1	6	1	5	2
10	10	1	—	—	6	3
11	—	—	7	4	—	—
12	—	—	8	3	1	7
13	—	—	9	2	8	2
14	—	—	10	1	9	3
15	—	—	—	—	10	4

So for J_1 we have $i = 10$, and $J_{10,10} = 1$, for J_2 we did not get non-zero value, and for J_3 we have $i = 6$, and $J_{6,10} = 3$. Overall we have the following:

Then J_1, J_2, and J_3 are obtained immediately. Say,

$$J_2 = \begin{pmatrix} 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&2&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&3&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&2&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&4&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&3&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&2&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \end{pmatrix}.$$

The next thing is to express (46) into an easily solvable linear algebraic equation of β_i.

Express g_i in a vector form with respect to the natural basis as

$$g_i = (B_n^{k-1}(\mu_1^i, \dots, \mu_n^i))^{\mathrm{T}},\ i = 1, \dots, m \tag{48}$$

where $\mu_j^i \in R^{s_n^{k-1}}$.

Note that the expression of $\phi(z)$ in (44) is in normal basis. In natural basis it becomes $T\phi(z)$. Differentiate it with respect to x_{C_i} is $(J_{C_i} \otimes I_n)(T\phi)$. Hence (46) can be expressed as

$$[(J_{C_i} \otimes I_n)(T\phi)]_j = \mu_j^i, \quad j \in U, \quad i = 1, \dots, m. \tag{49}$$

Now denote by $T = (T^{ij})$, $i, j = 1, \dots, n$. Where $T^{i,j}$ are $(s_n^k) \times (s_n^k)$ matrices. For notational ease we denote $X = \mathrm{col}(\beta_1, \dots, \beta_m)$. Then (49) can be rewritten as

$$(J_{C_i} T^{jD_1}, J_{C_i} T^{jD_2}, \dots, J_{C_i} T^{jD_m})X = J_{C_i} \sum_{l \in V} T^{jl} \alpha_{l-1} + \mu_j^i$$
$$j \in U, \quad i = 1, \dots, m. \tag{50}$$

Summarizing the above argument yields the following.

Theorem 4.5

System (42) is kth degree linearizable if and only if Equation (50) has a solution, $X = \mathrm{col}(\beta_1, \dots, \beta_m)$. Moreover, if the solution exists, putting any such solution into Equation (44), a diffeomorphism $x = z + \phi(z)$ and a proper state feedback transforms (42) into a kth degree linearized form.

Remark 2

1. In fact, Theorem 4.5 may be considered as an algebraic realization of the corresponding characterization conditions for a linearizable systems in References [6, 7]. (Also refer to References [16, 17].) The advantage of Theorem 4.5 is that it is linearly comparable.

2. If (1) does not have solution, we may find the least-square approximate solution. Such an approximate solution provides a feedback kth linear approximation to the system (4.3). In a later example it will be seen that this kind of approximation is also meaningful. This approximation is co-ordinate depending.

5. SYSTEM OF BALL AND BEAM

In this section we apply the algorithm introduced above to the ball and beam example, which was introduced and discussed in Reference [12]. After the Taylor expansion, a simple linear transformation the system of ball and Beam can be written as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_1 x_4^2 - x_3^3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u + o(\|x\|^4). \tag{51}$$

Since Equation (51) is a second linearized form, our question is: Is it third linearizable? We have only to check whether Equation (50) has a solution.

We figure the indeces first. For (51) we have

$$C = \{4\}, \quad U = \{1, 2, 3\}, \quad D = \{1\}, \quad V = \{2, 3\ , 4\}.$$

Then system (50) becomes

$$J_4 T^{j1} x = J_4 \sum_{l=2}^{4} J^{jl} \alpha_{l-1} + \mu_j, \quad i = 1, 2, 3. \tag{52}$$

The corresponding entries are

$$J_4 = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3
\end{pmatrix}$$

$$T = \begin{pmatrix} I & 0 & 0 & 0 \\ D & I & 0 & 0 \\ D^2 & 2D & I & 0 \\ D^3 & 3D^2 & 3D & I \end{pmatrix}$$

$$D = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&2&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&2&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&2&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&3&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&2&2&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&3&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&2&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0
\end{pmatrix}$$

$\alpha_1 = \alpha_3 = \alpha_4 = 0$ and all elements but two of α_2 are zero:

$$\alpha_2 = \delta_{10} - \delta_{17}$$

$\mu_i = 0,\ i = 1, 2, 3, 4.$

Equation (52) can be further simplified as

$$\begin{pmatrix} J_4 \\ J_4 D \\ J_4 D^2 \end{pmatrix} X = \begin{pmatrix} 0 \\ \alpha_1 \\ 2D\alpha_1 + \alpha_2 \end{pmatrix}. \tag{53}$$

The program verified Equation (53). The conclusion is: The system is not third degree linearizable.

The program also showed that the least square solution is

$$\beta = \delta_{20}.$$

Using Equation (44), the transformation becomes

$$x = z + \begin{pmatrix} 0 \\ 0 \\ -z_1 z_4^2 + z_3^3 \\ -z_2 z_4^2 + 3z_3^2 z_4 \end{pmatrix}. \tag{54}$$

Now a straightforward computation shows that in the co-ordinate z, after a suitable state feedback, that system (51) can be transformed to

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{pmatrix} = \begin{pmatrix} z_2 \\ z_3 \\ z_4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} v + \begin{pmatrix} 0 \\ 0 \\ 2z_1 z_4 \\ 0 \end{pmatrix} + o(\|x\|^4). \tag{55}$$

Equation (55) is the same as the approximated linear form obtained in Reference [12].

6. CONCLUSION

Using the modified normal form this paper presented a normal form representation of a class of affine nonlinear systems which have a controllable linear approximation. An algorithm has been developed to approximately linearize such a nonlinear system via solving an algebraic equation. In fact, it can be proved that by using the normal form approximation the arguments in the algebraic equations have been reduced to a minimum. Least square linear approximation is proposed for a system, which is not kth degree linearizable, to obtain the best kth degree linear approximation. The main purpose in this paper is to provide a mechanically computable formula procedure for the linearization of nonlinear systems. So it can be realized in computer.

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