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Generalized normal form and stabilization of non-linear systems

DAIZHAN CHENG†* and LIJUN ZHANG†

This paper considers the normal form of non-linear control systems. First we propose a generalized relative degree (relative degree vector) for non-linear single (respectively, multiple) input control system, which is called the point relative degree (respectively, point relative degree vector). For the systems without output, the concepts of essential relative degree (respectively, essential relative degree vector) and the essential point relative degree (respectively, essential point relative degree vector) are defined. Unlike the classical definition which requires regularity, the point relative degree (vector) is always well defined.

Using these new concepts the generalized normal form is obtained. Its relationship with the Jacobian linearization is investigated. Using it, a straightforward computation algorithm is provided to achieve the generalized normal form.

Based on the generalized normal form we prove that with an additional condition, if the zero-dynamics is stable the overall system is stabilizable by using pseudo-linear state feedback control.

For the systems under generalized normal form with unstable zero dynamics, the centre manifold approach is applied. It is shown that the stabilization technique via a designed centre manifold is still applicable to this kind of general non-linear control system.

1. Introduction

The stabilization problem of control systems is one of the most fundamental topics in control theory. The centre manifold theory has been used to solve the stabilization problem of Aeyels (1985) and Behtash and Dastry (1988). The concept of minimum phase zero dynamics and its fundamental results provide a systematic method to solve the problem (Byrnes *et al*. 1991, Isidori 1995). A convenient way to use this method is to convert the control system into a canonical form. Then the zero dynamics can be obtained directly. So this approach is closely related to the canonical form of non-linear control systems. We briefly review this approach.

Consider a non-linear control system

$$\left.\begin{aligned}&\dot{x}=f(x)+\sum_{i=1}^{m}g_i(x)u_i:=f(x)+g(x)u, \qquad x\in\mathbb{R}^n, \quad u\in R^m\\ &y=h(x), \qquad y\in R^m\end{aligned}\right\} \tag{1}$$

where $f(x)$ and $g_i(x)$ are smooth vector fields and $f(0)=0$, $g(0)$ has full column rank.

A non-linear normal form, called the Byrnes–Isidori normal form in some literature, plays a fundamental role in non-linear control (Isidori 1995). First, we recall some basic concepts related to this normal form.

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* Author for correspondence. e-mail: dcheng@iss03.iss.ac.cn

† Institute of Systems Science, Chinese Academy of Sciences, Beijing 100080, PR China.

Assume there exists a neighbourhood U of the origin and a vector $\rho=(\rho_1,\dots,\rho_m)$ of positive integers, such that

$$\left.\begin{aligned}&L_gL_f^kh_i(x)=0, \qquad x\in U, \quad k<\rho_i-1\\ &L_gL_f^{\rho_i-1}h_i(0)\neq 0, \qquad i=1,\dots,m\end{aligned}\right\} \tag{2}$$

then $(\rho_1,\cdots,\rho_m)$ is called the *relative degree vector*.

When SISO systems are considered, the relative degree vector is degenerated to relative degree, which is defined as a particular case of (2) for $m=1$.

Using the relative degree vector, the *decoupling matrix*, $W(x)$, is defined as

$$W(x)=\begin{pmatrix}L_{g_1}L_f^{\rho_1-1}h_1(x) & \cdots & L_{g_m}L_f^{\rho_1-1}h_1(x)\\ & \vdots & \\ L_{g_1}L_f^{\rho_m-1}h_m(x) & \cdots & L_{g_m}L_f^{\rho_m-1}h_m(x)\end{pmatrix} = \begin{pmatrix}\mathrm{d}L_f^{\rho_1-1}h_1(x)\\ \vdots\\ \mathrm{d}L_f^{\rho_m-1}h_m(x)\end{pmatrix}g(x) \tag{3}$$

The last equality in (3) is an immediate consequence of the definition of the Lie derivative. It is important in later discussion.

For system (1) we give two fundamental assumptions:

H1. $W(0)$ is invertible;

H2. $g_1(0),\dots,g_m(0)$ are linearly independent and $\mathrm{Span}\{g(x)\}$ is involutive near the origin.

Remark: In SISO case, the decoupling matrix becomes a scalar, $W(x)=L_gL_f^{\rho-1}h(x)$. By the definition

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of relative degree, $W(0) \neq 0$. Moreover, a distribution spanned by a single vector is always involutive. Therefore, for SISO systems the assumptions H1 and H2 are automatically satisfied.

Theorem 1 (Byrnes and Isidori 1988): *For system (1) assume the relative degree vector is well defined at the origin and the assumptions H1 and H2 hold, then there exists a local coordinate transformation* $(z, \xi) = (z(x), \xi(x))$ *with* $z(0) = 0$ *and* $\xi(0) = 0$, *such that system (1) can be expressed locally around the origin as*

$$\left.\begin{aligned} \dot{z}_1^i &= z_2^i \\ &\vdots \\ \dot{z}_{\rho_i-1}^i &= z_{\rho_i}^i \\ \dot{z}_{\rho_i}^i &= c_i(z,\xi) + d_i(z,\xi)u \quad i = 1, \dots, m \\ \dot{\xi} &= p(z,\xi), \quad \xi \in \mathbb{R}^{n-\|\rho\|} \\ y_i &= z_1^i, \quad i = 1, \dots, m \end{aligned}\right\} \tag{4}$$

where

$$\|\rho\| = \sum_{i=1}^{m} \rho_i,$$
$$d_i(z,\xi) = (d_{i1}(z,\xi), \dots, d_{im}(z,\xi)), \quad i = 1, \dots, m$$

In some later literature this normal form is called the Byrnes–Isidori normal form. Based on the theory of centre manifold (Carr 1981), a useful stabilization result is the following.

Proposition 1 (Byrnes *et al.* 1991, Isidori 1995): *If system* (4) *has minimum phase, i.e.*

$$\dot{\xi} = p(0, \xi) \tag{5}$$

is asymptotically stable at the origin, then the system is stabilizable via a pseudo-linear control

$$u = -\begin{pmatrix} d_1 \\ \vdots \\ d_m \end{pmatrix}^{-1} \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} + \begin{pmatrix} d_1 \\ \vdots \\ d_m \end{pmatrix}^{-1} \begin{pmatrix} \sum_{j=1}^{\rho_1} a_j^1 z_j^1 \\ \vdots \\ \sum_{j=1}^{\rho_m} a_j^m z_j^m \end{pmatrix} \tag{6}$$

Remark:

(1) In (6) a_j^i are chosen such that each linear block is Hurwitz. Precisely

$$\lambda^{\rho_i} - \sum_{j=1}^{\rho_i} a_j^i \lambda^{j-1}, \qquad i = 1, \dots, m$$

are Hurwitz polynomials.

(2) In (4), the coefficients of the controls form the decoupling matrix under the new coordinates (z, ξ), that is

$$W(x(z,\xi)) = \begin{pmatrix} d_1 \\ \vdots \\ d_m \end{pmatrix} = \begin{pmatrix} d_{11}(z,\xi) & \cdots & d_{1m}(z,\xi) \\ & \vdots & \\ d_{m1}(z,\xi) & \cdots & d_{mm}(z,\xi) \end{pmatrix} \tag{7}$$

where $x(z, \xi)$ is the inverse coordinate transformation of $(z, \xi) = (z(x), \xi(x))$.

Equation (7) can be obtained from (4) via a straightforward computation.

Proposition 1 is an immediate consequence of the following lemma, which will be used in a later discussion.

Lemma 1 (Isidori 1995, p. 511): *Consider a system*

$$\left.\begin{aligned} \dot{z} &= Az + p(z,w) \\ \dot{w} &= f(z,w) \end{aligned}\right\} \tag{8}$$

where $p(0, w) = 0$ *for all* w *near* 0 *and*

$$\frac{\partial p}{\partial z}(0,0) = 0$$

If $\dot{w} = f(0, w)$ *has an asymptotically stable equilibrium at* $w = 0$ *and* A *is Hurwitz, then system* (8) *has an asymptotically stable equilibrium at* $(z, w) = (0, 0)$.

This approach via normal form with the relative degree (vector) and minimum phase property has been proved as a power tool in dealing with the stabilization of non-linear control systems, and became a standard method in this field (Nijmeijer and Van der Schaft 1990, Isidori 1995, Khalil 1996). But it has some obvious shortages:

- the relative degree (vector) is not always well defined because of its regularity requirement;
- it may not provide the largest linearizable subsystem just because the output is 'improperly chosen';
- the zero dynamics may not have minimum phase.

We give a simple example to describe this.

Example 1: Consider the following system

$$\left.\begin{aligned} \dot{x}_1 &= x_2 + x_3^3 \\ \dot{x}_2 &= x_3 + x_2^2 x_3 \\ \dot{x}_3 &= x_4 + x_3^2 x_4 \\ \dot{x}_4 &= u \end{aligned}\right\} \tag{9}$$

Case 1: Set the output as

$$y = h(x) = \sin(x_2 + x_4^2) \tag{10}$$

Then

$$L_g h(x) = 2x_4 \cos(x_2 + x_4^2)$$

Since $L_g h(0) = 0$ and there isn't a neighbourhood, U, of the origin such that $L_g h(x) = 0$, $x \in U$. So the relative degree is not defined because of the singularity of the Lie derivative $L_g h(x)$ at the origin.

Case 2: Set the output as

$$y = x_3 \tag{11}$$

Since $L_g h(x) = 0$, $L_f(x) = x_4 + x_3^2 x_4$ and $L_g L_f h(x) = 1 + x_3^3 \neq 0$, $\|x\| < 1$. The relative degree is 2. So we can get a linearizable sub-system of dimension dim $= 2$.

But this is not the largest one. If we choose $y = x_2$, the relative degree will be 3, then we can get a linearizable sub-system of dimension dim $= 3$. (Later on we will see that under the classical definition, $y = x_2$ is the best choice. But the new modified definition will do better.)

Moreover, consider the output (11) again. According to the above calculation, we can follow a standard procedure by now choosing coordinates as

$$z_1 = h(x) = x_3, \qquad z_2 = L_f h(x) = x_4(1 + x_3^2),$$
$$z_3 = x_1, \qquad z_4 = x_2$$

and then convert system (9) and (11) into the Byrnes–Isidori normal form as follows:

$$\left.\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= \frac{2z_1 z_2^2}{1 + z_1^2} + (1 + z_1^2)u \\ \dot{z}_3 &= z_4 + z_1^3 \\ \dot{z}_4 &= z_1(1 + z_4^2) \\ y &= z_1 \end{aligned}\right\} \tag{12}$$

.

The zero dynamics is

$$\dot{z}_3 = z_4 \tag{13}$$

$$\dot{z}_4 = 0 \tag{14}$$

Obviously, it is not stable at the origin. So the standard technique for stabilizing minimum phase non-linear control systems is not applicable. This is obviously a weakness of the method because system (9) is stabilizable by linear state feedback.

Recently, to overcome the above-mentioned third shortage of the normal form approach, Cheng (2000) and Cheng and Martin (2001) considered the stabilization of (4) with non-minimum phase. The concept of a Lyapunov function with a homogeneous derivative was proposed.

The main motivation of this paper is to overcome the first and the second shortages as much as possible. Roughly speaking, since the restrictions of converting a non-linear control system into the normal form are rigorous, to apply the normal form analysis to more general systems, we have to generalize the normal form to include as many systems as possible.

In this paper the relative degree (vector) is extended to the essential relative degree to get the largest linearizable sub-system, and then to the (essential) point relative degree (vector), which is always well defined. Then the Byrnes–Isidori normal form is generalized to a generalized normal form. Finally, the stabilization problem for generalized norm form is considered. For the case of minimum phase, a result similar to the one for Byrnes–Isidori normal form is obtained under an additional condition. For the case of non-minimum phase, the technique of Lyapunov function with homogeneous derivative, developed in Cheng and Martin (2001), is applied to stabilizing systems.

The rest of the paper is organized as follows. Section 2 extends the relative degree (vector) to an essential and point relative degree (vector). Section 3 generalizes the Byrnes–Isidori normal form to get the generalized normal form. The relationship between the generalized normal form and the Jacobian linearization is revealed, which provides a simple way to calculate the generalized canonical form. Section 4 considers the stabilization of non-linear systems under the generalized normal form via a designed centre manifold. Some illustrative examples are presented in §5. The last section shows some conclusions.

2. Essential and point relative degrees

We start with SISO systems. Recalling the definition of relative degree, it is obvious that the relative degree depends on the output. When we are only interested in the state equations, then for the state equation of (1) with $m = 1$, renumber it as

$$\dot{x} = f(x) + g(x)u, \qquad x \in \mathbb{R}^n \tag{15}$$

We may look for an auxiliary output $h(x)$ such that the relative degree for the state equation with respect to this $h(x)$ can be the maximum one.

Definition 1: The essential relative degree of (15) is the largest relative degree related to an arbitrary chosen smooth output function.

It is obvious that the essential relative degree is closely related to the largest feedback linearizable subsystem (Marino 1986). Denote by

$$\Delta_i = \mathrm{Span}\{g, \mathrm{ad}_f g, \ldots, \mathrm{ad}_f^{i-1} g\}, \qquad i = 1, 2, \ldots \tag{16}$$

The following result is from Isidori (1995) with a slightly different statement.

Proposition 2 (Isidori 1995): *The essential relative degree ρ^e is the largest k, such that on some open neighbourhood U of* 0

$$\left.\begin{aligned} &\dim\{\Delta_k(0)\}_{LA} = n \\ &\dim\{\Delta_{k-1}(x)\}_{LA} = l \le n-1, \qquad x \in U \end{aligned}\right\} \tag{17}$$

where $\{\Delta_i(x)\}_{LA}$ is the Lie-algebra generated by $\Delta_i(x)$.

From differential geometry (Boothby 1986) we know that since $g(0) \neq 0$, Δ_1 is always non-singular and involutive locally. That means, the essential relative degree ρ^e of the system (15) is always greater than or equal to 1.

Consider $(f, g) \in V(\mathbb{R}^n) \times V(\mathbb{R}^n)$, where $V(\mathbb{R}^n)$ is the space of C^∞ vector fields on $\mathbb{R}^n$. Now if we use Whitney C^∞ topology to $V(\mathbb{R}^n)$ (Golubitsky and Guillemin 1973), it is easy to see that the set of (f, g) satisfying (17) is a zero measure set. That means only a zero measure set of SISO systems have non-trivial essential relative degree. (But we should not be too pessimistic, because many practically useful dynamic systems do have non-trivial essential relative degree.)

We give the following definition about the *point relative degree*. A more or less related concept may be found in Xia and Gao (1997).

Definition 2:

(1) For system (1) with $m = 1$, the point relative degree ρ^p at the origin is defined as

$$\left.\begin{aligned} &L_g L_f^k h(0) = 0, \qquad k < \rho^p - 1 \\ &L_g L_f^{\rho^p - 1} h(0) \neq 0 \end{aligned}\right\} \tag{18}$$

(2) For system (15) the essential point relative degree is the point relative degree for an auxiliary output $h(x)$ such that the point relative degree for (15) with respect to this $h(x)$, denoted by ρ^{ep}, is the maximum one.

We give a simple example to describe these concepts.

Example 2: Recall system (9). Assume the output is

$$y = x_4 e_1^x \tag{19}$$

According to the definitions, a straightforward computation shows that the different relative degrees at the origin are as follows.

(1) Since $L_g y(0) \neq 0$, the relative degree is $\rho = 1$.

(2) Since $\mathrm{Span}\{g, ad_f g\}$ is non-singular and involutive, moreover Δ_3 is not involutive and $\dim\{\Delta_3\}_{LA} = 4$, the essential relative degree is $\rho^e = 3$. One of the corresponding h is $h(x) = x_2$.

(3) The point relative degree is also $\rho^p = 1$. In fact, it follows from the definition that if the relative degree is well defined at a point, then the point relative degree should be the same as the relative degree.

To tell the difference between relative degree and the point relative degree, we now assume the output is

$$y = \sin(x_2 + x_4^2) \tag{20}$$

Then it is ready to verify that with this output the relative degree of the composed system of (19) and (20) is not defined. But the point relative degree is $\rho^p = 3$.

(4) Now if we choose $h(x) = x_1$, it is easy to check that the essential point relative degree for the composed system of (19) and this output is $\rho^{ep} = 4$.

Now we turn to the MIMO case. Denote the state equation of (1) as

$$\dot{x} = f(x) + g(x)u, \qquad x \in \mathbb{R}^n, \qquad u \in \mathbb{R}^m \tag{21}$$

Definition 3:

(1) For system (1) the point relative degree vector $(\rho_1, \ldots, \rho_m)$ is defined as

$$\left.\begin{aligned} &L_g L_f^k h_i(0) = 0, \qquad k < \rho_i - 1 \\ &L_g L_f^{\rho_i - 1} h(0) \neq 0, \qquad i = 1, \ldots, m \end{aligned}\right\} \tag{22}$$

(2) The essential relative degree vector, $\rho^e = (\rho_1^e, \ldots, \rho_m^e)$, (the essential point relative degree vector, $\rho^{ep} = (\rho_1^{ep}, \ldots, \rho_m^{ep})$) , for the state equation (21) is defined as the largest one of relative degree (respectively, point relative degree), ρ^*, for all possible auxiliary outputs, which makes the decoupling matrix, W^{ρ^e} (respectively, $W^{\rho^{ep}}$), non-singular. That is

$$\|\rho^{*e}\| = \sum_{i=1}^m \rho_i^{*e} = \max\{\|\rho^e\| \mid W^{\rho^e} \text{ is non-singular}\}; \tag{23}$$

and

$$\|\rho^{*ep}\| = \sum_{i=1}^m \rho_i^{*ep} = \max\{\|\rho^{ep}\| \mid W^{\rho^{ep}} \text{ is non-singular}\} \tag{24}$$

Motivated by the equation (19), we may ask that if system (15) has the essential point relative degree ρ^{ep} (relative degree vector $\rho^{ep} = (\rho_1^{ep}, \ldots, \rho_m^{ep})$), can we always find a canonical controllable linear part with dimension ρ^{ep} (or $\|\rho^{ep}\|$ for multi-input case)?

The answer is 'yes'. In fact, in the next section we will use the essential point relative degree to convert the state equation of system (1) into a so-called the generalized normal form, which has the largest linearly controllable subsystem of dimension $\|\rho^{ep}\|$.

Remark: Later on, we will show that the essential point relative degree (vector) always exists. But in the multi-input case we still do not know whether the essential (point) relative degree vector is unique.

3. Generalized normal form

Definition 4: For system (1) the following form is called the generalized normal form

$$\left.\begin{aligned} &\dot{z}^i = A_i z^i + b_i u_i + \begin{pmatrix} 0 \\ \alpha_i(z,w) \end{pmatrix} + p^i(z,w)u, \qquad z^i \in \mathbb{R}^{\rho_i} \\ &\qquad i = 1,\ldots,m \\ &\dot{w} = q(z,w), \qquad w \in \mathbb{R}^r \\ &y_i = z_1^i, \qquad i = 1,\ldots,m \end{aligned}\right\} \tag{25}$$

where $r + \sum_{i=1}^m \rho_i = n$, $\alpha_i(z,w)$ are scalars, $p^i(z,w)$ are $\rho_i \times m$ matrices, $q(z,w)$ is a $r \times 1$ vector field, and (A_i, b_i) are Brunovsky canonical form in $\mathbb{R}^{\rho_i}$ with the form

$$A_i = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad b_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

and $p^i(0,0) = 0$.

Comparing (25) with (4), the only difference between them is that in (4) $g_i = (0 \ \cdots \ 0 \ d_i(x,z))^{\mathrm{T}}$, $i = 1,\ldots,m$, and in (25) there exist $p^i(z,w)$ which are higher degree input channels.

The following proposition is essential for SISO normal form.

Proposition 3: *If system* (1) *with* $m = 1$ *has point relative degree* $r = \rho^p$ *at* $x = 0$*, then there exists a suitable local coordinate change, which converts system* (1) *into system* (25).

Proof: We first claim that $dh(0),\ldots,dL_f^{r-1}h(0)$ are linearly independent. Let c_i be real numbers such that

$$\sum_{i=1}^r c_i dL_f^{i-1}h(0) = 0$$

constructing a one-form $\omega = \sum_{i=1}^r c_i dL_f^{i-1}h(x)$, we have

$$L_g\omega = \sum_{i=1}^r c_i L_g L_f^{i-1}h(x).$$

Then

$$0 = L_g\omega(0) = \sum_{i=1}^r c_i L_g L_f^{i-1}h(0) = c_r L_g L_f^{r-1}h(0)$$

By definition $L_g L_f^{r-1}h(0) \neq 0$, so $c_r = 0$. So now $\omega = \sum_{i=1}^{r-1} c_i dL_f^{i-1}h(x)$. Considering $L_g L_f \omega$ in a similar way, it follows that $c_{r-1} = 0$. Continuing this procedure, we finally have that all $c_i = 0$, $i = 1,\ldots,r$, which proves the claim. Now we can choose partial coordinate variables as $(z_1,\ldots,z_r) = (h, L_f h,\ldots,L_f^{r-1}h)$. Then we can locally find $n-r$ functions $w_1,\ldots,w_{n-r}$ such that (z,w) is a set of local coordinate variables. Moreover, since $L_g L_f^{r-1}h(0) \neq 0$, w can be chosen in such a way that

$$L_g w_i = 0, \quad i = 1,\ldots,n-r$$

Under this coordinate frame we can, through a straightforward computation, convert the original system into

$$\left.\begin{aligned} \dot{z}_1 &= z_2 + L_g h(z,w)u \\ &\vdots \\ \dot{z}_{r-1} &= z_r + L_g L_f^{r-2}h(z,w)u \\ \dot{z}_r &= L_f^r h + L_g L_f^{r-1}h(z,w)u \\ \dot{w} &= q(z,w) \\ y &= z_1 \end{aligned}\right\} \tag{26}$$

which is the generalized normal form as (25) with $m = 1$. □

An immediate consequence is the following:

Proposition 4: *Assume system* (15) *has an essential point relative degree* $r = \rho^{ep}$ *at the origin, then it can be expressed as the state equations of* (26).

We will call the state equations of (25) the generalized normal state form.

Proposition 5: *Consider system (1).*

(1) *Assume H1 and H2 and if the system has point relative degree vector* $\rho^p = (\rho_1,\ldots,\rho_m)$*, then there exists a local coordinate frame such that the system can be converted into the generalized normal form* (25).

(2) *Assume H2 and if system* (21) *has essential point relative degree vector* $\rho^{ep} = (\rho_1,\ldots,\rho_m)$*, then there exists a local coordinate frame such that the system can be converted into the generalized normal state form as the state equation of* (25).

Proof: We prove (1). Then (2) is an immediate consequence of (1).

Similar to the SISO case, we assume

$$\sum_{i=1}^{m}\sum_{j=1}^{\rho_i} c_{i,j} dL_f^{j-1} h_i(0) = 0$$

and define

$$\omega(x) = \sum_{i=1}^{m}\sum_{j=1}^{\rho_i} c_{i,j} dL_f^{j-1} h_i(x)$$

From

$$L_{g_k}\omega(0) = 0, \qquad k = 1, \dots, m$$

we can get

$$W(0)\begin{pmatrix} c_{1\rho_1} \\ \vdots \\ c_{m\rho_m} \end{pmatrix} = 0$$

By assumption H1, $c_{i\rho_i} = 0$, $i = 1, \dots, m$. Keep going as for the SISO case yields that all $c_{ij} = 0$. Hence we can choose $h_i, L_f h_i, \dots, L_f^{\rho_i - 1} h_i$, $i = 1, \dots, m$ as part of coordinate variables.

Recall H2, we know that $G := \mathrm{Span}\{g(x)\}$ is an involutive distribution of dimension m. According to the Frobinius' Theorem (Boothby 1986)), there exist $n - m$ functions $\xi_1, \dots, \xi_{n-m}$, such that

$$G^{\perp} = \mathrm{Span}\{d\xi_i \mid i = 1, \dots, n - m\}$$

Denote

$$\Omega = \mathrm{Span}\{dL_f^j h_i \mid i = 1, \dots, m; j = 0, \dots, \rho_i - 1\}$$

and $\rho = \sum_{i=1}^{m} \rho_i$. Since $W(0)$ is non-singular and recalling the last equality of (3), it is clear than

$$\dim(\Omega \cap G^{\perp}) \le \rho - m$$

In fact, since ρ co-vectors in Ω are linearly independent, the '$\le$' should be '$=$'. We, therefore, are able to choose $(n - m) - (\rho - m) = n - \rho := r$ closed one-forms from $G^{\perp}$, which are linearly independent with Ω. Say, they are $dw_1, \dots, dw_r$.

Now we can use $L_f^j h_i$, $i = 1, \dots, m$, $j = 0, \dots, \rho_i - 1$ and w_k, $k = 1, \dots, r$ as a complete set of coordinate variables. Then a straightforward computation for coordinate transformation shows that under this coordinate frame system (1) becomes (25). □

In fact, everybody believes that the essential relative degree (vector) is much better than the relative degree (vector), because it can provide the largest linear part. The problem is the relative degree (vector) can provide the required coordinate variables by using output (outputs) and its (their) Lie derivatives. But to get the required coordinate variables, which convert the system into the canonical form with largest linear part, we have to solve a set of partial differential equations. So it is not practically applicable.

One of the significant advantages of the essential point relative degree is that it is easily computable, and then the related generalized normal form, which contains the largest linear part, can be obtained via straightforward computation. We construct it in the rest of this section.

For SISO systems, we show an analogue to Proposition 2 for essential point relative degree.

Proposition 6: Let Δ_i be defined as in (16). The essential point relative degree ρ^{ep} is the largest k, such that

$$\dim(\Delta_i(0)) = i, \qquad i \le k \tag{27}$$

Proof: Assume k is the largest one for (27) to be true. Then we can find a local coordinate frame z such that

$$\left(g(0), \dots, \mathrm{ad}_f^{k-1} g(0)\right) = \begin{pmatrix} I_k \\ 0 \end{pmatrix}$$

Now choose $h(z) = z_k$. It follows that $L_g h(0) = 0$, and

$$L_g L_f h(0) = L_{\mathrm{ad}_f g} h(0) - L_f L_g h(0)$$

Since $L_g h(0) = 0$ and $f(0) = 0$, then $L_f L_g h(0) = 0$. It follows that $L_g L_f h(0) = 0$. Similarly, $L_g L_f^i h(0) = 0$, $i < k - 1$. We also have $L_g L_f^{k-1} h(0) \ne 0$. As an immediate consequence of the definition, $k \le \rho^{ep}$. Conversely, assume the system has the essential point relative degree ρ^{ep}, then the system has the form as (26). Through a straightforward computation one sees easily that (27) holds for ρ^{ep}. Hence $k \ge \rho^{ep}$. We conclude that $k = \rho^{ep}$. □

For MIMO system (21), we denote

$$A = \frac{\partial f}{\partial x}(0), \qquad B = (b_1, \dots, b_m) = g(0)$$

Then a set of linear subspaces can be defined as

$$\Delta_i = \mathrm{Span}\{B, AB, \dots, A^{i-1}B\}, \qquad i = 1, 2, \dots$$

Assume Δ_k is the controllable subspace of (A, B). We can arrange the bases of Δ_i, $i = 1, \dots, k$ in table 1 (with $k_1 = k$).

The vectors in the table are linearly independent. Moreover, the vectors in the first i columns form the

Δ_1	Δ_2				Δ_k
b_1	Ab_1	$\cdots$			$A^{k_1-1}b_1$
b_2	Ab_2	$\cdots$		$A^{k_2-1}b_2$	
$\vdots$					
b_m	Ab_m	$\cdots$	$A^{k_m-1}b_m$		

Table 1. The bases of Δ_i.

basis of Δ_i. It is very easy to get the table by choosing linearly independent vectors column by column. To assure that $k_1 \geq k_2 \geq \cdots \geq k_m$, we may need to reorder g_i.

Proposition 7: *For system* (21), $(k_1, \ldots, k_m)$ *is an essential point relative degree vector.*

Proof: First we choose a linear transformation to convert the linear part of (21), say (A, B), into canonical form as

$$\left.\begin{aligned} \dot{x}_1^i &= x_2^i + \alpha_1^i(x) + \beta_1^i(x)u \\ &\vdots \\ \dot{x}_{k_i-1}^i &= x_{k_i}^i + \alpha_{k_i-1}^i(x) + \beta_{k_i-1}^i(x)u \\ \dot{x}_{k_i}^i &= \alpha_{k_i}^i(x) + \beta_{k_i}^i(x)u \\ & i = 1, \ldots, m \\ \dot{x}^{m+1} &= p(x) + q(x)u \end{aligned}\right\} \tag{28}$$

where $\alpha_j^i(0) = 0$, $j = 1, \ldots, k_i$, $\partial\alpha_j^i/\partial x^k = 0$, $j = 1, \ldots, k_i - 1$, $k = 1, \ldots, m$, and $\beta_j^i(0) = 0$, $j = 1, \ldots, k_i - 1$, $i = 1, \ldots, m$.

Choosing $y_i = z_1^i$, it is easy to see that the essential point relative degrees are $\rho_i = k_i$ and the decoupling matrix is non-singular. Hence,

$$\|\rho^{ep}\| \geq \sum_{i=1}^{m} k_i$$

Conversely, let $\rho^{ep} = (\rho_1, \ldots, \rho_m)$ be a point relative degree vector with a non-singular decoupling matrix, then from Proposition 5, system (21) can be expressed as (25). Then its Jacobian linearization has a controllable subsystem of dimension $\|\rho^p\|$. But $k_1 + \cdots + k_m$ is the dimension of the controllable subspace of the Jacobian linearization of system (21), then $\|\rho^p\| \leq k_1 + \cdots + k_m$. Therefore

$$\|\rho^{ep}\| = k_1 + \cdots + k_m \qquad \square$$

Next, we show how to get the pseudo-normal form.

For (28), we choose $x_1^1, \ldots, x_m^1$ as the auxiliary outputs to generate part of new coordinates z. Precisely speaking, set

$$\begin{aligned} z_1^1 &= x_1^1, \quad \cdots, \quad z_{\rho_1}^1 = L_f^{\rho_1-1} x_1^1, \quad \cdots \\ z_1^m &= x_m^1, \quad \cdots, \quad z_{\rho_m}^m = L_f^{\rho_m-1} x_1^m \end{aligned}$$

And then, as in the proof of Proposition 7, we can choose $r = n - \rho$ function ξ_i, $i = 1, \ldots, r$, such that on a neighbourhood U of origin

$$L_{g_j}\xi_i(x) = 0, \quad x \in U, \quad j = 1, \ldots, m; \quad i = 1, \ldots, r.$$

Moreover, (z, ξ) is a new local coordinate frame.

Then we express the system into this new coordinate frame (z, ξ). It becomes the pseudo-normal form as

$$\left.\begin{aligned} \dot{z}_1^i &= z_2^i + \beta_1^i(z, \xi)v \\ &\vdots \\ \dot{z}_{\rho_{i-1}}^i &= z_{\rho_i}^i + \beta_{\rho_i-1}^i(z, \xi)v \\ \dot{z}_{\rho_i}^i &= c_i(z, \xi) + d_i(z, \xi)v \\ & z^i \in \mathbb{R}^{\rho_i}, \qquad i = 1, \ldots, m, \\ \dot{\xi} &= q(z, \xi), \qquad \xi \in \mathbb{R}^r \end{aligned}\right\} \tag{29}$$

where $c_i(0,0) = 0$, $\beta_j^i(0,0) = 0$, $j = 1, \ldots, \rho_i - 1$, $i = 1, \ldots, m$, and $q(0,0) = 0$.

4. Stabilization

Now we consider the stabilization problem for the systems of generalized normal form. Recall system (25), we assume

H3. $\alpha_i(0, w)p^i(0, w) = 0$, $i = 1, \ldots, m$.

Using Lemma 1, the following stabilization property is obvious, which is a generalization of its counterpart Proposition 1 for Byrnes–Isidori normal form.

Proposition 8: *Assume H3. For the generalized normal state form (state equation of* (25)*) if the pseudo-zero dynamics*

$$\dot{w} = q(0, w) \tag{30}$$

is asymptotically stable at zero, then (25) *is stabilizable by a pseudo-linear state feedback control.*

Proof: Choosing

$$u_i = a_1^i z_1^i + \cdots + a_{\rho_i}^i z_{\rho_i}^i - \alpha_i(z, w), \qquad i = 1, \cdots, m$$

the closed-loop system becomes

$$\left.\begin{aligned} \dot{z}^i &= A_i z^i + p^i(z, w)u(z, w) := A_i z^i + \delta^i(z, w) \\ & i = 1, \ldots, m \\ \dot{w} &= q(z, w) \end{aligned}\right\} \tag{31}$$

According to H3, $\delta^i(0, w) = 0$, $i = 1, \ldots, m$. Using Lemma 1, if we can prove $(\partial\delta^i/\partial z)(0,0) = 0$ we are done. Note that $\alpha_i(0,0) = 0$ and $p^i(0,0) = 0$, the conclusion follows. $\square$

Proposition 8 may give some reason for proposing the point relative degree.

Next, we consider the case when the system is of non-minimum phase. Recall the generalized normal form

$$\left.\begin{aligned}&\dot{z}^i = A_i z^i + b_i u_i + \begin{pmatrix} 0 \\ \alpha_i(z,w) \end{pmatrix} + p^i(z,w)u, \qquad z^i \in \mathbb{R}^{\rho_i}\\ &\qquad\qquad i = 1,\dots,m\\ &\dot{w} = q(z,w), \qquad w \in \mathbb{R}^r\end{aligned}\right\} \tag{32}$$

where $p^i(0,0) = 0$, $\alpha_i(0,0) = 0$, and (A_i, b_i) have Brunovsky canonical form. We discussed in a previous section that under the assumption H2 any affine non-linear system can be locally expressed as (32). In fact (32) is nothing but the state equation of (25).

For the non-minimum phase case, we need another assumption.

H4. $q(z,w)$ with its first order derivatives vanish at the origin.

In fact, the canonical form (32) is not unique, but if we assume another coordinate frame $(\tilde{z}, \tilde{w})$ is such that system (32) keeps its structure unchanged. Then it is easy to see that the Jacobian matrix

$$J_{\tilde{z},\tilde{w}}(0) = \begin{pmatrix} J_{11} & J_{12} \\ 0 & J_{22} \end{pmatrix}$$

Then it is easy to see that the assumption H4 is independent of the different coordinate frames.

The assumption H4 means that the zero dynamics has zero linear part. It is sometimes called that the system has zero centre. In fact a necessary condition for the stabilizability is that the zero dynamics should have its linear part with zero real part eigenvalues.

In the following the notations and conventions used in Cheng and Martin (2001) will be inherited. We briefly give the following:

- $z_1 = (z_1^1, z_1^2, \dots, z_1^m)$,
  $\bar{z}_1 = (z_2^1, \dots, z_{\rho_1}^1, \ \dots, z_2^m, \dots, z_{\rho_m}^m)$
- For a multi-index, say $T = (t_1, \dots, t_r)$

$$\|T\| = \sum_{j=1}^{r} t_i$$

- Let $x = (x_1, \dots, x_\rho) \in \mathbb{R}^\rho$, $y = (y_1, \dots, y_r) \in \mathbb{R}^r$, $K = (k_1, \dots, k_\rho)$, and $T = (t_1, \dots, t_r)$. Then

$$x^K y^T = \prod_{i=1}^{\rho} x_i^{k_i} \prod_{j=1}^{r} y_j^{t_j}$$

- For an analytic function $q(w)$, the lowest degree of $q(w)$, denoted by $LD(q)$, is the lowest degree of the no-vanishing terms of its Taylor expansion.

Then we define the injection degrees as

$$\lambda_i = \min\{2\|K\| + \|T\| \mid \|K\| > 0, z_1^K w^T \text{ is in } q_i \text{ with non-zero coefficient}\}, \quad i = 1,\dots,r \tag{33}$$

The leading degrees are defined as

$$L_i = \begin{cases} \lambda_i, & \lambda_i \text{ is odd;} \\ \lambda_i + 1, & \lambda_i \text{ is even} \quad i = 1,\dots,r \end{cases} \tag{34}$$

Next, we give an example to explain the above notations.

Example 3: Consider the system

$$\left.\begin{aligned}\dot{x}_1 &= x_2 + w_1^2 u_1\\ \dot{x}_2 &= u_1 + w_2^3 u_2\\ \dot{x}_3 &= x_4 + w_1 \sin(x_1) u_1\\ \dot{x}_4 &= x_3^2 u_1 + e^{w_1} u_2\\ \dot{w}_1 &= x_1 w_2 + w_2^3\\ \dot{w}_2 &= x_4 w_1 + x_3^2 w_1 w_2 + x_1^2 x_3^4 w_1\end{aligned}\right\} \tag{35}$$

This is a generalized normal form. Now $z_1 = (x_1, x_3)$, $\bar{z}_1 = (x_2, x_4)$. In q_1 we have only one term of the form of $z_1^K w^T$, that is, $z_1 w_2$. For this term we have $K = (1,0)$ and $T = (0,1)$, so the injection degree is

$$\lambda_1 = 2\|K\| + \|T\| = 3.$$

For q_2 we have two terms with the injection form: for $z_3^2 w_1 w_2$ we have $K = (0,2)$ and $T = (1,1)$, then $2\|K\| + \|T\| = 6$. For the second one $z_1^2 z_3^4 w_1$ we have $K = (2,4)$ and $T = (1,0)$, so $2\|K\| + \|T\| = 13$. Choosing the smallest one, we have the injection degree

$$\lambda_2 = \min\{6, 13\} = 6$$

Now for the leading degrees, we have

$$L_1 = \lambda_1 = 3, \qquad L_2 = \lambda_2 + 1 = 7$$

The motivation is to convert all the dynamics on centre manifold into an odd leading system.

Definition 5: A system

$$\dot{w}_i = f_i(w), \qquad i = 1,\dots,r$$

is said to be $L = (L_1, \dots, L_r)$ approximately asymptotically stable at the origin, if the corresponding uncertain system

$$\dot{w}_i = f_i(w) + 0(\|w\|^{L_i+1}), \qquad i = 1,\dots,r$$

is asymptotically stable at the origin.

Note that as in Carr (1981) or other centre manifold applications, we use $0(\|x\|^t)$ for the set (or any) of smooth functions, $f(x)$, which have Taylor expansion, say $f(x) = \sum_i c_i x^{K_i}$, with

$$c_i = 0, \qquad \text{while } \|K_i\| < t$$

In other words, any non-vanishing term of the Taylor expansion of $f(x) \in 0(\|x\|^t)$ has degree greater than or equal to t.

Theorem 2: *For system* (32), *assume there exist m homogeneous quadratic functions*

$$\phi(w) = (\phi_1(w), \ldots, \phi_m(w))$$

and m homogeneous cubic functions

$$\psi(w) = (\psi_1(w), \ldots, \psi_m(w))$$

such that the following hold:

(1) *There exists an integer $s > 3$, such that*

$$LD(L_{q(\phi+\psi,\underbrace{0,\ldots,0}_{\|\rho\|-m},w)}(\phi+\psi)) \geq s \tag{36}$$

$$LD(p(\phi+\psi,\underbrace{0,\ldots,0}_{\|\rho\|-m},w)\alpha(\phi+\psi,\underbrace{0,\ldots,0}_{\|\rho\|-m},w)) \geq s \tag{37}$$

(2)

$$LD(q_i(\phi+\psi,\underbrace{0,\ldots,0}_{\|\rho\|-m},w)) = L_i, \qquad i = 1,\ldots,r \tag{38}$$

(3)

$$\dot{w} = q(\phi+\psi,\underbrace{0,\ldots,0}_{\|\rho\|-m},w) \tag{39}$$

is $L = (L_1, \ldots, L_r)$ approximately asymptotically stable at the origin, and

(4)

$$\begin{aligned} &q_i((\phi+\psi+0(\|w\|^s),\underbrace{0(\|w\|^s),\ldots,0(\|w\|^s)}_{\|\rho\|-m},w) \\ &= q_i(\phi+\psi,\underbrace{0,\cdots,0}_{\|\rho\|-m},w) + 0(\|w\|^{L_i+1}), \qquad i = 1,\ldots,r \end{aligned} \tag{40}$$

Then the overall system (32) *is state feedback stabilizable.*

Note that in the above for notational ease we denote $q(z,w) = q(z_1, \bar{z}_1, w)$, etc. So

$$q(\phi+\psi,\underbrace{0,\cdots,0}_{\|\rho\|-m},w)$$

means $z_1 = \phi+\psi$ and $\bar{z}_1 = 0$.

Proof: In fact it is a particular case of the general results in Cheng and Martin (2001). We just outline the proof. Choosing control as

$$\begin{aligned} u_i = &-\alpha_i(z,w) \\ &+ \left(a_1^i z_1^i + \cdots + a_n^i z_{\rho_i}^i - a_1^i(\phi_i(w)+\psi_i(w))\right), \\ &\qquad i = 1,\ldots,m \end{aligned} \tag{41}$$

and using

$$\left.\begin{aligned} z_1 &= \phi(w)+\psi(w) \\ \bar{z}_1(w) &= 0 \end{aligned}\right\} \tag{42}$$

as the approximation of the centre manifold of the closed-loop system. Then the approximation error is

$$\begin{aligned} &\frac{\partial z}{\partial w} q_{(z(w),w)} - \begin{pmatrix} A_i z^i(w) + b_i u_i(z(w),w) + \begin{pmatrix} 0 \\ \alpha_i(z(w),w) \end{pmatrix} + p^i(z(w),w)u(z(w),w) \\ i = 1,\ldots,m \end{pmatrix} \\ &= \frac{\partial z}{\partial w} q_{(z(w),w)} + p(\phi(w)+\psi(w),\underbrace{0,\ldots,0}_{\|\rho\|-m},w)\alpha(\phi(w)+\psi(w),\underbrace{0,\ldots,0}_{\|\rho\|-m},w) \\ &= 0(\|w\|^s) \end{aligned}$$

Note that the first term is $L_{q(\phi+\psi,0,\cdots,0,w)}(\phi+\psi)$. So (36) and (37) assure that the approximation error is of $0(\|w\|^s)$. That is the true centre manifold equation can be expressed as

$$\left.\begin{aligned} z_1 &= h_1(w) = \phi(w)+\psi(w)+0(\|w\|^s) \\ z_2 &= h_2(w) = 0(\|w\|^s) \end{aligned}\right\} \tag{43}$$

Now according to (40), the $0(\|w\|^s)$ centre manifold approximation error does not affect the leading degree terms of each equations of the dynamic equations of the centre manifold.

Note that the approximately asymptotical stability of (39) assures that

$$\dot{w} = q(\phi+\psi,\underbrace{0,\ldots,0}_{\|\rho\|-m},w) + 0(\|w\|^s) \tag{44}$$

is asymptotically stable at the origin.

Using (43), the true dynamics on the centre manifold is

$$\dot{w} = q(h_1(w),h_2(w),w) = q(\phi+\psi+0(\|w\|^s),0(\|w\|^s),w) \tag{45}$$

Recalling (40), one sees that the true dynamics on the centre manifold, (45), is of the form (44), and therefore it is asymptotically stable.

Finally, we choose a_j^i such that each linear block is Hurwitz, i.e.

$$\lambda^{\rho_i} - \sum_{j=1}^{\rho_i} a_j^i \lambda^{j-1}, \qquad i = 1,\ldots,m$$

are Hurwitz polynomials. Then by equivalence theorem of the centre manifold, the stability of the dynamics on centre manifold assures the asymptotical stability of the overall closed-loop system. □

Remark:

(1) *It seems that the condition* (38) *in the Theorem 2 is redundant. It is true that this condition is not necessary for the theorem itself. But in practical use, we require that the leading degrees are from the injection degrees and are all odd. In fact, we can only deal with this kind of systems* (*Cheng and Martin* 2001).

(2) *The control has to be chosen to meet two requirements. First of all, for even injection degree subsystems, the even leading terms have to be eliminated. Secondly, the final dynamics for the approximate centre manifold should be approximately asymptotically stable.*

(3) *The technique of Lyapunov function with homogeneous derivative* (*LFHD*)*, developed in* (*Cheng and Martin* 2001) *will be used to designing the control to assure the approximately asymptotical stability of the final dynamics on the center manifold.*

5. Illustrating examples

The first example demonstrates the algorithm to convert a system into the generalized normal form.

Example 4: Consider a single-input affine non-linear system

$$\left.\begin{aligned} \dot{x}_1 &= x_2 + x_3^2 + x_3^2 u \\ \dot{x}_2 &= x_3 - 2e^{x_3} u \\ \dot{x}_3 &= e^{x_3} u \\ \dot{\bar{w}}_1 &= -2x_1\bar{w}_1 - \bar{w}_1^2\bar{w}_2 \\ \dot{\bar{w}}_2 &= -\bar{w}_2^3 x_1 - \bar{w}_1^3 x_3 \end{aligned}\right\} \tag{46}$$

The system (46) can be approximated by Taylor expansion as

$$\left.\begin{aligned} \dot{x}_1 &= x_2 + x_3^2 + x_3^2 u \\ \dot{x}_2 &= x_3 - 2(1+x_3)u + 0(\|x\|^2)u \\ \dot{x}_3 &= (1+x_3)u + 0(\|x\|^2)u \\ \dot{\bar{w}}_1 &= -2x_1\bar{w}_1 - \bar{w}_1^2\bar{w}_2 \\ \dot{\bar{w}}_2 &= -\bar{w}_2^3 x_1 - \bar{w}_1^3 x_3 \end{aligned}\right\} \tag{47}$$

Then we take a coordinate change, say

$$(x, \bar{w})^{\mathrm{T}} = P(\tilde{x}, \tilde{w})^{\mathrm{T}}$$

where $x = (x_1, x_2, x_3)$, $\bar{w} = (\bar{w}_1, \bar{w}_2)$, $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$, $\tilde{w} = (\tilde{w}_1, \tilde{w}_2)$, the invertible matrix P is

$$P = \begin{pmatrix} 1 & -2 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

System (47) can be converted into the following form in which the controllable linear part is expressed as a Brunovsky canonical form

$$\left.\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_2 + \tilde{x}_3^2 + 0(\|\tilde{x}\|^2)u \\ \dot{\tilde{x}}_2 &= \tilde{x}_3 \\ \dot{\tilde{x}}_3 &= u + \tilde{x}_3 u + 0(\|\tilde{x}\|^2)u \\ \dot{\tilde{w}}_1 &= -2\tilde{x}_1\tilde{w}_1 + 4\tilde{x}_2\tilde{w}_1 - \tilde{w}_1^2\tilde{w}_2 \\ \dot{\tilde{w}}_2 &= -\tilde{w}_2^3\tilde{x}_1 + 2\tilde{w}_2^3\tilde{x}_2 - \tilde{w}_1^3\tilde{x}_3 \end{aligned}\right\} \tag{48}$$

According to system (48), it is easy to find a function, say $h(x) = \tilde{x}_1 = x_1 + 2x_2 + 4x_3$, and a straightforward computation to show that the essential point relative degree of system (46) is $\rho^{ep} = 3$, which is equal to the dimension of the controllable linear part of system (48). Consequently, we can obtain the generalized normal form by a non-linear coordinate change.

Denote

$$f = \begin{pmatrix} x_2 + x_3^2 \\ x_3 \\ 0 \\ -2x_1\bar{w}_1 - \bar{w}_1^2\bar{w}_2 \\ -\bar{w}_2^3 x_1 - \bar{w}_1^3 x_3 \end{pmatrix}, \quad g = \begin{pmatrix} x_3^2 \\ -2\mathrm{e}^{x_3} \\ \mathrm{e}^{x_3} \\ 0 \\ 0 \end{pmatrix}$$

A feasible coordinate change is calculated as

$$(z, w)^{\mathrm{T}} = (h, L_f h, L_f^2 h, \bar{w}_1, \bar{w}_2)^{\mathrm{T}} = \begin{pmatrix} x_1 + 2x_2 + 4x_3 \\ x_2 + 2x_3 + x_3^2 \\ x_3 \\ \bar{w}_1 \\ \bar{w}_2 \end{pmatrix}$$

where $z = (z_1, z_2, z_3), w = (w_1, w_2)$, and the Jacobian matrix $J_{(z,w)}|_{(0,0)}$ is non-singular. Under this coordinate frame system (46) can be converted into the generalized normal form as

$$\left.\begin{aligned} \dot{z}_1 &= z_2 + z_3^2 u + 0(\|x\|^3)u \\ \dot{z}_2 &= z_3 + 2z_3 u + 2z_3^2 u + 0(\|x\|^3)u \\ \dot{z}_3 &= u + z_3 u \\ \dot{w}_1 &= -2z_1 w_1 + 4z_2 w_1 - 4z_3^2 w_1 - w_1^2 w_2 \\ \dot{w}_2 &= -w_2^3 z_1 + 2w_2^3 z_2 - 2w_2^3 z_3^2 - w_1^3 z_3 \end{aligned}\right\} \tag{49}$$

It is easy to check that this system is non-minimum. We use the technique developed in Cheng and Martin (2001) to stabilize the dynamics on the designed centre manifold, which then also stabilizes the overall system.

Note that in system (49) the leading degrees are $L_1 = 3$, $L_2 = 5$, and the condition H3 holds. According to (41), the control is constructed as

$$u = -6z_1 - 11z_2 - 6z_3 + 12w_1^2 + 6w_2^2$$

Then the linear part of (49) is Hurwitz. Let

$$\varphi(w) = \begin{pmatrix} \varphi_1(w) \\ \varphi_2(w) \\ \varphi_3(w) \end{pmatrix} = \begin{pmatrix} 2w_1^2 + w_2^2 \\ 0 \\ 0 \end{pmatrix} = 0(\|w\|^2)$$

be used to approximate the centre manifold. Then we have

$$M\varphi(w) = D\varphi(w)\dot{w} = 0(\|w\|^4)$$

According to the approximation theorem (Carr 1981), the centre manifold can be expressed as

$$\left.\begin{aligned} z_1 &= 2w_1^2 + w_2^2 + 0(\|w\|^4) \\ z_2 &= 0(\|w\|^4) \\ z_3 &= 0(\|w\|^4) \end{aligned}\right\} \tag{50}$$

The dynamics on the centre manifold becomes

$$\left.\begin{aligned} \dot{w}_1 &= -4w_1^3 - 2w_1w_2^2 - w_1^2w_2 + 0(\|w\|^5) \\ \dot{w}_2 &= -w_2^5 - 2w_2^3w_1^2 + 0(\|w\|^7) \end{aligned}\right\} \tag{51}$$

For the certain part of (51), choose a LFHD (see Cheng and Martin 2001) as $V = \frac{1}{4}w_1^4 + \frac{1}{2}w_2^2$, then

$$\begin{aligned} \dot{V} &= w_1^3\dot{w}_1 + w_2\dot{w}_2 \\ &= -4w_1^6 - 2w_1^4w_2^2 - 2w_1^2w_2^4 - w_1^5w_2 - w_2^6 \\ &\le -4w_1^6 - w_1^5w_2 - w_2^6 \\ &\le -4w_1^6 - w_2^6 + \tfrac{5}{6}w_1^6 + \tfrac{1}{6}w_2^6 \\ &\le 0 \end{aligned}$$

$\dot{V}$ is negative. So system (51) is asymptotically stable. By Theorem 2, system (46) is also asymptotically stable.

The second example shows the design of stabilizing control for the multi-input case.

Example 5: Consider the following system

$$\left.\begin{aligned} \dot{x}_1 &= x_2 + \sin(z_1)u_1 \\ \dot{x}_2 &= \tan(x_1) + e^{x_2}u_1 \\ \dot{x}_3 &= x_4 + (x_3 + z_2)u_1 + z_2u_2 \\ \dot{x}_4 &= x_5 + \ln(1 + x_1z_1)u_2 \\ \dot{x}_5 &= x_3z_2 + u_2 \\ \dot{z}_1 &= \sin(z_1)x_1 + z_2^3 \\ \dot{z}_2 &= z_2x_1x_3 \end{aligned}\right\} \tag{52}$$

It is obvious that the system is already in a generalized normal form.

First, we use two quadratic functions $h_1(z)$ and $h_2(z)$ to approximate the centre manifold as

$$\left.\begin{aligned} &x_1(z) \sim h_1(z) = \alpha_1z_1^2 + \beta_1z_1z_2 + \gamma_1z_2^2 \\ &x_3(z) \sim h_2(z) = \alpha_2z_1^2 + \beta_2z_1z_2 + \gamma_2z_2^2 \\ &x_2(z) \sim 0, \qquad x_4(z) \sim 0, \qquad x_5(z) \sim 0 \end{aligned}\right\} \tag{53}$$

Then we design the controls as

$$\left.\begin{aligned} u_1 &= -\mathrm{e}^{-x_2}\tan(x_1) + \mathrm{e}^{-x_2} \\ &\quad\times [a_{11}x_1 + a_{12}x_2 - a_{11}(\alpha_1z_1^2 + \beta_1z_1z_2 + \gamma_1z_2^2)] \\ u_2 &= -x_3z_2 + [a_{21}x_3 + a_{22}x_4 + a_{23}x_5 \\ &\quad - a_{21}(\alpha_2z_1^2 + \beta_2z_1z_2 + \gamma_2z_2^2)] \end{aligned}\right\} \tag{54}$$

Now for the closed-loop system we verify the approximation error

$$\left.\begin{aligned} &\begin{pmatrix} \partial h_1/\partial z_1 & \partial h_1/\partial z_2 \\ \partial 0/\partial z_1 & \partial 0/\partial z_2 \\ \partial h_2/\partial z_1 & \partial h_2/\partial z_2 \\ \partial 0/\partial z_1 & \partial 0/\partial z_2 \\ \partial 0/\partial z_1 & \partial 0/\partial z_2 \end{pmatrix} \begin{pmatrix} \sin(z_1)h_1(z) + z_2^3 \\ z_2h_1(z)h_2(z) \end{pmatrix} \\ &\quad - \begin{pmatrix} -\sin(z_1)\tan(h_1(z)) \\ 0 \\ -(h_2(z) + z_2)\tan(h_1(z)) \\ -\ln(1 + h_1(z)z_1)h_2(z)z_2 \\ 0 \end{pmatrix} = 0(\|z\|^3) \end{aligned}\right\} \tag{55}$$

Hence the approximate order is $s = 3$.

Next, we consider the dynamics on the approximated centre manifold. We have

$$\left.\begin{aligned}\dot{z}_1 &= z_1(\alpha_1 z_1^2 + \beta_1 z_1 z_2 + \gamma_1 z_2^2) + z_2^3 \\ \dot{z}_2 &= z_2(\alpha_1 z_1^2 + \beta_1 z_1 z_2 + \gamma_1 z_2^2)(\alpha_2 z_1^2 + \beta_2 z_1 z_2 + \gamma_2 z_2^2)\end{aligned}\right\} \tag{56}$$

As a convention, we choose $L_1 = 3$ and $L_2 = 5$ (Cheng and Martin 2001).

The technique developed in Cheng and Martin (2001) can be used to design controls to make (56) approximately stable. To be independent, we simply choose a Lyapunov function with homogeneous derivative as

$$V = z_1^4 + z_2^2 \tag{57}$$

and choose $\alpha_1 = -1$, $\beta_1 = 0$, $\gamma_1 = -1$, $\alpha_2 = 0$, $\beta_2 = 0$ and $\gamma_2 = 1$. Then

$$\dot{V}|_{(56)} = -4z_1^6 - 4z_1^4 z_2^2 + 4z_1^3 z_2^3 - 2z_1^2 z_2^4 - 2z_2^6$$

which is negative definite. So (56) is approximately stable with homogeneous degrees $(3, 5)$ (Cheng and Martin 2001).

Finally, we have to check condition (39), which means the error does not affect the asymptotical stability. Since

$$\begin{aligned}&z_1(h_1(z) + 0(\|z\|^3) + z_2^3 - [z_1 h_1(z) + z_2^3] = 0(\|z\|^4) \\ &z_2(h_1(z) + 0(\|z\|^3)(h_2(z) + 0(\|z\|^3) \\ &\qquad\qquad - z_2 h_1(z) h_2(z) = 0(\|z\|^6)\end{aligned}$$

(39) follows.

To construct the controls we have to choose a_{ij} to make the linear part stable. Say, let $a_{11} = -1$, $a_{12} = -2$, $a_{21} = -1$, $a_{22} = -3$ and $a_{23} = -3$. Then the controls become

$$\left.\begin{aligned}u_1 &= -\mathrm{e}^{-x_2}\tan(x_1) + e^{-x_2}[-x_1 - 2x_2 + (-z_1^2 - z_2^2)] \\ u_2 &= -x_3 z_2 + (-x_3 - 3x_4 - 3x_5 + z_2^2)\end{aligned}\right\} \tag{57}$$

6. Conclusion

The paper proposed three new concepts: essential relative degree, point relative point and point essential relative degree. Based on them the Byrnes–Isidori normal form has been extended to generalized normal form. Two ways for solving the stabilization problems have been proposed. For the systems of minimum phase, a sufficient condition is given for stabilizing the system by pseudo-linear control. For the systems with non-minimum phase, the method of Lyapunov function with homogeneous derivative is used to stabilize the systems.

The generalized normal form has been proved to have several advantages.

- The essential relative degree (the essential point relative degree) is more powerful than the relative degree (the point relative degree respectively) because they provide the largest linearized sub-system (pseudo-linearized sub-system).
- Calculating the (essential) point relative degree (vector) is much easier than calculating the (essential) relative degree (vector).
- Even if the essential relative degree is known, in general, to get the corresponding Byrnes–Isidori normal form is difficult because it involves solving a set of partial differential equations. But it is a straightforward computation to get the generalized normal form with an easily calculated essential point relative degree.

Then it was shown that when the stabilization problem is considered, some basic properties for the standard normal form remain true for the generalized normal form. Particularly, we have the following facts:

- When the system is of minimum phase, with assumption H3 the overall system is stabilizable by pseudo-linear feedback control.
- The technique of Lyapunov function with homogeneous derivative is still applicable to the generalized normal form without any additional requirement.

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