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# Attitude Control of Missile Via Fliess Expansion

Yu Yao, Baoqing Yang, Fenghua He, Yupeng Qiao, and Daizhan Cheng, *Fellow, IEEE*

***Abstract*—Motivated by the attitude control of missiles, we consider the tracking problem of nonlinear systems via control with finite admissible values. First, the Fliess functional expansion is used to describe the outputs. Using piecewise admissible value control and the quadratic criteria, the optimal control is obtained. Then for certain model uncertainties, the model predictive control is considered. Online predicted uncertain parameters are used to update the system's dynamics. The method developed is used to the attitude control of missile. Simulation results show the effectiveness of the proposed method.**

***Index Terms*—Attitude control, Fliess functional expansion, model predictive tracking control, nonlinear system.**

## I. Introduction

ATTITUDE control is a typical nonlinear control problem. It is also very important in practice. Missiles concerned in this paper are sailing in the exoatmosphere, and advanced attitude control techniques are the key to realize a direct hit. Coincidence of a line-of-sight coordinate with a body coordinate is required in the process of the interception, which assures that the target cannot escape from field of view. Attitude regulation can be realized via attitude-control thrusters for the lack of aerodynamics in an exoatmospheric environment. The attitude-control thruster generates constant force, therefore on–off switching laws should be designed to maintain the angle errors within a required range. The problem of how to assure attitude control precision and prevent too frequent on–off thruster operations are essential problems to be solved [3]. During the interception, the moment of inertia and center of mass vary according to the expending of fuel carried by the missile continuously. Therefore, a disturbance torque is exerted on the attitude control system while the divert thrusters are working. From the above argument, one sees that attitude control system design of the missile can be attributed to a nonlinear control system design under uncertainties, disturbances, and with limited feasible controls.

Because of its practical importance, the attitude control has attracted a wide attention and various nonlinear control techniques have been used to design its control; see, e.g., [1] and [2]. Some design techniques have been proved as efficient ones. For instance, for a class of uncertain nonlinear systems which can be decoupled by state feedback, variable structure control law is applied by Singh and Iyer [4], so that the outputs of the closed-loop system asymptotically track the desired output trajectories. Egeland and Godhavn [5] adopt an adaptive control scheme for the attitude control of a rigid spacecraft, which is derived by using a linear parameterization of the equation of motion. The tracking error is described by the Euler parameter vector. Global convergence of the tracking error is obtained by using passivity theory. Show *et al.* [6] present a nonlinear control law for large-angle attitude control of spacecraft. The nonlinear controller contains linear feedback terms for stabilizing the system and nonlinear terms for performance enhancement. One salient feature of the proposed approach is that the nonlinear controller parameters are designed using the method of linear matrix inequality (LMI). Manikonda *et al.* [7] present a model predictive control to the design of a controller for formation keeping and formation attitude control. Control laws for formation keeping and attitude control are designed using a combination of feedback linearization and model predictive control. Actuator saturation is incorporated into the controller design. Switching between coordinated frames is incorporated to overcome singularities associated with local feedback linearization. Luo and Chu [8] apply the state-dependent Riccati equation (SDRE) of nonlinear suboptimal regulation to the design of attitude control of a rigid spacecraft. The saturated control is also considered by gain scheduling when using the SDRE method. For nonlinear spacecraft systems with unknown or uncertain inertia matrix and external disturbances, Bor-Sen *et al.* [9] adopt adaptive fuzzy mixed $\mathrm{H}_2$ and $\mathrm{H}_\infty$ attitude control law. Using an adaptive fuzzy approximation method, an uncertain nonlinear model is estimated. Then, by a mixed $\mathrm{H}_2$ and $\mathrm{H}_\infty$ attitude control design, the effect of external disturbance and fuzzy approximation error on spacecraft attitude can be restrained and the tracking error as well as consumed energy of the controller are minimized.

In this paper, the attitude control is considered as a particular form of an output tracking problem. We start from a general nonlinear control system to investigate some general design principles for a tracking problem. Consider the following nonlinear control system:

$$\begin{cases} \dot{x} = f(x) + \Delta f(x) + \sum_{i=1}^{m}[g^i(x) + \Delta g^i(x)]u^i \\ \quad := f(x) + \Delta f(x) + [g(x) + \Delta g(x)]u \\ y = h(x) = (h^1(x), \ldots, h^p(x))^T \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p, \Delta f(x)$, and $\Delta g^i(x)$ are uncertainties of the system, which could be system unmodeled dynamics and disturbance, which may be caused by exo-system, etc.

Let the reference output trajectory be described as

$$y^r(t) = (h^{r_1}(t), \ldots, h^{r_p}(t)). \tag{2}$$

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Y. Yao, B. Yang, and F. He are with the Harbin Institute of Technology, Science Park, Harbin 150080, China (e-mail: yaoyu@hit.edu.cn; yang_bao_qing@yahoo.com.cn; hefenghua@hit.edu.cn).

Y. Qiao and D. Cheng are with the Institute of Systems Science, Chinese Academy of Sciences, Beijing 100080, China (e-mail: qiaoyupeng@amss.ac.cn; dcheng@iss.ac.cn).

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The tracking problem is to find a suitable control $u$ such that the outputs can track the reference outputs. That is

$$\lim_{t\to\infty} \|y(t) - y^r(t)\| = 0. \tag{3}$$

Since the model has some uncertainties, certain control techniques have been developed according to different uncertainties. When the uncertainty is generated by an exosystem, it can be considered as an output regulation problem.

The output regulation problem is a fundamental topic in control theory. It has been studied widely for the last three decades. A novel comprehensive introduction can be found in [10]. In general, to solve an output regulation problem, one of the key issues is solving regulation equation, which is first proposed in [11]. The regulation equation is a partial differential equation, and hence is, in general, very difficult to get a closed-form solution. This is a bottleneck of solving the output regulation problem.

When the uncertainty is a random noise, robust control techniques maybe used to solve such a problem [12]. Robust control is designed to deal with all possible "bad" situations, it is, therefore, conservative.

When the uncertainty is precisely determined by some parameters, the model predictive control technique is a suitable tool for solving the problem [13]. In fact, it can provide much precise solution than a robust control technique, because the uncertainties can be gradually predicted and finally overcome.

In this paper, the available control set is defined as

$$u = (u^1, \ldots, u^m) \in U_{\mathrm{PC}}^m \quad \text{and} \quad u^i \in U_{\mathrm{PC}}. \tag{4}$$

With $u^i(t) \in \{c_1, \ldots, c_N\}, \forall t > 0, 1 \le i \le m$. That is, controls are piecewise constant with finite possible constant values.

We state the tracking problem considered in this paper as follows.

*Definition 1:* The tracking problem of system (1)–(4) is to find a control (4) solving

$$\min_{u^i \in U_{\mathrm{PC}}} \int_0^T \|y(t) - y^r(t)\|^2 \, dt. \tag{5}$$

*Remark:*

1) In practice, the norm in (5) may be weighted Euclid norm.
2) The restriction on the controls is from the engineering requirement. In fact, in our missile guide problem, as we explained in the first paragraph, the feasible controls are of Bang–Bang type, i.e., $\{-1, 0, 1\}$. (4) is more general and it can be used to approximate a large class of control sets.

What we are looking for is a numerical design technique via computer-aided realization, which can reach the optimal or suboptimal tracking performance and be realized online.

The basic design technique in this paper is as follows. Choose a time duration $T > 0$ as the sampling time. Assume within a sampling duration the controls are constant, i.e.,

$$u^i(t) = u_k^i, \quad kT < t < (k+1)T, \quad i = 1, \ldots, m. \tag{6}$$

That is, the possible switching (discontinuous) moments are at $\{kT \mid k = 1, 2, \ldots\}$, with $T > 0$ as a period. Denote by

$$x_k^i = x^i(kT), \quad i = 1, 2, \ldots, n$$
$$y_k^j = y^j(kT), \quad j = 1, \ldots, m; \quad k = 1, 2, \ldots.$$

Then within a sampling duration, the controls have only finite choices (precisely, according to (4) there are $N^m$ choices). Then we try to find a best $u^*$, which minimizes the tracking error $e$, that is

$$e_{k+1}(u_k^*) = \min_{u_k^i \in \{c_1, \ldots, c_N\}} \int_{kT}^{(k+1)T} \|y(t) - y^r(t)\|^2 \, dt. \tag{7}$$

A key point in our approach is obtaining $y_{k+1}$ by using Fliess functional expansion with admissible constant controls.

The Fliess expansion for piecewise constant controls has been employed previously in the context of numerical methods for nonlinear control system [14], redesign of continuous time nonlinear control systems for sampled data operation [15], and path planning [16].

The rest of this paper is organized as follows. Section II provides some formulas for Fliess functional expansion of constant controls. Section III considers the problem of optimal tracking control for nominal systems. Section IV studies attitude control of a missile with a precise model. In Section V, the parameter predictive control of missile attitude is investigated. Section VI develops time-varying Fliess functional expansion, and then it is used to predict time-varying models. Some simulations are presented in Section VII. Finally, some concluding remarks are given in Section VIII.

## II. Fliess Functional Expansion With Constant Controls

For notational ease, in this section we consider only the nominal form of system (1), which has no uncertainties. That is

$$\begin{cases} \dot{x} = f(x) + \sum_{i=1}^m g^i(x) u^i := f(x) + g(x) u \\ y = h(x) = (h^1(x), \ldots, h^p(x))^T \end{cases} \tag{8}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$.

An efficient way to express the input–output relation of system (8) is the Fliess functional expression [17]. For convenience, we briefly cite it here: for compactness, we denote

$$g^0 = f, \quad u^0 = 1.$$

Then we define a set of multi-integrals inductively as

$$\begin{cases} \int_0^t d\xi_i = \int_0^t u^i(\tau) d\tau \\ \int_0^t d\xi_{i_{k+1}} d\xi_{i_k} \cdots d\xi_{i_1} \\ \quad = \int_0^t u^{i_{k+1}}(\tau) \int_0^\tau d\xi_{i_k} \cdots d\xi_{i_1}, \quad k \ge 1. \end{cases} \tag{9}$$

Note that the multi-index integrations in (9) are defined recursively [17]. Then we have the following expression.

*Theorem 2:* The input output response of the system (8) can be expressed as

$$y^j(t) = h^j(x_0) + \sum_{k=0}^{\infty} \sum_{i_k,\ldots,i_0=0}^{m} L_{g^{i_0}} \cdots L_{g^{i_k}} h^j(x_0) \int_0^t d\xi_{i_k} \cdots d\xi_{i_0} \quad (10)$$

where $j = 1, \ldots, p$.

Note that Fliess series converges only locally. Since in later use we assume the sampling time is short enough, so the convergence can also be reasonably assumed. Next, we work out the Fliess functional expansion for system (8) with constant controls.

*Lemma 3:* Assume constant controls are used. Then for $t = T$ the integral (10), corresponding to the coefficient

$$L_f^{k_1} L_{g^{i_1}}^{s_1} L_f^{k_2} L_{g^{i_2}}^{s_2} \cdots L_f^{k_\ell} L_{g^{i_\ell}}^{s_\ell} h^j(x_0)$$

is

$$\underbrace{\int_0^T u_{i_\ell} d\tau_1 \int_0^{\tau_1} u_{i_\ell} d\tau_2 \cdots \int_0^{\tau_{s_\ell}-1} u_{i_\ell} d\tau_{s_\ell}}_{s_\ell} \cdots \underbrace{\int_0^{\tau_\mu - 1} d\tau_\mu \int_0^{\tau_\mu} d\tau_{\mu+1} \cdots \int_0^{\tau_{\mu+k_1-1}} d\tau_{\tau_{\mu+k_1}}}_{k_1} = (u^{i_1})^{s_1} (u^{i_2})^{s_2} \cdots (u^{i_\ell})^{s_\ell} \frac{T^r}{r!} \quad (11)$$

where $\mu = s_\ell + k_\ell + s_{\ell-1} + k_{\ell-1} + \cdots + s_2 + k_2 + s_1$, $r = k_1 + k_2 + \cdots + k_\ell + s_1 + s_2 + \cdots + s_\ell$.

*Proof:* First, using mathematical induction, it is easy to prove that corresponding to the coefficient $L_f^r h^j(0)$ the integral of controls in (11) for $t = T$ is

$$\underbrace{\int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{r-1}} d\tau_r}_{r} = \frac{T^r}{r!}. \quad (12)$$

Now since the controls are constant, replacing $f$ in the Lie derivative $L_f^r h(x_0)$ by $g_i$ at any place of the $r$ folds of Lie derivatives results on that the corresponding integrand of 1 is replaced by a constant control $u_i$, which turns out to be a coefficient of the integral. (11) follows immediately. ■

Using Lemma 3, we have the following.

*Theorem 4:* Consider the Fliess functional expansion of system (8) at time $T$. Assume:

**H 1**: the constant controls, $u_i(t) = u_i$;

**H 2**: $p = 1$ and the relative degree is $\rho$.

Then 1) the $\rho$th order approximation of Fliess functional expansion is

$$\begin{aligned} h(u,T) &= h(x_0) + L_f h(x_0) T + \cdots + L_f^\rho h(x_0) \frac{T^\rho}{\rho!} \\ &\quad + \sum_{i=1}^m u^i L_{g^i} L_f^{\rho-1} h(x_0) \frac{T^\rho}{\rho!} + R_1(u,T) \\ &:= W_1(u) + R_1(u,T) \end{aligned} \quad (13)$$

where $W_1(u)$ is a linear function on $u$, and $R_1(u,T) = O(|T|^{\rho+1})$.

2) The $(\rho+1)$th order approximation of Fliess functional expansion is

$$\begin{aligned} &h(u,T) \\ &= h(x_0) + L_f h(x_0) T + \cdots + L_f^{\rho+1} h(x_0) \frac{T^{\rho+1}}{(\rho+1)!} \\ &\quad + \sum_{i=1}^m u^i L_{g^i} L_f^{\rho-1} h(x_0) \frac{T^\rho}{\rho!} \\ &\quad + \sum_{i=1}^m u^i \left[ L_f L_{g^i} L_f^{\rho-1} h(x_0) + L_{g^i} L_f^\rho h(x_0) \right] \frac{T^{\rho+1}}{(\rho+1)!} \\ &\quad + \sum_{i=1}^m \sum_{j=1}^m u^i u^j L_{g^i} L_{g^j} L_f^{\rho-1} h(x_0) \frac{T^{\rho+1}}{(\rho+1)!} \\ &\quad + R_2(u,T) \\ &:= W_2(u) + R_2(u,T) \end{aligned} \quad (14)$$

where $W_2(u)$ is a quadratic polynomial on $u$, and $R_2(u,T) = O(|T|^{\rho+2})$.

3) The $(\rho+2)$th order approximation of Fliess functional expansion is

$$\begin{aligned} h(u,T) &= h(x_0) + L_f h(x_0) T + \cdots + L_f^{\rho+2} h(x_0) \frac{T^{\rho+2}}{(\rho+2)!} \\ &\quad + \sum_{i=1}^m u^i L_{g^i} L_f^{\rho-1} h(x_0) \frac{T^\rho}{\rho!} \\ &\quad + \sum_{i=1}^m u^i \Big[ L_f L_{g^i} L_f^{\rho-1} h(x_0) \\ &\quad + L_{g^i} L_f^\rho h(x_0) \Big] \frac{T^{\rho+1}}{(\rho+1)!} \\ &\quad + \sum_{i=1}^m \sum_{j=1}^m u^i u^j L_{g^i} L_{g^j} L_f^{\rho-1} h(x_0) \frac{T^{\rho+1}}{(\rho+1)!} \\ &\quad + \sum_{i=1}^m u^i \Big[ L_f^2 L_{g^i} L_f^{\rho-1} h(x_0) + L_f L_{g^i} L_f^\rho h(x_0) \\ &\quad + L_{g^i} L_f^{\rho+1} h(x_0) \Big] \frac{T^{\rho+2}}{(\rho+2)!} \\ &\quad + \sum_{i=1}^m \sum_{j=1}^m u^i u^j \Big[ L_{g^i} L_{g^j} L_f^\rho h(x_0) \\ &\quad + L_{g^i} L_f L_{g^j} L_f^{\rho-1} h(x_0) \\ &\quad + L_f L_{g^i} L_{g^j} L_f^{\rho-1} h(x_0) \Big] \frac{T^{\rho+2}}{(\rho+2)!} \\ &\quad + \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^m u^i u^j u^k L_{g^i} L_{g^j} L_{g^k} \\ &\quad \times L_f^{\rho-1} h(x_0) \frac{T^{\rho+2}}{(\rho+2)!} + R_3(u,T) \\ &:= W_3(u) + R_3(u,T) \end{aligned} \quad (15)$$

where $W_3(u)$ is a cubic polynomial on $u$, and $R_3(u,T) = O(|T|^{\rho+3})$.

It is not difficult to write even higher order approximations. However, they are not very useful practically, because they are too complicated to be used for online control design.

*Remark:* Since in our later approach the sampling time $T$ is very small, we are interested in the order of $T$. Note that though $W_i$ is the $i$th degree polynomial of $u$, it is not the $i$th approximation of $h(u,T)$ because $R_i(u,T)$ still contains terms of $u$ with degree lower than $i+1$. We refer to [15] for the truncation of $h(x(T),u)$ with respect to $u$.

## III. Design of Optimal Tracking Control

This section considers the design of optimal control strategy for tracking. For simplicity, linear approximation of Fliess functional expansion is considered first.

Assume the sampling duration $T>0$ is given, and then in each duration $[kT,(k+1)T]$ we try to find optimal control to minimize the tracking error at $(k+1)T$.

Now assume we are at stage $k$. We have to choose a performance criteria for optimization. When the tracking error is the only consideration, a quadratic form for the error can be chosen as

$$E_{k+1} = \left(h_{k+1} - h^r_{k+1}\right)^T P \left(h_{k+1} - h^r_{k+1}\right) \tag{16}$$

where $h_{k+1} = h(x((k+1)T)), h^r_{k+1} = h^r(x((k+1)T))$, and $P>0$ is a positive definite matrix. Different $P$ effects the performance. In this paper, we choose it from simulations. When the energy consumption is also considered, the performance criteria could be chosen as

$$J_{k+1} = \left(h_{k+1} - h^r_{k+1}\right)^T P \left(h_{k+1} - h^r_{k+1}\right) + u_k^T R u_k \tag{17}$$

where $R>0$ should be chosen related to $P$. If it is too large with respect to $P$, it may hurt the tracking efficiency. Sometimes we are concerning about the switching frequency. Then we may also add a punishment term for switching as

$$\begin{aligned} J^s_{k+1} = \left(h_{k+1} - h^r_{k+1}\right)^T P \left(h_{k+1} - h^r_{k+1}\right) + u_k^T R u_k \\ + \left(u_k - u_{k-1}\right)^T S(u_k - u_{k-1}) \end{aligned} \tag{18}$$

where $S>0$.

Our goal is to design a control

$$u(t) := u_k, \quad kT < t < (k+1)T$$

to minimize $E_{k+1}$ (or $J_{k+1}$, or $J^s_{k+1}$, respectively).

Recall system (16). Using (13) to each $y^i$, we have

$$\begin{aligned} h^i_{k+1} = h^i_k + TL_f h^i\,|_k + \frac{T^2}{2!}L_f^2 h^i\,|_k + \cdots + \frac{T^{\rho_i}}{\rho_i!}L_f^{\rho_i} h^i\,|_k \\ + \frac{T^{\rho_i}}{\rho_i!}L_g L_f^{\rho_i-1} h^i\,|_k u_k + O(|T|^{\rho_i+1}) \end{aligned} \tag{19}$$

where $i=1,\ldots,p$.

Define

$$\xi_k = \left.\begin{bmatrix} (h^r)^1_{k+1} - H_1 \\ \vdots \\ (h^r)^p_{k+1} - H_p \end{bmatrix}\right|_{x_k}$$

$$\eta_k = \left.\begin{bmatrix} \frac{T^{\rho_1}}{\rho_1!}L_{g^1}L_f^{\rho_1-1}h^1(x)\cdots\frac{T^{\rho_1}}{\rho_1!}L_{g^m}L_f^{\rho_1-1}h^1(x) \\ \vdots \\ \frac{T^{\rho_p}}{\rho_p!}L_{g^1}L_f^{\rho_p-1}h^p(x)\cdots\frac{T^{\rho_p}}{\rho_p!}L_{g^m}L_f^{\rho_p-1}h^p(x) \end{bmatrix}\right|_{x_k} \tag{20}$$

where

$$\begin{aligned} H_j = h^j(x) + TL_f h^j(x) + \frac{T^2}{2!}L_f^2 h^j(x) + \cdots \\ + \frac{T^{\rho_1}}{\rho_1!}L_f^{\rho_1} h^j(x), \qquad j=1,2,\ldots,p. \end{aligned}$$

Denote the optimal control by $u_k^*$. Using the $\rho$th-order approximation, we have

$$\begin{aligned} (\xi_k - \eta_k u_k^*)^T P\,(\xi_k - \eta_k u_k^*) \\ = \min_{u^k}(\xi_k - \eta_k u_k)^T P(\xi_k - \eta_k u_k). \end{aligned} \tag{21}$$

Assume $\eta_k$ is nonsingular. Then we can get the optimal control as

$$u_k^* = QT(\eta_k^{-1}\xi) \tag{22}$$

where $QT$ is the quantization function, which maps each component of a vector into their closest admissible values. For instance, let the set of admissible values of the $i$th component of $x$ be $\{-m_i, 0, m_i\}$ (which is the case in our missile tracking problem), and $y = QT(x)$. Then the $i$th component of $y$ is

$$y_i = \begin{cases} -m_i, & x_i < -\frac{m_i}{2} \\ 0, & -\frac{m_i}{2} \le x_i \le \frac{m_i}{2} \\ m_i, & x_i > \frac{m_i}{2}. \end{cases}$$

When $\eta_k$ is singular, we can simply choose

$$u_k^* = QT([\eta_k + \epsilon I_m]^{-1}\xi) \tag{23}$$

where $\epsilon > 0$ is a small value (say, $1E-10$ or so).

*Remark:*

1) As the sampling period $T$ small enough, using $\rho$th order approximation is reasonable. If $T$ can be chosen as small as required, the approximation error can be as small as we required.
2) According to the principle of dynamic programming, if $u$ is optimal, then at each subinterval $u_k$ is also optimal, and vise versa. So our design technique, which assures the subinterval optimality assures the overall optimality.

Finally, if we use $(\rho+1)$th order or $(\rho+2)$th order approximation, i.e., use $W_i(u_k)$, ($i = 2$ or $3$) to approximate $h_{k+1}$, then it is difficult to get a simple expression as (21). Instead of it, we may use the following form directly:

$$\left[y^r_{k+1} - W_i(u^*_k)\right]^T P \left[y^r_{k+1} - W_i(u^*_k)\right] = \min_{u_k} \left[y^r_{k+1} - W_i(u_k)\right]^T P \left[y^r_{k+1} - W_i(u_k)\right]. \quad (24)$$

Though we cannot get a simple formula as (22)and (23) for optimal controls, since $u_k = (u^1_k, \ldots, u^m_k)$, and each component $u^i_k$ has only finite choices $(N)$, $u_k$ also has finite choices $(N^m)$. Hence, (24) is also easily solvable.

## IV. Missile Attitude Control

Consider the control of a missile, with the following dynamic model:

$$\begin{cases} J_x \frac{d\omega_x}{dt} + (J_z - J_y)\omega_y\omega_z = u_1 \\ J_y \frac{d\omega_y}{dt} + (J_x - J_z)\omega_x\omega_z = u_2 \\ J_z \frac{d\omega_z}{dt} + (J_y - J_x)\omega_x\omega_y = u_3 \\ \frac{d\gamma}{dt} = \omega_x - (\omega_y \cos\gamma - \omega_z \sin\gamma)\tan\theta \\ \frac{d\phi}{dt} = \frac{1}{\cos\theta}(\omega_y\cos\gamma - \omega_z\sin\gamma) \\ \frac{d\theta}{dt} = \omega_y\sin\gamma + \omega_z\cos\gamma \\ y_1 = \gamma(t) \\ y_2 = \phi(t) \\ y_3 = \theta(t) \end{cases} \quad (25)$$

where $J_x, J_y, J_z$ are the inertias with respect to $x$, $y$, and $z$ axes of the missile (for simplicity, they are tentatively considered as constants), $\omega_x, \omega_y$, and $\omega_z$ are angular velocities with respect to the axes respectively; $\gamma, \phi$, and $\theta$ are three attitude angles.

The admissible controls are

$$u^i(t) \in \{-M_i, 0, M_i\}, \quad i = 1, 2, 3.$$

The tracking signals are: $\gamma^r(t), \phi^r(t)$, and $\theta^r(t)$. The tracking purpose is to solve

$$\min_u \int_0^T (|\gamma(t) - \gamma^r(t)|^2 + |\phi(t) - \phi^r(t)|^2 + |\theta(t) - \theta^r(t)|^2)dt.$$

Denoting

$$x_1 = \omega_x, x_2 = \omega_y, x_3 = \omega_z$$
$$x_4 = \gamma, x_5 = \phi, x_6 = \theta$$

the system (25) can be converted into canonical form as

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \quad (26)$$

where

$$f(x) = \begin{bmatrix} \frac{J_y - J_z}{J_x}\omega_y\omega_z \\ \frac{J_z - J_x}{J_y}\omega_x\omega_z \\ \frac{J_x - J_y}{J_z}\omega_x\omega_y \\ \omega_x - (\omega_y\cos\gamma - \omega_z\sin\gamma)\tan\theta \\ \frac{1}{\cos\theta}(\omega_y\cos\gamma - \omega_z\sin\gamma) \\ \omega_y\sin\gamma + \omega_z\cos\gamma \end{bmatrix}$$

$$g(x) = (g^1, g^2, g^3) = \begin{bmatrix} \frac{1}{J_x} & 0 & 0 \\ 0 & \frac{1}{J_y} & 0 \\ 0 & 0 & \frac{1}{J_z} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$h(x) = [x_4 \quad x_5 \quad x_6]^T.$$

It is easy to calculate the relative degree vector as $\rho_1 = \rho_2 = \rho_3 := \rho = 2$. Then we have

$$H_0 := L_f h = \begin{bmatrix} \omega_x - (\omega_y\cos\gamma - \omega_z\sin\gamma)\tan\theta \\ \frac{1}{\cos\theta}(\omega_y\cos\gamma - \omega_z\sin\gamma) \\ \omega_y\sin\gamma + \omega_z\cos\gamma \end{bmatrix}. \quad (27)$$

For notational ease, we define $H_f := dL_f h$. Then

$$H_f = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} & h_{15} & h_{16} \\ h_{21} & h_{22} & h_{23} & h_{24} & h_{25} & h_{26} \\ h_{31} & h_{32} & h_{33} & h_{34} & h_{35} & h_{36} \end{bmatrix} \quad (28)$$

where

$$\begin{aligned} h_{11} &= 1 \\ h_{12} &= -\cos\gamma\tan\theta \\ h_{13} &= \sin\gamma\tan\theta \\ h_{14} &= (\omega_y\sin\gamma + \omega_z\cos\gamma)\tan\theta \\ h_{16} &= -(\omega_y\cos\gamma - \omega_z\sin\gamma)\sec^2\theta \\ h_{22} &= \frac{1}{\cos\theta}\cos\gamma \\ h_{23} &= -\frac{1}{\cos\theta}\sin\gamma \\ h_{24} &= \frac{1}{\cos\theta}(-\omega_y\sin\gamma - \omega_z\cos\gamma) \\ h_{26} &= \tan\theta\sec\theta(\omega_y\cos\gamma - \omega_z\sin\gamma) \\ h_{32} &= \sin\gamma \\ h_{33} &= \cos\gamma \\ h_{34} &= \omega_y\cos\gamma - \omega_z\sin\gamma \\ h_{15} &= h_{21} = h_{25} = h_{31} = h_{35} = h_{36} = 0. \end{aligned}$$

It follows that

$$L_f^2 h = H_f f, \quad L_g L_f h = H_f g. \quad (29)$$

Now

$$\begin{cases} \xi_k = h^r_{k+1} - \left[h_k + TH_0\,|_k + \frac{T^2}{2!}H_f f\,|_k\right] \\ \eta_k = \frac{T^2}{2!}H_f g\,|_k. \end{cases} \quad (30)$$

Plugging them into (22) and (23), the optimal control is obtained.

We may also use $(\rho+1)$th-order or $(\rho+2)$th-order approximations to raise the accuracy. Simulation shows that the improvement is not very significant, but the computation complexity could be rased a lot. So at least, in this model to reduce the average tracking error, shorten the sampling time $T$ is more efficient.

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## V. Parameter Predictive Tracking

Recall system (1) where uncertainties exist. We assume the uncertainties are caused by some unknown parameters. They are constant or at least varying slowly. Say, $\lambda \in \mathbb{R}^s$ is the parameter vector. Then we can express $\Delta f = \Delta f(\lambda)$ and $\Delta g = \Delta g(\lambda)$.

We propose two ways for on line parameter prediction by using output and state, respectively.

### A. Prediction via Outputs

We still use the Fliess functional expansion to preform the prediction. Either one of the following approximations could be used to predict the parameters.

$$h_{k+1} = h_k + T \sum_{i=0}^{m} \left[ L_{g^i} h_k + L_{\Delta g^i(\lambda)} h_k \right] + R_1 \tag{31}$$

where $R_1 = O(|T|^2)$.

We also have

$$\begin{aligned} h_{k+1} =& h_k + T \sum_{i=0}^{m} (L_{g^i} h_k + L_{\Delta g^i(\lambda)} h_k) \\ &+ \frac{T^2}{2!} \sum_{i=0}^{m} \sum_{j=0}^{m} \big( L_{g^i} L_{g^j} h_k + L_{\Delta g^i(\lambda)} L_{g^j} h_k \\ &+ L_{g^i} L_{\Delta g^j(\lambda)} h_k \big) + R_2 \end{aligned} \tag{32}$$

where $R_2 = O(|T|^3)$.

Back to the model of missile. The inertias' change may be caused by the fuel consumption. The uncertainties can be expressed as

$$\Delta f(\lambda) = \begin{bmatrix} -\frac{\Delta J_x (J_y - J_z) + J_x(\Delta J_z - \Delta J_y)}{J_x (J_x + \Delta J_x)} \omega_y \omega_z \\ -\frac{\Delta J_y (J_z - J_x) + J_y(\Delta J_x - \Delta J_z)}{J_y (J_y + \Delta J_y)} \omega_z \omega_x \\ -\frac{\Delta J_z (J_x - J_y) + J_z(\Delta J_y - \Delta J_x)}{J_z (J_z + \Delta J_z)} \omega_x \omega_y \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{33}$$

$$\Delta g(\lambda) = \begin{bmatrix} \frac{-\Delta J_x}{J_x(J_x+\Delta J_x)} & 0 & 0 \\ 0 & \frac{-\Delta J_y}{J_y(J_y+\Delta J_y)} & 0 \\ 0 & 0 & \frac{-\Delta J_z}{J_z(J_z+\Delta J_z)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{34}$$

Now the inertias' change is considered as unknown parameters, denoted by

$$\lambda = (\Delta J_x, \Delta J_y, \Delta J_z)^{\mathrm{T}}.$$

both $\Delta f$ and $\Delta g_i$ have relative degree vector $\rho_i = 2, i = 1, 2, 3$, we can only use (32) (or even higher order approximation). Because $\Delta f$ and $\Delta g^i$ will not appear in (31).

Now (32) becomes

$$\begin{aligned} h_{k+1} \approx h_k + T L_f h_k + \frac{T^2}{2!} \Bigg[ & L_f^2 h_k + \sum_{i=1}^{3} u^i L_{g^i} L_f h_k \\ & + L_{\Delta f} L_f h_k + \sum_{i=1}^{3} u^i L_{\Delta g^i} L_f h_k \Bigg]. \end{aligned} \tag{35}$$

In (33) and (34), since $\|\Delta J\| \ll \|J\|$, we may us some approximations as

$$\frac{1}{J_x(J_x+\Delta J_x)} \approx \frac{1}{J_x^2} \qquad \frac{1}{J_y(J_y+\Delta J_y)} \approx \frac{1}{J_y^2}$$
$$\frac{1}{J_z(J_z+\Delta J_z)} \approx \frac{1}{J_z^2}.$$

Using these approximations, we define a matrix $-\Psi$ as

$$\begin{bmatrix} \frac{J_y - J_z}{J_x^2} \omega_y \omega_z + \frac{u_1}{J_x^2} & -\frac{\omega_y \omega_z}{J_x} & \frac{\omega_y \omega_z}{J_x} \\ \frac{\omega_z \omega_x}{J_y} & \frac{J_z - J_x}{J_y^2} \omega_z \omega_x + \frac{u_2}{J_y^2} & -\frac{\omega_z \omega_x}{J_y} \\ -\frac{\omega_x \omega_y}{J_z} & \frac{\omega_x \omega_y}{J_z} & \frac{J_x - J_y}{J_z^2} \omega_x \omega_z + \frac{u_3}{J_z^2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{36}$$

Then we have

$$\Psi\,|_k \lambda \approx \Delta f(\lambda)\,|_{x=x_k} + \sum_{i=1}^{3} u_k^i \Delta g^i(\lambda) \bigg|_{x=x_k}. \tag{37}$$

Using (35), (37), and the formulas (27), (28), we have

$$\begin{aligned} \frac{T^2}{2!} H_f\,|_k \Psi\,|_k \lambda \approx & h_{k+1} - h_k - T H_0\,|_k \\ & - \frac{T^2}{2!} \left[ H_f f\,|_k + \sum_{i=1}^{3} u^i H_f g^i\,|_k \right]. \end{aligned} \tag{38}$$

Finally, the predicted parameters are

$$\begin{aligned} \hat{\lambda}_k = \left( \frac{T^2}{2!} H_f\,|_k \Psi\,|_k \right)^{-1} \Bigg[ & h_{k+1} - h_k - T L_f h_k \\ & - \frac{T^2}{2!} \left( H_f f\,|_k + \sum_{i=1}^{3} u^i H_f g^i\,|_k \right) \Bigg]. \end{aligned} \tag{39}$$

Assume $\lambda_k$ is constant or varying very slow. Then we can use several past data to estimate it. Denote by

$$b_k = h_{k+1} - h_k - T H_0\,|_k - \frac{T^2}{2!} \left[ H_f f\,|_k + \sum_{i=1}^{3} u^i H_f g^i\,|_k \right]. \tag{40}$$

And assume the filter length is chosen, say, to be 3. Then we can define

$$B_k := \begin{bmatrix} b_{k-1} \\ \beta_{k-2} \\ \beta_{k-3} \end{bmatrix} \tag{41}$$

and

$$A_k := \frac{T^2}{2!}\begin{bmatrix} H_f\,|_{k-1}\Psi\,|_{k-1} \\ H_f\,|_{k-2}\Psi\,|_{k-2} \\ H_f\,|_{k-3}\Psi\,|_{k-3} \end{bmatrix}. \tag{42}$$

Then we have

$$A_k\lambda_k \approx B_k. \tag{43}$$

Assuming $A_k^T A_k$ is invertible, the least square estimate is

$$\hat{\lambda}_k = \left(A_k^T A_k\right)^{-1} A_k^T B_k.$$

It can be used to predict the model at $k$th time duration.

## *B. Prediction via States*

When the outputs are not sensitive to uncertainties and the whole state is observable, whole state could be used for the parameter prediction. We discuss it as follows.

Using (32) with $h_i = x_i, i = 1, \ldots, 6$, yields

$$\begin{aligned} x_{k+1} = x_k &+ T\sum_{i=0}^{m}\left(L_{g^i}x_k + L_{\Delta g^i(\lambda)}x_k\right)u^i \\ &+ \frac{T^2}{2!}\sum_{i=0}^{m}\sum_{j=0}^{m}\left(L_{g^i}L_{g^j}x_k + L_{\Delta g^i(\lambda)}L_{g^j}x_k\right. \\ &\left. + L_{g^i}L_{\Delta g^j(\lambda)}x_k\right)u_i + R_2. \end{aligned} \tag{44}$$

Then we have

$$\begin{aligned} x_{k+1} \approx x_k &+ T\left(f + \sum_{i=1}^{3} g^i u^i + \Delta f(\lambda) + \sum_{i=1}^{3}\Delta g^i(\lambda)u^i\right) \\ &+ \frac{T^2}{2!}\left(L_f f + \sum_{i=1}^{3} L_{g^i} f u^i\right) \\ &+ \frac{T^2}{2!} Df\left(\Psi_f + \Psi_1 + \Psi_2 + \Psi_3\right)\lambda \\ &+ \frac{T^2}{2!} D\Delta f\left(f + \sum_{i=1}^{3} g^i u^i\right) \end{aligned} \tag{45}$$

where

$$\Psi_f = -\begin{bmatrix} \frac{J_y-J_z}{J_x^2}\omega_y\omega_z & -\frac{\omega_y\omega_z}{J_x} & \frac{\omega_y\omega_z}{J_x} \\ \frac{\omega_z\omega_x}{J_y} & \frac{J_z-J_x}{J_y^2}\omega_z\omega_x & -\frac{\omega_z\omega_x}{J_y} \\ -\frac{\omega_x\omega_y}{J_z} & \frac{\omega_x\omega_y}{J_z} & \frac{J_x-J_y}{J_z^2}\omega_x\omega_y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{46}$$

$$Df = \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & d_{16} \\ d_{21} & d_{22} & d_{23} & d_{24} & d_{25} & d_{26} \\ d_{31} & d_{32} & d_{33} & d_{34} & d_{35} & d_{36} \\ h_{11} & h_{12} & h_{13} & h_{14} & h_{15} & h_{16} \\ h_{21} & h_{22} & h_{23} & h_{24} & h_{25} & h_{26} \\ h_{31} & h_{32} & h_{33} & h_{34} & h_{35} & h_{36} \end{bmatrix}$$

$$\begin{aligned} d_{12} &= \frac{J_y - J_z}{J_x}\omega_z \\ d_{13} &= \frac{J_y - J_z}{J_x}\omega_y \\ d_{21} &= \frac{J_z - J_x}{J_y}\omega_z \\ d_{23} &= \frac{J_z - J_x}{J_y}\omega_x \\ d_{31} &= \frac{J_x - J_y}{J_z}\omega_y \\ d_{32} &= \frac{J_x - J_y}{J_z}\omega_x \\ d_{11} &= d_{22} = d_{33} = d_{14} = d_{15} = d_{16} = 0 \\ d_{24} &= d_{25} = d_{26} = d_{34} = d_{35} = d_{36} = 0. \end{aligned}$$

and

$$\Psi_1 = -\begin{bmatrix} \frac{u_1}{J_x^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \Psi_2 = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{u_2}{J_y^2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Psi_3 = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{u_3}{J_y^3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Define

$$\begin{aligned} \kappa_1 &= -\frac{\Delta J_x(J_y - J_z) + J_x(\Delta J_z - \Delta J_y)}{J_x^2} \\ \kappa_2 &= -\frac{\Delta J_y(J_z - J_x) + J_y(\Delta J_x - \Delta J_z)}{J_y^2} \\ \kappa_3 &= -\frac{\Delta J_z(J_x - J_y) + J_z(\Delta J_y - \Delta J_x)}{J_z^2} \end{aligned}$$

and $K = [\kappa_1\ \ \kappa_2\ \ \kappa_3]^{\mathrm{T}} = \tilde{K}\lambda$, in which $\tilde{K}$ is as follows:

$$\tilde{K} = \begin{bmatrix} \frac{J_z-J_y}{J_x^2} & \frac{1}{J_x} & -\frac{1}{J_x} \\ -\frac{1}{J_y} & \frac{J_x-J_z}{J_y^2} & \frac{1}{J_y} \\ \frac{1}{J_z} & -\frac{1}{J_z} & \frac{J_y-J_x}{J_z^2} \end{bmatrix}.$$

Then we have

$$D\Delta f = \begin{bmatrix} 0 & \kappa_1\omega_z & \kappa_1\omega_y & 0 & 0 & 0 \\ \kappa_2\omega_z & 0 & \kappa_2\omega_x & 0 & 0 & 0 \\ \kappa_3\omega_y & \kappa_3\omega_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{47}$$

We define a matrix as

$$\tilde{\Psi}_f = \begin{bmatrix} \psi_{11} & 0 & 0 \\ 0 & \psi_{22} & 0 \\ 0 & 0 & \psi_{33} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{48}$$

with

$$\begin{aligned} \psi_{11} &= \omega_z \left( \frac{J_z - J_x}{J_y} \omega_x \omega_z + \frac{u_2}{J_y} \right) \\ &\quad + \omega_y \left( \frac{J_x - J_y}{J_z} \omega_x \omega_y + \frac{u_3}{J_z} \right) \\ \psi_{22} &= \omega_z \left( \frac{J_y - J_z}{J_x} \omega_y \omega_z + \frac{u_1}{J_x} \right) \\ &\quad + \omega_x \left( \frac{J_x - J_y}{J_z} \omega_x \omega_y + \frac{u_3}{J_z} \right) \\ \psi_{33} &= \omega_y \left( \frac{J_y - J_z}{J_x} \omega_y \omega_z + \frac{u_1}{J_x} \right) \\ &\quad + \omega_x \left( \frac{J_z - J_x}{J_y} \omega_x \omega_z + \frac{u_2}{J_y} \right). \end{aligned}$$

Then we have

$$\begin{aligned} &\left( T\Psi_f + T\sum_{i=1}^{3} \Psi_i + \frac{T^2}{2!} Df \left( \sum_{i=1}^{3} \Psi_i + \Psi_f \right) + \frac{T^2}{2!} \tilde{\Psi}_f \tilde{K} \right) \lambda \\ &\approx x_{k+1} - x_k - T\left( f + \sum_{i=1}^{3} g^i u^i \right) \\ &\quad - \frac{T^2}{2!} \left( L_f f + \sum_{i=1}^{3} L_{g^i} f \right). \end{aligned} \tag{49}$$

Let

$$A_k = T\Psi_f + T\sum_{i=1}^{3} \Psi_i + \frac{T^2}{2!} Df \left( \sum_{i=1}^{3} \Psi_i + \Psi_f \right) + \frac{T^2}{2!} \tilde{\Psi}_f \tilde{K}$$

and

$$\begin{aligned} B_k &= x_{k+1} - x_k - T\left( f + \sum_{i=1}^{3} g^i u^i \right) \\ &\quad - \frac{T^2}{2!} \left( L_f f + \sum_{i=1}^{3} L_{g^i} f \right) \end{aligned}$$

then we have

$$A_k \lambda = B_k.$$

Assuming $A_k^T A_k$ is invertible, the least square estimate is

$$\hat{\lambda} = \left( A_k^{\mathrm{T}} A_k \right)^{-1} A_k^{\mathrm{T}} B_k. \tag{50}$$

## VI. Time-Varying Model Predictive Tracking

We first consider Fliess expansion of time-varying systems. The purpose to consider it is that the uncertainties may be time-varying, i.e., as certain functions of time $t$. Consider a time-varying system, corresponding to (8), as

$$\begin{cases} \dot{x} = f(x,t) + \sum_{i=1}^{m} g^i(x,t) u^i \\ \quad := f(x,t) + g(x,t)u, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^m \\ y = h(x,t) = \left( h_1(x,t), \ldots, h_p(x,t) \right)^T, \quad y \in \mathbb{R}^p. \end{cases} \tag{51}$$

Adding one more variable, $x_{n+1} = t$, we have

$$\dot{x}_{n+1} = 1. \tag{52}$$

Then Fliess Expansion can be used for (51)–(52) directly. Denote by

$$\tilde{f} = \begin{bmatrix} f \\ 1 \end{bmatrix}, \quad \tilde{g} = \begin{bmatrix} g \\ 0 \end{bmatrix}.$$

Then it is easy to get the Fliess expansion for time-varying systems. Say, we have (for our purpose we write it up to $(t-t_0)^3$)

$$\begin{aligned} h(x,t) &= h(x_0,t_0) + (t-t_0) \Bigg[ L_f h(x_0,t_0) + \frac{\partial}{\partial t} h(x_0,t_0) \\ &\quad + \sum_{i=1}^{m} L_{g^i} h(x_0,t_0) u^i \Bigg] \\ &\quad + \frac{(t-t_0)^2}{2!} \Bigg[ L_f^2 h(x_0,t_0) + 2\frac{\partial}{\partial t} L_f h(x_0,t_0) \\ &\quad + \frac{\partial^2}{\partial t^2} h(x_0,t_0) + \sum_{i=1}^{m} L_f L_{g^i} h(x_0,t_0) u^i \\ &\quad + \sum_{i=1}^{m} L_{g^i} L_f h(x_0,t_0) u^i \\ &\quad + \sum_{i=1}^{m} \sum_{j=1}^{m} L_{g^i} L_{g^j} h(x_0,t_0) u^i u^j \\ &\quad + 2\sum_{i=1}^{m} \frac{\partial}{\partial t} L_{g^i} h(x_0,t_0) u^i \Bigg] \\ &\quad + \frac{(t-t_0)^3}{3!} \Bigg[ L_f^3 h(x_0,t_0) + 3\frac{\partial}{\partial t} L_f^2 h(x_0,t_0) \\ &\quad + 3\frac{\partial^2}{\partial t^2} L_f h(x_0,t_0) + \frac{\partial^3}{\partial t^3} h(x_0,t_0) \\ &\quad + \sum_{i=1}^{m} \Big( L_f^2 L_{g^i} h(x_0,t_0) + L_f L_{g^i} L_f h(x_0,t_0) \\ &\quad + L_{g^i} L_f^2 h(x_0,t_0) + 3\frac{\partial}{\partial t} L_f L_{g^i} h(x_0,t_0) \\ &\quad + 3\frac{\partial}{\partial t} L_{g^i} L_f h(x_0,t_0) + 3\frac{\partial^2}{\partial t^2} L_{g^i} h(x_0,t_0) \Big) u^i \\ &\quad + \sum_{i=1}^{m} \sum_{j=1}^{m} (L_f L_{g^i} L_{g^j} h(x_0,t_0) + \frac{\partial}{\partial t} L_{g^i} L_{g^j} h(x_0,t_0) \\ &\quad + L_{g^i} L_f L_{g^j} h(x_0,t_0) + 2\frac{\partial}{\partial t} L_{g^i} L_{g^j} h(x_0,t_0) \\ &\quad + L_{g^i} L_{g^j} L_f h(x_0,t_0)) u^i u^j \\ &\quad + \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} L_{g^i} L_{g^j} L_{g^k} h(x_0,t_0) u^i u^j u^k \Bigg] \\ &\quad + O(|t-t_0|^4). \end{aligned} \tag{53}$$

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Next, we consider the same problem as discussed in Section V. The only difference is assume now the uncertainties are certain kinds of functions of time $t$.

Back to the system (25) which is mostly concerned by us. Combining (25) with (33) and (34), one sees easily that as $\Delta f$ and/or $\Delta g$ are time-varying, the system (25) becomes a time-varying one. Practically, $\Delta J_x = \Delta J_x(t), \Delta J_y = \Delta J_y(t)$, and $\Delta J_z = \Delta J_z(t)$ are all time-varying. In such circumstances, we are facing the time-varying model predictive tracking problem.

For simplicity, we consider only the prediction via outputs. The prediction via states can be done similarly.

Using (53) and the deducting procedure in Section V, one can see, with a little surprise, that (35) remains true. Then we can still use (36)-(43).

Assume we use a linear approximation to estimate $\Delta J_x(t), \Delta J_y(t)$, and $\Delta J_z(t)$ as

$$\begin{cases} \Delta J_x(t) = \alpha_x + \beta_x t \\ \Delta J_y(t) = \alpha_y + \beta_y t \\ \Delta J_z(t) = \alpha_z + \beta_z t \end{cases}. \tag{54}$$

Then our purpose is to identify $\mu := (\alpha_x, \beta_x, \alpha_y, \beta_y, \alpha_z, b_z)^T$.

Since the number of unknowns has been doubled, we may increase the filter length. Say we choose it to be 4. Note that smaller length may hurt the accuracy and larger length will increase the complexity of the computation. So choosing 4 is a compromise. Then (44) becomes

$$B_k := \begin{bmatrix} b_{k-1} \\ \beta_{k-2} \\ \beta_{k-3} \\ \beta_{k-4} \end{bmatrix}. \tag{55}$$

It is easy to see that (with $t_k = kT$)

$$\lambda_k = [I_3 \otimes (t_k, 1)]\mu_k. \tag{56}$$

Using passed information, we can get the matrix $A_k$, which is corresponding to (42), as

$$A_k := \frac{T^2}{2!} \begin{bmatrix} H_f\,|_{k-1}\Psi\,|_{k-1} \\ H_f\,|_{k-2}\Psi\,|_{k-2} \\ H_f\,|_{k-3}\Psi\,|_{k-3} \\ H_f\,|_{k-4}\Psi\,|_{k-4} \end{bmatrix} \begin{bmatrix} I_3 \otimes (t_{k-1}, 1) \\ I_3 \otimes (t_{k-2}, 1) \\ I_3 \otimes (t_{k-3}, 1) \\ I_3 \otimes (t_{k-4}, 1) \end{bmatrix}. \tag{57}$$

Then we have

$$A_k \mu_k \approx B_k.$$

Assuming $A_k^T A_k$ is invertible, the least square estimate is

$$\hat{\mu}_k = \left(A_k^T A_k\right)^{-1} A_k^T B_k. \tag{58}$$

## VII. Simulations

We design the attitude control laws for certain kind of missiles using the method presented in this paper, the parameters are assumed to be as follows.

Nominal moments of inertia are

$$J_x = 0.4 \text{ kg} \cdot \text{m}^2$$
$$J_y = 4 \text{ kg} \cdot \text{m}^2$$
$$J_z = 4 \text{ kg} \cdot \text{m}^2.$$

Reference output trajectory is as follows:

$$\gamma_r = 5^\circ \sin(2\pi \cdot 0.8t)$$
$$\phi_r = 5^\circ \sin(2\pi \cdot 0.8t)$$
$$\theta_r = 5^\circ \sin(2\pi \cdot 0.8t).$$

Available control torques are

$$u_1 = \begin{cases} +2.5 \text{ N} \cdot \text{m} \\ 0 \\ -2.5\text{N} \cdot \text{m} \end{cases} \quad u_2 = \begin{cases} +35 \text{ N} \cdot \text{m} \\ 0 \\ -35 \text{ N} \cdot \text{m} \end{cases}$$
$$u_3 = \begin{cases} +35 \text{ N} \cdot \text{m} \\ 0 \\ -35 \text{ N} \cdot \text{m} \end{cases}.$$

The performance criteria is chosen as

$$J(k) = [y(k+1) - y^r(k+1)]^{\mathrm{T}} P [y(k+1) - y^r(k+1)] + u^{\mathrm{T}}(k) Q u(k).$$

The weight matrixes are chosen to be

$$P = \begin{bmatrix} 0.00008 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0003 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0005 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$Q = 10^{-8} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The initial attitude angles are

$$\gamma(0) = 2^\circ$$
$$\phi(0) = 2^\circ$$
$$\theta(0) = 2^\circ.$$

The initial angular velocities are

$$\omega_x(0) = 0 \text{ rad/s}$$
$$\omega_y(0) = 0 \text{ rad/s}$$
$$\omega_z(0) = 0 \text{ rad/s}.$$

The sampling time $T$ is 50 ms.

The following three simulations are made to illustrate the validity of the design method: case 1: nominal values of the moments of inertia; case 2: a constant offset from the nominal values of the moments of inertia (where $J_x$, $J_y$, and $J_z$ exist 3% offset from nominal values); case 3: the moments of inertia decreasing with linear rules (where $J_x$, $J_y$, and $J_z$ decrease 5%, 10%, and 5%, respectively). Case 1 is the ideal case, in which control laws are designed to assure the tracking accuracy using the method presented in Section 3 and 4. In case 2, time invariant uncertainties are considered. The uncertainties are estimated and then control laws are designed for nominal model and adjusted model (e.g., model with the estimated values of the moments of inertia), respectively. In Case 3, the model with time-varying uncertainties is considered, in which uncertainties

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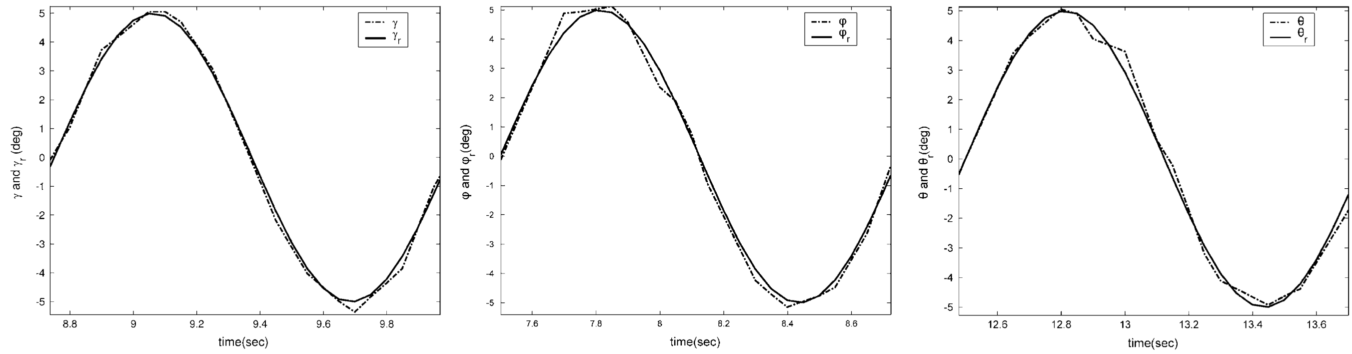
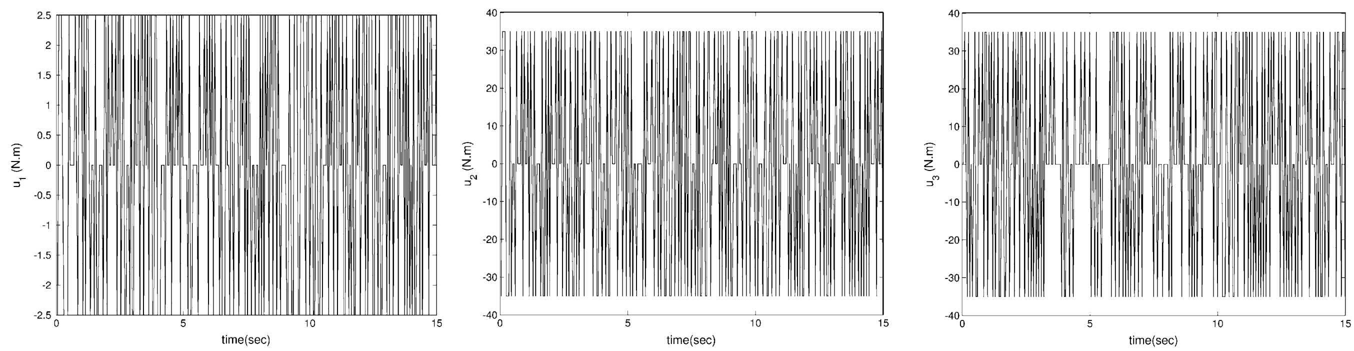


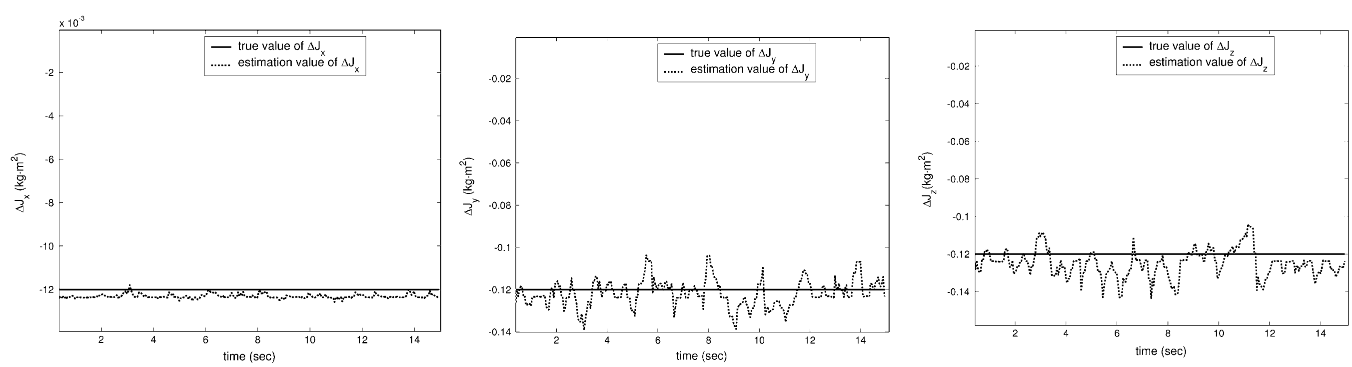

(a) (b) (c)

Fig. 1. Tracking for nominal model. (a) Roll. (b) Yaw. (c) Pitch.




(a) (b) (c)

Fig. 2. Control for nominal model. (a) $u_1$. (b) $u_2$. (c) $u_3$.




(a) (b) (c)

Fig. 3. Prediction of parameters $\Delta_J$. (a) $\Delta J_x$. (b) $\Delta J_y$. (c) $\Delta J_z$.

are estimated, and then control laws are designed for nominal model and adjusted model, respectively.

### A. Nominal Case

The tracking curves of $\gamma, \phi$, and $\theta$ are shown in Fig. 1. Control torques are shown in Fig. 2.

$\gamma_r, \phi_r$, and $\theta_r$ denote Reference output trajectory. It is shown in the simulation results that attitude tracking errors are less than $1^\circ$, and the attitude control accuracy is satisfied.

### B. Parameter Predictive Tracking

The true values and estimated values of $\Delta J_x, \Delta J_y$, and $\Delta J_z$ are shown in Fig. 3.

The estimate error mainly results from that first-order approximation in the estimation of moments of inertia. Tracking curves for $\gamma, \phi$, and $\theta$ are shown in Fig. 4.

The results show that the attitude tracking errors are less than $1^\circ$, which meet the requirements of attitude control.

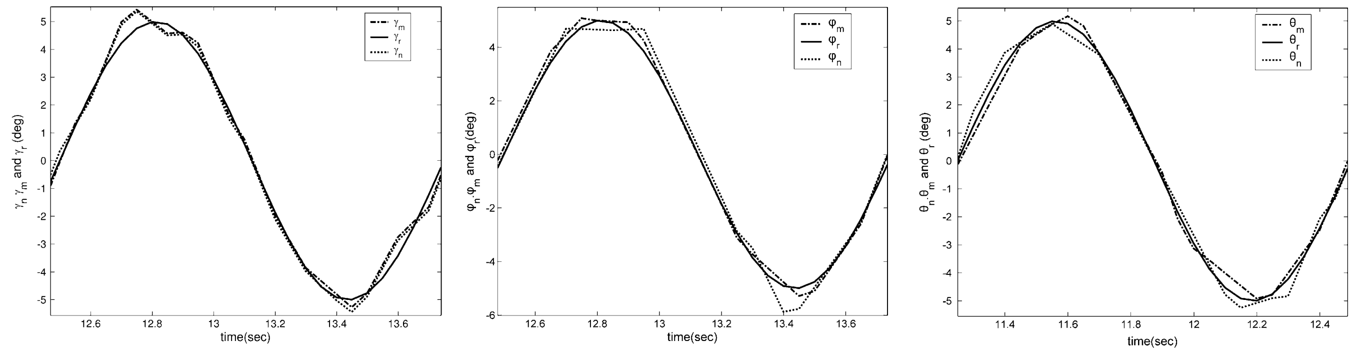
### C. Time-Varying Model Predictive Tracking

The true values and estimation values of $\Delta J_x, \Delta J_y$, and $\Delta J_z$ are shown in Fig. 5.

Tracking curves for $\gamma, \phi$, and $\theta$ are shown in Fig. 6.

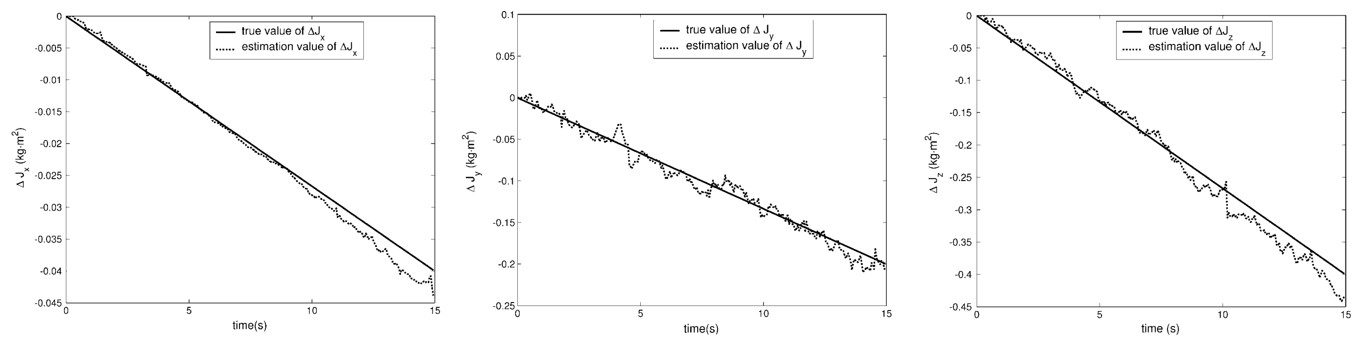
## VIII. Concluding Remarks

In this paper, the tracking problem of nonlinear systems is investigated by using Fliess functional expansion. Taking tracking



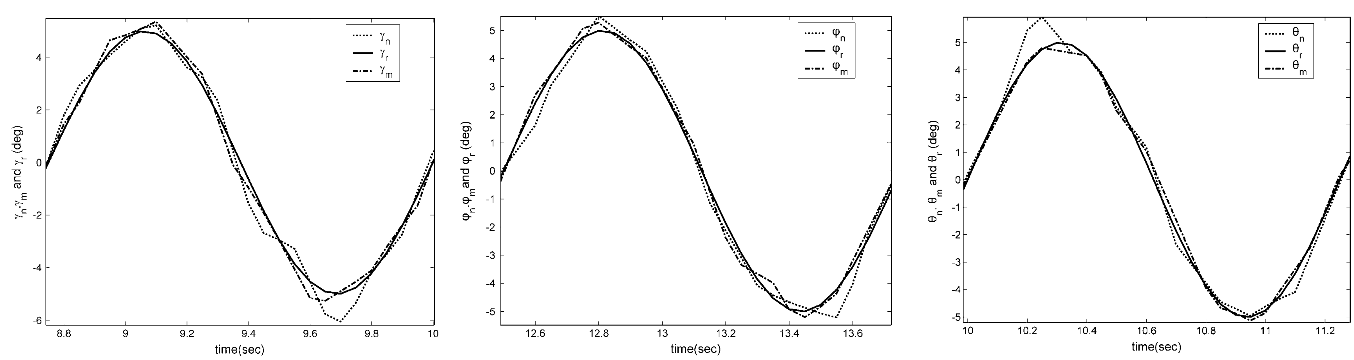

(a) (b) (c)

Fig. 4. Parameter Predictive Tracking. (a) Roll, (b) Yaw, (c) Pitch.




(a) (b) (c)

Fig. 5. Prediction of parameters $\Delta_J$. (a) $\Delta J_x$. (b) $\Delta J_y$. (c) $\Delta J_z$.




(a) (b) (c)

Fig. 6. Time-varying model predictive tracking. (a) Roll. (b) Yaw. (c) Pitch.

error as the criteria, the problem becomes an optimization one. First of all, the nominal system is considered. Within a short period, an optimal control with finite admissible values is obtained. Second, assume the system model has unknown parameters, a method to predict the parameters on line and using the predicted parameters to produce the optimal control is proposed. Third, when the model has uncertainties as certain functions of time, we first developed the time-varying Fliess functional expansion and then using it to predict the uncertainties on time and then using it to generate the optimal control.

The problem of attitude control of missiles is considered as a special tracking problem and the methods developed are used to it. Comparing the Fliess expansion approach with several existing linearization-based approaches, our approach has much higher accuracy. Moreover, unlike (approximate) linearization approaches, which treat state variables first, we treat the output directly, which avoids accumulated errors. Comparing it with the approaches based on receding horizon control, our approach is more robust because it can adapt to varying models. Particularly, when the system has switching, receding horizon control has a problem in finding optimal control (within the horizon), but the Fliess expansion approach can work as well as the non-switching case. In fact, above considerations are also our initial motivations for choosing Fliess expansion.

Before ending this paper, we would like to discuss the choice of the weighting matrix and the length of filter. The weighting matrix affects the performance. We have also used different weighting matrices in simulations. The differences are not so significant that a particular choice becomes unacceptably conservative. We leave the problem of finding principle for choosing a weighting matrix in a further study. As for the length of filter, principally, the longer the better, but practically, a long length of filter is time-consuming. To realize online control, $\mathrm{length} = 3$ is almost maximum. Then we may use a shorter sampling time to further improve the accuracy.

## REFERENCES


[1] S. Wang, M. Yang, and X. P. Shi, "Variable structure control law for space interceptor attitude control systems," *Aerospace Control*, no. 1, pp. 41–46, 2002.

[2] H. Oz and L. Meriovitch, "Optimal distributed control of flexible spacecraft during a large angle maneuver," *Guidance, Control, Dynamics*, vol. 3, no. 6, pp. 218–226, 1980.

[3] Y. H. Luo, G. J. Tang, and L. N. Zhou, "Similated annealing based global optimization of the general bilevel nonlinear programming problems," *Syst. Eng. Electron.*, vol. 26, no. 12, pp. 1922–1926, 2004.

[4] S. N. Singh and A. Iyer, "Nonlinear decoupling sliding mode control and attitude control of spacecraft," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 25, no. 5, pp. 621–633, Sep. 1989.

[5] O. Egeland and J. M. Godhavn, "Passivity-based adaptive attitude control of a rigid spacecraft," *IEEE Trans. Autom. Control*, vol. 39, no. 4, pp. 842–846, Apr. 1994.

[6] L. L. Show, J. C. Juang, and Y. W. Jan, "An LMI-based nonlinear attitude control approach," *IEEE Trans. Control Syst. Technol.*, vol. 11, no. 1, pp. 73–83, Jan. 2003.

[7] V. Manikonda, P. O. Arambel, and M. Gopinathan, "A model predictive control-based approach for spacecraft formation keeping and attitude control," in *Proc. Amer. Control Conf.*, 1999, vol. 6, pp. 4258–4262.

[8] W. C. Luo and Y. C. Chu, "Attitude control using the SDRE technique," in *Proc. ICARCV*, 2002, vol. 3, pp. 1281–1286.

[9] B. S. Chen, C. S. Wu, and Y. W. Jan, "Adaptive fuzzy mixed $H_2$ and $H_\infty$ attitude control of spacecraft," *IEEE Trans. Aerospace Electron. Syst.*, vol. 36, no. 4, pp. 1343–1359, Oct. 2000.

[10] J. Huang, *Nonlinear Output Regulation, Theory and Applications*. Philadelphia, PA: SIAM, 2004.

[11] A. Isidori and C. I. Byrnes, "Output regulation of nonlinear systems," *IEEE Trans. Autom. Control*, vol. 35, no. 2, pp. 131–140, Jul. 1990.

[12] U. Mackenroth, *Robust Conrol Systems: Theory and Case Studies*. New York: Springer, 2004.

[13] D. Q. Mayne, J. B. Rawlngs, C. V. Rao, and P. O. M. Scokaert, "Constrained model predictive control: Stability and optimality," *Automatica*, vol. 36, pp. 789–814, 2000.

[14] L. Gruene and P. E. Kloeden, "Higher order numerical schemes for affinely controlled nonlinear system," *Numer. Math.*, vol. 89, pp. 669–690, 2001.

[15] D. Nesic and L. Gruene, "Lyapunov-based continuous-time nonlinear controller redesign for sampled-data implementation," *Automatica*, vol. 41, pp. 1143–1156, 2005.

[16] G. Lafferriere and H. Sussmann, "Motion planning for controllable systems without drift," in *Proc. IEEE Int. Conf. Robot. Autom.*, 1991, pp. 1148–1153.

[17] A. Isidori, *Non-linear Control Systems*, 3nd ed. New York: Springer-Verlag, 1995.



**Yu Yao** received the B.S.. M.D., and Ph.D. degrees from the Harbin Institute of Technology, Harbin, China, in 1983, 1986, and 1990, respectively.

He is currently with the Department of Control Science and Engineering, Harbin Institute of Technology, Harbin, China, where he is the Dean of the School of Astronauts. He is also the Associate Chairman of Technical Committee on Chinese Astronaut Association Electronic and Photoelectrical Technology and Councilor of the Chinese System Simulation Association. His research interests include nonlinear control system, sampled-data control systems, hybrid systems, real-time control system, design and guidance for flight control system and simulation of aircraft, etc. He is the author or coauthor of over 100 papers.

**Baoqing Yang** received the B.S. and M.S. degrees in control science and engineering from the Harbin Institute of Technology, Harbin, China, in 2003 and 2005, respectively, where he is currently working towards the Ph.D. degree at the Control and Simulation Center.

His research interests are nonlinear control, model predictive control, and aircraft control.

**Fenghua He** received the B.S., M.D., and Ph.D. degrees from the Harbin Institute of Technology, Harbin, China, in 1998, 2000, and 2004, respectively.

She is currently a Lecturer in the Control and Simulation Center, Harbin Institute of Technology. Her research interests include nonlinear systems and hybrid system control.

**Yupeng Qiao** received the B.S. and M.S. degrees in science from the Harbin Institute of Technology, Harbin, China, in 2003 and 2005, respectively. She is currently pursuing the Ph.D. degree at the Institute of Systems Science, AMSS, Chinese Academy of Sciences, Beijing, China.

Her research interests are in nonlinear system control.

**Daizhan Cheng** (M'01-A'01–SM'01–F'06) received the B.S. degree from Tsinghua University, Beijing, China, in 1970, the M.S. degree from the Graduate School, Chinese Academy of Sciences, Beijing, in 1981, and the Ph.D. degree from Washington University, St. Louis, MO, in 1985.

Since 1990, he has been a Professor with the Institute of Systems Science, AMSS, Chinese Academy of Sciences, Beijing, China. His research interests include nonlinear control system, hybrid systems, and numerical methods in systems and control. He is currently the Chairman of the Technical Committee on Control Theory, Chinese Association of Automation, and Chairman of the IEEE CSS Beijing Chapter.