Brief paper

On partitioned controllability of switched linear systems

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When a switched linear system is not completely controllable, the controllability subspace is not enough to describe the controllability of the system over whole state space. In this case the state space can be divided into two or three control-invariant sub-manifolds, which form a control-related partition of the state space. This paper investigates when each component is a controllable sub-manifold. First, we consider when a sub-manifold is controllable for no control input case. Then the results are used to produce a necessary and sufficient condition assuring the controllability of the partitioned control-invariant sub-manifolds of a class of switched linear systems. An example is given to demonstrate the effectiveness of the results.

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1. Introduction

A switched linear system is a hybrid system which consists of several linear subsystems and a rule that orchestrates the switching among them. There are many studies on the controllability of switched linear systems. For instance, studies for low-order switched linear systems have been presented in Blondel and Tsitsiklis (1999) and Hu, Zhang, and Deng (2004). Sun and Zheng (2001), Sun, Ge, and Lee (2002), and Sun and Ge (2005) investigated the controllability and reachability issues for switched linear systems in detail.

Consider a switched linear system

\[ \dot{x}(t) = A_\sigma(t)x(t) + B_\sigma(t)u(t), \quad x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, \]  

(1)

where \( \sigma : [0, \infty) \rightarrow \Lambda = \{1, 2, \ldots, N\} \) is a piece-wise constant, right continuous mapping, called switching signal. As a particular case when there is no control input we have

\[ \dot{x}(t) = A_\sigma(t)x(t), \quad x(t) \in \mathbb{R}^n, \]  

(2)

which is called a switched linear system without control.

The reachable set of \( x_0 \), denoted by \( R(x_0) \), is defined as: \( y \in R(x_0) \), if there exist \( u, \sigma \) and \( T > 0 \), such that \( y = \varphi(u, \sigma, x_0, T) \). (Correspondingly, for system (2), \( y = \varphi(\sigma, x_0, t) \).)

Here \( \varphi(u, \sigma, x_0, t) \) is the trajectory of system (1) with initial point \( x(0) = x_0 \), control \( u(t) \) and switching signal \( \sigma(t) \). Similarly, we use \( \varphi(\sigma, x_0, t) \) to denote the trajectory of system (2).

For system (1) we define a subspace as

\[ C = \langle A_1, \ldots, A_N | B_1, \ldots, B_N \rangle, \]

which is the smallest subspace containing \( B_i \) and \( A_i \) invariant. The main result about the controllability of system (1) is the following:

\textbf{Theorem 1} (Sun et al. (2002)). For system (1), the largest reachable set from the origin is \( R(0) = C \). Moreover, for any two points \( x, y \in C, x \in R(y) \).

System (1) is completely controllable, if and only if, \( \dim(C) = n \).

We call \( C \) the controllable subspace of system (1). It is clear that the controllable subspace for system (2) is \( C = \{0\} \).

\textbf{Definition 2.} A sub-manifold \( U \subset \mathbb{R}^n \) is called a controllable sub-manifold if for any two points \( x, y \in U, x \in R(y) \).

From Theorem 1 one sees easily that the controllable subspace \( C \) is a controllable sub-manifold. Moreover, it is the largest subspace, which is also a controllable sub-manifold.

\textbf{Definition 3.} A sub-manifold \( U \subset \mathbb{R}^n \) is called a control invariant sub-manifold if for any two points \( x \in U \) and \( y \in U^c, x \not\in R(y) \), and \( y \not\in R(x) \).

Note that if \( U \) is a control invariant sub-manifold, then so is its complement \( U^c \). We also have (with mild revision).

\textbf{Proposition 4} (Cheng, Lin, and Wang (2006)). \( C \) is a control invariant sub-manifold.
Assume the controllable subspace, \( C \), of system (1) is not the whole space. Then \( C \) becomes a zero measure set. To describe the controllability of the system over whole state space, we are interested in finding (non-subspace type of) controllable sub-manifolds in \( C^c \). For block diagonal systems or symmetric systems the problem has been discussed in Cheng et al. (2006). This paper investigates the same problem for more general cases. Moreover, the procedure for designing controls and switching laws is also provided.

2. Controllability of switched linear systems without control

Consider system (2). It is obvious that \( \{0\} \) and \( \mathbb{R}^n \setminus \{0\} \) are control-invariant. So we ask when \( \mathbb{R}^n \setminus \{0\} \) is a controllable sub-manifold?

Before giving a useful sufficient condition, we need some preliminaries.

Definition 5. A point \( x_0 \neq 0 \) is called an interior point of system (2), if \( 0 \) is an interior point of the convex cone generated by \( \{A_i x_0 | \lambda \in \Lambda \} \).

The geometric meaning of the interior points is obvious, but we need a clear algebraic description for verification. We briefly cite some well known results as follows (Rotman, 1988; Massey, 1967).

- Let \( V_i, \ldots, V_m \in \mathbb{R}^n \). They are said to be affine independent if \( V_i - V_j, i = 2, \ldots, m \) are linearly independent.
- \( p \) is an interior point of a set of vectors \( \{V_i | \lambda \in \Lambda \} \), if there exist \( n + 1 \) vectors \( V_i \in \{V_i | \lambda \in \Lambda \}, i = 1, \ldots, n + 1 \), which form an affine independent set, such that

\[
\sum_{i=1}^{n+1} \mu_i V_i = p,
\]

where \( \mu_i > 0 \) and \( \sum_{i=1}^{n+1} \mu_i = 1 \).

The following lemma is an immediate consequence of the definition and the above comments.

Lemma 6. Assume \( 0 \) is an interior point of a set of vectors \( \{V_i | \lambda \in \Lambda \} \). Then there exist \( n + 1 \) affine independent vectors \( V_i \in \{V_i | \lambda \in \Lambda \} \), such that for any \( V \neq 0 \),

\[
V = -\sum_{i=1}^{n+1} a_i V_i,
\]

where \( a_i > 0, i = 1, \ldots, n + 1 \).

Theorem 7. 1. If a point \( x \neq 0 \) is an interior point of system (2), then there exists a neighborhood \( \mathcal{N}_x \) of \( x \), which is a controllable sub-manifold.

2. Let \( U \subset \mathbb{R}^n \setminus \{0\} \) be a path-wise connected open subset of \( \mathbb{R}^n \). If every point \( x \in U \) is an interior point of system (2), then \( U \) is a controllable sub-manifold.

Proof. See Appendix A.

Remark. It is easy to prove that when \( \text{codim}(C) = 1 \), \( C^c \) has two path-wise connected components, while \( \text{codim}(C) > 1 \), \( C^c \) is path-wise connected. In the following we assume \( C^c \) is path-wise connected. Otherwise, we have only to replace \( C^c \) by its each connected component.

Example 8. Consider the following system

\[
\dot{x} = A_0 x, \quad x \in \mathbb{R}^2,
\]

for which \( \Lambda = \{1, 2, 3\} \) and

\[
A_1 = I_2, \quad A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.
\]

It is easy to verify that as long as \( x \neq 0 \), \( V_i = A_0 x, i = 1, 2, 3 \) are affine independent. Moreover, let \( c_1 = c_2 = c_3 = 1/3 \). Then \( c_1 + c_2 + c_3 = 1 \), and for any \( x \neq 0 \), we have \( \sum_{i=1}^3 c_i A_0 x = 0 \). Thus every point \( x \in \mathbb{R}^2 \setminus \{0\} \) is an interior point of the system. Then Theorem 7 assures that for system (5), \( \mathbb{R}^2 \setminus \{0\} \) is a controllable sub-manifold.

3. Controllability of switched linear systems

Consider system (1). Denote \( C_\lambda = \{A_i | B_i \}, \lambda \in \Lambda \). Assume the controllable subspace of system (1), \( C \), is composed by the controllable subspaces of the switching modes. That is,

\[
C = C_1 \oplus C_2 \oplus \cdots \oplus C_N.
\]

Then system (1) can be expressed as

\[
\begin{align*}
\dot{z}^1 &= \sum_{j=1}^N A_{0j}^\mu z^1 + A_{1j}^\mu (N+1) z^2 + B_{0j}^\mu u, \\
\dot{z}^2 &= A_{j0}^\mu z^1, \\
\end{align*}
\]

where \( z^1 \) corresponds to \( C_1 \) respectively. An immediate consequence is

Lemma 9. Assumption A1 assures that \( (A_i^\mu, B_i^\mu), i = 1, \ldots, N \), are controllable.

For system (7), we have the following result:

Theorem 10. Consider system (7). Assume A1. Then \( C^c \) is a controllable sub-manifold, if and only if, for subsystem

\[
\dot{z}^2 = A_{j0}^\mu z^1,
\]

\( \mathbb{R}^{n-k} \setminus \{0\} \) is a controllable sub-manifold, where \( k \) is the dimension of \( C \).

Proof. See Appendix B.

Remark. The controllability of subsystem (8) may be verified by using Theorem 7.

4. An illustrative example

The proof of Theorem 10 is constructive, so it can be used to construct the control. In the following example, a detailed design process of the control is depicted.

Example 11. Consider the following system with \( n = 3, m = 1 \), \( \Lambda = \{1, 2\} \):

\[
\dot{x} = A_0 x + B_0 u
\]

where

\[
A_1 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \\
A_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]
Denote the controllable subspace of system (9) by \( \mathcal{C} \), the controllable subspace of every mode of system (9) by \( \mathcal{C}_1 \), \( \mathcal{C}_2 \) respectively. Then
\[
\mathcal{C} = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} = \{ x \in \mathbb{R}^3 | x_3 = 0 \}.
\]
\[
\mathcal{C}_1 = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}, \quad \mathcal{C}_2 = \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.
\]
Obviously we have
\[
\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2.
\]
Denote \( x = (x_1^T, x_2^T, x_3^T)^T \). It is easy to know that \( \mathbb{R} \setminus \{0\} \) is composed of two controllable sub-manifolds for the subsystem \( x_1^2 \). According to Theorem 10, \( \mathcal{C}^c \) is a controllable sub-manifold for system (9). Given two points \( a, b \in \mathcal{C}^c \), say \( a = \begin{pmatrix} 1 & 2 & -2 \end{pmatrix}^T \) and \( b = \begin{pmatrix} 0 & -1 & -1 \end{pmatrix}^T \), from the proof of Theorem 10, we can drive \( a \) to \( b \) in 3 steps with middle points \( \alpha, \beta \) as:
\[
a \to \alpha : \quad \sigma_0(t), \quad u_0(t), \quad t \in [0, t_1);
\]
\[
\alpha \to \beta : \quad \sigma_1(t), \quad u_1(t), \quad t \in [t_1, t_1 + t_2);
\]
\[
\beta \to b : \quad \sigma_2(t), \quad u_2(t), \quad t \in [t_1 + t_2, t_1 + t_2 + t_3).\]
Next we design \( u_1(t), \sigma(t), t \) to drive the trajectory from \( a \) to \( \alpha \) to \( \beta \) to \( b \) respectively.

We analyze the design process in a backward way:
\( 1 \) \( b \) to \( \beta \): Choose \( \sigma_2(t) \equiv 2, t_2 = 2 \) and \( u_2(t) \) to be designed. Then \( x_1 \), \( x_2 \) are free systems. So \( \beta_1 \), \( \beta_2 \) can be uniquely determined as \( \beta_1 = 3.6296 \) and \( \beta_2 = \begin{pmatrix} -7.3891 \\ 0 \end{pmatrix} \). Then \( x_2 \) will be determined later.\( \beta \) to \( \alpha \): Choose \( \sigma_1(t) \equiv 1, t_1 = 1 \) and \( u_1(t) \) to be designed. Then \( x_1 \) is a free system. So \( \alpha_1 \) and \( \alpha_2 \) can be uniquely determined as \( \alpha_1 = \begin{pmatrix} 2.7183 \\ 0 \end{pmatrix} \). Then \( \alpha \) will be determined later.

Now we are ready to design controls and switches.
\( 3 \) \( a \) to \( \alpha \): \( \{ x \in \mathbb{R} : x < 0 \} \) is a control systems for the subsystem \( x_1 \). Then we can find \( \sigma_0(t) \) to drive \( \beta \) to \( \alpha \) at time \( t_1 \). Because \( \alpha_2 \) is known, so \( \sigma_0(t) \equiv 1 \), we can calculate out that \( t_1 = 0.3069 \). Then we have \( \alpha = e^{At} a = \begin{pmatrix} 0.8481 \\ -2.7183 \end{pmatrix}^T \).

\( 4 \) \( \alpha \) to \( \beta \): Since \( \sigma_1(t) \equiv 1, t_1 = 1 \), we can design \( u_1(t) = K_1 x(t) = (1.8541, -1, -2) x(t) \) such that \( \alpha_1 \) can be driven to \( \beta_1 \). Then we have \( \beta_1 = e^{(A_1 + K_1 K_2) T} \alpha = \begin{pmatrix} 3.6296 \\ -2.8825 \end{pmatrix} \).

\( 5 \) \( \beta \) to \( b \): Since \( \sigma_2(t) \equiv 2, t_2 = 2 \), we can design \( u_2(t) = K_2 x(t) = (1, 0.4707, -2) x(t) \) such that \( \beta_2 \) can be driven to \( b_2 \). Then we have \( \beta = e^{(A_2 + K_2 K_2) T} \beta_2 = \begin{pmatrix} 0 & -1 & -1 \end{pmatrix}^T \).

Summarizing the above, and letting \( T = t_1 + t_2 + t_3 \), we obtain that under the switching law:
\[
\sigma(t) = \begin{cases} 
1, & t \in [0, t_1), \\
1, & t \in [t_1, t_1 + t_2), \\
2, & t \in [t_1 + t_2, T),
\end{cases}
\]
and the control
\[
u(t) = \begin{cases} 
0, & t \in [0, t_1), \\
(1.8541, -1, -2) x(t), & t \in [t_1, t_1 + t_2), \\
(1, 0.4707, -2) x(t), & t \in [t_1 + t_2, T),
\end{cases}
\]
where \( \mu_k > 0, \alpha_k > 0 \) and \( \sum_{k=1}^{n+1} \alpha_k = 1 \). Denote by
\[
\psi(x) = (A_{ij} - \alpha_{ij} x) \cdots (A_{in} - \alpha_{in} x) x,
\]
\section{Conclusion}
This paper considered when control-invariant sub-manifolds of switched linear systems are controllable. The main controllability results of the paper consisted of two parts. First, the controllability via switching law was investigated, a sufficient condition was obtained. Then in the case that the controllable subspace is partitioned by the controllable subsaces of switching models, a necessary and sufficient condition for \( \mathcal{C}^c \) being a controllable sub-manifold was obtained. The proof provided a procedure to construct the controls and switches. An illustrative example was constructed step by step to demonstrate the controllability and control design techniques.

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\section*{Appendix A. Proof of Theorem 7}
(1) If a point \( x \neq 0 \) is an interior point of system (2), then there exist \( n \) linearly independent vectors \( A_{i_1} x, A_{i_2} x, \ldots, A_{i_n} x \) where \( i_1, \ldots, i_n \in \Lambda \). Define a mapping
\[
\phi : t = (t_1 \cdots t_n) \rightarrow e^{\hat{A}_{i_1} t_1} \cdots e^{\hat{A}_{i_n} t_n} x.
\]
(1.1)
It is easy to see that \( \phi \) is a local diffeomorphism (Hermann, 1968). Therefore, we can find an \( \epsilon > 0 \), and \( U = \{ t : ||t|| < \epsilon \} \), such that \( \phi : U \rightarrow \phi(U) = V \) is a diffeomorphism, and \( V \) is a neighborhood of \( x \). Define
\[
K := \sup_{0 < ||t|| < \epsilon} \frac{||\phi(t) - x||}{||t||}.
\]
It is easy to see that \( K < \infty \) is well defined.

Using Lemma 6 (with a mild modification), there exist \( A_{i_1} x, \ldots, A_{i_n} x \), which are affine independent, such that
\[
A_{i_k} x = -\mu_k \sum_{k=1}^{n+1} \alpha_k A_{i_k} x, \quad k = 1, \ldots, n,
\]
(1.2)
where \( \mu_k > 0, \alpha_k > 0 \) and \( \sum_{k=1}^{n+1} \alpha_k = 1 \). Denote by
\[
\psi(x) = (A_{i_1} - \alpha_{i_1} x) \cdots (A_{i_n} - \alpha_{i_n} x) x,
\]
which is a nonsingular matrix. Then we have
\[
\begin{bmatrix}
\alpha_k^z(z) \\
\vdots \\
\alpha_k^z(z)
\end{bmatrix} = - \frac{1}{\lambda_k} \Psi^{-1}(z)(A_{h_k} + \lambda_k A_{h_{k+1}})z.
\] (A.3)

By continuity, we may choose \( \epsilon > 0 \) small enough such that when \( z \in \phi(U) \), \( \|z - x\| \) is also small enough such that \( \Psi(z) \) is invertible. Then define
\[
\begin{bmatrix}
\alpha_k^z(z) \\
\vdots \\
\alpha_k^z(z)
\end{bmatrix} = - \frac{1}{\lambda_k} \Psi^{-1}(z)(A_{h_k} + \lambda_k A_{h_{k+1}})z.
\] (A.4)

For \( z \in \phi(U) \), we can conclude the following
\[
A_{h_k} z = -\lambda_k \sum_{s=1}^{n+1} \alpha_k^z(s)A_{h_s} z;
\] (A.5)
\[
\alpha_k^z(z) > 0; \quad \sum_{s=1}^{n} \alpha_k^z(s) < 1;
\] (A.6)

Set \( \alpha^k(z) = (\alpha_k^z(z), \ldots, \alpha_k^z(z), 1 - \sum_{s=1}^{n} \alpha_k^z(z)) \). Then
\[
\|\alpha^k(z) - \alpha^k\| = O(\|z - x\|),
\] (A.7)

where \( O(\cdot) \) is an infinitesimal with the same order as \( \|\cdot\| \).

Using Taylor expansion, we have
\[
e^{A_{h_k} z} = (I + t_k A_{h_k} + O(|t_k|^2)) z,
\] (A.8)
\[
\prod_{s=1}^{n+1} e^{\lambda_s \alpha^z(s)(-\nu)} z
\]
\[
= \prod_{s=1}^{n+1} \left( \left( I - \frac{\lambda_s \alpha^z(s)}{t_k} z A_{h_s} + O(|t_k|^2) \right) z \right)
\]
\[
= \prod_{s=1}^{n+1} \left( I - \frac{t_k \lambda_s}{t_s} \sum_{j=1}^{n} \alpha^z(j)(z) A_{h_j} + O(|t_k|^2) \right) z.
\] (A.9)

From \( z \in \phi(U) \), we have
\[
\|z - x\| \leq K\|t\|.
\] (A.10)

Comparing (A.8) with (A.9) and using (A.7), we can conclude that
\[
e^{A_{h_k} z} = \prod_{s=1}^{n+1} e^{\lambda_s \alpha^z(s)(-\nu)} z + R,
\] (A.11)

where \( R = O(\|t\|^2) \). Now we can choose \( \|t\| \leq \epsilon_0 \), where \( 0 < \epsilon_0 < \epsilon \) is small enough such that
\[
R \ll \|t\|.
\] (A.12)

Define \( U = \{ t : \|t\| < \epsilon_0 \}, U_0 = \{ t : \|t\| < \epsilon_0/2 \}\). As \( \epsilon_0 \) being small enough, \( \phi : U \to V = \psi(U) \) is a diffeomorphism.

Denote \( V_0 = \phi(U_0) \subset V \). Now we claim each point \( y \in V_0 \) satisfies \( y \in R(x) \). Let
\[
y = \phi(t_1^0, \ldots, t_n^0) = \prod_{j=1}^{n} c_{j0}(t_j^0) x.
\]

We have to treat the problem of negative-time, which is not physically realizable. Construct the following mapping
\[
\psi_k := \left\{ \prod_{j \in A'} c_{j0}(t_j^0) \right\} \quad \text{if} \quad k_0 > 0,
\] (A.13)
\[
\text{and}
\]
\[
\phi(t_1, t_2, \ldots, t_n) := \psi_0 \circ \psi_2 \circ \cdots \circ \psi_n x.
\]

From the definition one sees easily that \( \phi \) is a local diffeomorphism from a neighborhood of the origin to a neighborhood of \( y \). This comes from the following consideration: Since \( y \in V_0 \), then we have \( \|y - x\| \leq K\|t\| \).

From (A.11) and (A.12), the replacement of \( e^{A_{h_k} z} \) by \( \psi_k \) may cause an error \( O(|t_k|^2) \), that is,
\[
\|\tilde{\phi}(0) - y \| = O(\|t\|^2),
\]
but the new mapping allows \( |t_k| \leq |t_k| \).

Similarly, when \( k = 2, \text{system (7)} \) becomes
\[
\frac{t_1}{2^1} = A^2_1 z^1_1 + A^2_2 z^2_2 + A^2_3 z^3_2 + B^1_1 u^1,
\]
\[
\frac{t_2}{2^2} = A^2_{21} z^1_2 + A^2_{22} z^2_2 + A^2_{23} z^3_2 + B^2_2 u^2,
\]
\[
\frac{t_3}{2^3} = A^2_{31} z^1_3 + A^2_{32} z^2_3 + A^2_{33} z^3_3 + B^3_3 u^3.
\]

Denote the starting point and the destination as \( x = (x_1^0, x_2^0, x_3^0) \) and \( y = (y_1^0, y_2^0, y_3^0)^T \). We design the control in three steps.
Denote the switching law, the control and the duration of the three steps by \((\sigma_1, u_1, T_1), (\sigma_2, u_2, T_2), (\sigma_3, u_3, T_3)\) respectively. Assume \(T_2 = \text{constant} \), \(T_3 = \text{constant} \), \(\sigma_2 \equiv 1 \), \(\sigma_3 \equiv 2 \), \(u_1 \equiv 0 \).

Since \(T_3 = \text{constant} \), \(\sigma_3 \equiv 2 \), and \(y\) is known, we have \(\beta_2 = e^{-A_{22}T_3}y^2\), and

\[
\beta_1^2 = e^{-A_{11}T_1}y_1^2 - \int_0^T e^{-A_{11}(T-\tau)}A_{13}e^{A_{21}\tau}y_1^2 d\tau.
\]

Since \(T_2 = \text{constant} \), \(\sigma_2 \equiv 1 \), we know \(\alpha^2 = e^{-A_{11}T_2}\beta_2\). Because \(z^2 = A_{33}z^2\) is controllable over \(z^2 \setminus \{0\}\), then for a given \(T_1 > 0\), we can find a switching law \(\sigma_1(t)\) such that \(\alpha^2 = \psi(\sigma_1(t), x^2, T_1)\). Letting \(u_1 \equiv 0\), we have \(\alpha_1^1 = \psi(\sigma_1(t), x^1, T_1)\). From Lemma 9, \((A_{11}^1, B_1)\) and \((A_{12}^2, B_2)\) are controllable. So for \(T_2 = \text{constant} \), \(0 < T_2\), we can find a control \(u_2(t)\) such that \(\beta_1^2 = \psi(u_2(t), \sigma_2(t), \alpha_1^1, T_2)\). Then \(\beta_1^2 = \psi(\sigma_2(t), u_2^1, T_2)\). As \((A_{12}^2, B_2)\) is controllable, then we can find a control \(u_3(t)\) such that \(y_2^2 = \psi(u_3(t), \sigma_3(t), \beta_2, T_3)\). So letting

\[
\sigma(t) = \begin{cases} 
\sigma_1(t), & t \in [0, T_1), \\
1, & t \in [T_1, T_1 + T_2), \\
2, & t \in [T_1 + T_2, T_1 + T_2 + T_3),
\end{cases}
\]

and

\[
u(t) = \begin{cases} 
0, & t \in [0, T_1), \\
u_2(t), & t \in [T_1, T_1 + T_2), \\
u_3(t), & t \in [T_1 + T_2, T_1 + T_2 + T_3),
\end{cases}
\]

we have \(y = \psi(u(t), \sigma(t), x, T_1 + T_2 + T_3)\). □

References


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