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Brief paper

On partitioned controllability of switched linear systems[☆]

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ABSTRACT

When a switched linear system is not completely controllable, the controllability subspace is not enough to describe the controllability of the system over whole state space. In this case the state space can be divided into two or three control-invariant sub-manifolds, which form a control-related partition of the state space. This paper investigates when each component is a controllable sub-manifold. First, we consider when a sub-manifold is controllable for no control input case. Then the results are used to produce a necessary and sufficient condition assuring the controllability of the partitioned control-invariant sub-manifolds of a class of switched linear systems. An example is given to demonstrate the effectiveness of the results.

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1. Introduction

A switched linear system is a hybrid system which consists of several linear subsystems and a rule that orchestrates the switching among them. There are many studies on the controllability of switched linear systems. For instance, studies for low-order switched linear systems have been presented in Loparo, Aslanis, and Ilajek (1987) and Xu and Antsaklis (1999). Some sufficient conditions and necessary conditions for controllability were presented in Ezzine and Haddad (1989) and Szigeti (1992) for switched linear systems under the assumption that the switching sequence is fixed. The complexity of stability and controllability of hybrid systems was addressed in Blondel and Tsitsiklis (1999) and Hu, Zhang, and Deng (2004). Sun and Zheng (2001), Sun, Ge, and Lee (2002), and Sun and Ge (2005) investigated the controllability and reachability issues for switched linear systems in detail.

Consider a *switched linear system*

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \quad x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, \tag{1}$$

where $\sigma : [0, \infty) \to \Lambda = \{1, 2, \ldots, N\}$ is a piece-wise constant, right continuous mapping, called switching signal. As a particular case when there is no control input we have

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad x(t) \in \mathbb{R}^n, \tag{2}$$

which is called a *switched linear system without control*.

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The *reachable set* of x_0, denoted by $R(x_0)$, is defined as: $y \in R(x_0)$, if there exist u, σ and $T > 0$, such that $y = \varphi(u, \sigma, x_0, T)$. (Correspondingly, for system (2), $y = \varphi(\sigma, x_0, T)$.)

Here $\varphi(u, \sigma, x_0, t)$ is the trajectory of system (1) with initial point $x(0) = x_0$, control $u(t)$ and switching signal $\sigma(t)$. Similarly, we use $\varphi(\sigma, x_0, t)$ to denote the trajectory of system (2).

For system (1) we define a subspace as

$$\mathcal{C} = \langle A_1, \ldots, A_N | B_1, \ldots, B_N \rangle ,$$

which is the smallest subspace containing B_i and A_i invariant. The main result about the controllability of system (1) is the following:

Theorem 1 (*Sun et al. (2002)*). *For system (1), the largest reachable set from the origin is $R(0) = \mathcal{C}$. Moreover, for any two points $x, y \in \mathcal{C}$, $x \in R(y)$.*

System (1) is completely controllable, if and only if, $\dim(\mathcal{C}) = n$.

We call $\mathcal{C}$ the *controllable subspace* of system (1). It is clear that the controllable subspace for system (2) is $\mathcal{C} = \{0\}$.

Definition 2. A sub-manifold $U \subset \mathbb{R}^n$ is called a controllable sub-manifold if for any two points $x, y \in U$, $x \in R(y)$.

From Theorem 1 one sees easily that the controllable subspace $\mathcal{C}$ is a controllable sub-manifold. Moreover, it is the largest subspace, which is also a controllable sub-manifold.

Definition 3. A sub-manifold $U \subset \mathbb{R}^n$ is called a control invariant sub-manifold if for any two points $x \in U$ and $y \in U^c$, $x \notin R(y)$, and $y \notin R(x)$.

Note that if U is a control invariant sub-manifold, then so is its complement U^c. We also have (with mild revision).

Proposition 4 (*Cheng, Lin, and Wang (2006)*). *$\mathcal{C}$ is a control invariant sub-manifold.*

Assume the controllable subspace, $\mathcal{C}$, of system (1) is not the whole space. Then $\mathcal{C}$ becomes a zero measure set. To describe the controllability of the system over whole state space, we are interested in finding (non-subspace type of) controllable sub-manifolds in $\mathcal{C}^c$. For block diagonal systems or symmetric systems the problem has been discussed in Cheng et al. (2006). This paper investigates the same problem for more general cases. Moreover, the procedure for designing controls and switching laws is also provided.

2. Controllability of switched linear systems without control

Consider system (2). It is obvious that $\{0\}$ and $\mathbb{R}^n \setminus \{0\}$ are control-invariant. So we ask when $\mathbb{R}^n \setminus \{0\}$ is a controllable sub-manifold?

Before giving a useful sufficient condition, we need some preliminaries.

Definition 5. A point $x_0 \neq 0$ is called an interior point of system (2), if 0 is an interior point of the convex cone generated by $\{A_\lambda x_0 | \lambda \in \Lambda\}$.

The geometric meaning of the interior points is obvious, but we need a clear algebraic description for verification. We briefly cite some well known results as follows (Rotman, 1988; Massey, 1967).

- Let $V_1, \ldots, V_m \in \mathbb{R}^n$. They are said to be affine independent if $V_i - V_1$, $i = 2, \ldots, m$ are linearly independent.
- p is an interior point of a set of vectors $\{V_\lambda | \lambda \in \Lambda\}$, iff there exist $n+1$ vectors $V_{\lambda_i} \in \{V_\lambda | \lambda \in \Lambda\}$, $i = 1, \ldots, n+1$, which form an affine independent set, such that

$$\sum_{i=1}^{n+1} \mu_i V_{\lambda_i} = p, \tag{3}$$

where $\mu_i > 0$ and $\sum_{i=1}^{n+1} \mu_i = 1$.

The following lemma is an immediate consequence of the definition and the above comments.

Lemma 6. *Assume 0 is an interior point of a set of vectors $\{V_\lambda | \lambda \in \Lambda\}$. Then there exist $n+1$ affine independent vectors $V_{\lambda_i} \in \{V_\lambda | \lambda \in \Lambda\}$, such that for any $V \neq 0$,*

$$V = -\sum_{i=1}^{n+1} \alpha_i V_{\lambda_i}, \tag{4}$$

where $\alpha_i > 0$, $i = 1, \ldots, n+1$.

Theorem 7. 1. *If a point $x \neq 0$ is an interior point of system (2), then there exists a neighborhood $\mathcal{N}_x$ of x, which is a controllable sub-manifold.*
2. *Let $U \subset \mathbb{R}^n \setminus \{0\}$ be a path-wise connected open subset of $\mathbb{R}^n$. If every point $x \in U$ is an interior point of system (2), then U is a controllable sub-manifold.*

Proof. See Appendix A.

Remark. It is easy to prove that when $codim(\mathcal{C}) = 1$, $\mathcal{C}^c$ has two path-wise connected components, while $codim(\mathcal{C}) > 1$, $\mathcal{C}^c$ is path-wise connected. In the following we assume $\mathcal{C}^c$ is path-wise connected. Otherwise, we have only to replace $\mathcal{C}^c$ by its each connected component.

Example 8. Consider the following system

$$\dot{x} = A_{\sigma(t)} x, \quad x \in \mathbb{R}^2, \tag{5}$$

for which $\Lambda = \{1, 2, 3\}$ and

$$A_1 = I_2, \qquad A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}.$$

It is easy to verify that as long as $x \neq 0$, $V_i = A_i x$, $i = 1, 2, 3$ are affine independent. Moreover, let $c_1 = c_2 = c_3 = 1/3$. Then $c_1 + c_2 + c_3 = 1$, and for any $x \neq 0$, we have $\sum_{i=1}^{3} c_i A_i x = 0$. Thus every point $x \in \mathbb{R}^2 \setminus \{0\}$ is an interior point of the system. Then Theorem 7 assures that for system (5), $\mathbb{R}^2 \setminus \{0\}$ is a controllable sub-manifold.

3. Controllability of switched linear systems

Consider system (1). Denote $\mathcal{C}_\lambda = \langle A_\lambda | B_\lambda \rangle$, $\lambda \in \Lambda$. Assume the controllable subspace of system (1), $\mathcal{C}$, is composed by the controllable subspaces of the switching modes. That is,

A1

$$\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \cdots \oplus \mathcal{C}_N. \tag{6}$$

Then system (1) can be expressed as

$$\begin{cases} \dot{z}_i^1 = \sum_{j=1}^{N} A_{\sigma(t)}^{ij} z_j^1 + A_{\sigma(t)}^{i(N+1)} z^2 + B_{\sigma(t)}^i u^i, \\ \quad i = 1, 2, \ldots, N, \\ \dot{z}^2 = A_{\sigma(t)}^{(N+1)(N+1)} z^2, \end{cases} \tag{7}$$

where z_i^1 corresponds to $\mathcal{C}_i$ respectively. An immediate consequence is

Lemma 9. *Assumption* A1 *assures that (A_i^{ii}, B_i^i), $i = 1, \ldots, N$, are controllable.*

For system (7), we have the following result:

Theorem 10. *Consider system (7). Assume* A1. *Then $\mathcal{C}^c$ is a controllable sub-manifold, if and only if, for subsystem*

$$\dot{z}^2 = A_{\sigma(t)}^{(N+1)(N+1)} z^2, \tag{8}$$

$\mathbb{R}^{n-k} \setminus \{0\}$ is a controllable sub-manifold, where k is the dimension of $\mathcal{C}$.

Proof. See Appendix B.

Remark. The controllability of subsystem (8) may be verified by using Theorem 7.

4. An illustrative example

The proof of Theorem 10 is constructive, so it can be used to construct the control. In the following example, a detailed design process of the control is depicted.

Example 11. Consider the following system with $n = 3$, $m = 1$, $\Lambda = \{1, 2\}$:

$$\dot{x} = A_{\sigma(t)} x + B_{\sigma(t)} u \tag{9}$$

where

$$A_1 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix};$$

$$A_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Denote the controllable subspace of system (9) by $\mathcal{C}$, the controllable subspace of every mode of system (9) by $\mathcal{C}_1$, $\mathcal{C}_2$ respectively. Then

$$\mathcal{C} = span\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\end{pmatrix}\right\} = \{x \in \mathbb{R}^3 | x_3 = 0\},$$

$$\mathcal{C}_1 = span\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}, \qquad \mathcal{C}_2 = span\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}.$$

Obviously we have

$$\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2.$$

Denote $x = (x_1^1\ x_2^1\ x^2)^{\mathrm{T}}$. It is easy to know that $\mathbb{R} \setminus \{0\}$ is composed of two controllable sub-manifolds for the subsystem x^2. According to Theorem 10, $\mathcal{C}^c$ is a controllable sub-manifold for system (9). Given two points $a, b \in \mathcal{C}^c$, say $a = \begin{pmatrix}1 & 2 & -2\end{pmatrix}^{\mathrm{T}}$ and $b = \begin{pmatrix}0 & -1 & -1\end{pmatrix}^{\mathrm{T}}$, from the proof of Theorem 10, we can drive a to b in 3 steps with middle points α, β as:

$$\begin{array}{llll} a \to \alpha: & \sigma_0(t), & u_0(t), & t \in [0, t_1); \\ \alpha \to \beta: & \sigma_1(t), & u_1(t), & t \in [t_1, t_1 + t_2); \\ \beta \to b: & \sigma_2(t), & u_2(t), & t \in [t_1 + t_2, t_1 + t_2 + t_3). \end{array}$$

Next we design $u_i(t), \sigma(t), t_i$ to drive the trajectory from a to α to β to b respectively.

We analyze the design process in a backward way:

① b to β: Choose $\sigma_2(t) \equiv 2, t_2 = 2$ and $u_2(t)$ to be designed. Then x_1^1, x^2 are free systems. So β_1^1, β^2 can be uniquely determined as $\beta_1^1 = 3.6296$ and $\beta^2 = -7.3891$. β_2^1 will be determined later.

② β to α: Choose $\sigma_1(t) \equiv 1, t_1 = 1$ and $u_1(t)$ to be designed. Then x^2 is a free system. So α^2 can be uniquely determined as $\alpha^2 = -2.7183$. α_1^1 and α_2^1 will be determined later.

Now we are ready to design controls and switches.

③ a to α: $\{x \in \mathbb{R} : x < 0\}$ is a controllable sub-manifold for the subsystem x^2, and $a^2, \alpha^2 \in \{x \in \mathbb{R} : x < 0\}$. Setting $u_0(t) \equiv 0$, we can find $\sigma_0(t)$ to drive a^2 to α^2 at time t_1. Because a^2 and α^2 are known, choosing $\sigma_0(t) \equiv 1$, we can calculate out that $t_1 = 0.3069$. Then we have $\alpha = \mathrm{e}^{A_1 t_1} a = \begin{pmatrix}0.2089 & 0.8481 & -2.7183\end{pmatrix}^{\mathrm{T}}$.

④ α to β: Since $\sigma_1(t) = 1, t_1 = 1$, we can design $u_1(t) = K_1 x(t) = (1.8541, -1, -2)x(t)$ such that α_1^1 can be driven to β_1^1. Then we have $\beta = \mathrm{e}^{(A_1+B_1K_1)t_2}\alpha = \begin{pmatrix}3.6269 & -2.8825 & -7.3891\end{pmatrix}^{\mathrm{T}}$.

⑤ β to b: Since $\sigma_2(t) = 2, t_2 = 2$, we can design $u_2(t) = K_2 x(t) = (-1, 0.4707, -2)x(t)$ such that β_2^1 can be driven to b_2^1. Then we have $b = \mathrm{e}^{(A_2+B_2K_2)t_3}\beta = \begin{pmatrix}0 & -1 & -1\end{pmatrix}^{\mathrm{T}}$.

Summarizing the above, and letting $T = t_1 + t_2 + t_3$, we obtain that under the switching law

$$\sigma(t) = \begin{cases} 1, & t \in [0, t_1), \\ 1, & t \in [t_1, t_1 + t_2), \\ 2, & t \in [t_1 + t_2, T), \end{cases}$$

and the control

$$u(t) = \begin{cases} 0, & t \in [0, t_1), \\ (1.8541, -1, -2)x(t), & t \in [t_1, t_1 + t_2), \\ (-1, 0.4707, -2)x(t), & t \in [t_1 + t_2, T), \end{cases}$$

a can be driven to b at time $T = 3.3069$.

That is, $\mathcal{C}^c$ is a controllable sub-manifold for system (9).

The trajectory $x(t)$ is depicted in Fig. 1.

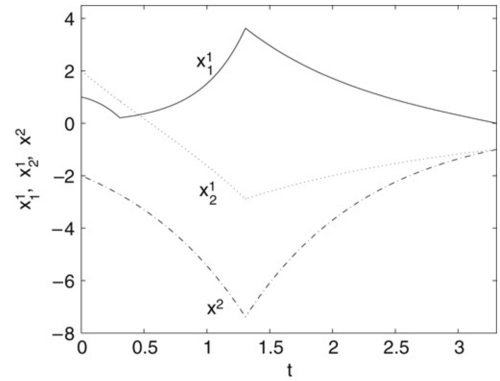

Fig. 1. The trajectories.

5. Conclusion

This paper considered when control-invariant sub-manifolds of switched linear systems are controllable. The main controllability results of the paper consisted of two parts. First, the controllability via switching law was investigated, a sufficient condition was obtained. Then in the case that the controllable subspace is partitioned by the controllable subspaces of switching models, a necessary and sufficient condition for $\mathcal{C}^c$ being a controllable sub-manifold was obtained. The proof provided a procedure to construct the controls and switches. An illustrative example was constructed step by step to demonstrate the controllability and control design techniques.

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Appendix A. Proof of Theorem 7

(1) If a point $x \neq 0$ is an interior point of system (2), then there exist n linearly independent vectors $A_{i_1}x, A_{i_2}x, \ldots, A_{i_n}x$, where $i_1, \ldots, i_n \in \Lambda$. Define a mapping

$$\phi : t = (t_1 \cdots t_n) \to \mathrm{e}^{A_{i_1}t_1} \cdots \mathrm{e}^{A_{i_n}t_n} x. \tag{A.1}$$

It is easy to see that ϕ is a local diffeomorphism (Hermann, 1968). Therefore, we can find an $\epsilon > 0$, and $U = \{t : \|t\| < \epsilon\}$, such that $\phi : U \to \phi(U) := V$ is a diffeomorphism, and V is a neighborhood of x. Define

$$K := \sup_{0 \neq \|t\| < \epsilon} \frac{\|\phi(t) - x\|}{\|t\|}.$$

It is easy to see that $K < \infty$ is well defined.

Using Lemma 6 (with a mild modification), there exist $A_{j_1}x, \ldots, A_{j_{n+1}}x$, which are affine independent, such that

$$A_{i_k}x = -\mu_k \sum_{s=1}^{n+1} \alpha_s^k A_{j_s} x, \quad k = 1, \ldots, n, \tag{A.2}$$

where $\mu_k > 0$, $\alpha_s^k > 0$ and $\sum_{s=1}^{n+1} \alpha_s^k = 1$. Denote by

$$\Psi(x) = \left[(A_{j_1} - A_{j_{n+1}})x \cdots (A_{j_n} - A_{j_{n+1}})x\right],$$

which is a nonsingular matrix. Then we have

$$\begin{bmatrix}\alpha_1^k\\ \vdots\\ \alpha_n^k\end{bmatrix} = -\frac{1}{\lambda_k}\Psi^{-1}(x)(A_{i_k}+\lambda_k A_{j_{n+1}})x. \tag{A.3}$$

By continuity, we may choose $\epsilon > 0$ small enough such that when $z \in \phi(U)$, $\|z-x\|$ is also small enough such that $\Psi(z)$ is invertible. Then define

$$\begin{bmatrix}\alpha_1^k(z)\\ \vdots\\ \alpha_n^k(z)\end{bmatrix} = -\frac{1}{\lambda_k}\Psi^{-1}(z)(A_{i_k}+\lambda_k A_{j_{n+1}})z. \tag{A.4}$$

For $z \in \phi(U)$, we can conclude the following

$$A_{i_k}z = -\lambda_k \sum_{s=1}^{n+1}\alpha_s^k(z)A_{j_s}z; \tag{A.5}$$

$$\alpha_s^k(z) > 0; \quad \sum_{s=1}^{n}\alpha_s^k(z) < 1; \tag{A.6}$$

Set $\alpha^k(z) = (\alpha_1^k(z),\ldots,\alpha_n^k(z),\ 1-\sum_{s=1}^n \alpha_s^k(z))^{\mathrm{T}}$. Then

$$\|\alpha^k(z)-\alpha^k\| = O(\|z-x\|), \tag{A.7}$$

where $O(\|\cdot\|)$ is an infinitesimal with the same order as $\|\cdot\|$.

Using Taylor expansion, we have

$$\mathrm{e}^{A_{i_k}t_k}z = (I + t_k A_{i_k} + O(|t_k|^2))z, \tag{A.8}$$

$$\begin{aligned}&\prod_{s=1}^{n+1}\mathrm{e}^{\lambda_k\alpha_s^k A_{j_s}(-t_k)}z\\ &= \prod_{s=1}^{n+1}(I-\lambda_k\alpha_s^k t_k A_{j_s} + O(|t_k|^2))z\\ &= \Bigg(I - t_k\lambda_k\sum_{j=1}^{n+1}\alpha_s^k(z)A_{j_s}\\ &\quad + t_k\lambda_k\sum_{j=1}^{n+1}\left(\alpha_s^k(z)-\alpha_s^k\right)A_{j_s} + O(|t_k|^2)\Bigg)z.\end{aligned} \tag{A.9}$$

From $z \in \phi(U)$, we have

$$\|z-x\| \le K\|t\|. \tag{A.10}$$

Comparing (A.8) with (A.9) and using (A.7), we can conclude that

$$\mathrm{e}^{A_{i_k}t_k}z = \prod_{s=1}^{n+1}\mathrm{e}^{\lambda_k\alpha_s^k A_{j_s}(-t_k)}z + R, \tag{A.11}$$

where $R = O(\|t\|^2)$. Now we can choose $\|t\| < \epsilon_0$, where $0 < \epsilon_0 < \epsilon$ is small enough such that

$$R \ll \|t\|. \tag{A.12}$$

Define $U = \{t : \|t\| < \epsilon_0\}$, $U_0 = \{t : \|t\| < \epsilon_0/2\}$. As ϵ_0 being small enough, $\phi : U \to V = \psi(U)$ is a diffeomorphism. Denote $V_0 = \phi(U_0) \subset V$. Now we claim each point $y \in V_0$ satisfies $y \in R(x)$. Let

$$y = \phi(t_1^0,\ldots,t_n^0) = \prod_{j=1}^{n}\mathrm{e}^{A_{i_j}t_j^0}x.$$

We have to treat the problem of negative-time, which is not physically realizable. Construct the following mapping

$$\psi_k := \begin{cases}\mathrm{e}^{A_{i_k}(t_k^0+t_k)}, & t_k^0 > 0\\ \prod_{j\in\Lambda'}\mathrm{e}^{\alpha_j A_j(-t_k^0+t_k)}, & t_k^0 < 0\end{cases}, \tag{A.13}$$

and

$$\tilde{\phi}(t_1,t_2,\ldots,t_n) := \psi_1 \circ \psi_2 \circ \cdots \circ \psi_n x.$$

From the definition one sees easily that $\tilde{\phi}$ is a local diffeomorphism from a neighborhood of the origin to a neighborhood of y. This comes from the following consideration: Since $y \in V_0$, then we have $\|y-x\| \le K\|t^0\|$. From (A.11) and (A.12), the replacement of $\mathrm{e}^{A_{i_k}t_k^0}$ by ψ_k may cause an error $O(|t_k^0|^2)$, that is, $\|\tilde{\phi}(0)-y\| = O(\|t^0\|^2)$, but the new mapping allows $|t_k| \le |t_k^0|$ freedom to go. Similar to the argument for ϕ, we can define a

$$\tilde{K} := \sup_{|t_k|\le|t_k^0|}\frac{\|\tilde{\phi}(t)-\tilde{\phi}(0)\|}{\|t\|}.$$

Define $W = \{z : \|z-\tilde{\phi}(0)\| \le \tilde{K}\|t^0\|\}$, then $W \subset R(\tilde{\phi}(0))$. Since $\|t^0\| < \epsilon_0/2$, as ϵ_0 small enough, we have $\|\tilde{\phi}(0)-y\| \ll \tilde{K}\|t^0\|$. Hence $y \in W$ and then $y \in R(x)$. Since $y \in V_0$ is arbitrary, V_0 is reachable from x.

Next, we have to show that starting from any $y \in V_0$ there is a switching law, which drives the system from y back to x. Let $y = \mathrm{e}^{A_{i_1}t_1^0}\cdots\mathrm{e}^{A_{i_n}t_n^0}x$. Then $x = \mathrm{e}^{A_{i_n}(-t_n^0)}\cdots\mathrm{e}^{A_{i_1}(-t_1^0)}y$. Using a similar argument as for $x \to y$, we can construct a non-negative time mapping that goes from y back to a neighborhood of x, which shows that $x \in R(y)$.

Then for any $y_1, y_2 \in V_0$, we have $y_1 \in R(y_2)$ and $y_2 \in R(y_1)$. It follows that V_0 is a controllable sub-manifold for system (2).

(2) Let $U \subset \mathbb{R}^n \setminus \{0\}$ be a path-wise connected open subset of $\mathbb{R}^n$. Then for any two points $x, y \in U$, we can connect them by a path $c(t)$, $0 \le t \le 1$ with $c(0) = x$ and $c(1) = y$. According to the proof in (1), each point $c(t)$ has an open controllable neighborhood, denoted by $U(x_t)$, where $x_t = c(t)$. Since $c(t)$ is the continuous image of a compact set $[0, 1]$, it is compact. Now $\{U(x_t), 0 \le t \le 1\}$ is an open covering of $c(t)$, so it has a finite sub-covering $\{U_1 = U(x), U_2, \ldots, U_j = U(y)\}$. Ordering them by corresponding times, we can assume $U_i \cap U_{i+1} \ne \emptyset$, and $p_i \in U_i \cap U_{i+1}$. Then $x \in R(p_i)$, $\forall i$, which means $x \in R(y)$. Then we can conclude that U is a controllable sub-manifold for system (2). □

Appendix B. Proof of Theorem 10

The necessity is trivial. We prove the sufficiency.

Let $x, y \in \mathcal{C}^c$. For system (7), to prove $\mathcal{C}^c$ is a controllable sub-manifold, we have to find a $T > 0$, a switching law $\sigma(t)$ and a control $u(t)$ such that $y = \varphi(u, \sigma, x, T)$. For simplicity, we only prove the case when $N = 2$ (When $N > 2$, the proof is essentially the same). That is, $\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2$, where $\mathcal{C}_\lambda = \langle A_\lambda|B_\lambda\rangle$, $\lambda = 1, 2$.

Then when $\lambda = 1$, using Kalman's decomposition, system (7) can be written as

$$\begin{cases}\dot{z}_1^1 = A_1^{11}z_1^1 + A_1^{12}z_2^1 + A_1^{13}z^2 + B_1^1u^1,\\ \dot{z}_2^1 = A_1^{22}z_2^1 + A_1^{23}z^2,\\ \dot{z}^2 = A_1^{33}z^2.\end{cases}$$

Similarly, when $\lambda = 2$, system (7) becomes

$$\begin{cases}\dot{z}_1^1 = A_2^{11}z_1^1 + A_2^{13}z^2,\\ \dot{z}_2^1 = A_2^{12}z_1^1 + A_2^{22}z_2^1 + A_2^{23}z^2 + B_2^2u^2,\\ \dot{z}^2 = A_2^{33}z^2.\end{cases}$$

Denote the starting point and the destination as $x = (x_1^1, x_2^1, x^2)^{\mathrm{T}}$ and $y = (y_1^1, y_2^1, y^2)^{\mathrm{T}}$. We design the control in three steps.

(a) $x = (x_1^1 \quad x_2^1 \quad x^2)^{\mathrm{T}} \to \alpha = (\alpha_1^1 \quad \alpha_2^1 \quad \alpha^2)^{\mathrm{T}}$.
(b) $\alpha = (\alpha_1^1 \quad \alpha_2^1 \quad \alpha^2)^{\mathrm{T}} \to \beta = (\beta_1^1 \quad \beta_2^1 \quad \beta^2)^{\mathrm{T}}$.
(c) $\beta = (\beta_1^1 \quad \beta_2^1 \quad \beta^2)^{\mathrm{T}} \to y = (y_1^1, y_2^1, y^2)^{\mathrm{T}}$.

Denote the switching law, the control and the duration of the three steps by (σ_1, u_1, T_1), (σ_2, u_2, T_2), (σ_3, u_3, T_3) respectively. Assume $T_2 = constant$, $T_3 = constant$, $\sigma_2 \equiv 1$, $\sigma_3 \equiv 2$, $u_1 \equiv 0$. Since $T_3 = constant, \sigma_3 \equiv 2$, and y is known, we have $\beta^2 = \mathrm{e}^{-A_2^{33}T_3}y^2$, and

$$\beta_1^1 = \mathrm{e}^{-A_2^{11}T_3}y_1^1 - \int_0^{\mathrm{T}} \mathrm{e}^{-A_2^{11}(T-\tau)}A_2^{13}\mathrm{e}^{A_2^{33}\tau}\beta_2 d\tau.$$

Since $T_2 = constant, \sigma_2 \equiv 1$, we know $\alpha^2 = \mathrm{e}^{-A_1^{33}T_2}\beta^2$. Because $\dot{z}^2 = A_{\sigma(t)}^{33}z^2$ is controllable over $z^2 \setminus \{0\}$, then for a given $T_1 > 0$, we can find a switching law $\sigma_1(t)$ such that $\alpha^2 = \varphi(\sigma_1(t), x^2, T_1)$. Letting $u_1 \equiv 0$, we have $\alpha_1^1 = \varphi(\sigma_1(t), x_1^1, T_1)$, $\alpha_2^1 = \varphi(\sigma_1(t), x_2^1, T_1)$. From Lemma 9, (A_1^{11}, B_1) and (A_2^{22}, B_2) are controllable. So for $T_2 = constant > 0$, we can find a control $u_2(t)$ such that $\beta_1^1 = \varphi(u_2(t), \sigma_2(t), \alpha_1^1, T_2)$. Then $\beta_2^1 = \varphi(\sigma_2(t), \alpha_2^1, T_2)$. As (A_2^{22}, B_2) is controllable, then we can find a control $u_3(t)$ such that $y_2^1 = \varphi(u_3(t), \sigma_3(t), \beta_2^1, T_3)$. So letting

$$\sigma(t) = \begin{cases} \sigma_1(t), & t \in [0, T_1), \\ 1, & t \in [T_1, T_1 + T_2), \\ 2, & t \in [T_1 + T_2, T_1 + T_2 + T_3), \end{cases}$$

and

$$u(t) = \begin{cases} 0, & t \in [0, T_1), \\ u_2(t), & t \in [T_1, T_1 + T_2), \\ u_3(t), & t \in [T_1 + T_2, T_1 + T_2 + T_3), \end{cases}$$

we have $y = \varphi(u(t), \sigma(t), x, T_1 + T_2 + T_3)$. $\square$

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