Chapter 19
Stability Region of Dynamic System

Consider a dynamic system. The stability region of a stable equilibrium plays very important role in practice, because the stability region is the allowed working area of an engineering system. Particularly, consider a power system, there are many working points (stable equilibriums), and investigating their stability region is fundamental for the safety of the system.

19.1 Stability Region

Consider the following nonlinear dynamic system,

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{19.1}$$

where $f(x)$ is an analytic field.

Definition 19.1. Let x_e be an equilibrium of (19.1).

1. The stable and unstable sub-manifold of x_e, denoted by $W^s(x_e)$, is defined as

$$W^s(x_e) = \left\{ p \in \mathbb{R}^n \,\middle|\, \lim_{t \to \infty} x(t,p) \to x_e \right\}. \tag{19.2}$$

2. The unstable sub-manifold of x_e, denoted by $W^u(x_e)$, is defined as

$$W^u(x_e) = \left\{ p \in \mathbb{R}^n \,\middle|\, \lim_{t \to -\infty} x(t,p) \to x_e \right\}. \tag{19.3}$$

Definition 19.2. 1. Let x_s be a stable equilibrium of (19.1). The region of attraction of x_s is defined as

$$A(x_s) = \left\{ p \in \mathbb{R}^n \,\middle|\, \lim_{t \to \infty} x(t,p) \to x_s \right\}. \tag{19.4}$$

The boundary of $A(x_s)$ is denoted by $\partial A(x_s)$.

2. An equilibrium x_e is said to be hyperbolic, if the Jacobi matrix of f at x_e, $J_f(x_e)$ has bo zero real part eigenvalues.
3. A hyperbolic equilibrium is said to be of type-k, , if $J_f(x_e)$ has k positive real part eigenvalues.

The following result is fundamental for our approach.

Theorem 19.1 ([6, 3]). *Consider system (19.1). Assume x_s is a stable equilibrium, satisfying the following three assumptions*

(i) the equilibriums on $\partial A(x_s)$ are all hyperbolic;
(ii) the stable and unstable sub-manifolds of the equilibriums on $\partial A(x_s)$ are transversal;
1. (iii) each trajectory on $\partial A(x_s)$ converges to an equilibrium as $t \to \infty$.

Then the boundary of the stability region consists of the unstable sub-manifolds of the equilibriums on the boundary.

Fig. 19.1 illustrates this.

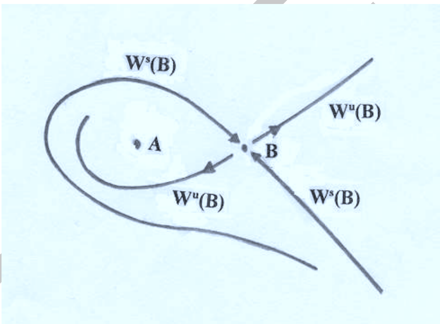

Fig. 19.1 Boundary of Stability Region

Note that two sub-manifolds N and S of a manifold M is said to be transversal, if for any $x \in N \cap S$, the union of then tangent spaces of these two sub-manifolds is the tangent space of M. Precisely,

$$T_x(N) \cup T_x(S) = T_x(M).$$

It is well known that [3] if the state manifold is of dimension n, then the boundary of the stability region is of dimension $n-1$. Hence, the boundary is basically generated by the stable sub-manifolds of type-1 equilibriums. Based on this consideration, the stable sub-manifolds of type-1 equilibriums are particularly important. There are many algorithms to calculate approximations of the stable sun-manifolds of type-1 equilibriums.

The purpose of this chapter is the explore the Taylor expansion of the equation of the sub-manifolds. Particularly, it can be used to obtain a best quadratic approximation, comparing previously existing results.

19.2 Stable Sub-Manifold

In this section we search a function to describe the stable sub-manifolds of type-1 equilibrium.

Without loss of generality, we assume $x_u = 0$ is a type-1 equilibrium. Wright down the Taylor series expansion of the $f(x)$ in (19.1) as

$$f(x) = \sum_{i=1}^{\infty} F_i x^i = Jx + F_2 x^2 + \cdots, \tag{19.5}$$

where $F_1 = J = J_f(0)$, and $F_i = \frac{1}{i!} D^i f(0)$ are known $n \times n^i$ matrices.

We use $A^{-\mathrm{T}}$ for the inverse of A^{T}. Matrix A is said to be hyperbolic is it has no zero real part eigenvalue.

Lemma 19.1. *Let A be a hyperbolic matrix. Denote by V_s and V_u the stable and unstable sub-manifolds of A respectively, and by U_s and U_u the stable and unstable sub-manifolds of $A^{-\mathrm{T}}$. Then*

$$V_s^{\perp} = U_u, \quad V_u^{\perp} = U_s. \tag{19.6}$$

Proof. Assume A is of the type-k, then we can convert A into a Jordan canonical form as

$$Q^{-1}AQ = \begin{bmatrix} J_s & 0 \\ 0 & J_u \end{bmatrix},$$

where J_s and J_u are stable and unstable blocks respectively. Splitting $Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}$, where Q_1 and Q_2 are consisted by the first $n-k$ and the last k columns of Q. Then

$$V_s = \mathrm{Span}\, col\{Q_1\}, \quad V_u = \mathrm{Span}\, col\{Q_2\}.$$

It is easy to see that

$$Q^{\mathrm{T}} A^{-\mathrm{T}} Q^{-\mathrm{T}} = \begin{bmatrix} J_s^{-\mathrm{T}} & 0 \\ 0 & J_u^{-\mathrm{T}} \end{bmatrix}.$$

Similarly, splitting $Q^{-\mathrm{T}} = \begin{bmatrix} \tilde{Q}_1 & \tilde{Q}_2 \end{bmatrix}$, where $\tilde{Q}_1$ and $\tilde{Q}_2$ consist of the first $n-k$ and last k columns of $Q^{-\mathrm{T}}$ respectively, we have

$$U_s = \mathrm{Span}\, col\{\tilde{Q}_1\}, \quad U_u = \mathrm{Span}\, col\{\tilde{Q}_2\}.$$

The conclusion follows from $Q^{-1}Q = I$. □

The following corollary is an immediate consequence of the above lemma.

Corollary 19.1. *Let A be a matrix of type-1 with its unique unstable eigenvalue μ. Assume η is the eigenvector of A^{T} with respect to μ, then η is perpendicular to the stable subspace of A.*

Proof. Since the only unstable eigenvalue of $A^{-\mathrm{T}}$ is $\frac{1}{\mu}$, denote by η the eigenvector of $A^{-\mathrm{T}}$ corresponding to this unstable eigenvalue. Then by Lemma 19.1 $\mathrm{Span}\{\eta\} =$

$U_u = V_s^{\perp}$. Hence we need only to prove that η is also the eigenvector of A^{T} with respect to μ. Since

$$A^{-T}\eta = \frac{1}{\mu}\eta \Rightarrow A^{\mathrm{T}}\eta = \mu\eta,$$

the claim follows. □

Without loss of generality, we assume the unstable equilibrium $x_u = 0$ of the system (19.1), concerned in the sequel, is of type-1.

The following theorem provides a necessary and sufficient condition for the stable sub-manifold of a type-1 equilibrium.

Theorem 19.2. *Let $x_u = 0$ be an equilibrium of type-1 of the system (19.1).*

$$W^s(e_u) = \{x \,|\, h(x) = 0\}. \tag{19.7}$$

Then $h(x)$ is uniquely determined by the following equations (19.8)–(19.10).

$$h(0) = 0, \tag{19.8}$$

$$h(x) = \eta^{\mathrm{T}}x + O(\|x\|^2), \tag{19.9}$$

$$L_f h(x) = \mu h(x), \tag{19.10}$$

where $L_f h(x)$ is the Lie derivative of $h(x)$ with respect to f, η is the eigenvector of $J_f^{\mathrm{T}}(0)$ with respect to its unique positive eigenvalue μ.

Proof. (Necessity) The necessity of (19.8) and (19.9) are obvious. We need only to prove the necessity of (19.10). First, note that

$$\frac{\partial h}{\partial x} = \eta^{\mathrm{T}} + O(\|x\|). \tag{19.11}$$

Hence, there exists a neighborhood U of the origin, such that

$$\operatorname{rank}(h(x)) = 1, \quad x \in U. \tag{19.12}$$

Since $W^s(e_u)$ is f invariant, we have

$$\begin{cases} h(x) = 0, \\ L_f h(x) = 0, \quad x \in W^s(e_u). \end{cases} \tag{19.13}$$

Since $\dim(W^s(e_u)) = n-1$, we have

$$\operatorname{rank}\left(\begin{bmatrix} h(x) \\ L_f h(x) \end{bmatrix}\right) = 1,$$

this implies that $h(x)$ and $L_f h(x)$ are linearly dependent. A straightforward computation shows that

$$L_f h(x) = \eta^{\mathrm{T}} J_f(0) x + O(\|x\|^2) = \mu \eta^{\mathrm{T}} x + O(\|x\|^2).$$

Hence for $x \in U$, the linearly dependence of $h(x)$ and $L_f h(x)$ yields (19.10). Finally, because of the analyticity of the system, we conclude that (19.10) is globally correct.

(Sufficiency) First, we prove that if $h(x)$ satisfies (19.8)−(19.10), then locally we have

$$\{x \in U \,|\, h(x) = 0\}$$

is the stable sub-manifold over U. According to the rank condition (19.12), we know that (refer to [1], Theorem 5.8)

$$V := \{x \in U \,|\, h(x) = 0\}$$

is an $(n-1)$-dimensional sub-manifold.

Next, since $L_f h(x) = 0$, V is locally f invariant. Finally, (19.9) shows that zero is locally the asymptotically stable equilibrium of $f|_V$, which is the restriction of f on V. Hence, locally V is the stable sub-manifold of (19.1). But the stable sub-manifold is unique [2], it follows that locally $V = W^s(e_u)$.

Since the system is analytic, $\{x \,|\, h(x) = 0\}$ conincides globally with $W^s(e_u)$. □

19.3 Quadratic Approximation

In general, it is not easy to figure out the equation $h(x)$ of the stable sub-manifold. The quadratic approximation of the boundary of the stability region has been investigated by several authors [5, 4]. This section provides a quadratic approximation of $h(x)$. The precise formula is provided, which is the unique approximation with error $O(\|x\|^3)$.

Denote the Taylor series expansion of $h(x)$ as

$$h(x) = H_1 x + H_2 x^2 + H_3 x^3 + \cdots = H_1 x + \frac{1}{2} x^{\mathrm{T}} \Psi x + H_3 x^3 + \cdots. \tag{19.14}$$

In the above we use two forms to express the quadratic terms: semi-tensor product form $H_2 x^2$ and standard quadratic form $\frac{1}{2} x^{\mathrm{T}} \Psi x$, where $\Psi = \mathrm{Hess}(h(0))$ is the Hessian matrix of $h(x)$ at $x = 0$, , and $H_2 = V_c^{\mathrm{T}}(\frac{1}{2}\Psi)$ is the row stacking form of $\frac{1}{2}\Psi$.

Note that for a real function $f(x,y) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ its Hessian matrix is

$$\mathrm{Hess}(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial y_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial y_m} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial y_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial y_m} \end{bmatrix}.$$

Lemma 19.2. *Assume* 0 *is the type-1 equilibrium of (19.1). Then the quadratic terns of (19.14) satisfies*

$$\Psi\left(\frac{\mu}{2}I-J\right)+\left(\frac{\mu}{2}I-J^{\mathrm{T}}\right)\Psi=\sum_{i=1}^{n}\eta_i \mathrm{Hess}(f_i(0)), \tag{19.15}$$

where μ and η are as in Corollary 19.1, $\mathrm{Hess}(f_i)$ *is the Hessian matrix of the i-th component of f.*

Proof. First, the linear approximation of $h(x)=0$ is

$$H_1x=0,$$

which is the tangent space of the stable sub-manifold $W^s(x_u)$. Since η is perpendicular to $W^s(x_u)$ at x_u, we have $H_1=\eta$.

According to Theorem 19.2, the Lie derivative satisfying

$$L_fh(x)=0.$$

Using (15.74), we have

$$Dh(x)=H_1+H_2\Phi_1x+H_3\Phi_2x^2+\cdots=H_1+x^{\mathrm{T}}\Psi+H_3\Phi_2x^2+\cdots.$$

Note that the vector field f can be expressed as

$$f(x)=Jx+\frac{1}{2}\begin{bmatrix}x^{\mathrm{T}}\mathrm{Hess}(f_1(0))x\\ \vdots\\ x^{\mathrm{T}}\mathrm{Hess}(f_n(0))x\end{bmatrix}+O(\|x\|^3).$$

Calculating L_fh out yields

$$\begin{aligned}L_fh&=\eta^{\mathrm{T}}Jx+x^{\mathrm{T}}\left(\tfrac{1}{2}\sum_{i=1}^{n}\eta_i\mathrm{Hess}(f_i(0))+\Psi J\right)x+O(\|x\|^3)\\&=\mu\eta^{\mathrm{T}}x+x^{\mathrm{T}}\left(\tfrac{1}{2}\sum_{i=1}^{n}\eta_i\mathrm{Hess}(f_i(0))+\Psi J\right)x+O(\|x\|^3).\end{aligned} \tag{19.16}$$

Observing that as the invariant sub-manifold of f, we have

$$W^s(e_u)=\left\{x\,\middle|\,h(x)=0,\ L_fh(x)=0\right\}. \tag{19.17}$$

Applying (19.14) and (19.17) to $W^s(e_u)$ yields

$$x^{\mathrm{T}}\left(\frac{1}{2}\sum_{i=1}^{n}\eta_i\mathrm{Hess}(f_i(0))+\Psi(J-\frac{\mu}{2}I)\right)x+O(\|x\|^3)=0. \tag{19.18}$$

Expressing the quadratic form into the symmetric form, we then have (19.15). □

Lemma 19.3. *Equation (19.15) has unique symmetric solution.*

Proof. Express (19.15) into a linear system as

$$(A \otimes I_n + I_n \otimes A)V_c(\Psi) = V_c\left(\sum_{i=1}^{n} \eta_i \mathrm{Hess}(f_i(0))\right), \tag{19.19}$$

where

$$A = \frac{\mu}{2} I - J^{\mathrm{T}}.$$

(19.19) is the linear form of Lyapunov mapping. Hence, let $\lambda_i \in \sigma(A)$, $i = 1, \cdots, n$ be the eigenvalues of A. Then the eigenvalues of $A \otimes I_n + I_n \otimes A$ are

$$\{\lambda_i + \lambda_j \,|\, 1 \le i, j \le n,\ \lambda_i \in \sigma(A)\}.$$

(We refer to Chapter 3 for Lyapunov mapping and its properties.)

To show $A \otimes I_n + I_n \otimes A^{\mathrm{T}}$ is nonsingular, it suffices to show that all $\lambda_i + \lambda_j \neq 0$. Let $\xi_i \in \sigma(J)$, $i = 1, \cdots, n$ be the eigenvalues of J. Then

$$\lambda_i = \frac{\mu}{2} - \xi_i, \quad i = 1, \cdots, n.$$

Observing the eigenvalues of J, it is easy to see that the only negative eigenvalue of A is $-\frac{\mu}{2}$, and all other eigenvalues of A have positive real parts, which are greater than $\frac{\mu}{2}$. It follows that

$$\lambda_i + \lambda_j \neq 0, \quad 1 \le i,\ j \le n.$$

Hence (19.15) has unique solution. Finally, we prove the solution is symmetric. It is ready to verify that

$$(A \otimes I_n + I_n \otimes A)W_{[n]} = W_{[n]}(A \otimes I_n + I_n \otimes A). \tag{19.20}$$

Using (19.20), we have

$$\begin{aligned}
&(A \otimes I_n + I_n \otimes A)V_r(\Psi) = (A \otimes I_n + I_n \otimes A)W_{[n]}V_c(\Psi) \\
&= W_{[n]}(A \otimes I_n + I_n \otimes A)V_c(\Psi) = W_{[n]}V_c\left(\sum_{i=1}^{n} \eta_i \mathrm{Hess}(f_i(0))\right) \\
&= V_r\left(\sum_{i=1}^{n} \eta_i \mathrm{Hess}(f_i(0))\right) = V_c\left(\sum_{i=1}^{n} \eta_i \mathrm{Hess}(f_i(0))\right).
\end{aligned} \tag{19.21}$$

The last equality comes from the fact that $\sum_{i=1}^{n} \xi_i \mathrm{Hess}(f_i(0))$ is a symmetric matrix, hence its row and column stacking forms are the same. (19.21) shows that $V_r(\Psi)$ is the other solution of (19.19). But the solution of (19.19) is unique, which leads to

$$V_r(\Psi) = V_c(\Psi).$$

That is, Ψ is symmetric. □

Denote by V_c^{-1} the inverse mapping of V_c, which retrieves A from its column stacking form $V_c(A)$.

Summarizing the Lemmas 19.1, 19.2, 19.3, we have the following result about the quadratic approximation of the stable sub-manifold.

Theorem 19.3. *Assume $x_u = 0$ is the type-1 equilibrium of the system (19.1), and its stable sub-manifold is determined by $h(x) = 0$. Then*

$$h(x) = H_1 x + \frac{1}{2} x^{\mathrm{T}} \Psi x + O(\|x\|^3), \tag{19.22}$$

where

$$\begin{cases} H_1 = \eta^{\mathrm{T}} \\ \Psi = V_c^{-1} \left\{ \left[(\frac{\mu}{2} I_n - J^{\mathrm{T}}) \otimes I_n + I_n \otimes (\frac{\mu}{2} I_n - J^{\mathrm{T}}) \right]^{-1} V_c \left(\sum_{i=1}^{n} \eta_i \operatorname{Hess}(f_i(0)) \right) \right\}, \end{cases}$$

μ and η are defined as in Corollary 19.1 with respect to $J = F_1$, Hess(f_i) *is the Hessian matrix of the i-th component, f_i, of f.*

Remark 19.1. If e_u is an equilibrium of type-$n-1$, μ is the unique negative eigenvalue, and its corresponding eigenvector is η, then all the above arguments remain available for describing the unstable sub-manifold. Particularly, (19.22) is the quadratic approximation of the function for unstable sub-manifold.

Observing (19.18), the following corollary is an immediate consequence, which is sometimes useful for simplifying computations.

Corollary 19.2. *Assume*

$$\sum_{i=1}^{n} \eta_i \operatorname{Hess}(f_i(0)) \left(\frac{\mu}{2} I_n - J \right)^{-1}$$

is symmetric, then the quadratic approximation of the equation of stable sub-manifold is

$$h(x) = \eta^{\mathrm{T}} x + \frac{1}{4} x^{\mathrm{T}} \sum_{i=1}^{n} \eta_i \operatorname{Hess}(f_i(0)) \left(\frac{\mu}{2} I_n - J \right)^{-1} x = 0. \tag{19.23}$$

Example 19.1. Consider the system

$$\begin{cases} \dot{x}_1 = x_1, \\ \dot{x}_2 = -x_2 + x_1^2, \quad x \in \mathbb{R}^2. \end{cases} \tag{19.24}$$

Its stable and unstable sub-manifolds are respectively (reported in [4])

$$W^s(0) = \{x \in \mathbb{R}^2 \,|\, x_1 = 0\},$$
$$W^u(0) = \{x \in \mathbb{R}^2 \,|\, x_2 = \tfrac{1}{3} x_1^2\}.$$

We use them to verify formula (19.23). For (19.24), we have

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

For stable sub-manifold $W^s(0)$, it is easy to verify that its stable eigenvalue is $\mu = 1$, its corresponding eigenvector is $\eta = (1\ 0)^{\mathrm{T}}$. Moreover,

$$\mathrm{Hess}(f_1(0)) = 0, \quad \mathrm{Hess}(f_2(0)) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence

$$\frac{1}{4}\sum_{i=1}^{2} \eta_i \,\mathrm{Hess}(f_i(0)) \left(\frac{1}{2}I - J\right)^{-1} = 0,$$

that is,

$$h_s(x) = (1\ 0)x + 0 + O(\|x\|^3) = x_1 + O(\|x\|^3).$$

For unstable sub-manifold $W^u(0)$, it is easy to check that its unstable eigenvalue is $\mu = -1$, its corresponding eigenvector is $\eta = (0\ 1)^{\mathrm{T}}$. Hence

$$\frac{1}{4}\sum_{i=1}^{2} \eta_i \,\mathrm{Hess}(f_i(0))(\frac{-1}{2}I - J)^{-1} = \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 0 \end{bmatrix}.$$

That is,

$$h_u(x) = (0\ 1)x + x^{\mathrm{T}} \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 0 \end{bmatrix} x + O(\|x\|^3) = x_2 - \frac{1}{3}x_1^2 + O(\|x\|^3).$$

In fact, if we use the conclusion in the next section, we can prove that the errors for the approximations $h_s(x)$ and $h_u(x)$ are both 0. Alternatively, we can also use Theorem 19.2 to verify this directly. For instance, we verify $h_u(x)$: Assume $h_u(x) = x_2 - \frac{1}{3}x_1^2$, then $W^u(e_u) = \{x \,|\, h_u(x) = 0\}$, if and only if $h_u(x) = 0$ implies $L_f h_u(x) = 0$. This is true, because

$$L_f(h_u(x)) = \begin{bmatrix} -\frac{2}{3}x_1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ -x_2 + x_1^2 \end{bmatrix} = -x_2 + \frac{1}{3}x_1^2 = -h_u(x).$$

19.4 Higher Order Approximation

This section considers the Taylor series expansion of the equation of stability sub-manifold. In the following calculation we need Φ_k. To calculate it, the following proposition is necessary.

Proposition 19.1.

$$W_{[n^s,n]} = \prod_{i=0}^{s-1} \left(I_{n^i} \otimes W_{[n,n]} \otimes I_{n^{s-i-1}}\right). \tag{19.25}$$

Proof. Using Proposition 2.12, we have

$$W_{[n^s,n]} = \left(W_{[n^{s-1},n]} \otimes I_n\right)\left(I_{n^{s-1}} \otimes W_{[n,n]}\right).$$

Using the first decomposition of Proposition 2.12 again, and again, we finally obtain (19.25). Note that as a convention we have $I_{n^0} = 1$, $\Phi_0 = I_n$. □

Using (19.25), it is easy to calculate Φ_k. We use an example to depict it.

Example 19.2. Assume $n = 2$, then

$$\Phi_0 = I_n,$$

$$\Phi_1 = W_{[n,n]} + I \otimes W_{[1,n]} = W_{[n]} + I_{n^2} = \begin{bmatrix} 2&0&0&0\\ 0&1&1&0\\ 0&1&1&0\\ 0&0&0&2 \end{bmatrix},$$

$$\begin{aligned} \Phi_2 &= W_{[n^2,n]} + I_n \otimes W_{[n,n]} + I_{n^2} \otimes W_{[1,n]} \\ &= \left(W_{[n]} \otimes I_n\right)\left(I_n \otimes W_{[n]}\right) + I_n \otimes W_{[n]} + I_{n^3} \\ &= \begin{bmatrix} 3&0&0&0&0&0&0&0\\ 0&1&2&0&0&0&0&0\\ 0&1&1&0&1&0&0&0\\ 0&0&0&2&0&0&1&0\\ 0&1&0&0&2&0&0&0\\ 0&0&0&1&0&1&1&0\\ 0&0&0&0&0&2&1&0\\ 0&0&0&0&0&0&0&3 \end{bmatrix}, \quad \cdots. \end{aligned}$$

Next, we proceed to solve H_k from equations (19.8) - (19.10). First problem is: since x^k is a redundant generator of k, we are not able to get unique solution from (19.8)−(19.10). To overcome this difficulty we have to convert the equations to natural basis. Recall Chapter 15, let $S \in \mathbb{Z}_+^n$. The natural basis is defined as

$$N_n^k = \{x^S \mid S \in \mathbb{Z}_+^n,\ |S| = k\}.$$

We arrange the elements in N_n^k in the alphabetic order. That is, for $S^1 = (s_1^1, \cdots, s_n^1)$ and $S^2 = (s_1^2, \cdots, s_n^2)$ we use order $x^{S^1} \prec x^{S^2}$, if there exists a t, $1 \le t \le n-1$, such that

$$s_1^1 = s_1^2,\ \cdots,\ s_t^1 = s_t^2,\ s_{t+1}^1 > s_{t+2}^2.$$

In this way we arrange the elements of N_n^k as a column and denote it as $x_{(k)}$.

Example 19.3. Let $n = 3$ and $k = 2$. Then

$$x^2 = (x_1^2, x_1x_2, x_1x_3, x_2x_1, x_2^2, x_2x_3, x_3x_1, x_3x_2, x_3^2)^{\mathrm{T}},$$

and

$$x_{(2)} = (x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2)^{\mathrm{T}}.$$

In Chapter 15 it has been proved that the size of B_n^k is

$$|B_n^k| := d = \frac{(n+k-1)!}{k!(n-1)!}, \quad k \geq 0, \quad n \geq 1. \tag{19.26}$$

Recall that in Chapter 15 we have defined two matrices: $T_N(n,k) \in M_{n^k \times d}$ and $T_B(n,k) \in M_{d \times n^k}$, which can convert two generators x^k and $x_{(k)}$ back and forth. Precisely,

$$x^k = T_N(n,k)x_{(k)}, \quad x_{(k)} = T_B(n,k)x^k.$$

Moreover,

$$T_B(n,k)T_N(n,k) = I_d.$$

We write a special pair as follows:

Example 19.4. Let $n = 2$, $k = 3$. Then the $T_B(n,k)$ is

$$T_B(2,3) = \begin{array}{c} \begin{array}{cccccccc} (111) & (112) & (121) & (122) & (211) & (212) & (221) & (222) \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ (111) \\ (112) \\ (122) \\ (222) \end{array}. \tag{19.27}$$

Meanwhile, $T_N(2,3)$ is

$$T_N(2,3) = \begin{array}{c} \begin{array}{cccc} (111) & (112) & (122) & (222) \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ (111) \\ (112) \\ (121) \\ (122) \\ (211) \\ (212) \\ (221) \\ (222) \end{array}. \tag{19.28}$$

Recall (19.14), instead of solving H_k, we will try to solve G_k, which satisfies

$$H_k x^k = G_k x_{(k)}.$$

Recall that H_k is a symmetric coefficient matrix, if any two equal elements in x^k have the equal coefficients in $H_k x^k$. We use the following example to explain it.

Example 19.5. Let $n = 3$ and $k = 2$. Then x^2 is as in Example 19.3. For a given second order homogeneous polynomial $p(x) = x_1^2 + 2x_1x_2 - 3x_1x_3 + x_2^2 - x_3^2$, we can express it as

$$p(x) = H_1 x^2 = (1,\ 2,\ -3,\ 0,\ 1,\ 0,\ 0,\ 0,\ -1)x^2.$$

Alternatively, we can also express it as

$$p(x) = H_2 x^2 = \left(1,\ 1,\ -\frac{3}{2},\ 1,\ 1,\ 0,\ -\frac{3}{2},\ 0,\ -1\right) x^2.$$

It is easy to see that H_1 is not symmetric, while H_2 is.

We also know the following.

Proposition 19.2. *1. The symmetric coefficient matrix H_k is unique.*
2.

$$H_k = G_k T_B(n,k), \quad G_k = H_k T_N(n,k). \tag{19.29}$$

Now we are ready to construct the higher terms of the $h(x)$ in the equation of stable sub-manifold. Denote by

$$f(x) = F_1 x + F_2 x^2 + \cdots;$$

and

$$h(x) = H_1 x + H_2 x^2 + \cdots.$$

Note that we already known that $F_1 = J_f(0) = J$, $H_1 = \eta^{\mathrm{T}}$, and H_2 can be uniquely determined by (19.17).

Proposition 19.3. *The coefficients H_k, $k \geq 2$, of $h(x)$ satisfy the following equations*

$$\left[\sum_{i=1}^{k} H_i \Phi_{i-1}(I_{n^{i-1}} \otimes F_{k-i+1}) - \mu H_k\right] x^k = 0, \quad k \geq 2. \tag{19.30}$$

Proof. Note that $h(x) = 0$ is invariant with respect to vector field $f(x)$, that is the Lie derivative

$$L_f h(x) = 0. \tag{19.31}$$

Using Proposition 15.4, we have

$$Dh(x) = H_1 + H_2 \Phi_1 x + H_3 \Phi_2 x^2 + \cdots = H_1 + 2x^{\mathrm{T}} \Psi + H_3 \Phi_2 x^2 + \cdots.$$

A straightforward computation shows

$$\begin{aligned} L_f h(x) &= \mu \eta^{\mathrm{T}} x + [H_2 \Phi_1 (I_n \otimes F_1) + H_1 F_2] x^2 + \cdots \\ &\quad + \left[\sum_{i=1}^{k} H_i \Phi_{i-1}(I_{n^{i-1}} \otimes F_{k+1-i})\right] x^k + \cdots. \end{aligned}$$

Note that $h(x)$ satisfies

$$\begin{cases} h(x) = 0, \\ L_f h(x) = 0. \end{cases} \tag{19.32}$$

Subtracting μ from the second equation of (19.32), and then multiply it with the first equation of (19.32), we can prove, inductively on k, that

$$\left[\sum_{i=1}^{k} H_i \Phi_{i-1} (I_{n^{i-1}} \otimes F_{k-i+1}) - \mu H_k \right] x^k + O(\|x\|^{k+1}) = 0, \quad k \geq 2,$$

The conclusion follows. □

Observing (19.30), according to Proposition 19.2, it can be expressed as

$$\begin{aligned} & G_k \left[\mu I_d - T_B(n,k) \Phi_{k-1} (I_{n^{k-1}} \otimes F_1) T_N(n,k) \right] x_{(k)} \\ & \equiv \left[\sum_{i=1}^{k-1} G_i T_B(n,i) \Phi_{i-1} (I_{n^{i-1}} \otimes F_{k-i+1}) \right] T_N(n,k) x_{(k)}, \quad k \geq 3. \end{aligned} \tag{19.33}$$

The following theorem is a summarization of the above arguments, which can be used for general case.

Theorem 19.4. *Assume the matrix*

$$C_k := \mu I_d - T_B(n,k) \Phi_{k-1} (I_{n^{k-1}} \otimes F_1) T_N(n,k), \quad k \geq 3 \tag{19.34}$$

is non-singular, then

$$G_k = \left[\sum_{i=1}^{k-1} G_i T_B(n,i) \Phi_{i-1} (I_{n^{i-1}} \otimes F_{k-i+1}) \right] T_N(n,k) C_k^{-1}. \tag{19.35}$$

Remark 19.2. In fact, H_2 can also be solved in this way. (19.15) and (19.35) can produce the same result. In fact, when H_2 is solved from (19.15), since the symmetric quadratic equation is used, the symmetry of the coefficients has been automatically assured.

It is obvious that the efficiency of (19.35) depends on whether C_i is singular. Unfortunately, in quadratic case, we are not able to assure it in certain way. It is discussed in the following example.

Example 19.6. Consider the following system

$$\begin{cases} \dot{x}_1 = -c x_1, \quad c > 0, \\ \dot{x}_2 = x_2 - 2x_1^2 + x_1^3, \end{cases} \tag{19.36}$$

where $c > 0$ is a parameter.

We calculate the equation of the stable sub-manifold. It is easy to calculate that $\mu = 1$, $\eta = (0\ 1)^{\mathrm{T}}$,

$$J = \begin{bmatrix} -c & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$\text{Hess}(f_1(0)) = 0, \quad \text{Hess}(f_2(0)) = \begin{bmatrix} -4 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence we can use (19.23) to calculate that

$$h(x) = (0\ 1)x + x^{\mathrm{T}} \begin{bmatrix} -\frac{2}{2c+1} & 0 \\ 0 & 0 \end{bmatrix} x + O(\|x\|^3). \tag{19.37}$$

Using (19.26), (19.27), and the Φ_2 calculated in Example 19.2, we can calculate C_3 as

$$C_3 = \begin{bmatrix} 3c+1 & 0 & 0 & 0 \\ 0 & 2c & 0 & 0 \\ 0 & 0 & c-1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}. \tag{19.38}$$

Assume $c \neq 1$, then C_3 is invertible. Then we have

$$H_1 = (0,\ 1), \quad H_2 = \left(-\frac{2}{2c+1},\ 0,\ 0,\ 0 \right),$$

$$F_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}, \quad F_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Plugging them into (19.34) yields

$$G_3 = \left(\frac{1}{3c+1},\ 0,\ 0,\ 0 \right).$$

Hence we have

$$h(x) = x_2 - \frac{2}{2c+1} x_1^2 + \frac{1}{3c+1} x_1^3 + O(\|x\|^4).$$

In fact, it is easy to verify that

$$h(x) = x_2 - \frac{2}{2c+1} x_1^2 + \frac{1}{3c+1} x_1^3 = 0,$$

and

$$L_f h(x) = h(x) = 0.$$

We conclude that

$$W^s(0) = \left\{ x \in \mathbb{R}^2 \,\middle|\, x_2 - \frac{2}{2c+1} x_1^2 + \frac{1}{3c+1} x_1^3 = 0 \right\}.$$

According to Theorem 19.4 and Example 19.6, we give the following algorithm:

Algorithm 4. Step 1. If $C_3, \cdots, C_{k-1}$ are nonsingular, we continue to search H_k to approximate $h(x)$ till the accuracy is satisfied.

Step 2. If C_k is singular, we search for the least square solution G_k via

$$\begin{aligned} & G_k \left[\mu I_d - T_B(n,k)\Phi_{k-1}(I_{n^{k-1}} \otimes F_1)T_N(n,k)\right] \\ & = \left[\sum_{i=1}^{k-1} G_i T_B(n,i)\Phi_{i-1}(I_{n^{i-1}} \otimes F_{k-i+1})\right] T_N(n,k) \end{aligned} \tag{19.39}$$

Then use $G_3, \cdots, G_k$ to construct a k-th order approximation of $h(x)$.

Step 3. (possible further improvement) If the least square solution is a real number solution for (19.39), solve the following system:

$$\begin{cases} G_k \left[\mu I_d - T_B(n,k)\Phi_{k-1}(I_{n^{k-1}} \otimes F_1)T_N(n,k)\right] \\ \quad = \left[\sum_{i=1}^{k-1} G_i T_B(n,i)\Phi_{i-1}(I_{n^{i-1}} \otimes F_{k-i+1})\right] T_N(n,k), \\ 0 = \left[\sum_{i=1}^{k} G_i T_B(n,i)\Phi_{i-1}(I_{n^{i-1}} \otimes F_{k-i+1})\right] T_N(n,k+1). \end{cases} \tag{19.40}$$

In fact, considering the k-th and $k+1$-th order terms leaders to (**??**).

Recall Example 19.6. When $c = 1$, the least square solution is

$$G_3 = \left(\frac{1}{3c+1},\ 0,\ t,\ 0\right),$$

where t is an arbitrary parameter. It is ready to verify that G_3 is a real number solution of (19.39). Hence, we can try to solve (19.40). A careful calculation shows that (19.40) has a solution $G_3 = \left(\frac{1}{3c+1},\ 0,\ 0,\ 0\right)$. It is easy to check that this G_3 is a real number solution.

In the following we consider another more general example.

Example 19.7. Consider the following system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1 - 2x_2, \\ \dot{x}_3 = 2x_3 - x_2(e^{x_1} - 1). \end{cases} \tag{19.41}$$

It is easy to show that $\mu = 2$, $\eta = (0\,0\,1)^{\mathrm{T}}$,

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

$$A = \frac{\mu}{2} I_3 - J^{\mathrm{T}} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\mathrm{Hess}(f_1(0)) = \mathrm{Hess}(f_2(0)) = 0,$$

$$\mathrm{Hess}(f_3(0)) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Using formula (19.22), we have

$$\begin{aligned} h(x) &\approx \eta^{\mathrm{T}} x + x^{\mathrm{T}} (\tfrac{1}{2}\Psi) x \\ &= (0\,0\,1)x + x^{\mathrm{T}} \begin{bmatrix} 0.09375 & -0.09375 & 0 \\ -0.09375 & -0.03125 & 0 \\ 0 & 0 & 0 \end{bmatrix} x \\ &= x_3 + 0.09375 x_1^2 - 0.1875 x_1 x_2 - 0.03125 x_2^2. \end{aligned} \tag{19.42}$$

To calculate the third order terms, we verify C_3. Using (19.34), we have

$$C_3 = \begin{bmatrix} 2 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 6 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}.$$

It can be calculated and verified to be invertible via computer. From the quadratic part of $h(x)$ we have

$$H_1 = \eta^{\mathrm{T}} = (0,\ 0,\ 1),$$

$$H_2 = (0.09375,\ -0.09375,\ 0,\ -0.09375,\ -0.03125,\ 0,\ 0,\ 0,\ 0),$$

$F_2 \in M_{3\times 9}$ has all zero components except $F_2(3,2)$ and $F_2(3,4)$ which are

$$F_2(3,2) = F_2(3,4) = -\frac{1}{2},$$

$F_3 \in M_{3\times 29}$ has all zero components except 3 elements: $F_3(3,2)$, $F_3(3,4)$ $F_3(3,10)$, which are

$$F_3(3,2) = F_3(3,4) = F_3(3,10) = -\frac{1}{6}.$$

Plugging them into (19.35) yields

$$G_3 = (0.0408,\ -0.0816,\ 0,\ -0.0256,\ 0,\ 0,\ -0.0032,\ 0,\ 0,\ 0).$$

Hence the equation of the stable sub-manifold, approximated to third order terms, is

$$\begin{aligned} h(x) \approx\, & x_3 + 0.09375x_1^2 - 0.1875x_1x_2 - 0.03125x_2^2 + 0.0408x_1^3 \\ & - 0.0816x_1^2x_2 - 0.0256x_1x_2^2 - 0.0032x_2^3. \end{aligned} \tag{19.43}$$

Continuing this process, we can calculate the even higher order terms of $h(x)$.

In fact, for this special system the stable sub-manifold can be obtained by a suitable coordinate transformation. Hence the above result can be confirmed.

19.5 Differential-Algebraic System

This section considers the stability region of a differential-algebraic system. Such systems exist widely. For instance, the power network is of this type. Consider the following system

$$\begin{cases} \dot{x} = f(x,y), & x \in \mathbb{R}^n,\ y \in \mathbb{R}^m, \\ \Phi(x,y) = 0, & \Phi(x,y) \in \mathbb{R}^m, \end{cases} \tag{19.44}$$

where $f(0,0) = 0$, $\Phi(0,0) = 0$. Moreover, the dynamics determined by this set of equations is unique, hence we require that

$$\text{rank}\left(\frac{\partial \Phi}{\partial y}(0,0)\right) = m. \tag{19.45}$$

Based on the aforementioned reason , we assume $(0,0)$ is type-1 unstable equilibrium. We will use the result obtained in the previous sections to deduce the equation of the stable sub-manifold. For convenience, we consider only the quadratic approximation of the (19.44). Higher order terms can be calculated in a similar way.

According to the Implicit Function Theory, (19.45) implies that y can be solved from the second equation of (19.44) as $y = y(x)$. Substituting it into the first equation of (19.44) yields an equation of the form of (19.1) as

$$\dot{x} = f(x, y(x)). \tag{19.46}$$

Of course, equation (19.46) is locally true. But the Taylor series expansion requires only local information, hence local expression is enough. Now the only obstacle is solving $y = y(x)$, which is, in general, impossible. Recall (19.15), what do we need is only

$$\begin{aligned} & J := \frac{\partial f(x,y(x))}{\partial x}, \\ & H_i := Hess(f_i(0,0)), \quad i = 1, \cdots, n. \end{aligned} \tag{19.47}$$

Hence instead of solving y, we can calculate J and H_i, and then the formula (19.15) can be used to find the quadratic approximation.

Since

$$\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{\partial y}{\partial x} = 0,$$

then

$$\frac{\partial y}{\partial x} = -\left(\frac{\partial \Phi}{\partial y}\right)^{-1}\frac{\partial \Phi}{\partial x}. \tag{19.48}$$

Using Chain Rule, we have

$$J = \frac{\partial f}{\partial x}(0,0) - \frac{\partial f}{\partial y}(0,0)\left(\frac{\partial \Phi}{\partial y}(0,0)\right)^{-1}\frac{\partial \Phi}{\partial x}(0,0). \tag{19.49}$$

Recall Corollary 15.1, Let $A(x)$ and $B(x)$ be $p \times q$ and $q \times r$ functional matrices. Then

$$DA(x)B(x) = DA(x)B(x) + A(x)DB(x). \tag{19.50}$$

Moreover, according the Chain Rule, we have

$$DA(x,y(x)) = D_x A(x,y) + D_y A(x,y)\left(I_n \otimes \frac{\partial y}{\partial x}\right). \tag{19.51}$$

Here we use D_x to express the differential with respect to x only.

Now we calculate H_i. First, the gradient of f_i can be expressed as

$$\begin{aligned}\nabla f_i(x,y(x)) &= \nabla_x f_i(x,y) + \left(d_y f_i(x,y)\tfrac{\partial y}{\partial x}\right)^{\mathrm{T}}\\ &= \nabla_x f_i(x,y) - \left(\tfrac{\partial \Phi}{\partial x}\right)^{\mathrm{T}}\left(\tfrac{\partial \Phi}{\partial y}\right)^{-\mathrm{T}}\nabla_y f_i(x,y).\end{aligned} \tag{19.52}$$

Since $y = y(x)$ is a function of x, we use $\nabla_x f_i(x,y)$ and $\nabla_y f_i(x,y)$ for the gradients with respect to x and y respectively.

Then, by definition we have

$$H_i = D(\nabla f_i)\big|_{(0,0)}, \quad i = 1,\cdots,n. \tag{19.53}$$

Applying (19.51) to the first term of (19.52), we have

$$D(\nabla_x f_i) = \frac{\partial^2 f_i}{\partial x \partial x} + \frac{\partial^2 f_i}{\partial x \partial y}\left(\frac{\partial y}{\partial x}\right). \tag{19.54}$$

Similarly, we have

$$D(\nabla_y f_i) = \frac{\partial^2 f_i}{\partial y \partial x} + \frac{\partial^2 f_i}{\partial y \partial y}\left(\frac{\partial y}{\partial x}\right). \tag{19.55}$$

Note that hereafter for any function $\xi(x,y)$, we use $\frac{\partial^2\xi(x,y)}{\partial x\partial y}$ to represent an $n\times m$ matrix, which has its (i,j)-th element as $\frac{\partial^2\xi}{\partial x_i\partial y_j}$. Hence in general,

$$\frac{\partial^2\xi(x,y)}{\partial x\partial y} \neq \frac{\partial^2\xi(x,y)}{\partial y\partial x}.$$

Applying (19.51) to the second term of (19.52), we have

$$\begin{aligned} D\left[d_y f_i(x,y)\frac{\partial y}{\partial x}\right]^{\mathrm{T}} &= D\left\{\left(\frac{\partial y}{\partial x}\right)^{\mathrm{T}}(\nabla_y f_i(x,y))\right\} \\ &= D\left(\frac{\partial y}{\partial x}\right)^{\mathrm{T}}(\nabla_y f_i(x,y)\otimes I_n)+\left(\frac{\partial y}{\partial x}\right)^{\mathrm{T}} D(\nabla_y f_i(x,y)). \end{aligned} \tag{19.56}$$

Next, we calculate (19.56) term by term. Using (19.48), we have

$$\left(\frac{\partial y}{\partial x}\right)^{\mathrm{T}}\left(\frac{\partial \Phi}{\partial y}\right)^{\mathrm{T}}+\left(\frac{\partial \Phi}{\partial x}\right)^{\mathrm{T}}=0,$$

Differentiate both sides of the above equation and applying (19.50) yield

$$D\left(\frac{\partial y}{\partial x}\right)^{\mathrm{T}}\left[\left(\frac{\partial \Phi}{\partial y}\right)^{\mathrm{T}}\otimes I_n\right]+\left(\frac{\partial y}{\partial x}\right)^{\mathrm{T}} D\left(\frac{\partial \Phi}{\partial y}\right)^{\mathrm{T}}+D\left(\frac{\partial \Phi}{\partial x}\right)^{\mathrm{T}}=0. \tag{19.57}$$

Each terms are calculated as follows:

$$\begin{aligned} X &:= D\left(\frac{\partial \Phi}{\partial x}\right)^{\mathrm{T}}\Big|_{(0,0)} \\ &= \left[\left(\frac{\partial^2\Phi_1}{\partial x\partial x}+\frac{\partial^2\Phi_1}{\partial x\partial y}\cdot\frac{\partial y}{\partial x}\right),\cdots,\left(\frac{\partial^2\Phi_m}{\partial x\partial x}+\frac{\partial^2\Phi_m}{\partial x\partial y}\cdot\frac{\partial y}{\partial x}\right)\right]\Big|_{(0,0)}. \end{aligned} \tag{19.58}$$

$$\begin{aligned} Y &:= D\left(\frac{\partial \Phi}{\partial y}\right)^{\mathrm{T}}\Big|_{(0,0)} \\ &= \left[\left(\frac{\partial^2\Phi_1}{\partial y\partial x}+\frac{\partial^2\Phi_1}{\partial y\partial y}\cdot\frac{\partial y}{\partial x}\right),\cdots,\left(\frac{\partial^2\Phi_m}{\partial y\partial x}+\frac{\partial^2\Phi_m}{\partial y\partial y}\cdot\frac{\partial y}{\partial x}\right)\right]\Big|_{(0,0)}. \end{aligned} \tag{19.59}$$

Substituting (19.58) and (19.59) into (19.57) yields

$$D\left(\frac{\partial y}{\partial x}\right)^{\mathrm{T}}\Big|_{(0,0)}=-\left[\left(\frac{\partial y}{\partial x}(0,0)\right)^{\mathrm{T}}Y+X\right]\left[\left(\frac{\partial \Phi}{\partial y}(0,0)\right)^{-\mathrm{T}}\otimes I_n\right]. \tag{19.60}$$

Finally, substituting (19.54), (19.55), and (19.60) into (19.53), we can get the expression of H_i as follows:

$$H_i = \frac{\partial^2 f_i}{\partial x \partial x} + \frac{\partial^2 f_i}{\partial x \partial y}\left(\frac{\partial y}{\partial x}\right) - \left[\left(\frac{\partial y}{\partial x}\right)^{\mathrm{T}} Y + X\right]\left[\left(\frac{\partial \Phi}{\partial y}\right)^{-\mathrm{T}} \otimes I_n\right](\nabla_y f_i \otimes I_n)$$
$$+ \left(\frac{\partial y}{\partial x}\right)^{\mathrm{T}}\left[\frac{\partial^2 f_i}{\partial y \partial x} + \frac{\partial^2 f_i}{\partial y \partial y}\left(\frac{\partial y}{\partial x}\right)\right], \tag{19.61}$$

where X, Y and the detailed expression of $\frac{\partial y}{\partial x}$ are (19.58), (19.59), and (19.48) respectively.

Exercise 19

1. (to be completed).

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