Chapter 18
Linearization of Nonlinear Control Systems

Since the dynamics of a linear system is much simpler than that of a nonlinear system, if a nonlinear (control) system can be converted into a linear system, the analysis and design tools for linear (control) systems can then be used. Hence different kinds of linearizations become an interesting topic in Physics and System and Control. In this chapter we consider some special linearizations, in which semi-tensor product players an important role.

18.1 Carleman Linearization

Consider a dynamic system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{18.1}$$

where $f(x)$ is an analytic vector field with $f(0) = 0$.

J. Carleman proposed a method to merge the system into an infinite dimensional linear system. In this section its basic form and the realization are discussed. Its original form is rather complicated. Semi-tensor product makes it much more simple.

Choosing $x, x^2, \cdots$ as a set of basis, the system (18.1) can be expressed as

$$\dot{x} = F_1 x + F_2 x^2 + F_3 x^3 + \cdots, \tag{18.2}$$

where F_1 is an $n \times n$ matrix, and F_2 is an $n \times n^2$ matrix, and so on.

We may consider $x, x^2, x^3 \cdots$ as a set of independent variables and then calculate their derivatives to get a linear form, called the Carleman linearization, as follows:

$$\begin{bmatrix} \dot{x} \\ \dot{x^2} \\ \dot{x^3} \\ \vdots \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & \cdots \\ 0 & A_{22} & A_{23} & A_{24} & \cdots \\ 0 & 0 & A_{33} & A_{34} & \cdots \\ \vdots & & & & \end{bmatrix} \begin{bmatrix} x \\ x^2 \\ x^3 \\ \vdots \end{bmatrix}. \tag{18.3}$$

Theorem 18.1. *In Carleman linearization form (18.3) the coefficients A_{ij} are determined by the following equations.*

$$\begin{cases} A_{1i} = F_i, \quad i \geq 1, \\ A_{k,k+s} = \sum_{i=0}^{k-1} I_{n^i} \otimes F_{s+1} \otimes I_{n^{k-1-i}}. \end{cases} \tag{18.4}$$

Proof. According to chair rule, we have

$$\frac{d}{dt}(x^k) = \sum_{i=0}^{k-1} x^i \dot{x} x^{k-i-1} = \sum_{s=0}^{\infty} \sum_{i=0}^{k-1} x^i F_{s+1} x^{k-i+s}.$$

Using (2.56), we have

$$x^i F_{s+1} x^{k-i+s} = (I_{n^i} \otimes F_{s+1}) \ltimes x^{k+s} = (I_{n^i} \otimes F_{s+1} \otimes I_{n^{k-i-1}}) x^{k+s}.$$

(18.4) follows. □

We can express (18.3) as the following linear form

$$\dot{X} = AX, \tag{18.5}$$

where A is an infinite dimensional block upper triangular matrix.

The infinite dimensional block upper triangular matrices have some special properties, which make (18.5) meaningful. We give a brief discussion here.

Denote the k-th left-upper block of A by A_k, i.e.,

$$A_k = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ 0 & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_{kk} \end{bmatrix}.$$

A square block upper triangular matrix is said to have a set of structure parameters $(k_1, k_2, \cdots)$, if the diagonal blocks have dimensions $\dim(A_{ii}) = k_i \times k_i$. For instance, consider (18.3), its structure parameters are $(n, n^2, n^3, \cdots)$. For statement ease, we identify A_k with its infinite-dimensional extension A_k^e, which is an infinite-dimensional matrix with A_k as its left upper minor and all other elements are zero. Using this convention, the coefficient matrix A can be expressed as

$$A = \lim_{k \to \infty} A_k.$$

This limit is well defined in the following sense: Denote the (i,j)-th element of A_k by a_{ij}^k. Then the sequence $\{a_{ij}^k,\ k=1,2,\cdots\}$ has the following form

$$(a_{ij}^1, a_{ij}^2, \cdots, a_{ij}^k, \cdots) = (0, \cdots, 0, c_{ij}, c_{ij}, c_{ij}, \cdots),$$

That is, after first finite terms, it becomes a constant sequence. Hence this sequence converges. Based on the same reason, the following operations are also well defined.

Definition 18.1. 1. Let A and B be two infinite dimensional upper triangular matrices with same structure matrices. The product of A and B is defined as

$$AB := \lim_{k\to\infty} A_k B_k.$$

2. Assume $A_{ii},\ i=1,2,\cdots$ are invertible. Then its inverse is defined as

$$A^{-1} := \lim_{k\to\infty} A_k^{-1}.$$

3.

$$e^A := \lim_{k\to\infty} e^{A_k}.$$

Now it is natural to use the solution of the linearized system (18.5)

$$X = e^{At} X_0$$

as a solution of (18.1). In fact, we can use only finite terms to approximate the real solution.

Denote by $E_{ij}^k(t)$ the (i,j)-th block of $e^{A_k t}$. It follows from the structure of A that

$$E_{ij}^s(t) = E_{ij}^k(t), \quad s > k, \quad i,j \le k.$$

Hence we can define

$$X^n(t) = \sum_{k=1}^n E_{1k}^k(t) X_0^k.$$

Observing (18.3), one sees easily that if

$$X(t) = \lim_{n\to\infty} X^n(t)$$

exists, then it is the solution of (18.1) with initial value $X(0) = X_0$.

We are particularly interested in the upper triangular matrix generated by Carleman linearization. In Carleman linearized form (18.3). We assume $F_1 = A_{11}$ is stable (anti-stable), that is, all the eigenvalues of A_{11} have negative real part $\mathrm{Re}\,\sigma(A_{11}) < 0$ (correspondingly, $\mathrm{Re}\,\sigma(A_{11}) > 0$), then A is invertible. In fact, we have

Theorem 18.2. *Assume $F_1 = A_{11}$ has eigenvalues $\sigma(A_{11}) = \{\lambda_1, \cdots, \lambda_n\}$, then $A_{ii},\ i \ge 2$ has eigenvalues*

$$\sigma(A_{ii}) = \{\lambda_{k_1} + \cdots + \lambda_{k_i} \,|\, k_1, \cdots, k_i = 1, \cdots, n\}.$$

Proof. First, we assume the eigenvalues of A_{11} are distinct, and their corresponding eigenvectors are

$$\{\xi_1, \cdots, \xi_n\}.$$

A straightforward computation shows

$$A_{ii}(\xi_{k_1} \ltimes \cdots \ltimes \xi_{k_i}) = (\lambda_{k_1} + \cdots + \lambda_{k_i})(\xi_{k_1} \ltimes \cdots \ltimes \xi_{k_i}).$$

To avoid notational confusion, we consider it only for the case of $i = 2$. Using (18.4), we have

$$A_{22} = I_n \otimes A_{11} + A_{11} \otimes I_n.$$

Hence,

$$\begin{aligned} A_{22}(\xi_i \ltimes \xi_j) &= (I_n \otimes A_{11} + A_{11} \otimes I_n)(\xi_i \ltimes \xi_j) \\ &= (I_n \otimes A_{11})(\xi_i \ltimes \xi_j) + (A_{11} \otimes I_n)(\xi_i \ltimes \xi_j) \\ &= \lambda_j \xi_i \ltimes \xi_j + \lambda_i \xi_i \ltimes \xi_j. \end{aligned}$$

It follows that $\lambda_i + \lambda_j$, $i, j = 1, \cdots, n$ are eigenvalues of A_{22}. To prove that they are the complete set of eigenvalues it is enough to prove that

$$\{\xi_i \ltimes \xi_j \mid i, j = 1, \cdots, n\}$$

are linearly independent. Note that since all λ_i, $i = 1, \cdots, n$, are distinct, it follows that all ξ_i $i = 1, \cdots, n$, are linearly independent. Assume

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_{i,j} \xi_i \xi_j = 0.$$

Rewrite it as

$$\sum_{i=1}^{n} \xi_i [\sum_{j=1}^{n} c_{ij} \xi_j] = 0.$$

Using Proposition 2.14, we have

$$\sum_{j=1}^{n} c_{ij} \xi_j = 0, \quad i = 1, \cdots, n.$$

Hence, it is clear that $c_{i,j} = 0$, $i = 1, \cdots, n$, $j = 1, \cdots, n$.

We conclude that the eigenvalues of A_{22} are

$$\sigma(A_{22}) = \{\lambda_i + \lambda_j \mid i, j = 1, \cdots, n\}. \tag{18.6}$$

Finally, by continuity we know that even multi-eigenvalues exist, the structure of eigenvalues of A_{22}, precisely, (18.6) remains true. □

18.2 First Integral

This section considers the first integral of a vector field.

Definition 18.2. Let $f(x)$, $x \in \mathbb{R}^n$, be a smooth vector field. A smooth time-varying function $\phi(t,x)$ is said to be the first integral of $f(x)$ if it satisfies

$$\frac{d}{dt}\phi(t,x) = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} f(x) = 0.$$

Consider a polynomial system

$$\dot{x} = F_1 x + F_2 x^2 + \cdots + F_k x^k. \tag{18.7}$$

We search for the following type of first integral

$$H(t,x) = e^{-\xi t} P(x).$$

Carleman linearization technique can be used to investigate it.

Assume $P(x) = P_0 + P_1 x + \cdots + P_s x^s$, with its symmetric coefficients as $P_1, \cdots, P_s$. (Where the "symmetry" means the coefficients for same items with different factor orders are the same. For instance, $x_1^2 x_2$, $x_1 x_2 x_1$, and $x_2 x_1^2$ have the same coefficients.) It is easy to see that if $\xi \neq 0$ then $P_0 = 0$. Hence, we can simply assume $P_0 = 0$.

Setting $dH(t,x)/dt = 0$, we have

$$\begin{aligned} &\begin{bmatrix} P_1 & \cdots & P_s \end{bmatrix} \begin{bmatrix} A_{11} & \cdots & \cdots & A_{1k} & & \\ & \ddots & & & \ddots & \\ & & A_{ss} & \cdots & \cdots & A_{s,s+k-1} \end{bmatrix} \begin{bmatrix} x \\ x^2 \\ \vdots \\ x^{s+k-1} \end{bmatrix} \\ &= \xi \begin{bmatrix} P_1 & \cdots & P_s \end{bmatrix} \begin{bmatrix} x \\ x^2 \\ \vdots \\ x^s \end{bmatrix}. \end{aligned} \tag{18.8}$$

Since x^k is redundant, the coefficients are not unique. Using this form to search first integral is conservative, and the result obtained is, in general, not necessity. Because under other equivalent coefficients may produce other first integrals.

To get necessary and sufficient condition we convert it into natural basis. Set

$$P_i = \tilde{P}_i T_B(n,i), \quad \tilde{A}_{ij} = T_B(n,i) A_{ij} T_N(n,j).$$

Plugging it into (18.8) yields

$$\begin{bmatrix}\tilde{P}_1 \cdots \tilde{P}_s\end{bmatrix}\begin{bmatrix}\tilde{A}_{11} & \cdots & \cdots & \tilde{A}_{1k} & & \\ & \ddots & & & \ddots & \\ & & \tilde{A}_{ss} & \cdots & \cdots & \tilde{A}_{s,s+k-1}\end{bmatrix}\begin{bmatrix}x\\ x_{(2)}\\ \vdots \\ x_{(s+r-1)}\end{bmatrix}$$
$$= \xi \begin{bmatrix}\tilde{P}_1 \cdots \tilde{P}_s\end{bmatrix}\begin{bmatrix}x\\ x_{(2)}\\ \vdots \\ x_{(s)}\end{bmatrix}.$$

Theorem 18.3. *Denote $h_i = \tilde{P}_i^{\mathrm{T}}$ and $B_{ij} = \tilde{A}_{ji}^{\mathrm{T}}$. System (18.7) has first integral $H(t,x)$, if and only if there exists ξ such that the following system has non-zero solution $(h_1,\cdots,h_s)$.*

$$\begin{cases}\begin{bmatrix}B_{11} & 0 & 0\cdots & 0\\ B_{21} & B_{22} & 0\cdots & 0\\ \vdots & & & \\ B_{s1} & B_{s2} & \cdots & B_{s,s}\end{bmatrix}\begin{bmatrix}h_1\\ h_2\\ \vdots\\ h_s\end{bmatrix} = \xi \begin{bmatrix}h_1\\ h_2\\ \vdots\\ h_s\end{bmatrix},\\ \\ \begin{bmatrix}B_{s+1,1} & B_{s+1,2} & \cdots & B_{s+1,s}\\ \vdots & & & \\ B_{k,1} & B_{k,2} & \cdots & B_{k,s}\\ 0 & B_{k+1,2} & \cdots & B_{k+1,s}\\ \vdots & & & \\ 0 & \cdots & 0 & B_{s+k-1,s}\end{bmatrix}\begin{bmatrix}h_1\\ h_2\\ \vdots\\ h_s\end{bmatrix} = 0.\end{cases} \tag{18.9}$$

Next, we consider the solution of (18.9).

The following lemma is itself interesting. Let $I = (i_1,\cdots,i_k)$. We use notation $\mathrm{id}(I;n^k)$ for $\mathrm{id}(i_1,\cdots,i_k;n,\cdots,n)$.

Lemma 18.1. *Assume the row vector $h \in \mathbb{R}^{n^k}$ is labeled by multi-index $Id(I;n^k)$, and it is symmetric with respect to $\mathrm{id}(I;n^k)$, $F \in M_{n\times n}$. Define*

$$A = F\otimes I_{n^{k-1}} + I_n \otimes F \otimes I_{n^{k-2}} + \cdots + I_{n^{k-1}} \otimes F.$$

Then hA is also symmetric with respect to $Id(I;n^k)$.

Proof. Since interchange any two indices can be realized by swapping two adjacent indices, we need only to prove that hA is invariant under the swap of two adjacent indices. Define

$$\Phi = I_{n^{j-1}} \otimes W_{[n]} \otimes I_{n^{k-j-1}}.$$

It is clear that exchanging the j-th and $(j+1)$-th indices yields a new vector $h\Phi$. Since h is symmetric with respect to $\mathrm{id}(I;n^k)$, then $h\Phi = h$, and $h\Phi A = hA$. To prove

hA is symmetric, i.e., $hA\Phi = hA$, it suffices to prove

$$A\Phi = \Phi A.$$

Note that the terms of A has the following form

$$I \otimes \cdots \otimes I \otimes F \otimes I \otimes \cdots \otimes I. \tag{18.10}$$

If F does not lie on the j-th or $j+1$-th position, then it is obvious that Φ and (18.10) are commutative. hence we need only to consider the two related terms of A. This means

$$W_{[n]}(F \otimes I + I \otimes F) = (F \otimes I + I \otimes F)W_{[n]}.$$

Note that $W_{[n]}^{-1} = W_{[n]}$, the above equality is true. □

Proposition 18.1. *Assume the F_1 in (18.7) has eigenvalues $\sigma = \{\lambda_1, \cdots, \lambda_n\}$, then the eigenvalues of B_{kk} in (18.9) are*

$$\sigma(B_{kk}) = \{\lambda_{i_1} + \cdots + \lambda_{i_k} \,|\, i_1, \cdots, i_k = 1, \cdots, n\}.$$

Proof. Since $B_{kk} = \tilde{A}_{kk}^{\mathrm{T}}$, and the eigenvalues of A_{kk} are σ_k, it is enough to prove that the $\tilde{A}_{kk}$ and A_{kk} have the same eigenvalues. Assume μ is an eigenvalue of $\tilde{A}_{kk}$, then there exists a $\tilde{P} \neq 0$ such that

$$\tilde{P}\tilde{A}_{kk} = \mu \tilde{P}.$$

By definition we have $\tilde{A}_{kk} = T_B(n,k)A_{kk}T_N(n,k)$. Note that $T_B(n,k)T_N(n,k) = I$, hence

$$\tilde{P}T_B(n,k)A_{kk}T_N(n,k)\begin{bmatrix} x_{(1)} \\ x_{(2)} \\ \vdots \\ x_{(k)} \end{bmatrix} = \mu \tilde{P}T_B(n,k)T_N(n,k)\begin{bmatrix} x_{(1)} \\ x_{(2)} \\ \vdots \\ x_{(k)} \end{bmatrix}. \tag{18.11}$$

Let $P = \tilde{P}T_B(n,k)$. Then $P \neq 0$ is a symmetric set. For P, (18.11) becomes

$$PA_{kk}\begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^k \end{bmatrix} = \mu P \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^k \end{bmatrix}. \tag{18.12}$$

According to Lemma 18.1, PA_{kk} is a symmetric set. Note that the symmetric coefficients are unique, we have

$$PA_{kk} = \mu P.$$

Hence, μ is also an eigenvalue of A_{kk}.

Conversely, assume μ is an eigenvalue of A_{kk}, then $\mu = \lambda_{i_1} + \cdots + \lambda_{i_k}$. Denote by Y_j the eigenvector corresponding to λ_{i_j} of F_1, then we construct

$$Y = \sum_{\sigma \in \mathbf{S}_k} Y_{\sigma(1)} \otimes \cdots \otimes Y_{\sigma(k)},$$

where $\mathbf{S}_k$ is the k-th order symmetric group. Then we have

$$YA_{kk} = \mu Y.$$

Since Y is symmetric, there exists $\tilde{Y} \neq 0$ such that $Y = \tilde{Y} T_B(n,k)$. Hence, we have

$$\tilde{Y} T_B(n,k) A_{kk} = \mu \tilde{Y} T_B(n,k).$$

Right-multiplying both sides of the above equality by $T_N(n,k)$ yields

$$\tilde{Y} \tilde{A}_{kk} = \mu \tilde{Y}.$$

That is, μ is also an eigenvalue of $\tilde{A}_{kk}$. □

Proposition 18.2. *(1) If (18.9) has solution $h \neq 0$, then*

$$\xi = c_1 \lambda_{i_1} + \cdots + c_s \lambda_{i_s},$$

where $\lambda_{i_1}, \cdots, \lambda_{i_s} \in \sigma(F_1)$; $c_1, \cdots, c_s$ take value 1 or 0.

(2) If h has a component $h_j \neq 0$, then $\xi \in \sigma^j$. If h has t non-zero components, then σ^s has at least one t fold element, where $\sigma^t = \{c_1 \lambda_{i_1} + \cdots + c_t \lambda_{i_t} \,|\, c_1, \cdots, c_t \in \{0,1\}\}$.

(3) If (18.7) has a linear first integral $H(t,x) = e^{-\xi t} h^{\mathrm{T}} x$, then for arbitrary integer $j > 0$, $H_j(t,x) = e^{-j\xi t} (h^{\mathrm{T}})^j x^j$ is a first integral is (18.7).

Proof. (1) and (2) are the immediate consequences of Proposition 18.1. We prove (3) If (18.7) has a linear first integral $H(t,x) = e^{-\xi t} h^{\mathrm{T}} X$, then

$$\begin{cases} F_1 h = \xi h, \\ F_i h = 0, \quad i = 2, \cdots, k. \end{cases}$$

Assume $p = (0_n, 0_{n^2}, \cdots, 0_{n^{j-1}}, h^j)$, where 0_k is the zero vector in $\mathbb{R}^k$. Since

$$A_{j,j+s-1} = I_{n^{j-1}} \otimes F_s + I_{n^{j-2}} \otimes F_s \otimes I + \cdots + F_s \otimes I_{n^{j-1}},$$

we have

$$\begin{cases} A_{jj} h^j = j \xi h^j, \\ A_{jt} h^j = 0, \quad t = j+1, \cdots, j+k-1, \end{cases}$$

which means p satisfies (18.8) with ξ being replaced by $j\xi$. □

Remark 18.1. Proposition 18.2 provides a convenient way to find first integer. In fact, after fix ξ the problem becomes solving linear algebraic system. For Lorenz system, the (1) and (2) of Proposition 18.2 have been proved in [2]. Here the statement is a generalization.

Example 18.1. Lotka-Volterra equation established the interactive relation in Chemistry or for co-existence of spices. Lotka-Volterra equation is

$$\dot{x}_i = x_i\left(a_i + \sum_{j=1}^{n} b_{ij}x_j\right), \quad i = 1, \cdots, n. \tag{18.13}$$

Consider $n = 2$. Set

$$A_{11} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} b_{11} & b_{12} & 0 & 0 \\ 0 & 0 & b_{21} & b_{22} \end{bmatrix}.$$

Then

$$\begin{aligned} A_{22} &= A_{11} \otimes I_2 + I_2 \otimes A_{11} = \begin{bmatrix} 2a_1 & 0 & 0 & 0 \\ 0 & a_1 + a_2 & 0 & 0 \\ 0 & 0 & a_1 + a_2 & 0 \\ 0 & 0 & 0 & 2a_2 \end{bmatrix}, \\ A_{23} &= A_{12} \otimes I_2 + I_2 \otimes A_{12} \\ &= \begin{bmatrix} 2b_{11} & b_{12} & b_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{11} & b_{21} & b_{12} + b_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{11} + b_{21} & b_{12} & b_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{21} & b_{21} & 2b_{22} \end{bmatrix}. \end{aligned}$$

Carleman linearized form becomes

$$\begin{bmatrix} \dot{x} \\ \dot{x}^2 \\ \dot{x}^3 \\ \vdots \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & \cdots \\ 0 & A_{22} & A_{23} & 0 & \cdots \\ 0 & 0 & A_{33} & A_{34} & \cdots \\ \vdots & & & & \end{bmatrix} \begin{bmatrix} x \\ x^2 \\ x^3 \\ \vdots \end{bmatrix}.$$

Assume we look for the first integral with the form as $H(t,x) = e^{-\xi t}(P_1 x + P_2 x^2)$. Then

$$T_B(2,2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_N(2,2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$T_N(2,3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B_{11} = A_{11}^{\mathrm{T}}, \quad B_{21} = \tilde{A}_{12}^{\mathrm{T}} = T_B(2,1)A_{12}T_N(2,2) = \begin{bmatrix} b_{11} & b_{12} & 0 \\ 0 & b_{21} & b_{22} \end{bmatrix},$$

$$B_{22}^{\mathrm{T}} = T_B(2,2)A_{22}T_N(2,2) = \begin{bmatrix} 2a_1 & 0 & 0 \\ 0 & a_1+a_2 & 0 \\ 0 & 0 & 2a_2 \end{bmatrix},$$

$$B_{32}^{\mathrm{T}} = T_B(2,2)A_{23}T_N(2,3) = \begin{bmatrix} 2b_1 & 2b_{12} & 0 & 0 \\ 0 & b_{11}+b_{21} & b_{12}+b_{22} & 0 \\ 0 & 0 & 2b_{21} & 2b_{22} \end{bmatrix}.$$

We conclude that the second order first integral exists, if and only if the following equation has solution.

$$\begin{cases} \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \xi \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}, \\ B_{32}h_2 = 0, \end{cases}$$

where $\xi \in \{a_1, a_2, 2a_1, 2a_2, a_1+a_2\}$.

18.3 Invariance of Polynomial System

Definition 18.3. Consider a smooth dynamic system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n. \tag{18.14}$$

A smooth time-varying function $h(t,x)$ is said to be an invariance of (18.14) if

$$\left.\frac{dh}{dt}\right|_{(18.14)} = 0. \tag{18.15}$$

In this section we consider a polynomial system

$$\dot{x} = F_0 + F_1 x + \cdots + F_k x^k, \quad x \in \mathbb{R}^2. \tag{18.16}$$

We look for an invariance with the following form:

$$H(x,t) = e^{\xi t} x_1^{\alpha} x_2^{\beta} (P_0 + P_1 x + \cdots + P_l x^l). \tag{18.17}$$

This form of invariance has been investigated in [2]. Recently, Darboux method has also been used for this investigation [3].

Our purpose is the convert it into an algebraic system. Using formula 15.74, the time derivative of $H(x,t)$ can be calculated as

$$\begin{aligned}\frac{dH}{dt} &= \frac{\partial H}{\partial t} + DH \cdot \dot{x}\\ &= \xi e^{\xi t} x_1^{\alpha} x_2^{\beta} (P_0 + P_1 x + \cdots + P_l x^l)\\ &\quad + e^{\xi t} (\alpha x_1^{\alpha-1} x_2^{\beta},\ \beta x_1^{\alpha} x_2^{\beta-1}) (P_0 + P_1 x + \cdots + P_l x^l)(F_0 + F_1 x + \cdots + F_k x^k)\\ &\quad + e^{\xi t} x_1^{\alpha} x_2^{\beta} (P_1 + P_2 \Phi_1 x + \cdots + P_l \Phi_{l-1} x^{l-1})(F_0 + F_1 x + \cdots + F_k x^k).\end{aligned}$$

Setting

$$\frac{dH}{dt} = 0,$$

and observing that

$$x_1 x_2 = (0\ 1\ 0\ 0) x^2, \quad (\alpha x_2,\ \beta x_1) = (0\ \beta\ \alpha\ 0) x,$$

we have

$$\begin{aligned}&\xi (0\ 1\ 0\ 0) x^2 (P_0 + P_1 x + \cdots + P_l x^l)\\ &\quad + (0\ \beta\ \alpha\ 0) x (P_0 + P_1 x + \cdots + P_l x^l)(F_0 + F_1 x + \cdots + F_k x^k)\\ &\quad + (0\ 1\ 0\ 0) x^2 (P_1 + P_2 \Phi_1 x + \cdots + P_l \Phi_{l-1} x^{l-1})(F_0 + F_1 x + \cdots + F_k x^k)\\ &= 0.\end{aligned}$$

Using (2.56), it can be expressed as

$$\begin{aligned}&\xi (0\ 1\ 0\ 0)[(I_4 \otimes P_0) x^2 + (I_4 \otimes P_1) x^3 + \cdots + (I_4 \otimes P_l) x^{l+2}]\\ &\quad + (0\ \beta\ \alpha\ 0)[(I_2 \otimes P_0) x + (I_2 \otimes P_1) x^2\\ &\quad + \cdots + (I_2 \otimes P_l) x^{l+1}](F_0 + F_1 x + \cdots + F_k x^k)\\ &\quad + (0\ 1\ 0\ 0)[(I_4 \otimes P_1) x^2 + (I_4 \otimes P_2 \Phi_1) x^3\\ &\quad + \cdots + (I_4 \otimes P_l \Phi_{l-1}) x^{l+1})(F_0 + F_1 x + \cdots + F_k x^k)\\ &= 0.\end{aligned}$$

Using (2.56) again, we can multiply the above form out as

$$\begin{aligned}&\sum_{s=1}^{l+1} \xi (0\ 1\ 0\ 0)[I_4 \otimes P_{s-1}] x^{s+1}\\ &\quad + \sum_{s=0}^{k+l} (0\ \beta\ \alpha\ 0) \sum_{i=0,j=0}^{i+j=s} [(I_2 \otimes P_i)(I_{2^{i+1}} \otimes F_{s-i})] x^{s+1}\\ &\quad + \sum_{s=1}^{k+l} (0\ 1\ 0\ 0) \sum_{i=1,j=0}^{i+j=s} [(I_4 \otimes P_i \Phi_{i-1})(I_{2^{i+1}} \otimes F_{s-i})] x^{s+1}\\ &= 0.\end{aligned}$$

Converting each terms into the forms with naturel basis, we have the following result.

Theorem 18.4. *System (18.16) has the invariance of the form (18.17), if and only if the following algebraic system has solution* $(\xi, \alpha, \beta, P_0, \cdots, P_l)$.

$$
\begin{aligned}
&(0\ \beta\ \alpha\ 0)[(I_2 \otimes P_0)(I_2 \otimes F_0)] = 0\\
&\Big\{\xi(0\ 1\ 0\ 0)(I_4 \otimes P_{s-1}) + (0\ \beta\ \alpha\ 0)\sum_{i=0,j=0}^{i+j=s}[(I_2 \otimes P_i)(I_{2^{i+1}} \otimes F_{s-i})]\\
&\quad + (0\ 1\ 0\ 0)\sum_{i=1,j=0}^{i+j=s}[(I_4 \otimes P_i \Phi_{i-1})(I_{2^{i+1}} \otimes F_{s-i})]\Big\} T_N(2, s+1) = 0,\\
&\qquad s = 1, \cdots, l+1 \\
&\Big\{(0\ \beta\ \alpha\ 0)\sum_{i=0,j=0}^{i+j=s}[(I_2 \otimes P_i)(I_{2^{i+1}} \otimes F_{s-i})]\\
&\quad + (0\ 1\ 0\ 0)\sum_{i=1,j=0}^{i+j=s}[(I_4 \otimes P_i \Phi_{i-1})(I_{2^{i+1}} \otimes F_{s-i})]\Big\} T_N(2, s+1) = 0,\\
&\qquad s = l+2, \cdots, l+k.
\end{aligned}
\tag{18.18}
$$

Remark 18.2. The advantage of this approach lies on

(i) the solution is easily solvable via computer;
(ii) it can easily be generated into higher dimensional cases.

18.4 Feedback Linearization of Nonlinear Control System

Consider an affine nonlinear system

$$
\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \quad f(0) = 0, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, \tag{18.19}
$$

where $f(x)$, $g_i(x)$, $i = 1, \cdots, m$ are analytic vector fields. The feedback linearization is defined as following.

Definition 18.4. (18.19) is locally state feedback linearizable, if there exists a state feedback control

$$
u = \alpha(x) + \beta(x)v, \tag{18.20}
$$

and a local diffeomorphism $z = \xi(x)$ on a neighborhood U of the origin, such that the closed-loop system under the new coordinate frame becomes a controllable linear system

$$
\dot{z} = Az + \sum_{i=1}^{m} b_i v_i, \quad z \in U, v \in \mathbb{R}^m. \tag{18.21}
$$

If in control (18.20) $\beta(x)$ is an $m\times m$ nonsingular matrix, the linearization is called a regular feedback linearization; otherwise, say, $\beta(x)$ is an $m\times k$ matrix ($k<m$), it is called a non regular feedback linearization. Particularly, when $k=1$ it is called the single input feedback linearization.

Heymann's Lemma [7] says that for a completely controllable linear system there exists a linear feedback

$$u = Hv, \quad v\in\mathbb{R}$$

such that the closed-loop system becomes a single-input complete controllable system. The following lemma is an immediate consequence of the Heymann Lemma.

Lemma 18.2. *System (18.19) is state feedback linearizable, if and only if it is single-input feedback linearizable. That is it is linearizable via (18.20), where $\beta(x)$ is an $m\times 1$ vector.*

The following lemma is useful and easily verifiable.

Lemma 18.3 ([9]). *Set $A=J_f(0)$, which is the Jacobi matrix of f at the origin, and $B=g(0)$. If the system (18.19) is linearizable, then (A,B) is completely controllable.*

In the following investigation we need one more concept, called the normal form, which has been used to investigate many nonlinear (control) systems [8, 4, 5]. We first introduce it briefly [6]: Let H_n^k be the set of k-th order homogeneous polynomial vector fields. Then

1. H_n^k is an $\mathbb{R}$-linear vector space;
2. Let $Ax\in H_n^1$ be a given linear vector field, where A is an $n\times n$ constant matrix. Then the derivative $\mathrm{ad}_{Ax}: H_n^k\to H_n^k$ is a linear mapping.

The following normal form expression [1] and its application in linearization [5] are the starting point of our study.

Definition 18.5. Assume $A\in\mathscr{M}_{n\times n}$, and $\sigma(A)=\lambda=(\lambda_1,\cdots,\lambda_n)$ is the set of int eigenvalues. A is an resonant matrix, , if there exist $m=(m_1\cdots,m_n)\in\mathbb{Z}_+^n$ and $|m|\geq 2$, i.e., $m_i\geq 0$ and $\sum_{i=1}^n m_i\geq 2$, such that for an $1\leq s\leq n$, $\lambda_s=\langle m,\lambda\rangle$. Otherwise, A is called an non-resonant matrix.

Theorem 18.5 (Poincaré Theorem [1]). *Consider an analytic system*

$$\dot{x}=Ax+f_2(x)+f_3(x)+\cdots, \quad x\in\mathbb{R}^n, \tag{18.22}$$

where $f_i(x)$, $i\geq 2$ are i-th order homogeneous polynomial vector fields. If A is non-resonant, then there exists a coordinate transformation

$$x=y+h(y), \tag{18.23}$$

where $h(y)=h_2(y)+h_3(y)+\cdots$ with $h_i(y)$ the i-th homogeneous vectors, such that system (18.22) can be expressed as $\dot{y}=Ay$.

The following lemma gives a sufficient condition for the non-resonant matrix.

Proposition 18.3 ([5]). *Let $\lambda = (\lambda_1, \cdots, \lambda_n)$ be the set of eigenvalues of a Hurwitz matrix A. A is non-resonant, if*

$$\max\{|\mathrm{Re}(\lambda_i)|\,|\,\lambda_i \in \sigma(A)\} \le 2\min\{|\mathrm{Re}(\lambda_i)|\,|\,\lambda_i \in \sigma(A)\}. \tag{18.24}$$

18.5 Single Input Feedback Linearization

First, we give a normal form for non-regular feedback linearization.

A constant vector $b = (b_1, \cdots, b_n)^{\mathrm{T}} \in \mathbb{R}^n$ is called a component non-zero vector, if $b_i \neq 0$, $\forall i$.

Proposition 18.4. *A linear control system*

$$\dot{x} = Ax + \sum_{i=1}^{m} b_i u_i := Ax + Bu, \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \tag{18.25}$$

is completely controllable, if and only if there exist two matrices F, G such that the closed-loop system

$$\dot{x} = (A + BF)x + BGv$$

can be converted, via a linear coordinate transformation, into the following form

$$\dot{z} = Az + bv := \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} z + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} v, \tag{18.26}$$

where d_i, $i = 1, \cdots, n$ are distinct, and b is a non-zero component vector.

The proof is simple. The key is for such a system the controllability matrix C has its determinant as

$$\det(C) = \prod_{i=1}^{n} b_i \prod_{i<j} (d_j - d_i) \neq 0. \tag{18.27}$$

It is essentially a Vandermonde matrix. Because of Proposition 18.4 we call (18.26) the non-regular single input feedback A-diagonal (NRSIFAD) normal form. Moreover, we give the following assumption:

A1. A is diagonal with distinct diagonal elements, d_i, $i = 1, \cdots, n$, and A is non-resonant.

Lemma 18.4. *Assume A satisfies A1, g is a k-th order homogeneous polynomial vector fields, $k \ge 2$. Then there exists a k-th order homogeneous vector field η such that*

$$\mathrm{ad}_{Ax}\eta = g. \tag{18.28}$$

Proof. For a given η, assume $f = ad_{Ax}\eta$. Then a straightforward computation shows that the i-th component f_i of f depends only on the i-th component η_i of η. Now assume $x_1^{r_1}\cdots x_n^{r_n}$ is a term of η_i, computation shows that

$$\mathrm{ad}_{Ax}\eta = \left[\begin{pmatrix} d_1x_1 \\ \vdots \\ d_ix_i \\ \vdots \\ d_nx_n \end{pmatrix}, \begin{pmatrix} \times \\ \vdots \\ x_1^{r_1}\cdot x_n^{r_n} \\ \vdots \\ \times \end{pmatrix}\right] = \begin{bmatrix} \times \\ \vdots \\ \mu_i x_1^{r_1}\cdot x_n^{r_n} \\ \vdots \\ \times \end{bmatrix}, \tag{18.29}$$

where

$$\mu_i = d_1r_1 + \cdots + d_nr_n - d_i, \tag{18.30}$$

Since A in non-resonant and $\mu_i \neq 0$, then for every term of g_i, say, $x_1^{r_1}\cdots x_n^{r_n}$, we can construct a corresponding term of η_i, say, $\frac{1}{\mu_i}x_1^{r_1}\cdots x_n^{r_n}$, such that $\mathrm{ad}_{Ax}\eta = g$. □

Note that since all the vector fields and functions involved are analytic, then all the functions and their derivatives concerned have convergent Taylor expansions.

Note also that if A satisfies A1, then for vector field $g = g_kx^k + g_{k+1}x^{k+1} + \cdots \in O(\|x\|^k)$, we can apply Lemma 18.4 to its each components, then we can construct a vector field $\eta \in O(\|x\|^k)$, such that $\mathrm{ad}_{Ax}\eta = g$.

Go back to the linearization problem. We consider the following system:

$$\dot{x} = Ax + \xi(x) + \sum_{i=1}^{m} g_i(x)u_i, \tag{18.31}$$

where A satisfies A1, and $\xi(x) = O(\|x\|^2)$. An immediate conclusion from the above argument is

Proposition 18.5. *Consider system (18.31). It is non-regular feedback linearizable, if*

1. *$\xi(x) \in \mathrm{Span}\{g_1,\cdots,g_m\}$;*
2. *there exists a non-zero component vector b, such that*

$$b \in \mathrm{Span}\{g_1,\cdots,g_m\}.$$

When one condition of Proposition 18.5 is not satisfied, we may use the normal form to investigate the linearization problem directly.

According to Lemma 18.4, we can always find a vector field $\eta(x)$, such that

$$\mathrm{ad}_{Ax}\eta(x) = \xi(x). \tag{18.32}$$

Define a local diffeomorphism $z_1 = x - \eta(x)$. Then on coordinate chart z_1 the system (18.31) can be expressed as

$$\dot{z}_1 = Az_1 - J_0(x)\xi(x) + \sum_{i=1}^{m} g_i^1(x)u_i, \tag{18.33}$$

where $J_0(x)$ is the Jacobi matrix of $\eta(x)$. Moreover, $g_i^1(x) = (I - J_0(x))g_i(x)$.

For notational convenience, we denote by $x := z_0$, $\xi(x) := \xi_0(x)$, $\eta(x) := \eta_0(x)$, $g_i(x) := g_i^0(x)$. Hence, we can continue the previous procedure to define new coordinate chart.

$$\mathrm{ad}_{Ax}(\eta_k) = \xi_k, \quad z_{k+1} = z_k - \eta_k(x), \quad k \geq 0,$$

and a new vector field

$$g_i^{k+1}(x) = (I - J_k(x))\, g_i^k(x), \quad 1 \leq i \leq m, \quad k \geq 0,$$

where $J_k(x)$ is the Jacobi matrix of $\eta_k(x)$. Using it, one sees easily that under the coordinate chart z_k the system can be expressed as

$$\dot{z}_k = Az_k + \xi_k(x) + \sum_{i=1}^{m} g_i^k(x)u_i, \quad k \geq 1. \tag{18.34}$$

Summarizing the above argument, we have

Corollary 18.1. *System (18.31) is non-regular state feedback linearizable, if there exists $k \geq 0$, such that (18.34) verify the conditions (1) and (2) of Proposition 18.5.*

From the recursive calculation one sees that

$$deg(\xi_i) = c_{i+1} + 1, \quad i = 0, 1, \cdots,$$

where $\{c_i\}$ is the Fibonacci series, i.e., $(c_1, c_2, \cdots) = (1, 1, 2, 3, 5, 8, \cdots)$. Hence when $k \to \infty$ we have $\xi_k(x) \to 0$, because we assume it converges. Hence, we have

Corollary 18.2. *System (18.31) is non-regular state feedback linearizable, if there exist a non-zero component of constant vector b, such that*

$$b \in \mathrm{Span}\left\{ \prod_{i=0}^{\infty} (I - J_i(x))g_j(x),\ j = 1, \cdots, m \right\}.$$

18.6 Numerical Realization of Non-regular Feedback Linearization

This section first provides a formula to realize the Poincaré coordinate transformation (18.23), then the necessary and sufficient conditions will be given for (approximate) linearization.

To begin with, applying Taylor expansion to $f(x)$, system (18.22) can be expressed as

$$\dot{x} = Ax + F_2 x^2 + F_3 x^3 + \cdots, \tag{18.35}$$

where F_k are $n \times n^k$ constant matrices.

Next, we assume

$$\mathrm{ad}_{Ax}\, \eta_k = F_k x^k.$$

Using Lemma 18.4, we can obtain that

$$\eta_k = (\Gamma_k^n \odot F_k)\, x^k, \quad x \in \mathbb{R}^n. \tag{18.36}$$

Here $\odot$ is the Hadamard product. (We refer to Chapter 1 for it.) According to (18.30), Γ_k^n can be constructed as

$$(\Gamma_k^n)_{ij} = \frac{1}{\left(\sum\limits_{s=1}^{n} \alpha_s^j \lambda_s\right) - \lambda_i}, \quad i = 1, \cdots, n;\ j = 1, \cdots, n^k. \tag{18.37}$$

where $\alpha_1^j, \cdots, \alpha_n^j$ are the powers of $x_1, \cdots, x_n$ respectively in the j-th component of x^k.

Then we have the following main result.

Theorem 18.6. *Assume A satisfies A1, then the system (18.35) can be transformed via the coordinate transformation*

$$z = x - \sum_{i=2}^{\infty} E_i x^i \tag{18.38}$$

to the linear form

$$\dot{z} = Az, \tag{18.39}$$

where E_i can be determined by the following recursive formula.

$$\begin{aligned} E_2 &= \Gamma_2 \odot F_2, \\ E_s &= \Gamma_s \odot \left(F_s - \sum_{i=2}^{s-1} E_i \Phi_{i-1} (I_{n^{i-1}} \otimes F_{s+1-i}) \right),\ s \geq 3. \end{aligned} \tag{18.40}$$

(In the formula Φ_i is defined in (15.75).)

Proof. Applying (18.38) to the system yields

$$
\begin{aligned}
\dot{z} &= \left(Ax + \sum_{i=2}^{\infty} F_i x^i\right) - \sum_{i=2}^{\infty} \frac{\partial E_i x^i}{\partial x}\left(Ax + \sum_{i=2}^{\infty} F_i x^i\right) \\
&= Az + \sum_{i=2}^{\infty} F_i x^i + A\sum_{i=2}^{\infty} E_i x^i - \sum_{i=2}^{\infty} \frac{\partial E_i x^i}{\partial x} Ax \\
&\quad - \left(\sum_{i=2}^{\infty} \frac{\partial E_i x^i}{\partial x}\right)\left(\sum_{j=2}^{\infty} F_j x^j\right) \\
&= Az - \sum_{i=2}^{\infty} \mathrm{ad}_{Ax}(E_i x^i) + F_2 x^2 + \sum_{s=3}^{\infty} \left(F_s x^s - \right. \\
&\quad \left. \sum_{i=2}^{s-1} \frac{\partial E_i x^i}{\partial x} F_{s+1-i} x^{s+1-i}\right) \\
&:= Az - \sum_{i=2}^{\infty} \mathrm{ad}_{Ax}(E_i x^i) + \sum_{s=2}^{\infty} L_s,
\end{aligned}
\tag{18.41}
$$

where

$$
\begin{aligned}
L_2 &= F_2 x^2 \\
L_s &= F_s x^s - \sum_{i=2}^{s-1} \frac{\partial E_i x^i}{\partial x} \ltimes F_{s+1-i} x^{s+1-i} \\
&= \left(F_s - \sum_{i=2}^{s-1} E_i \Phi_{i-1}(I_{n^{i-1}} \otimes F_{s+1-i})\right) x^s, \ s \geq 3.
\end{aligned}
\tag{18.42}
$$

Because of assumption *A1* we can define

$$
E_s x^s = \mathrm{ad}_{Ax}^{-1}(L_s), \quad s = 2, 3, \cdots.
$$

Hence, (18.35) becomes (18.39). □

The advantage of this Taylor series expansion is that we can get the linear form directly, without calculating the infinite times of coordinate transformations z_i, $i = 1, 2, 3, \cdots$.

Next, we consider the linearization of system (18.19). Denote $A = \frac{\partial f}{\partial x}|_0$, $B = g(0)$, and assume (A, B) is completely controllable. Then we can find feedback coefficient matrix K and a linear coordinate transformation T, such that $\tilde{A} = T^{-1}(A + BK)T$ satisfies *A1*. For statement ease, the above transformation is called a non-resonant (NR) transformation. .

Using the above notations and computations, the following result is obvious.

Theorem 18.7. *System (18.19) is single input feedback linearizable, if and only if there exists an NR transformation and a component non-zero constant vector b such that*

$$
b \in \mathrm{Span}\left\{ \left(I - \sum_{i=2}^{\infty} E_i \Phi_{i-1} x^{i-1}\right) g_j \,\middle|\, j = 1, \cdots, m \right\}. \tag{18.43}
$$

In the following we consider approximate linearization, which is practically useful.

Definition 18.6. System (18.19) is said to be k-th order non-regular (NR-k) feedback approximately linearizable, if there exist a state feedback, a local coordinate chart z, such that within this chart the closed-loop system becomes

$$\dot{z} = Az + O(\|z\|^{k+1}) + (b + O(\|z\|^k))v, \tag{18.44}$$

where (A,b) is completely controllable.

For approximate linearization, the non-resonant requirement can be relaxed a little bit.

Definition 18.7. Let $\lambda = (\lambda_1, \cdots, \lambda_n)$ be the set of eigenvalues of A. A is k-th order resonant, if there exists $m = (m_1 \cdots, m_n) \in \mathbb{Z}_+^n$ with $2 \le |m| \le k$, such that for certain $1 \le s \le n$, we have $\lambda_s = \langle m, \lambda \rangle$.

From equation (18.37) it is ready to verify the following result, which is a corollary Poincaré's theorem.

Corollary 18.3. *Consider an analytic system (18.22). If A is k-th order non-resonant, then there exists a coordinate transformation (18.23), which transforms the system (18.22) into an approximately linear system*

$$\dot{z} = Az + O(\|z\|^{k+1}). \tag{18.45}$$

If we consider the k-th order approximate linearization of system (18.35), we need only to adjust (18.38) to

$$z = x - \sum_{i=2}^{k} E_i x^i. \tag{18.46}$$

Then the formulas in (18.40) remain available for $s \le k$. Moreover, (18.39) becomes

$$\dot{z} = Az + O(\|x\|^{k+1}). \tag{18.47}$$

We call a transformation k-th order non-resonant transformation, if it is the same as NR transformation except the condition of non-resonant is replaced by the one of k-th order non-resonant.

Theorem 18.8. *System (18.4) is k-th order single input feedback approximately linearizable, if and only if there exists a NR-k transformation and a component non-zero constant vector b, such that*

$$b \in \operatorname{Span}\left\{ \left(I - \sum_{i=2}^{k} E_i \Phi_{i-1} x^{i-1} \right) g_j \,\middle|\, \forall j \right\} + O(\|x\|^k). \tag{18.48}$$

We use the following example to depict the linearization process.

Example 18.2. Consider the 4-th order feedback approximate linearization of the following system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-4\sin x_1 - \frac{2}{3}x_1^3 + 5x_2^2 + 6x_2^3\\ -5x_2 - 3x_3^2\\ -6x_3\end{bmatrix} + \begin{bmatrix}0\\ 6(1+x_3)\\ 7\end{bmatrix} u_1 + \begin{bmatrix}1\\ 0\\ 0\end{bmatrix} u_2. \tag{18.49}$$

Using Taylor series expansion, we can express system (18.49) as

$$\dot{x} = \begin{bmatrix}-4 & 0 & 0\\ 0 & -5 & 0\\ 0 & 0 & -6\end{bmatrix} x + \begin{bmatrix}5x_2^2\\ -3x_3^2\\ 0\end{bmatrix} + \begin{bmatrix}6x_2^3\\ 0\\ 0\end{bmatrix} + O(\|x\|^5) + \begin{bmatrix}0\\ 6+6x_3\\ 7\end{bmatrix} u_1 + \begin{bmatrix}1\\ 0\\ 0\end{bmatrix} u_2. \tag{18.50}$$

It is easy to calculate that

$$\begin{aligned}
&L_2 = (5x_2^2,\ -3x_3^2,\ 0)^{\mathrm{T}},\\
&E_2x^2 = ad_{Ax}^{-1}(L_2) = (-\tfrac{5}{6}x_2^2,\ \tfrac{3}{7}x_3^2,\ 0)^{\mathrm{T}};\\
&L_3 = (6x_2^3 - 5x_2x_3^2,\ 0,\ 0)^{\mathrm{T}},\\
&E_3x^3 = ad_{Ax}^{-1}(L_3) = \left(-\tfrac{6}{11}x_2^3 + \tfrac{5}{13}x_2x_3^2,\ 0,\ 0\right)^{\mathrm{T}}.
\end{aligned}$$

The expected coordinate transformation is

$$z = x - \begin{bmatrix}-\frac{5}{6}x_2^2 - \frac{6}{11}x_2^3 + \frac{5}{13}x_2x_3^2\\ \frac{3}{7}x_3^2\\ 0\end{bmatrix}, \tag{18.51}$$

Under this new coordinate frame (18.49) can be expressed as

$$\dot{z} = \begin{bmatrix}-4 & 0 & 0\\ 0 & -5 & 0\\ 0 & 0 & -6\end{bmatrix} z + O(\|x\|^5) + \begin{bmatrix}h(x) & 1\\ 6 & 0\\ 7 & 0\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}, \tag{18.52}$$

where $h(x) = (6+6x_3)(\frac{5}{3}x_2 + \frac{18}{11}x_2^2 - \frac{5}{13}x_3^2) - \frac{70}{13}x_2x_3$. Since

$$\begin{bmatrix}1\\ 6\\ 7\end{bmatrix} = \begin{bmatrix}h(x)\\ 6\\ 7\end{bmatrix} \times 1 + \begin{bmatrix}1\\ 0\\ 0\end{bmatrix} \times (-h(x)+1),$$

Theorem 18.8 assures that the system is 4-th order single input feedback approximately linearizable.

Choosing state feedback control as

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -h(x)+1 \end{bmatrix} v. \tag{18.53}$$

Plugging it into (18.52), we have

$$\dot{z} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -6 \end{bmatrix} z + O(||x||^5) + \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix} v, \tag{18.54}$$

which is the 4-th order single input approximately linearized system of the system (18.49).

Exercise 18

1. (to be completed).

References

1. Arnold, V.: Geometrical Methods on the Theory of Ordinary Differential Equations. Springer-Verlag, New York (1983)
2. Cairo, L., Feix, M.: Families of invariants of the motion for Lotka-Volterra 1st family. J. Math. Phys. **33**(7), 2240–2455 (1992)
3. Cairo, L., Feix, M., Llibre, J.: Darboux method and rearch of invariats for the Lotka-Volterra and complex quadratic systems. J. Math. Phys. **40**(4), 2074–2091 (1999)
4. Cheng, D.: Matrix and Polynomial Approach to Dynamic Control Systems. Science Press, Beijing (2002)
5. Devanathan, R.: Linearization condition through state feedback. IEEE Trans. Aut. Contr. **46**(8), 1257–1260 (2001)
6. Guckenheimer, J., Holmes, P.: Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields. Springer, Berlin (1983)
7. Heymann, M.: Pole assignment in multi-input linear systems. IEEE Trans. Aut. Contr. **13**(6), 748–749 (1968)
8. Krener, A., Kang, W.: Extended normal forms of quadratic systems. In: Proc. 29th IEEE Conference on Decision and Control, pp. 2091–2096 (1990)
9. Sun, Z., Xia, X.: On nonregular feedback linearization. Automatica **33**(7), 1739–1744 (1997)