Chapter 17
Morgan's Problem

Consider a control system, which has multiple inputs and multiple outputs. Morgan's problem is also called an input-output decoupling problem. It plays an important role in control design. When input number equals the output number, the problem has been solved perfectly [3, 7]. But as input number is greater than the output number, it becomes a long standing problem. The problem has been claimed to be solved several times, but the conclusions have then been proved wrong. In this chapter we consider only the static feedback control. Some later developments can be seen in [5, 2]. As for the dynamic feedback case, a necessary and sufficient condition has been provided in [1].

Using semi-tensor product, this chapter provides a numerical solution for the problem.

17.1 Input-Output Decomposition

Consider a linear control system

$$\begin{cases} \dot{x} = Ax + \sum_{i=1}^{m} b_i u_i := Ax + Bu, \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m, \\ y = Cx, \quad y \in \mathbb{R}^p, \end{cases} \tag{17.1}$$

where u_i, $i = 1, \cdots, m$ are controls and y_j, $j = 1, \cdots, p$ are outputs, and $m \geq p$. In addition we assume $\text{rank}(B) = m$, $\text{rank}(C) = p$. The input-output decomposition problem, which is also called the Morgan's problem, means to find a partition of inputs, such that each block of inputs control the corresponding output without affecting other outputs. We give a precise definition.

Definition 17.1. Consider the system (17.1). The input-output decomposition problem (Morgan's problem) is solvable, if there exists a state feedback

$$u = kx + Hv, \quad v \in \mathbb{R}^p, \tag{17.2}$$

and a partition of v, say $v^1, \cdots, v^p$, such that v^i controls y_i and v^i does not affect y_j, $j \neq i$. Equivalently, the transfer matrix from v to y is block diagonal and nonsingular [7].

For statement ease, we introduce some concepts.

Let $y_j = c_j x$, its relative degree, denoted by ρ_j, is defined as

$$\rho_j = \min\{i \,|\, c_j A^{i-1} B \neq 0\}, \quad j = 1, \cdots, p. \tag{17.3}$$

Using relative degree vector $\rho = (\rho_1, \cdots, \rho_p)$, we define a $p \times m$ matrix, called the decoupling matrix, as

$$D = \begin{bmatrix} c_1 A^{\rho_1 - 1} B \\ c_2 A^{\rho_2 - 1} B \\ \vdots \\ c_p A^{\rho_p - 1} B \end{bmatrix}. \tag{17.4}$$

When $m = p$, we have the following classical result as:

Theorem 17.1 ([3]). *When $m = p$, the Morgan's problem is solvable, if and only if the decoupling matrix D is nonsingular.*

Corresponding to linear case, the Morgan's problem for nonlinear control systems has also discussed widely. Consider a nonlinear control system

$$\begin{cases} \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i := f(x) + G(x) u, \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m, \\ y_j = h_j(x), \quad j = 1, \cdots, p, \end{cases} \tag{17.5}$$

where, $f(x)$, $g_i(x)$ $i = 1, \cdots, m$ are smooth vector fields, $h_j(x)$, $j = 1, \cdots, p$ are smooth functions. For system (17.5), the relative degree vector $\rho = (\rho_1, \cdots, \rho_p)^{\mathrm{T}}$ is defined as

$$\begin{aligned} & L_{g_i} L_f^k h_j(x) = 0, \quad x \in U, \\ & \quad i = 1, \cdots, m; \ k = 0, 1, \cdots, \rho_j - 2. \\ & L_{g_i} L_f^{\rho_j - 1} h_j(x_0) \neq 0, \quad \exists i \in \{1, 2, \cdots, m\}, \end{aligned}$$

where U is a neighborhood of x_0, which is the concerned point. For simplicity, set $x_0 = 0$. Assume the relative degree vector is well defined, we define the relative degree as

$$D(x) = \begin{bmatrix} L_{g_1} L_f^{\rho_1 - 1} h_1(x) & \cdots & L_{g_m} L_f^{\rho_1 - 1} h_1(x) \\ L_{g_1} L_f^{\rho_2 - 1} h_2(x) & \cdots & L_{g_m} L_f^{\rho_2 - 1} h_2(x) \\ \vdots & & \vdots \\ L_{g_1} L_f^{\rho_p - 1} h_p(x) & \cdots & L_{g_m} L_f^{\rho_p - 1} h_p(x) \end{bmatrix}. \tag{17.6}$$

As a generalization of linear case, we have the following result:

Theorem 17.2 ([4]). *When $m = p$, the Morgan's problem is locally solvable at x_0, if and only if the decoupling matrix is nonsingular at x_0.*

In this chapter, we consider only the linear case.

17.2 Problem Formulation

From the previous section it is clear that the challenging case for Morgan's problem is when $m > p$. According to Theorem 17.1, we have the following lemma.

Lemma 17.1. *Morgan's problem is solvable, if and only if there exist $K \in M_{m\times n}$, $H \in M_{m\times p}$, $1 \le \rho_i \le n$, $i = 1,\cdots,p$ such that*

$$c_i(A+BK)^{t_i}BH = 0, \quad t_i = 0,\cdots,\rho_i - 2, \quad i = 1,\cdots,p. \tag{17.7}$$

Moreover, the decoupling matrix for the closed-loop system,

$$D = \begin{bmatrix} c_1(A+BK)^{\rho_1-1}BH \\ \vdots \\ c_p(A+BK)^{\rho_p-1}BH \end{bmatrix} \tag{17.8}$$

is nonsingular.

According to the above lemma, there are two designable feedback matrices K and H. The purpose of this section is to give an equivalent condition, which reduce the unknown matrices to one.

First, note that when $\rho_i = 1$ (17.7) disappears. We then denote

$$\Lambda = \{i \,|\, \rho_i = 2\}, \quad C_\Lambda = col\{c_i \,|\, i \in \Lambda\}.$$

By definition of relative degree, when $1 \le \rho_i \le 2$, $i = 1,\cdots,p$, (17.7) becomes

$$H \subset (C_\Lambda B)^\perp.$$

Denote

$$\Lambda^c = \{1,\cdots,p\}\backslash\Lambda.$$

Then $i \in \Lambda^c$ implies that $\rho_i = 1$.

Hence we have

Corollary 17.1. *For $1 \le \rho_i \le 2$, $i = 1,\cdots,p$, Morgan's problem is solvable, if and only if there exists $K \in M_{m\times n}$, such that*

$$D = \begin{bmatrix} C_{\Lambda^c}B \\ c_\Lambda(A+BK)B \end{bmatrix} (C_\Lambda B)^\perp \tag{17.9}$$

has full row rank.

In general case, it is not so easy to eliminate H. Further works are necessary. Define

$$W(K) := \begin{bmatrix} c_1 B \\ c_1(A+BK)B \\ \vdots \\ c_1(A+BK)^{\rho_1-2}B \\ \vdots \\ c_p(A+BK)^{\rho_p-2}B \end{bmatrix}, \quad T(K) := \begin{bmatrix} c_1(A+BK)^{\rho_1-1}B \\ \vdots \\ c_p(A+BK)^{\rho_p-1}B \end{bmatrix}.$$

Then (17.7) becomes

$$W(K)H = 0, \tag{17.10}$$

and (17.8) becomes

$$D = T(K)H. \tag{17.11}$$

Since $1 \le \rho_i \le n$, $i = 1, \cdots, p$, for fixed ρ_i, $i = 1, \cdots, p$ we may consider the solvability of Morgan's problem, Because we need to check finite (precisely, n^p) cases. In the remaining part of this chapter we consider the solvability of Morgan's problem under a set of fixed ρ_i, unless elsewhere stated.

Lemma 17.2. *Morgan's problem is solvable, if and only if there exists $K \in M_{m\times n}$ such that*

(1)

$$\mathrm{Im}(T^{\mathrm{T}}(K)) \cap \mathrm{Im}(W^{\mathrm{T}}(K)) = \{0\}; \tag{17.12}$$

(2) $T(K)$ has full row rank.

Proof. We prove the following statements are equivalent:

(i) there exists H such that $T(K)H$ is nonsingular and $W(K)H = 0$;
(ii) $T(K)((W(K)^{\mathrm{T}})^{\perp}) = \mathbb{R}^p$;
(iii) $[(T^{\mathrm{T}}(K))^{-1}(W(K)^{\mathrm{T}})]^{\perp} = \mathbb{R}^p$;
(iv) $(T^{\mathrm{T}}(K))^{-1}(W(K)^{\mathrm{T}}) = 0$;
(v) Conditions (1) and (2) in Lemma 17.2.

Where $T(K)$ and $(T^{\mathrm{T}}(K))^{-1}$ are considered as functional mappings [6].

(i)⇒(ii): If $\dim(T(k)(W(K)^{\mathrm{T}})^{\perp}) < p$, since $\mathrm{Im}(H) \subset (W(K)^{\mathrm{T}})^{\perp}$, then $\mathrm{rank}(T(K)H) < p$, which leads to a contradiction;

(ii)⇒(i): Choosing p vectors $h_i \in (W(K)^{\mathrm{T}})^{\perp}$, such that $T(K)\mathrm{Im}(h_1, \cdots, h_p) = \mathbb{R}^p$. Then we can set $H = (h_1, \cdots, h_p)$;

(ii)⇔(iii): Refer to [6] page 23;

(iii)⇔(iv): It is obvious;
(iv)⇔(v): It is easy to verify that both (iv) and (v) are equivalent to the following statement: If $Y \in \mathbb{R}^p$ and $T^{\mathrm{T}}(K)Y \in \mathrm{Im}((W(K))^{\mathrm{T}})$, then $Y = 0$. □

From the above lemma we can prove the following theorem easily.

Theorem 17.3. *For fixed ρ_i's the Morgan's problem is solvable, if and only if there exists $K_0 \in M_{m\times n}$, such that*

$$\mathrm{rank}\left(\begin{bmatrix} T(K_0) \\ W(K_0) \end{bmatrix}\right) = p + \mathrm{rank}(W(K_0)). \tag{17.13}$$

Consider the case when $\rho_i \le 2$, $\forall i$, if the following assumption holds:
A1. $C_\Lambda B = 0$, then we need not consider $W(K_0)$. Hence we have

Corollary 17.2. *Assume $1 \le \rho_i \le 2$, $i = 1, \cdots, p$, and A1 holds. Moreover, if there exists $K_0 \in M_{m\times n}$ such that $T(K_0)$ has full row rank, then Morgan's problem is solvable.*

Definition 17.2. Assume $A(K)$ is a matrix with its entries $a_{ij}(K)$ as the polynomial of K, where $K \in M_{m\times n}$. Define the essential rank of $A(K)$, denoted by $\mathrm{rank}_e(A(K))$, as

$$\mathrm{rank}_e(A(K)) = \max_{K\in M_{m\times n}} \mathrm{rank}(A(K)).$$

Now under the fixed ρ_i denote

$$\mathrm{rank}_e(T(K)) = t, \quad \mathrm{rank}_e(W(K)) = s, \quad \mathrm{rank}_e\left(\begin{bmatrix} T(K) \\ W(K) \end{bmatrix}\right) = q.$$

Because the essential rank is easily computable, the following corollary is convenient in certain cases.

Corollary 17.3. *Morgan's problem is solvable, if $q = p + s$.*

Since both $T(K)$ and $W(K)$ are polynomial matrices of K, the essential rank can be reached on all K except a zero-measure set of K, it is easy to calculate the essential rank via computer. Hence, Corollary 17.3 is easily verifiable.

17.3 Numerical Expression of Solvability

We first calculate $T(K)$ and $W(K)$. Denote by $Z = V_r(K) \in \mathbb{R}^{mn}$, We first express $T(K)$ and $W(K)$ as polynomials of Z via semi-tensor product.

Lemma 17.3. *Given a matrix $A \in M_{n\times m}$.*

1. *If $x \in \mathbb{R}^n$ is a row vector, then*

$$xA = V_r^{\mathrm{T}}(A) \ltimes x^{\mathrm{T}}. \tag{17.14}$$

2. *If* $Y \in M_{p\times n}$, *then*

$$YA = (I_p \otimes V_r^{\mathrm{T}}(A)) \ltimes V_r(Y). \tag{17.15}$$

Proof. A straightforward computation shows that

$$\begin{aligned} xA &= V_r^{\mathrm{T}}(A)x^{\mathrm{T}} \\ &= \left(\sum_{i=1}^{n} a_{i1}x_i, \cdots, \sum_{i=1}^{n} a_{im}x_i \right), \end{aligned}$$

Using (17.14), we have

$$YA := \begin{bmatrix} Y^1 \\ \vdots \\ Y^p \end{bmatrix} A = \begin{bmatrix} V_r^{\mathrm{T}}(A)(Y^1)^{\mathrm{T}} \\ \vdots \\ V_r^{\mathrm{T}}(A)(Y^p)^{\mathrm{T}} \end{bmatrix} = (I_p \otimes V_r^{\mathrm{T}}(A))V_r(Y).$$

□

Next, we expand $(A+BK)^t$ as follows:

$$(A+BK)^t = \sum_{i=0}^{2^t-1} P_i(A,BK),$$

where P_i is used to replace the i-th product of t elements, which are either A or BK. P_i can be figure out as follows: Convert i into a binary number of length t. Then use "A" to replace "0" and use "BK" to replace "1". Then we collect terms with respect to different orders of "K". Finally, we can have the following expression.

$$c_k(A+BK)^t B = \sum_{i=0}^{t} \sum_{j=1}^{T_i} S_0^{ij} K S_1^{ij} K \cdots S_{t-1}^{ij} K S_t^{ij}, \quad k = 1, \cdots, p, \tag{17.16}$$

where $T_i = \begin{bmatrix} i \\ t \end{bmatrix}$. Using Lemma 17.3 and equation (17.14), (17.16) can be expressed as

$$\begin{aligned} & c_k(A+BK)^t B \\ &= \sum_{i=0}^{t} \sum_{j=1}^{T_i} S_0^{ij} \ltimes (I_m \otimes V_r^{\mathrm{T}}(S_1^{ij})) \ltimes Z \ltimes \cdots \ltimes (V_r^{\mathrm{T}}(I_m \otimes S_t^{ij})) \ltimes Z \\ &= \sum_{i=0}^{t} \sum_{j=1}^{T_i} S_0^{ij} \ltimes (I_m \otimes V_r^{\mathrm{T}}(S_1^{ij})) \ltimes (I_{m^2 n} \otimes V_r^{\mathrm{T}}(S_2^{ij})) \\ &\qquad \ltimes \cdots \ltimes (I_{m^t n^{t-1}} \otimes V_r^{\mathrm{T}}(S_t^{ij})) \ltimes Z^t, \quad k = 1, \cdots, p. \end{aligned} \tag{17.17}$$

Using (17.17), $W(K)$ and $T(K)$ can be expressed in canonical form as:

$$W(K) = W_0 + W_1 \ltimes Z + \cdots + W_{l-1} \ltimes Z^{l-1} \in M_{d\times m}, \quad d = \sum_{i=1}^{p} \rho_i - p,$$
$$T(K) = T_0 + T_1 \ltimes Z + \cdots + T_l \ltimes Z^l \in M_{p\times m}. \tag{17.18}$$

where $l = \max\{\rho_i - 1 \,|\, i = 1, \cdots, p\}$.

Denote by $W^s = \mathrm{Row}_s(W(K))$, then the size of W^s is

$$|W^s| = \frac{d!}{s!(d-s)!}.$$

Now Morgan's problem can be converted into an algebraic form as

Proposition 17.1. *Morgan's problem is solvable, if and only if there exists an* $1 \le s \le m - p + 1$ *such that*

$$R(Z) := \sum_{L\in W^s} \det\left(L(Z)L^{\mathrm{T}}(Z)\right) = 0 \tag{17.19}$$

and

$$J(Z) := \sum_{L\in W^{s-1}} \det\left(\begin{bmatrix} T(Z) \\ L(Z) \end{bmatrix} \left(T^{\mathrm{T}}(Z)\, L^{\mathrm{T}}(Z)\right)\right) > 0 \tag{17.20}$$

has solution Z.

Proof. If the system (17.19)–(17.20) has solution $Z = V_r(K)$, then from (17.19) we have $\mathrm{rank}(W(K)) < s$, and from (17.20) we have

$$\mathrm{rank}\begin{pmatrix} T(K) \\ W(K) \end{pmatrix} = p + s - 1.$$

According to Theorem 17.3, the conclusion follows. □

Now the Morgan's problem becomes a numerical problem: For each set of fixed $1 \le \rho_i \le n$, $i = 1, \cdots, p$ and $1 \le s \le m - p + 1$, solve system (17.19)–(17.20). Since there are only finite possible cases, the Morgan's problem is solvable as long as there is a case (a pair of $(\rho_i,\ s)$), under which system (17.19)–(17.20) has solution.

There are many numerical methods, which are suitable to solve this numerical problem.

For instance, we may convert it the the so called Wu's problem [8]: Is polynomials $R(z) = 0$ implies $J(z) = 0$? If for all cases the answer is "yes", then the Morgan's problem is not solvable. Otherwise, if at least there is one case, where the answer is "no", then the Morgan's problem is solvable.

An alternative approach is to convert it into an optimization problem:

$$\max_{R(z)=0} J(z).$$

If the maximum value is zero, then the Morgan's problem is not solvable. Otherwise, it is solvable.

Note that if every element of $A(Z)$ can be expressed as a polynomial of the form $a_0 + a_1 \ltimes Z + \cdots + a_L \ltimes Z^L$, the determinant $\det(A(Z))$ can be calculated directly. Hence, to get (17.19) and (17.20) we need to calculate the following product: Let

$$A = A_0 + A_1 \ltimes z + \cdots + A_s \ltimes z^s \in M_{m\times n},$$
$$B = B_0 + B_1 \ltimes z + \cdots + B_t \ltimes z^t \in M_{p\times n}.$$

Then

$$AB^{\mathrm{T}} = \sum_{i=0}^{s}\sum_{j=0}^{t} A_i \ltimes z^i \ltimes (z^{\mathrm{T}})^j \ltimes B_j^{\mathrm{T}}.$$

Using (17.14), we have

$$(z^{\mathrm{T}})^j = (z^{\mathrm{T}})^j I_{n^j} = V_r^{\mathrm{T}}(I_{n^j}) \ltimes z^j;$$

$$z^i \ltimes V_r^{\mathrm{T}}(I_{n^j}) = (I_{n^i} \ltimes V_r^{\mathrm{T}}(I_{n^j})) \ltimes z^i;$$

and

$$z^{i+j} \ltimes B_j^{\mathrm{T}} = (I_{n^{i+j}} \ltimes B_j^{\mathrm{T}}) \ltimes z^{i+j}.$$

Using them, we have

$$AB^{\mathrm{T}} = \sum_{i=0}^{s}\sum_{j=0}^{t} A_i \ltimes \left(I_{n^i} \ltimes V_r^{\mathrm{T}}(I_{n^j})\right) \ltimes \left(I_{n^{i+j}} \ltimes B_j^{\mathrm{T}}\right) \ltimes Z^{i+j}. \tag{17.21}$$

As immediate consequence of the above argument is: when $1 \le \rho_i \le 2$, $i = 1, \cdots, p$, we have the following result.

Corollary 17.4. *Assume* $1 \le \rho_i \le 2$, $i = 1, \cdots, p$, *and* $A1$ *holds. Then Morgan's problem is solvable, if and only if*

$$J(Z) := \det\left(T(Z)T^{\mathrm{T}}(Z)\right) > 0 \tag{17.22}$$

has solution Z.

In this case, the Morgan's problem converts to a free (i.e., without restriction) optimization problem:

$$\max J(z).$$

It the maximum value is zero, the Morgan's problem is not solvable. Otherwise, it is solvable.

Summarizing the above argument, we present the numerical algorithm for solving Morgan's problem.

Algorithm 3. Step 1. For $\rho_1, \cdots, \rho_p = 1, \cdots, n$, using (17.16) - (17.17) to express $T(K)$ and $W(K)$ into the standard polynomial form as (17.18).

Step 2. For $s = 1, \cdots, m-p+1$ and each $L \in W^s$, using (17.21) to calculate

$$L(z)L^{\mathrm{T}}(z), \quad \begin{bmatrix} T(z) \\ L(z) \end{bmatrix} \begin{bmatrix} T^{\mathrm{T}}(z) & L^{\mathrm{T}}(z) \end{bmatrix}.$$

Step 3. Using (17.19) to calculate $R(z)$, and using (17.20) to calculate $J(z)$ respectively.
Step 4. Solving the numerical problem

$$\begin{cases} R(z) = 0, \\ J(z) > 0, \end{cases} \tag{17.23}$$

where $R(z)$ and $J(z)$ are polynomials obtained from Step 3.

Finally, we give a numerical example to depict it.

Example 17.1. Consider a linear system

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 6 & -2 & -4 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & -2 & 1 & 1 & -1 \\ -1 & 1 & -1 & -1 & 1 \\ 0 & 2 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} -2 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \\ -1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} u, \\ y = \begin{bmatrix} 0 & 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix} x. \end{cases} \tag{17.24}$$

Note that $\rho_1 + \rho_2$ can not be great than the dimension 5, we have to verify the following possible cases: $\rho_1 = 1, \rho_2 = 1,2,3,4; \rho_1 = 2, \rho_2 = 1,2,3; \rho_1 = 3, \rho_2 = 1,2; \rho_1 = 4, \rho_2 = 1$. As an example we verify the case when $\rho_1 = 3, \rho_2 = 2$. In this case we have

$$W(K) = \begin{bmatrix} c_1 B \\ c_1(A+BK)B \\ c_2 B \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ p_1(Z) & p_2(Z) & p_3(Z) \\ 1 & 0 & -1 \end{bmatrix},$$

where

$$Z = V_r(K) = (k_{11}, \ldots, k_{15}, \ \ldots \ , k_{31}, \ldots, k_{35})^{\mathrm{T}},$$

Using (4.33), we have

$$\begin{bmatrix} p_1(Z) \\ p_2(Z) \\ p_3(Z) \end{bmatrix} = V_r(c_1AB + c_1BKB) = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} + ((c_1B) \otimes B^{\mathrm{T}})Z.$$

Then we can calculate that

$$\begin{aligned} p_1(Z) &= -1 + 2k_{11} - 2k_{13} + k_{14} + k_{15} - 2k_{31} + 2k_{33} - k_{34} - k_{35}, \\ p_2(Z) &= 1 - k_{11} - k_{15} + k_{31} + k_{35}, \\ p_3(Z) &= 1 - k_{11} + 2k_{13} - k_{14} - 2k_{15} + k_{31} - 2k_{33} + k_{34} + 2k_{35}. \end{aligned}$$

Moreover,

$$
\begin{aligned}
T(K) &= \begin{bmatrix} c_1(A+BK)^2B \\ c_2(A+BK)B \end{bmatrix} \\
&= \begin{bmatrix} c_1A^2B \\ c_2AB \end{bmatrix} + \begin{bmatrix} c_1ABKB + c_1BKAB \\ c_2BKB \end{bmatrix} + \begin{bmatrix} c_1BKBKB \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \end{bmatrix} + T_1Z + T_2Z^2,
\end{aligned}
$$

where

$$
\begin{aligned}
T_1 = &\left[\begin{matrix} 2 & -1 & 0 & -1 & 0 & -1 & -4 & 1 & 4 & 1 & 0 & -3 & 2 & -2 & -2 \\ -2 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & -2 & -1 & 0 & 1 & -1 & 1 & 1 \end{matrix}\right. \\
&\begin{matrix} -2 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & -2 & -1 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \\
&\left.\begin{matrix} -2 & 1 & 0 & 1 & 0 & 1 & 4 & -1 & -4 & -1 & 0 & 3 & -2 & 2 & 2 \\ 2 & -1 & 0 & 0 & 0 & 0 & -2 & 0 & 2 & 1 & 0 & -1 & 1 & -1 & -1 \end{matrix}\right],
\end{aligned}
$$

where T_2 is a 2×675 matrix, which is skipped here. (The reader can calculate it via computer easily.)

According to Proposition 17.1, we need to check it for two cases: $s = 1$ and $s = 2$. It is clear that when $s = 1$, $R(Z) > 0$. For $s = 2$, from (17.11) one sees that to make D nonsingular, $\operatorname{rank}(H) \geq 2$. Then, from (17.10), $\operatorname{rank} W(K) \leq 1$. Hence, we can assume

$$
p_1(Z) = 0, \quad p_2(Z) = 0, \quad p_3(Z) = 0, \tag{17.25}
$$

It is easy to find a set of solution as

$$
K = V_r^{-1}(Z) = \begin{bmatrix} 0 & 2 & -1 & -2 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -2 & -1 \end{bmatrix},
$$

and

$$
W(K) = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \qquad T(K) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.
$$

It is obvious that $R(Z) = 0$. Moreover,

$$
\begin{aligned}
J(Z) &= \det\left(\begin{bmatrix}1&0&-1\\0&1&0\\-1&0&1\end{bmatrix}\begin{bmatrix}1&0&-1\\0&1&0\\-1&0&1\end{bmatrix}^{\mathrm{T}}\right)\\
&\quad+\det\left(\begin{bmatrix}1&0&-1\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}1&0&-1\\0&1&0\\0&0&0\end{bmatrix}^{\mathrm{T}}\right)\\
&\quad+\det\left(\begin{bmatrix}1&0&-1\\0&1&0\\1&0&-1\end{bmatrix}\begin{bmatrix}1&0&-1\\0&1&0\\1&0&-1\end{bmatrix}^{\mathrm{T}}\right)\\
&=4>0
\end{aligned}
$$

According to Proposition 17.1, we conclude that the Morgan's problem for system (17.24) is solvable.

In the following we look for the required feedback matrix H. Since $\mathrm{Span}\{\mathrm{Col}(H)\}\subset\ker(W(K))$, we can choose

$$
H=\begin{bmatrix}0&1\\1&0\\0&1\end{bmatrix}.
$$

Using feedback control $u=Kx+Hv$, the closed-loop system becomes

$$
\begin{cases}
\dot{x}=\begin{bmatrix}0&1&0&0&0\\-1&0&0&0&1\\0&-2&1&1&1\\-1&1&-1&-1&0\\0&1&0&0&0\end{bmatrix}x+\begin{bmatrix}1&1\\0&0\\0&-2\\0&1\\1&1\end{bmatrix}v,\\
y=\begin{bmatrix}0&1&-1&-1&0\\0&1&0&-1&0\end{bmatrix}x.
\end{cases}
\tag{17.26}
$$

It is ready to verify that

$$
WH=\begin{bmatrix}0&0\\0&0\\0&0\end{bmatrix},\quad D=TH=\begin{bmatrix}0&2\\1&0\end{bmatrix}.
$$

We then have $\rho_1=3$ and $\rho_2=2$. Moreover, the decoupling matrix D is nonsingular.

Remark 17.1. 1. From the argument of the above example one sees that all the K, satisfying (17.25) are the solutions of the Morgan's problem.
2. Using the K and H obtained in the above example, we can verify that the equality (17.7) in Lemma 17.1 holds. In addition, (17.4) is nonsingular. We can also check that the equality (17.12) in Lemma 17.2 holds, and $T(K)$ has row full rank.
3. For Theorem 17.3, it is easy to check that

$$\operatorname{rank}\left(\begin{bmatrix} T(K_0) \\ W(K_0) \end{bmatrix}\right) = 3, \quad \operatorname{rank}(W(K_0)) = 1, \quad p = 2,$$

hence, (17.13) holds.

Exercise 17

1. (to be completed).

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