Chapter 13
Boolean Control System

Logical control systems considered in this chapter are Boolean control networks. Using the algebraic form of logical control systems obtained by semi-tensor product, some fundamental dynamic properties such as controllability and observability are investigated as well as some control problems such as disturbance decoupling, optimal control, etc.

13.1 Dynamics of Boolean Control Networks

It was pointed out by [8] that "Gene regulatory networks are defined by trans and cis logic. $\cdots$ Both of these types of regulatory networks have input and output." The investigation of control problem is essential in the study of cellular network.

A Boolean control network is defined as

$$\begin{cases} x_1(t+1) = f_1(x_1(t), x_2(t), \cdots, x_n(t), u_1(t), \cdots, u_m(t)) \\ x_2(t+1) = f_2(x_1(t), x_2(t), \cdots, x_n(t), u_1(t), \cdots, u_m(t)) \\ \vdots \\ x_n(t+1) = f_n(x_1(t), x_2(t), \cdots, x_n(t), u_1(t), \cdots, u_m(t)), \end{cases} \tag{13.1}$$

and

$$y_j(t) = h_j(x_1(t), x_2(t), \cdots, x_n(t)), \quad j = 1, 2, \cdots, p, \tag{13.2}$$

where $f_i : \mathscr{D}^{n+m} \to \mathscr{D}$, $i = 1, 2, \cdots n$, and $h_j : \mathscr{D}^n \to \mathscr{D}$, $j = 1, 2, \cdots p$ are logical functions; $x_i \in \mathscr{D}$, $i = 1, 2, \cdots n$ are states; $y_j \in \mathscr{D}$, $j = 1, 2, \cdots p$ are outputs; and $u_\ell \in \mathscr{D}$, $\ell = 1, 2, \cdots m$ are inputs (or controls).

Denote by $x = \ltimes_{i=1}^n x_i$, $u = \ltimes_{i=1}^m u_i$, and $y = \ltimes_{i=1}^p y_i$. Using vector form, (13.1) and (13.2) can be expressed by the following

$$\begin{cases} x(t+1) = Lu(t)x(t) \\ y(t) = Hx(t), \end{cases} \tag{13.3}$$

Chapter 12 have studied the topological structure of Boolean network and Boolean control network with given input network. In this chapter, we will firstly consider the topological structure of Boolean control network. We rigorously define the fixed points and cycles of Boolean control network.

Definition 13.1. Consider system (13.1). Denote the input-state (product) space by

$$\mathscr{S} = \{(U,X)\,|\,U = (u_1,\cdots,u_m) \in \mathscr{D}^p,\ X = (x_1,\cdots,x_n) \in \mathscr{D}^n\}.$$

Note that $|\mathscr{S}| = 2^{m+n}$.

1. Let $S_i = (U^i,X^i) \in \mathscr{S}$ and $S_j = (U^j,X^j) \in \mathscr{S}$. Denote by $U^i = (u^i_1,\cdots,u^i_m)$, $X^i = (x^i_1,\cdots,x^i_n)$, etc. (S_i,S_j), is said to be a directed edge, if X^i,U^i,X^j satisfy (13.3). Precisely,

$$x^j_k = f_k(x^i_1,\cdots,x^i_n,u^i_1,\cdots,u^i_m), \quad k = 1,\cdots,n.$$

The set of edges is denoted by $\mathscr{E} \subset \mathscr{S} \times \mathscr{S}$.
2. The pair $(\mathscr{S},\mathscr{E})$ forms a directed graph, which is called the input-state transfer graph (ISTG).
3. $(S_1,S_2,\cdots,S_\ell)$ is called a path, if $(S_i,S_{i+1}) \in \mathscr{E}$, $i = 1,2,\cdots,\ell-1$.
4. A path $(S_1,S_2,\cdots)$ is called a cycle, if $S_{i+\ell} = S_i$ for all i, the smallest ℓ is called the length of the cycle. Particular, the cycle of length 1 is called a fixed point.
5. A cycle $(S_1,S_2,\cdots,S_\ell)$ in which $S_i = (U^i,X^i)$ is called a simple cycle, if $X^i \neq X^j$ for $1 \le i < j \le \ell$.

Definition 13.2. Denote the vertexes of the ISTG of system (13.1) by $\{\delta^i_{2^{m+n}}|i = 1,\cdots,2^{m+n}\}$. The the input-state incidence matrix of the Boolean control network (13.1) is defined by

$$\mathscr{J}_{ij} = \begin{cases} 1, & \text{there exists an edge from } \delta^j_{2^{m+n}} \text{ to } \delta^i_{2^{m+n}}, \\ 0, & \text{otherwise.} \end{cases} \tag{13.4}$$

We give a simple example to describe the input-state transfer graph.

Example 13.1. Consider a Boolean control network Σ as

$$\Sigma : \begin{cases} x_1(t+1) = (x_1(t) \vee x_2(t)) \wedge u(t) \\ x_2(t+1) = x_1(t) \leftrightarrow u(t). \end{cases} \tag{13.5}$$

Setting $x(t) = x_1(t) \ltimes x_2(t)$, it is easy to calculate that the algebraic form of Σ is

$$\Sigma : x(t+1) = Lu(t)x(t),$$

where

$$L = \delta_4 \left[1\ 1\ 2\ 4\ 4\ 4\ 3\ 3\right]. \tag{13.6}$$

According to the dynamic equation (13.5) (equivalently, (13.6)), we can draw the flow of $(u(t),(x_1(t),x_2(t)))$ on the product space $\mathscr{U} \times \mathscr{X}$, called the input-state dynamic graph, as in Fig. 13.1.

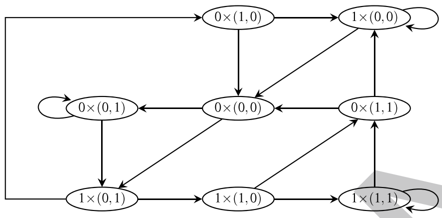

Fig. 13.1 Input-state dynamic graph

Then, we can find the input-state incidence matrix of (13.5), is

$$\mathscr{J} = \begin{bmatrix} 1&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&1\\ 0&0&0&1&1&1&0&0\\ 1&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&1\\ 0&0&0&1&1&1&0&0 \end{bmatrix}. \tag{13.7}$$

Comparing (13.6) with (13.7), one might be surprised to find that

$$\mathscr{J} = \begin{bmatrix} L \\ L \end{bmatrix}.$$

In fact, this is also true for general case. Consider equation (13.3). Note that, the j-th column of L corresponds to the "output" $x(t+1)$ for "input" $u(t)x(t) = \delta_{2^{m+n}}^j$ of the dynamic system. If this column $\mathrm{Col}_j(L) = \delta_{2^n}^i$, then it means that the output $x(t+1)$ is exactly the i-th element of Δ_{2^n}. Now since $u(t+1)$ can be arbitrary, it follows that the input-state incidence matrix of system (13.3) is

$$\mathscr{J} = \left.\begin{bmatrix} L \\ L \\ \vdots \\ L \end{bmatrix}\right\} 2^m \in \mathscr{B}_{2^{m+n} \times 2^{m+n}}, \tag{13.8}$$

where the first block corresponds to $u(t+1) = \delta_{2^m}^1$, the second block corresponds to $u(t+1) = \delta_{2^m}^2$, and so on.

We call an $m \times m$ matrix A is row-periodic, if it can be expressed as $A = \mathbf{1}_\tau A_0$, where $A_0 \in \mathscr{M}_{n\times m}$ and $m = \tau n$, the such smallest τ is called the period of A, A_0 is call the basic block of A. By straightforward computation, it can be verified that if A is row-periodic with period τ, then so is A^s, where s is a positive integer.

The following proposition can help with the computation of powers of the incidence matrix $\mathscr{J}$.

Proposition 13.1.

$$\mathscr{J}_0^{s+1} = M^s L, \tag{13.9}$$

where

$$M = \sum_{i=1}^{2^m} \mathrm{Blk}_i(L).$$

Proof.

$$\begin{aligned}
\mathscr{J}_0^{s+1} &= (\delta_{2^m}^1)^T \mathscr{J}^{s+1} \\
&= \left((\delta_{2^m}^1)^T \mathscr{J}\right) \mathscr{J}^s \\
&= L \mathbf{1}_{2^m} \mathscr{J}_0^s \\
&= \sum_{i=1}^{2^m} \mathrm{Blk}_i(L) \mathscr{J}_0^s
\end{aligned}$$

□

We consider the physical meaning of $\mathscr{J}^s$. When $s = 1$ we know that $\mathscr{J}_{ij}$ means whether there exists a set of controls such that $\delta_{2^{m+n}}^i$ is reachable from $\delta_{2^{m+n}}^j$ in one step by judging if $\mathscr{J}_{ij} = 1$ or not. For $s > 1$, we have

Theorem 13.1. *Consider system (13.3). Assume that the (i, j)-th element of the s-th power of its input-state incidence matrix, $\mathscr{J}_{ij}^s = c$. Then there are c paths from point $\delta_{2^{m+n}}^i$ reach P_i at s-th step with proper controls.*

Then, similar to Theorem 12.2 and 12.3, we can get the following result about the topological structure of Boolean control networks.

Theorem 13.2. *Consider the state equation of system (13.1) with its input-state incidence matrix $\mathscr{J}$. The number of the fixed points in the input-state dynamic graph is The number of length s cycles can be calculated inductively as*

$$N_s = \frac{\mathrm{tr}(M^s) - \sum\limits_{k\in\mathscr{P}(s)} kN_k}{s}, \quad s \geq 1 \tag{13.10}$$

We use an example to depict it.

Example 13.2. Recall Example 13.1. Since

$$M = \mathrm{Blk}_1(L) + \mathrm{Blk}_2(L) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

We can calculate that

$$\begin{array}{ll} \mathrm{tr}\, M = 3, & \mathrm{tr}\, M^3 = 6, \\ \mathrm{tr}\, M^4 = 15, & \mathrm{tr}\, M^5 = 33, \\ \mathrm{tr}\, M^6 = 66, & \mathrm{tr}\, M^7 = 129, \\ \mathrm{tr}\, M^8 = 255. & \end{array}$$

Using Theorem 13.2, we conclude that $N_1 = 3$, $N_3 = 1$, $N_4 = 3$, $N_5 = 6$, $N_6 = 10$, $N_7 = 18$, $N_8 = 30$. It is not an easy job to count them from the graph directly.

13.2 Controllability

Controllability is a fundamental topic in modern control theory. Limited by the mathematical tools, there are few known results on control design of Boolean control networks [1, 6, 7]. However, the semi-tensor product makes it much easier to investigate the controllability of Boolean control networks. First, we give the definition of controllability.

Definition 13.3. Consider system (13.1). Denote its state space as $\mathscr{X} = \mathscr{D}^n$, and let $X_0 \in \mathscr{X}$.

1. $X \in \mathscr{X}$ is said to be reachable from X_0 at time $s > 0$, if we can find a sequence of controls $U(0) = \{u_1(0), \cdots, u_m(0)\}$, $U(1) = \{u_1(1), \cdots, u_m(1)\}$, $\cdots$, such that the trajectory of (13.3) with the initial value X_0 and the controls $\{U(t)\}$, $t = 0, 1, \cdots$ will reach X at time $t = s$. The reachable set at time s is denoted by $R_s(X_0)$. The overall reachable set is denoted by
$$R(X_0) = \cup_{s=1}^{\infty} R_s(X_0).$$
2. System (13.1) is said to be controllable at X_0 if $R(X_0) = \mathscr{X}$. The system is said to be controllable if it is controllable at every $X \in \mathscr{X}$.

Before the investigation of controllability, we should introduce the Boolean product and power of Boolean matrix.

Definition 13.4. 1. Let $A_k = (a_{ij}^k) \in \mathscr{B}_{m\times n}$, $k = 1, \cdots, r$, σ is an r-ary logical operator, then
$$\sigma(A_1, \cdots, A_r) := (\sigma(a_{ij}^1, \cdots, a_{ij}^r)).$$
2. Let $A \in \mathscr{B}_{m\times n}$, $b \in \mathscr{D}$, the scalar product is define as

$$bA = Ab := b \wedge A.$$

Particularly, if $A = a \in \mathscr{D}$, $ab = ba = a \wedge b$

3. Let $A = (a_{ij}), B = (b_{ij}) \in \mathscr{B}_{m \times n}$. Then we define Boolean addition as

$$A +_{\mathscr{B}} B := (a_{ij} +_{\mathscr{B}} b_{ij}) := (a_{ij} \vee b_{ij}).$$

4. Let $A \in \mathscr{B}_{m \times n}$ and $B \in \mathscr{B}_{n \times p}$. Then the Boolean product is defined as

$$A \times_{\mathscr{B}} B := C \in \mathscr{B}_{m \times p}$$

where

$$c_{ij} = \sum_{k=1}^{n}{}_{\mathscr{B}} a_{ik} b_{kj}.$$

If $A \prec_t B$ ($A \succ_t B$), the Boolean semi-tensor product is defined as

$$A \ltimes_{\mathscr{B}} B := (A \otimes I_t) \ltimes_{\mathscr{B}} B. \quad (A \ltimes_{\mathscr{B}} B := A \ltimes_{\mathscr{B}} (B \otimes I_t).)$$

Particularly, if $A \prec_t A$ ($A \succ_t A$),

$$A^{(k)} := \underbrace{A \ltimes_{\mathscr{B}} A \ltimes_{\mathscr{B}} \cdots \ltimes_{\mathscr{B}} A}_{k}.$$

We use a simple example to illustrate the Boolean operations.

Example 13.3. Assume

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Then

$$\neg A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad A +_{\mathscr{B}} B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix},$$

$$A \ltimes_{\mathscr{B}} B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad A^{(s)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, s \geq 1$$

From Theorem 13.1, we know that if $\mathscr{J}_{ij}^s = c$, there are c pathes from $\delta_{2^{m+n}}^j$ to $\delta_{2^{m+n}}^i$ at s-th steps. That means, if $\mathscr{J}_{ij}^s > 0$, at s-th step, $\delta_{2^{m+n}}^i$ is reachable from $\delta_{2^{m+n}}^j$. When the controllability is considered, we do not need to consider how many paths from one state to the other, but only want to know whether a state can be reached from another one. Hence, we simply use the Boolean power of $\mathscr{J}$, and we have the following conclusion.

Theorem 13.3. *Consider system (13.3) with its input-state incidence matrix $\mathscr{J}$. Define the controllability matrix as*

$$\mathscr{M}_{\mathscr{C}} := \sum_{s=1}^{2^{m+n}}{}_{\mathscr{B}} M^{(s)} \in \mathscr{B}_{2^n \times 2^n}, \tag{13.11}$$

and denote $\mathscr{M}_{\mathscr{C}} = (c_{ij})$. Then

(i) $\delta_{2^n}^i$ is reachable from $\delta_{2^n}^j$, iff $c_{ij} = 1$;
(ii) The system is controllable at $\delta_{2^n}^j$, iff $\mathrm{Col}_j(\mathscr{M}_{\mathscr{C}}) = \mathbf{1}_{2^n}$;
(iii) The system is controllable, iff $\mathscr{M}_{\mathscr{C}} = \mathbf{1}_{2^n \times 2^n}$, where $\mathbf{1}_{2^n \times 2^n}$ is a $2^n \times 2^n$ matrix with all entries are 1.

Proof. (i): By Cayley-Hamilton Theorem in linear algebra, it is easy to see that if $\mathscr{J}_{ij}^s = 0$, $s \le 2^{m+n}$, then so are $\mathscr{J}^s$, $\forall s$. Thus, we consider only $\{\mathscr{J}^s | s \le 2^{m+n}\}$. Next, we know that $\delta_{2^n}^i$ is reachable from $\delta_{2^n}^j$, iff there exist β and s such that $\alpha \delta_{2^n}^i$ is reachable from $\beta \delta_{2^n}^j$ at s-th step for all $1 \le \alpha \le 2^m$, which means

$$1 = \sum_{s=1}^{2^{m+n}}{}_{\mathscr{B}} \sum_{\beta=1}^{2^m}{}_{\mathscr{B}} \mathrm{Blk}_\beta(\mathscr{J}_0^{(s)})_{ij} = \sum_{s=1}^{2^{m+n}}{}_{\mathscr{B}} \mathscr{M}_{\mathscr{C}\,ij} = c_{ij}.$$

(ii) and (iii) can be obtained directly from (i). □

Example 13.4. Consider the following Boolean control network

$$\begin{cases} x_1(t+1) = (x_1(t) \leftrightarrow x_2(t)) \vee u_1(t) \\ x_2(t+1) = \neg x_1(t) \wedge u_2(t), \end{cases} \tag{13.12}$$
$$y(t) = x_1(t) \vee x_2(t).$$

Setting $x(t) = \ltimes_{i=1}^2 x_i(t)$, $u = \ltimes_{i=1}^2 u_i(t)$, we have

$$\begin{cases} x(t+1) = Lu(t)x(t) \\ y(t) = Hx(t), \end{cases} \tag{13.13}$$

where

$$L = \delta_4[2\ 2\ 1\ 1\ 2\ 2\ 2\ 2\ 2\ 4\ 3\ 1\ 2\ 4\ 4\ 2],$$
$$H = \delta_2[1\ 1\ 1\ 2].$$

For system (13.12), the basic block of its input-state incidence matrix $\mathscr{J}_0 = L$.

1. Is δ_4^1 reachable from $x(0) = \delta_4^2$?
   After a straightforward computation, we have

$$(M^{(1)})_{12} = 0, \quad (M^{(2)})_{12} = 1.$$

   That means that $x(2) = \delta_4^1$ is reachable from $x(0) = \delta_4^2$ at 2nd step.

2. Is the system controllable or controllable at any point?
   We check the controllability matrix:

$$\mathscr{C} = \sum_{s=1}^{2^4} M^{(s)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

   According to Corollary 13.3, we conclude that

   (i) The system is not controllable. It is controllable at $x_0 = \delta_4^3 \sim (0,1)$.
   (ii) $x_d = \delta_4^3 \sim (0,1)$ is not reachable from $x_0 = \delta_4^1 \sim (1,1)$, or $x_0 = \delta_4^2 \sim (1,0)$, or $x_0 = \delta_4^4 \sim (0,0)$.

13.3 Observability

In this section we consider the observability of system (13.1) and (13.2). We first give a definition.

Definition 13.5. Consider system (13.1) and (13.2). Denote by $Y(t) = (y_1(t), \cdots, y_p(t)) \in \mathscr{D}^p$.

1. X_1^0 and X_2^0 are said to be distinguishable, if there exists a control sequence $\{U(0), U(1), \cdots, U(s)\}$, where $s \geq 0$, such that

$$Y^1(s+1) = y^{s+1}(U(s), \cdots, U(0), X_1^0) \neq Y^2(s+1) = y^{s+1}(U(s), \cdots, U(0), X_2^0). \tag{13.14}$$

2. The system is said to be observable, if any two initial points $X_1^0, X_2^0 \in \mathscr{X}$ are distinguishable.

In the following we introduce a necessary and sufficient condition for observability of controllable Boolean control networks.

Split L into 2^m equal blocks as

$$\begin{aligned} L =& [\mathrm{Blk}_1(L), \mathrm{Blk}_2(L), \cdots, \mathrm{Blk}_{2^m}(L)] \\ :=& [B_1, B_2, \cdots, B_{2^m}], \end{aligned}$$

where $B_i \in \mathscr{L}_{2^n \times 2^n}$, $i = 1, \cdots, 2^m$.

Define a sequence of sets of matrices $\Gamma_i \in \mathscr{L}_{2^p \times 2^n}$, $i = 0, 1, 2, \cdots$, as

$$
\begin{cases}
\Gamma_0 = \{H\} \\
\Gamma_1 = \{HB_i | i = 1,2,\cdots,2^m\} \\
\Gamma_2 = \{HB_iB_j \big| i,j = 1,2,\cdots,2^m\} \\
\vdots \\
\Gamma_s = \{HB_{i_1}B_{i_2}\cdots B_{i_s} | i_1,i_2,\cdots,i_s = 1,2,\cdots,2^m\} \\
\vdots
\end{cases} \tag{13.15}
$$

Note that $\Gamma_s \subset \mathscr{L}_{2^p\times 2^n}$, $\forall s$, and $\mathscr{L}_{2^p\times 2^n}$ is a finite set. Then it is easy to see that there exists a smallest s^* such that

$$
\Gamma_j \subset \cup_{k=1}^{s^*}\Gamma_k, \quad \forall j > s^*.
$$

For notational ease, we rewrite $\{\Gamma_1,\cdots,\Gamma_{s^*}\}$ as matrices, i.e.

$$
\Gamma_k = \begin{bmatrix}
H\underbrace{B_1B_1\cdots B_1}_{k} \\
H\underbrace{B_1B_1\cdots B_2}_{k} \\
\vdots \\
H\underbrace{B_{2^m}B_{2^m}\cdots B_{2^m}}_{k}
\end{bmatrix}.
$$

And then we construct a matrix, called the observability matrix, as

$$
\mathscr{O} = \begin{bmatrix}
\Gamma_0 \\
\Gamma_1 \\
\vdots \\
\Gamma_{s^*}
\end{bmatrix}.
$$

Refer to [5] or [3], we have the following result about observability.

Theorem 13.4. *Assume system (13.1) and (13.2) is controllable. Then it is observable, iff*

$$
\operatorname{rank}(\mathscr{O}) = 2^n. \tag{13.16}
$$

We give an example to illustrate it.

Example 13.5. Consider the following Boolean control network

$$
\begin{cases}
x_1(t+1) = \neg x_1(t) \vee x_2(t) \\
x_2(t+1) = u(t) \wedge \neg x_1(t) \vee (\neg u(t) \wedge x_1(t) \wedge \neg x_2(t))
\end{cases} \tag{13.17}
$$

$$
y(t) = x_1 \bar{\vee} x_2. \tag{13.18}
$$

Its algebraic form is

$$\begin{cases} x(t+1) = Lu(t)x(t) \\ y(t) = H(t) \end{cases} \tag{13.19}$$

where

$$L = \delta_4[2\ 4\ 1\ 1\quad 2\ 3\ 2\ 2],$$
$$H = \delta_2[2\ 1\ 1\ 2].$$

Checking the controllability matrix, we have

$$\mathscr{M}_{\mathscr{C}} = \sum_{s=1}^{2^3} \sum_{i=1}^{2} (\mathrm{Blk}_i(\mathscr{J}_0^{(s)})) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} > 0.$$

Thus the system is controllable.

A straightforward computation shows the observability matrix is

$$\mathscr{O} = \begin{bmatrix} H \\ HB_1 \\ HB_2 \\ HB_1B_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ \vdots & & & \end{bmatrix}.$$

From part of $\mathscr{O}$, all the columns has already been different, it is enough to see $\mathrm{rank}(\mathscr{O}) = 4$. Thus, the system is also observable.

There is another necessary and sufficient condition of observability in [2] which is basically the same as Theorem 13.4. But Theorem 13.4 is more convenient to use. However, these conditions need the system to be controllable. Next, we give a sufficient condition of observability which have no assumption about controllability.

For these, we will use the input-state incidence matrix defined in Section 1. It has been known that $\mathrm{Blk}_i(\mathscr{J}_0^{(s)})$ corresponds to the input $u(0) = \delta_{2^m}^i$, and $\mathrm{Col}_j[\mathrm{Blk}_i(\mathscr{J}_0^{(s)})]$ corresponds to $x_0 = \delta_{2^n}^j$. We want to exchange the order of the indexes i and j, precisely, we define

$$\tilde{\mathscr{J}}_0^{(s)} := \mathscr{J}_0^{(s)} W_{[2^n,2^m]}, \tag{13.20}$$

and then split it into 2^n blocks as

$$\tilde{\mathscr{J}}_0^{(s)} = \left[\text{Blk}_1(\tilde{\mathscr{J}}_0^{(s)}) \;\; \text{Blk}_2(\tilde{\mathscr{J}}_0^{(s)}) \;\; \cdots \;\; \text{Blk}_{2^n}(\tilde{\mathscr{J}}_0^{(s)}) \right],$$

where $\text{Blk}_i(\tilde{\mathscr{J}}_0^{(s)}) \in \mathscr{B}_{2^n \times 2^m}$, $i = 1, \cdots, 2^n$. Now we can see that each block $\text{Blk}_i(\tilde{\mathscr{J}}_0^{(s)})$ corresponds to $x_0 = \delta_{2^n}^i$, and in each block $\text{Col}_j(\text{Blk}_i(\tilde{\mathscr{J}}_0^{(s)}))$ corresponds to $u(0) = \delta_{2^m}^j$. Using this matrix, we can get the following result.

Theorem 13.5. *Consider system (13.1)-(13.2) with its algebraic form (13.3). If*

$$\bigvee_{s=1}^{2^{m+n}} \left[\left(H \ltimes \text{Blk}_i(\tilde{\mathscr{J}}_0^{(s)}) \right) \bar{\vee} \left(H \ltimes \text{Blk}_j(\tilde{\mathscr{J}}_0^{(s)}) \right) \right] \neq 0, \quad 1 \leq i < j \leq 2^n, \tag{13.21}$$

then the system is observable.

Example 13.6. Recall Example 13.4. System (13.12) is not controllable, so we can't use Theorem 13.4 to verify wether it is observable.

Denote

$$O_{ij} = \vee_{s=1}^{2^4} \left[\left(H \ltimes \text{Blk}_i(\tilde{\mathscr{J}}_0^{(s)}) \right) \bar{\vee} \left(H \ltimes \text{Blk}_j(\tilde{\mathscr{J}}_0^{(s)}) \right) \right].$$

A straightforward computation yields

$$O_{12} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \; O_{13} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \; O_{14} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix},$$
$$O_{23} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \; O_{24} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \; O_{34} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Then by Theorem (13.5), we can conclude that the system is observable.

13.4 Disturbance Decoupling

Disturbance decoupling problem (DDP) is a classical problem for control theory. In practice, a control system is inevitably affected by disturbance. To analyze and control the system, we first want to design a control such that the disturbance on the system will not affect the output of the system, which is the purpose of DDP.

Consider the following system,

$$\begin{cases} x_1(t+1) = f_1(x_1(t), \cdots, x_n(t), u_1(t), \cdots, u_m(t), \xi_1(t), \cdots, \xi_q(t)) \\ \vdots \\ x_n(t+1) = f_n(x_1(t), \cdots, x_n(t), u_1(t), \cdots, u_m(t), \xi_1(t), \cdots, \xi_q(t)), \end{cases}$$
$$y_j(t) = h_j(x(t)), \quad j = 1, \cdots, p, \tag{13.22}$$

where $\xi_i(t) \in \mathscr{D}$, $i = 1, \cdots, q$ are disturbances. Let $x(t) = \ltimes_{i=1}^n x_i(t)$, $u(t) = \ltimes_{i=1}^m u_i(t)$, $\xi(t) = \ltimes_{i=1}^q \xi_i(t)$, and $y(t) = \ltimes_{i=1}^p y_i(t)$. Then the algebraic form of (13.22) is expressed as

$$x(t+1) = Lu(t)\xi(t)x(t),\\ y(t) = Hx(t), \tag{13.23}$$

where $L \in \mathscr{L}_{2^n \times 2^{n+m+q}}$, $H \in \mathscr{L}_{2^p \times 2^n}$.

Definition 13.6. Consider system (13.22). The DDP is solvable, if we can find a feedback control $u(t) = \phi(x(t))$, and a coordinate transformation $z = T(x)$, such that under z coordinate frame the closed-loop system becomes

$$\begin{cases} z^1(t+1) = F^1(z(t), \phi(x(t)), \xi(t)) \\ z^2(t+1) = F^2(z^2(t)), \\ y(t) = G(z^2(t)). \end{cases} \tag{13.24}$$

where $z(t) = ((z^1(t))^T, (z^2(t))^T)^T$.

From the definition we can see that the key issues of solving DDP is to find a regular sub-basis $z^2(t)$ by which the output can be expressed, and a proper feedback control such that the complement coordinate sub-basis z^1 and the disturbances ξ can be deleted from the dynamics of z^2. We begin with finding the coordinate transformation. For this, we should introduce a new kind of subspace.

Definition 13.7. Let $\mathscr{X} = \mathscr{F}_\ell\{x_1, \cdots, x_n\}$ be the state space, and $Y = \{y_1, \cdots, y_p\} \subset \mathscr{X}$. A regular subspace $\mathscr{Z} \subset \mathscr{X}$ is called a Y-friendly subspace, if $y_i \in \mathscr{Z}$, $i = 1, \cdots, p$. A Y-friendly subspace of minimum dimension is called a minimum Y-friendly subspace.

Next, we will discuss how to find out a minimum Y-friendly. Express the algebraic form of output as

$$y = Hx = \delta_{2^p}[i_1\ i_2\ \cdots\ i_{2^n}]x. \tag{13.25}$$

Denote by

$$n_j = \left|\{k \,|\, i_k = j, 1 \le k \le 2^n\}\right|, \quad j = 1, 2, \cdots, 2^p$$

where $|\cdot|$ is the cardinal number of the set. Then we have the following theorem.

Theorem 13.6. *Consider system (13.22). Assume $y = \ltimes_{i=1}^p y_i$ has its algebraic form (13.25).*

1. *There is a Y-friendly subspace of dimension r, iff n_j, $j = 1, \cdots, 2^p$ have a common factor 2^{n-r}.*
2. *Assume 2^{n-r} is the largest common 2-type factor of n_j, $j = 1, \cdots, 2^p$. Then the minimum Y-friendly subspace is of dimension r.*

The following algorithm for finding y can be considered as a proof of this theorem, we leave the rigorous proof for exercise.

Algorithm 1. 1. For system (13.22), find a common 2-type factor 2^{n-r} of n_j, $j = 1,\cdots,2^p$, and $n_j = 2^{n-r}m_j$. Denote $J_j = \{k|i_k = j\}$ and $I_j = \{1+\sum_{k=1}^{j-1} m_k,\cdots,\sum_{k=1}^{j} m_k\}$ for $j = 1,\cdots,2^p$. Split a $2^r \times 2^n$ matrix $T_0 = (t_{r,s})$ to $2^p \times 2^p$ minors as

$$T_0^{i,j} = \{t_{r,s}\,|\,r \in I_i, s \in J_j\}, \quad i,j = 1,\cdots,2^p.$$

2. Set

$$T_0^{i,j} = \begin{cases} I_{m_i} \otimes \mathbf{1}_{2^{n-r}}^T, & i = j \\ 0, & \text{otherwise}, \end{cases} \tag{13.26}$$

which is an $m_i \times m_j 2^{n-r}$ matrix.

3. Set

$$z = \ltimes_{i=1}^{r} z_i := T_0 x.$$

And then retrieve z_i, $i = 1,\cdots,r$ from T_0. (We refer to Chapter 6 for retrieving technique.) $\{z_i|i = 1,\cdots,r\}$ is a sub-basis of an r-dimensional Y-friendly subspace.

This algorithm provide a way to construct an r-dimensional Y-friendly subspace, but it is obvious that different assignments of 1 in $T_0^{i,i}$ can construct different sub-bases. One may expect that those sub-bases form a unique subspace. Unfortunately, the following example shows the minimum Y-friendly subspace is not unique.

Example 13.7. Let $\mathscr{X} = \mathscr{F}_\ell\{x_1,x_2,x_3,x_4\}$.

$$\begin{aligned} y_1 &= h_1(x_1,x_2,x_3,x_4) = (x_1 \leftrightarrow x_3) \wedge (x_2 \bar{\vee} x_4), \\ y_2 &= h_2(x_1,x_2,x_3,x_4) = x_1 \wedge x_3 \end{aligned} \tag{13.27}$$

Setting $x = \ltimes_{i=1}^{4} x_i$, $y = y_1 y_2$, we have its algebraic form as $y = Hx$, where

$$H = \delta_4[3\ 1\ 4\ 4\ 1\ 3\ 4\ 4\ 4\ 4\ 4\ 2\ 4\ 4\ 2\ 4].$$

Thus, $n_1 = n_2 = n_3 = 2$, $n_4 = 10$. Since the largest common 2-type factor is $2 = 2^{4-3}$, we can have the minimum Y-friendly subspace of dimension $r = 3$. To construct T_0 we have

$$\begin{aligned} &J_1 = \{2,5\}; \quad J_2 = \{12,15\}; \quad J_3 = \{1,6\}; \\ &J_4 = \{3,4,7,8,9,10,11,13,14,16\}; \\ &I_1 = \{1\}; \quad I_2 = \{2\}; \quad I_3 = \{3\}; \quad I_4 = \{4,5,6,7,8\}. \end{aligned}$$

By (13.26), T_0 is obtained as

$$T_0 = \delta_8[3\ 1\ 4\ 4\ 1\ 3\ 5\ 5\ 6\ 6\ 7\ 2\ 7\ 8\ 2\ 8].$$

Correspondingly, we can construct

$$G = \delta_4[1\ 2\ 3\ 4\ 4\ 4\ 4\ 4],$$

such that $GT_0 = H$.

Thus we have obtained a sub-basis $\{z_1, z_2, z_3\}$ such that $z = z_1 z_2 z_3 = T_0 x$ and $y = Gz$.

$T_0^{4,4}$ can also be assigned as

$$T_0^{4,4} = \begin{bmatrix} 1\,0\,0\,0\,0\,0\,1\,0 \\ 0\,1\,0\,0\,0\,0\,0\,1 \\ 0\,0\,1\,0\,1\,0\,0\,0 \\ 0\,0\,0\,1\,0\,1\,0\,0 \end{bmatrix},$$

then we can get

$$T_0' = \delta_8[3\;1\;5\;6\;1\;3\;7\;8\;7\;8\;4\;2\;4\;5\;2\;6],$$

which also satisfies $GT_0' = H$. Denoted by $\{z_1', z_2', z_3'\}$ the sub-basis satisfying $z' = z_1' z_2' z_3' = T_0' x$, and $\mathscr{Z} = \mathscr{F}_\ell\{z_1, z_2, z_3\}, \mathscr{Z}' = \mathscr{F}_\ell\{z_1', z_2', z_3'\}$.

Suppose $\mathscr{Z} = \mathscr{Z}'$, then there exists a nonsingular matrix $P \in \mathscr{L}_{8\times 8}$, such that $T_0 = PT_0'$. Hence

$$P = T_0 T_0'^T (T_0' T_0'^T)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 \end{bmatrix}$$

is not a logical matrix. Thus $\mathscr{Z} \neq \mathscr{Z}'$.

Assume a Y-friendly subspace is obtained as z^2. Then we can find z^1, such that $z = \{z^1, z^2\}$ form a new coordinate frame, and the system can be expressed in the new coordinate frame as

$$\begin{cases} z^1(t+1) = F^1(z(t), u(t), \xi(t)) \\ z^2(t+1) = F^2(z(t), u(t), \xi(t)), \end{cases} \\ \quad y(t) = G(z^2(t)). \tag{13.28}$$

We call it the Y-friendly form.

Then it is obvious that if we can find a feedback control $u(t) = u(z(t))$, such that F^2 only depends on z^2, we have solved the DDP.

Assume $z^2 = (z_1^2, \cdots, z_k^2)$ is of dimension k, and the feedback control $u(t) = Kz^1(t)$. Then $F^2(z(t), u(t), \xi(t))$ can be expressed as

$$\begin{aligned} F^2(z(t),u(t),\xi(t)) &= M_2 u(t)\xi(t)z(t) \\ &= M_2 K W_{[2^q,2^{n-k}]}(I_{w^q}\otimes M_r^{2^{n-k}})W_{[2^k,2^{q+n-k}]}z^2(t)\xi(t)z^1(t) \\ &:= Q z^2(t)\xi(t)z^1(t) \end{aligned} \tag{13.29}$$

where M_2 is the structure matrix of F^2.

Split Q to 2^k blocks

$$Q = [Q_1, Q_2, \cdots, Q_{2^k}]$$

It is obvious that $F^2(z(t),u(t),\xi(t)) = F^2(z^2(t))$, if and only if for each j, all the columns of Q_j are the same.

To verify whether the DDP is solvable, we should try every subspaces to see if all the columns of Q_j are the same. But there are lots of Y-friendly subspaces of system (13.22). However, the following proposition may help to reduce the complexity.

Proposition 13.2. *Let V be a $Y = \{y_1,\cdots,y_p\}$-friendly subspace. Then there exists a minimum Y-friendly subspace, $W \subset V$.*

Proof. Denote by

$$y = \ltimes_{i=1}^p y_i = Hx,$$

and $n_i = |\{j|\,\mathrm{Col}_j(H) = \delta_{2^p}^i\}|$, $i = 1,\cdots,2^p$. We know that 2^s is the largest common 2-type factor of $\{n_i\}$, iff the minimum Y-friendly subspace has dimension of $n-s$.

Let $\{v_1,\cdots,v_t\}$ be a basis of V. If $t = n-s$, we are done. So we assume $t > n-s$. Since V is a Y-friendly subspace, denoting $v = \ltimes_{i=1}^t v_i$, we can express

$$y = Gv, \quad \text{where } G \in \mathscr{L}_{2^p\times 2^t}.$$

Denote $r_i = |\{j|\,\mathrm{Col}_j(G) = \delta_{2^p}^i\}|$, $i = 1,\cdots,2^p$. Assume that 2^j is the largest 2-type common factor of r_i, and denote $r_i = m_i 2^j$. Since V is a regular subspace, we denote $v = Ux$, where $U \in \mathscr{L}_{2^t\times 2^n}$ and (12.26) holds for U. Note that

$$y = Gv = GUx,$$

we have to calculate GU. Using the construction of G and the property of U, it is easy to verify that each column of $\delta_{2^p}^i$ yields 2^{n-t} columns of $\delta_{2^p}^i$ in GU. Hence we have

$$r_i \cdot 2^{n-t} = m_i \cdot 2^{n-t+j}.$$

That means, the largest common 2-type factor of $\{n_i\}$ is 2^{n-t+j}. It follows that $n-t+j = s$. Equivalently,

$$j = t-(n-s).$$

Since the dimension of V is t and r_i have largest 2-type common fact 2^j, we can find out a minimum Y-friendly subspace of V of the dimension $t-j = t-[t-(n-s)] = n-s$. It follows from the dimension that this Y-friendly minimum subspace of V is also a minimum Y-friendly subspace of $\mathscr{X} = \mathscr{F}_\ell\{x_1,\cdots,x_n\}$. □

We can see that if for a minimum Y-friendly subspace W whose sub-basis is $z^2(t)$, there is no such u that $\xi(t)$ can be deleted from $F_2(z^1(t), z^2(t), u(t), \xi(t))$, then neither for other Y-friendly subspaces containing W. Hence, using this proposition, we can start from minimum Y-friendly subspaces to solve DDP.

We give an example to describe how to find a solution of DDP.

Example 13.8. Consider the following system

$$\begin{cases} x_1(t+1) = x_4(t) \bar{\vee} u_1(t) \\ x_2(t+1) = (x_2(t) \bar{\vee} x_3(t)) \wedge \neg \xi(t) \\ x_3(t+1) = [(x_2(t) \leftrightarrow x_3(t)) \vee \xi(t)] \bar{\vee} [(x_1 \leftrightarrow x_5) \vee u_2(t)] \\ x_4(t+1) = [u_1(t) \rightarrow (\neg x_2(t) \vee \xi(t))] \wedge (x_2(t) \leftrightarrow x_3(t)) \\ x_5(t+1) = (x_4(t) \bar{\vee} u_1(t)) \leftrightarrow [(u_2(t) \wedge \neg x_2(t)) \vee x_4(t)], \end{cases}$$
$$y(t) = x_4(t) \wedge (x_1(t) \leftrightarrow x_5(t)), \tag{13.30}$$

where $u_1(t), u_2(t)$ are controls, $\xi(t)$ is a disturbance, $y(t)$ is the output.

Setting $x(t) = \ltimes_{i=1}^5 x_i(t)$, $u = u_1(t) u_2(t)$, we express (**??**) into it algebraic form as

$$\begin{aligned} x(t+1) &= L u(t) \xi(t) x(t) \\ y(t) &= h(t), \end{aligned} \tag{13.31}$$

where

$$\begin{aligned} L = \delta_{32}[&30\ 30\ 14\ 14\ 32\ 32\ 16\ 16\ 32\ 32\ 15\ 15\ 30\ 30\ 13\ 13 \\ &30\ 30\ 14\ 14\ 32\ 32\ 16\ 16\ 32\ 32\ 15\ 15\ 30\ 30\ 13\ 13 \\ &32\ 32\ 16\ 16\ 20\ 20\ 4\ 4\ 20\ 20\ 3\ 3\ 30\ 30\ 13\ 13 \\ &32\ 32\ 16\ 16\ 20\ 20\ 4\ 4\ 20\ 20\ 3\ 3\ 30\ 30\ 13\ 13 \\ &30\ 26\ 14\ 10\ 32\ 28\ 16\ 12\ 32\ 28\ 16\ 12\ 30\ 26\ 14\ 10 \\ &26\ 30\ 10\ 14\ 28\ 32\ 12\ 16\ 28\ 32\ 12\ 16\ 26\ 30\ 10\ 14 \\ &32\ 28\ 16\ 12\ 20\ 24\ 4\ 8\ 20\ 24\ 4\ 8\ 30\ 26\ 14\ 10 \\ &28\ 32\ 12\ 16\ 24\ 20\ 8\ 4\ 24\ 20\ 8\ 4\ 26\ 30\ 10\ 14 \\ &13\ 13\ 29\ 29\ 15\ 15\ 31\ 31\ 15\ 15\ 32\ 32\ 13\ 13\ 30\ 30 \\ &13\ 13\ 29\ 29\ 15\ 15\ 31\ 31\ 15\ 15\ 32\ 32\ 13\ 13\ 30\ 30 \\ &13\ 13\ 29\ 29\ 3\ 3\ 19\ 19\ 3\ 3\ 20\ 20\ 13\ 13\ 30\ 30 \\ &13\ 13\ 29\ 29\ 3\ 3\ 19\ 19\ 3\ 3\ 20\ 20\ 13\ 13\ 30\ 30 \\ &13\ 9\ 29\ 25\ 15\ 11\ 31\ 27\ 15\ 11\ 31\ 27\ 13\ 9\ 29\ 25 \\ &9\ 13\ 25\ 29\ 11\ 15\ 27\ 31\ 11\ 15\ 27\ 31\ 9\ 13\ 25\ 29 \\ &13\ 9\ 29\ 25\ 3\ 7\ 19\ 23\ 3\ 7\ 19\ 23\ 13\ 9\ 29\ 25 \\ &9\ 13\ 25\ 29\ 7\ 3\ 23\ 19\ 7\ 3\ 23\ 19\ 9\ 13\ 25\ 29], \end{aligned}$$

and

$$h = \delta_2[1\ 2\ 2\ 2\ 1\ 2\ 2\ 2\ 1\ 2\ 2\ 2\ 1\ 2\ 2\ 2\ 2\ 1\ 2\ 2\ 2\ 1\ 2\ 2\ 2\ 1\ 2\ 2\ 2\ 1\ 2\ 2].$$

First, we find the minimum output-friendly subspace. From h we can see $n_1 = 8$ and $n_2 = 24$. Then we have the largest 2-type common factor $2^s = 2^3$, and $m_1 = 1$, $m_2 = 3$. Hence, we know that the minimum output-friendly subspace is of dimension

$n-s=5-3=2$. Using Algorithm 1, we may choose

$$T_0=\delta_4[1\ 2\ 3\ 4\ 1\ 2\ 3\ 4\ 1\ 2\ 3\ 4\ 1\ 2\ 3\ 4\ 2\ 1\ 4\ 3\ 2\ 1\ 4\ 3\ 2\ 1\ 4\ 3\ 2\ 1\ 4\ 3].$$

and

$$G=\delta_2[1\ 2\ 2\ 2].$$

From T_0 we can find the output-friendly sub-basis, denote it by $\{z_4,z_5\}$, with $z_4z_5=T_0x$. Add $\{z_1,z_2,z_3\}$ to make $z=z^1z^2=\ltimes_{i=1}^5 z_i=Tx$ is a coordinate transformation. Using Proposition 6.4, we simply set $z_i=M_ix$, $i=1,2,3$, where M_i are chosen as follows:

$$\begin{aligned}
M_1&=\delta_2[1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2];\\
M_2&=\delta_2[2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1];\\
M_3&=\delta_2[1\ 1\ 1\ 1\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 1\ 1\ 1\ 1].
\end{aligned}$$

It is easy to check that

$$\begin{aligned}
T=\delta_{32}[\ &9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 5\ 6\ 7\ 8\ 1\ 2\ 3\ 4\ 26\ 25\\
&28\ 27\ 30\ 29\ 32\ 31\ 22\ 21\ 24\ 23\ 18\ 17\ 20\ 19]
\end{aligned}$$

is a coordinate transformation. Conversely, we have $x=T^Tz$, with

$$\begin{aligned}
T^T=[\ &13\ 14\ 15\ 16\ 9\ 10\ 11\ 12\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 30\ 29\\
&32\ 31\ 26\ 25\ 28\ 27\ 18\ 17\ 20\ 19\ 22\ 21\ 24\ 23].
\end{aligned}$$

Now under the coordinate frame z,

$$\begin{aligned}
z(t+1)&=Tx(t+1)=TLu(t)\xi(t)x(t)=TLu(t)\xi(t)T^Tz(t)\\
&=TL(I_8\otimes T^T)u(t)\xi(t)z(t):=\tilde{L}u(t)\xi(t)z(t).
\end{aligned}$$

We have its Y-friendly form as

$$\begin{cases}
z^1(t+1)=\tilde{L}_1u(t)\xi(t)z(t)\\
z^2(t+1)=\tilde{L}_2u(t)\xi(t)z(t)
\end{cases}$$
$$y(t)=Gz^2(t),$$

where

$$\begin{aligned}
\tilde{L}_1=\delta_8[&5\ 5\ 1\ 1\ 5\ 5\ 1\ 1\ 5\ 5\ 1\ 1\ 5\ 5\ 1\ 1\ 5\ 5\ 1\ 1\ 5\ 5\ 1\ 1\ 5\ 5\ 1\ 1\ 5\ 5\ 1\ 1\\
&5\ 5\ 1\ 1\ 7\ 7\ 3\ 3\ 5\ 5\ 1\ 1\ 7\ 7\ 3\ 3\ 5\ 5\ 1\ 1\ 7\ 7\ 3\ 3\ 5\ 5\ 1\ 1\ 7\ 7\ 3\ 3\\
&5\ 6\ 1\ 2\ 5\ 6\ 1\ 2\ 5\ 6\ 1\ 2\ 5\ 6\ 1\ 2\ 5\ 6\ 1\ 2\ 5\ 6\ 1\ 2\ 5\ 6\ 1\ 2\ 5\ 6\ 1\ 2\\
&5\ 6\ 1\ 2\ 7\ 8\ 3\ 4\ 5\ 6\ 1\ 2\ 7\ 8\ 3\ 4\ 5\ 6\ 1\ 2\ 7\ 8\ 3\ 4\ 5\ 6\ 1\ 2\ 7\ 8\ 3\ 4\\
&1\ 1\ 5\ 5\ 1\ 1\ 5\ 5\ 1\ 1\ 5\ 5\ 1\ 1\ 5\ 5\ 1\ 1\ 5\ 5\ 1\ 1\ 5\ 5\ 1\ 1\ 5\ 5\ 1\ 1\ 5\ 5\\
&1\ 1\ 5\ 5\ 3\ 3\ 7\ 7\ 1\ 1\ 5\ 5\ 3\ 3\ 7\ 7\ 1\ 1\ 5\ 5\ 3\ 3\ 7\ 7\ 1\ 1\ 5\ 5\ 3\ 3\ 7\ 7\\
&1\ 2\ 5\ 6\ 1\ 2\ 5\ 6\ 1\ 2\ 5\ 6\ 1\ 2\ 5\ 6\ 1\ 2\ 5\ 6\ 1\ 2\ 5\ 6\ 1\ 2\ 5\ 6\ 1\ 2\ 5\ 6\\
&1\ 2\ 5\ 6\ 3\ 4\ 7\ 8\ 1\ 2\ 5\ 6\ 3\ 4\ 7\ 8\ 1\ 2\ 5\ 6\ 3\ 4\ 7\ 8\ 1\ 2\ 5\ 6\ 3\ 4\ 7\ 8];
\end{aligned}$$

$$\begin{aligned}\tilde{L}_2 = \delta_4[&1\ 1\ 1\ 1\ 3\ 3\ 3\ 3\ 1\ 1\ 2\ 2\ 3\ 3\ 4\ 4\ 1\ 1\ 1\ 1\ 3\ 3\ 3\ 3\ 1\ 1\ 2\ 2\ 3\ 3\ 4\ 4\\ &1\ 1\ 1\ 1\ 3\ 3\ 3\ 3\ 3\ 3\ 4\ 4\ 3\ 3\ 4\ 4\ 1\ 1\ 1\ 1\ 3\ 3\ 3\ 3\ 3\ 3\ 4\ 4\ 3\ 3\ 4\ 4\\ &1\ 1\ 2\ 2\ 3\ 3\ 4\ 4\ 1\ 1\ 2\ 2\ 3\ 3\ 4\ 4\ 1\ 1\ 2\ 2\ 3\ 3\ 4\ 4\ 1\ 1\ 2\ 2\ 3\ 3\ 4\ 4\\ &1\ 1\ 2\ 2\ 3\ 3\ 4\ 4\ 3\ 3\ 4\ 4\ 3\ 3\ 4\ 4\ 1\ 1\ 2\ 2\ 3\ 3\ 4\ 4\ 3\ 3\ 4\ 4\ 3\ 3\ 4\ 4\\ &1\ 1\ 1\ 1\ 3\ 3\ 3\ 3\ 1\ 1\ 2\ 2\ 3\ 3\ 4\ 4\ 1\ 1\ 1\ 1\ 3\ 3\ 3\ 3\ 1\ 1\ 2\ 2\ 3\ 3\ 4\ 4\\ &1\ 1\ 1\ 1\ 3\ 3\ 3\ 3\ 1\ 1\ 2\ 2\ 3\ 3\ 4\ 4\ 1\ 1\ 1\ 1\ 3\ 3\ 3\ 3\ 1\ 1\ 2\ 2\ 3\ 3\ 4\ 4\\ &1\ 1\ 2\ 2\ 3\ 3\ 4\ 4\ 1\ 1\ 2\ 2\ 3\ 3\ 4\ 4\ 1\ 1\ 2\ 2\ 3\ 3\ 4\ 4\ 1\ 1\ 2\ 2\ 3\ 3\ 4\ 4\\ &1\ 1\ 2\ 2\ 3\ 3\ 4\ 4\ 1\ 1\ 2\ 2\ 3\ 3\ 4\ 4\ 1\ 1\ 2\ 2\ 3\ 3\ 4\ 4\ 1\ 1\ 2\ 2\ 3\ 3\ 4\ 4];\end{aligned}$$

For all $K \in \mathscr{L}_{4,8}$, we can not find a K such that each 4×16 block Q_i of $Q = \tilde{L}_2 K W_{[2,8]}(I_2 \otimes M_r^8) W_{[4,16]}$ has the same columns.

However, when we set

$$K = \delta_4[4\ 4\ 4\ 4\ 4\ 4\ 4\ 4],$$

which means $u_1(t) = u_2(t) \equiv 0$, we have

$$\begin{aligned}Q = \delta_4[&1\ 3\ 1\ 3\ 1\ 3\ 1\ 3\ 1\ 3\ 1\ 3\ 1\ 3\ 1\ 3\ 1\ 3\ 1\ 3\ 1\ 3\ 1\ 3\ 1\ 3\ 1\ 3\ 1\ 3\ 1\ 3\\ &2\ 4\ 2\ 4\ 2\ 4\ 2\ 4\ 2\ 4\ 2\ 4\ 2\ 4\ 2\ 4\ 2\ 4\ 2\ 4\ 2\ 4\ 2\ 4\ 2\ 4\ 2\ 4\ 2\ 4\ 2\ 4].\end{aligned}$$

Then we can see that $z_4(t+1) = z_3(t)$, $z_5(t+1) = z_4(t)$. Thus we want to enlarge the output-friendly subspace to include z_3 to see whether z_3 is affected by ξ, z_1 and z_2. Retrieving the system under the controls $u_1(t) = u_2(t) \equiv 0$, we have

$$\begin{cases} z_1(t+1) = z_4(t) \\ z_2(t+1) = z_3(t) \vee \xi(t) \\ z_3(t+1) = z_5(t) \\ z_4(t+1) = z_3(t) \\ z_5(t+1) = z_4(t), \end{cases} \\ y(t) = z_4(t) \wedge z_5(t). \tag{13.32}$$

One sees that the closed-loop system is in such a form that the DDP is solved.

13.5 Some Other Control Problems

This section briefly introduces stabilization, optimal control and identification of Boolean control networks. Most of proofs are ignored, one can refer to [3], [4], [9] and [5] for details.

13.5.1 Stability and Stabilization

We give rigorous definition of stability and stabilization. Consider Boolean network

$$\begin{cases} x_1(t+1) = f_1(x_1(t), x_2(t), \cdots, x_n(t)) \\ x_2(t+1) = f_2(x_1(t), x_2(t), \cdots, x_n(t)) \\ \vdots \\ x_n(t+1) = f_n(x_1(t), x_2(t), \cdots, x_n(t)), \end{cases} \tag{13.33}$$

or Boolean control network

$$\begin{cases} x_1(t+1) = f_1(x_1(t), x_2(t), \cdots, x_n(t), u_1(t), \cdots, u_m(t)) \\ x_2(t+1) = f_2(x_1(t), x_2(t), \cdots, x_n(t), u_1(t), \cdots, u_m(t)) \\ \vdots \\ x_n(t+1) = f_n(x_1(t), x_2(t), \cdots, x_n(t), u_1(t), \cdots, u_m(t)). \end{cases} \tag{13.34}$$

Their algebraic forms are, respectively

$$x(t+1) = Lx(t) \tag{13.35}$$

and

$$x(t+1) = Lu(t)x(t). \tag{13.36}$$

Definition 13.8. 1. System (13.33) is said to be globally stable if it is globally convergent. In other words, it has a fixed point as a global attractor (equivalently, the only attractor).
2. The global stabilization problem of system (13.34) is to find, if possible, $u(t)$ such that the system becomes globally convergent.

The following results for global stability and stabilization are straightforward.

Theorem 13.7. *1. System (13.33) with its algebraic form (13.35) is global stable, iff there exist a state $x_0 = \delta_{2^n}^i$ and a positive integer k such that*

$$\mathrm{Col}_i(L) = \delta_{2^n}^i, \quad \mathrm{Col}_j(L^k) = \delta_{2^n}^i, \ j = 1, 2, \cdots, 2^n.$$

2. System (13.34) with its algebraic form (13.36) can be globally stabilized to $x_0 = \delta_{2^n}^i$ by a closed loop control $u(t) = Gx(t)$, iff there exists a positive integer k such that

$$\mathrm{Col}_i(LGM_r^{2^n}) = \delta_{2^n}^i, \quad \mathrm{Col}_j((LGM_r^{2^n})^k) = \delta_{2^n}^i, \ j = 1, 2, \cdots, 2^n.$$

3. System (13.34) with its algebraic form (13.36) can be globally stabilized to $x_0 = \delta_{2^n}^i$ by $u(t)$, iff there exists a positive integer k such that

$$\mathrm{Col}_i(M) = \delta_{2^n}^i, \quad M_{ij}^{(k)} = 1, \ j = 1, 2, \cdots, 2^n,$$

where M is defined in (13.9).

Although these results are necessary and sufficient, it is hard to calculate when n is large. Next, we will propose a sufficient condition of global stability which uses the incidence matrix of the network graph (it is different from the input-state incidence matrix). For this, we firstly introduce vector distance of Boolean matrices.

Definition 13.9. 1. Recall the network graph of system (13.33), the nodes $\mathscr{N} = \{x_1, x_2, \cdots, x_n\}$, the edges $\mathscr{E} = \{(x_i, x_j) | x_j \text{ is affected by } x_i\}$. Denoted $X = (x_1, x_2, \cdots, x_n)^T \in \mathscr{D}^n$, and the function of the dynamic by $X(t+1) = F(X(t))$. The incidence matrix of the network graph of is a $n \times n$ Boolean matrix $\mathscr{I}(F) = (I_{ij})$, where

$$I_{ij} = \begin{cases} 1, & (x_i, x_j) \in \mathscr{E} \\ 0, & \text{otherwise.} \end{cases} \tag{13.37}$$

2. Let $X = (x_{ij}), Y = (y_{ij}) \in \mathscr{B}_{m\times n}$. We said $X \le Y$ if $x_{ij} \le y_{ij}$, $\forall i, j$.
3. Let $X = (x_{ij}), Y = (y_{ij}) \in \mathscr{B}_{m\times n}$. The vector distance of X and Y, denoted by $D_v(X,Y)$, is defined as

$$D_v(X,Y) = X \bar{\nabla} Y. \tag{13.38}$$

We leave the following properties of vector distance for exercise.

Proposition 13.3. *1. $D_v(X,Y)$ defined in (13.38) is a distance.*
2. Let A, $B \in \mathscr{B}_{m\times n}$, and $C \in \mathscr{B}_{n\times p}$, $E \in \mathscr{B}_{q\times m}$. Then

$$D_v(AC, BC) \le D_v(A,B)C, \quad D_v(EA, EB) \le ED_v(A,B). \tag{13.39}$$

In the following, we use the scalar form of state, we can see that $X \in \mathscr{B}_{n\times 1}$, thus we can employ $D_v(X,Y)$ to describe the distance of two states.

Theorem 13.8. *If $\xi \in \mathscr{D}^n$ is a fixed point of Boolean network (13.33), and there exists an integer $k > 0$ such that*

$$[\mathscr{I}(F)]^{(k)} = 0, \tag{13.40}$$

then ξ is global attractor.

Proof. Firstly, we prove that

$$D_v(F(X), F(Y)) \le \mathscr{I}(F) \ltimes_{\mathscr{B}} D_v(X,Y). \tag{13.41}$$

Since $D_v(F(X), F(Y)) = (D_v(f_1(X), f_1(Y)), \cdots, D_v(f_n(X), f_n(Y)))$, using triangle inequality

$$
\begin{aligned}
D_v(f_i(X), f_i(Y)) \leq & D_v(f_i(x_1,\cdots,x_n), f_i(y_1,x_2,\cdots,x_n)) \\
& +_{\mathscr{B}} D_v(f_i(y_1,x_2,\cdots,x_n), f_i(y_1,y_2,x_3,\cdots,x_n)) \\
& +_{\mathscr{B}} \cdots \\
& +_{\mathscr{B}} D_v(f_i(y_1,\cdots,y_{n-1},x_n), f_i(y_1,\cdots,y_n)) \\
\leq & \sum_{k=1}^{n}{}_{\mathscr{B}} b_{i,k} D_v(x_k, y_k),
\end{aligned}
$$

then (13.41) follows.

If $\xi \in \mathscr{D}^n$ is a fixed point and $[\mathscr{I}(F)]^{(k)} = 0$, then using (13.41) we have

$$
D_v(F^k(X), \xi) \leq [\mathscr{I}(F)]^{(k)} \ltimes_{\mathscr{B}} D_v(X, \xi) = 0,
$$

for any $X \in \mathscr{D}^n$. That means for any initial state X_0, after at most k steps, the state will converge to ξ. □

Ones can see that the condition in Theorem 13.7 is coordinate-independent, since if L^k is a constant mapping (all the columns are the same), then so is $T^{-1}L^kT$ for any coordinate transformation T. But this is not true for the condition in Theorem 13.8. We give an example to illustrate these.

Example 13.9. Consider the following system

$$
\begin{cases}
x_1(t+1) = [x_1(t) \wedge (x_2(t) \bar{\vee} x_3(x))] \vee (\neg x_1(t) \wedge x_3(t)), \\
x_2(t+1) = [x_1(t) \wedge (\neg x_2(t))] \vee (\neg x_1 \wedge x_2), \\
x_3(t+1) = [x_1(t) \wedge (\neg (x_2(t) \wedge x_3(t)))] \vee [\neg x_1(t) \wedge (x_2(t) \vee x_3(t))].
\end{cases}
\tag{13.42}
$$

Setting $x(t) = x_1(t)x_2(t)x_3(t)$, we have $x(t+1) = Lx(t)$, where

$$
L = \delta_8[8\ 3\ 1\ 5\ 1\ 5\ 3\ 8].
$$

Then by Theorem 12.2, we know that the only fixed point of this system is $\delta_8^8 \sim (0,0,0)^T := \xi$. By direct calculation, we have

$$
L^3 = \delta_8[8\ 8\ 8\ 8\ 8\ 8\ 8\ 8].
$$

Thus, by Theorem 13.7 we can conclude that this system is globally stable.

Next, we try to use Theorem 13.8. The incidence matrix of network graph of this system is

$$
\mathscr{I}(F) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.
$$

We can see that $[\mathscr{I}(F)]^{(k)} = \mathbf{1}_{3\times 3}$ for $k \geq 2$, then Theorem 13.8 can not be employed directly.

However, we consider a coordinate transformation as

$$\begin{cases} z_1 = [x_1 \wedge \neg(x_3)] \vee [(\neg x_1) \wedge (x_2 \bar{\vee} x_3)] \\ z_2 = [x_1 \wedge (x_2 \bar{\vee} x_3)] \vee [(\neg x_1) \wedge x_3] \\ z_3 = x_2. \end{cases}$$

In the vector form, we can easily calculated that

$$z = z_1 z_2 z_3 = Tx,$$

where

$$T = \delta_8[7\ 1\ 6\ 4\ 5\ 3\ 2\ 8].$$

Then in coordinate frame z we have

$$z(t+1) = TLT^T z(t) := \tilde{L}z(t),$$

where

$$\tilde{L} = \delta_8[6\ 6\ 5\ 5\ 7\ 7\ 8\ 8].$$

Retrieve this back to logical form, we have

$$\begin{cases} z_1(t+1) = 0 \\ z_2(t+1) = z_1(t) \\ z_3(t+1) = z_1(t) \bar{\vee} z_2(t). \end{cases} \tag{13.43}$$

Then its incidence matrix is

$$\mathscr{I}(\tilde{F}) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix},$$

which is strictly lower triangular, hence $[\mathscr{I}(\tilde{F})]^{(3)} = 0$. And we can see $Z(t) = (z_1(t), z_2(t), z_3(t))^T = (0,0,0)^T$ is the only fixed point of system (13.43), thus we can conclude by Theorem 13.8 that system (13.43) globally converge to $Z_0 = (0,0,0)^T$. Equivalently, system (13.42) globally converge to $X_0 = (0,0,0)^T$.

For Boolean control networks, it is clear that if we can find a control sequence $u(t)$ (or feedback control $u(t) = Gx(t)$) and a proper coordinate transformation $z(t) = Tx(t)$ such that the incidence matrix of network graph of the system under frame $z(t)$ satisfies (13.40), then the system can be stabilized (or stabilized by feedback control). However, so far we have no good method to find out such $u(t)$. Thus in general case, when such control sequence can not be observed directly, Theorem 13.7 may be more usefull.

Next, we give an example for stabilization of Boolean control network.

Example 13.10. Consider the following system

$$
\begin{cases}
x_1(t+1) = (u_1(t) \leftrightarrow x_2(t)) \wedge x_3(t) \\
x_2(t+1) = \neq x_3(t) \\
x_3(t+1) = (u_2(t)\bar{\vee}x_1(t)) \to x_2(t).
\end{cases}
\tag{13.44}
$$

If we choose $u_1(t) = \neg x_2(t)$ and $u_2(t) = x_1(t)$, then under this feedback control, the system becomes

$$
\begin{cases}
x_1(t+1) = 0 \\
x_2(t+1) = \neq x_3(t) \\
x_3(t+1) = 1,
\end{cases}
$$

whose incidence matrix of network graph is

$$
\mathscr{I}(F) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.
$$

We conclude that the system is globally stabilized to $X = (0,0,1)^T \sim \delta_8^7$ by the given feedback control.

If we can not observe such feedback control, we can also use Theorem 13.7 to verify whether the system is globally stabilizable by open loop control or by feedback control. Here we take the case of open loop control as an example. Setting $x(t) = x_1(t)x_2(t)x_3(t)$ and $u(t) = u_1(t)u_2(t)$, we have

$$
x(t+1) = Lu(t)x(t),
$$

where

$$
L = \delta_8[3\ 5\ 7\ 5\ 3\ 5\ 8\ 6\ 3\ 5\ 8\ 6\ 3\ 5\ 7\ 5 \\ 1\ 5\ 3\ 5\ 7\ 5\ 4\ 6\ 7\ 5\ 4\ 6\ 7\ 5\ 3\ 5].
$$

Then,

$$
M = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0
\end{bmatrix},
$$

and

$$M^{(2)} = \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&1 \\ 1&0&1&0&1&0&1&0 \\ 0&0&1&1&0&0&1&1 \\ 0&0&1&0&0&0&1&0 \\ 1&1&1&1&1&1&1&1 \\ 1&0&1&0&1&0&1&0 \end{bmatrix}$$

Hence, we can conclude that the system can be stabilized to δ_8^3 or δ_8^7.

13.5.2 Optimal Control

The infinite horizon optimal control problem is considered. For Boolean control network (13.34), our purpose is to find a control sequence $u(t)$ to maximize the objective function,

$$J(u) = \overline{\lim_{T\to\infty}} \frac{1}{T} \sum_{t=1}^{T} P(x(t), u(t)). \tag{13.45}$$

The next result make the optimal control problem be converted to finding the cycles of Boolean control network, then we can use the technic developed in Section 1.

Theorem 13.9. *For the Boolean control network (13.34) with the objective function (13.45), there exists an optimal control $u^*(t)$ such that the objective function is maximized and the trajectory of $s^*(t) = u^*(t)x^*(t)$ will become periodic after a finite time. And there exists a logical matrix G^*, such that u^* can be expressed as*

$$u^*(t+1) = G^* u^*(t) x^*(t). \tag{13.46}$$

For a length-ℓ cycle, denote

$$P(C) = \frac{1}{\ell} \sum_{t=1}^{\ell} P(x(t), u(t)).$$

We have

Proposition 13.4. *Any cycle C contains a simple cycle C_s such that*

$$P(C_s) \geq P(C) \tag{13.47}$$

According to this proposition, we can find an optimal cycle C^* only from all the simple cycles which can be reached from the initial state. We give an example to describe how to find an optimal control.

Example 13.11. We consider infinitely repeated prisoners' dilemma. Assume player 1 is a machine and player 2 is a person. Their actions can be

0 : the player cooperates with the partner,
1 : the player defects the partner.

The payoff bi-matrix is assumed to be

Table 13.1 Payoff bi-matrix

$P_1 \backslash P_2$	0	1
0	3,3	0,5
1	5,0	1,1

Assume player 1 (the machine) fixes its strategy as "One Tit For One Tat", which means player 1 will defect only if player 2 defects in last step. Denote by $x(t)$ the action of player 1 at t-th step, and $u(t)$ the action of player 2. The game can be described as the following Boolean control network.

$$x(t+1) = Lu(t)x(t), \tag{13.48}$$

where

$$L = \delta_2[1\ 1\ 2\ 2].$$

Player 2 (the person) choose his actions $u(t)$, to maximize his own payoff

$$J(u) = \overline{\lim_{T\to\infty}} \frac{1}{T}\sum_{t=1}^{T} P_h(x(t),u(t)),$$

where

$$P_h(x(t),u(t)) = u'(t)\begin{bmatrix}3 & 0\\ 5 & 1\end{bmatrix}x(t).$$

Since the length of simple cycle can not be bigger than 2, we can only find all the fixed points and length-2 cycles. By Theorem 13.2 and checking $\mathscr{J}^{(s)}$, we can obtained the simple cycles that can be reached from the initial state $x(0) = \delta_2^1$ are

$$C_1 = \delta_2 \times \delta_2((1,1)), \quad C_2 = \delta_2 \times \delta_2((2,2)), \quad C_3 = \delta_2 \times \delta_2((1,2),(2,1)).$$

It is easy to see that $P(C_1)$ is the largest. Let $u_(0) = \delta_2^1$, the trajectory can enter C_1 directly from initial state.

Finally, we can see that player 2 can take such strategy

$$u^*(t+1) = G^*u^*(t)x^*(t),$$

where

$$G^* = \delta_2[1\ *\ *\ *], \quad * \text{ is arbitrary in } \mathscr{D}.$$

For example, we can take $G^* = \delta_2[1\ 1\ 1\ 1]$ which means player 2 always cooperates; or $G^* = \delta_2[1\ 2\ 1\ 2]$, which is also "One Tit for One Tat" strategy.

13.5.3 Identification

At the end of this chapter, we consider how to identify a Boolean control network from observed data. First, we give the rigorous definition of identification for Boolean control networks.

Definition 13.10. Assume we have a Boolean control network with its dynamic structure as (13.1)-(13.2). The identification problem is finding the functions f_i, $i = 1, \cdots, n$, h_j, $j = 1, \cdots, p$ (equivalently, (L,H) of the algebraic form), via certain input-output data $\{U(0), U(1), \cdots\}$, $\{Y(0), Y(1), \cdots\}$. The identification problem is said to be solvable if f_i and h_j can be uniquely determined by using designed inputs $\{U(0), U(1), \cdots\}$.

Here we use the following notations:

$$X(t) := (x_1(t), x_2(t), \cdots, x_n(t));$$
$$Y(t) := (y_1(t), y_2(t), \cdots, y_p(t));$$
$$U(t) := (u_1(t), u_2(t), \cdots, u_m(t)).$$

Note that if $z = Tx$ is a coordinate transformation, the system under the new frame becomes

$$\begin{cases} z(t+1) = \tilde{L}u(t)z(t) \\ y(t) = \tilde{H}z(t), \end{cases} \tag{13.49}$$

then (L,H) and $(\tilde{L},\tilde{H})$ can not be distinguishable by any input-output data. So precisely speaking, we should say the pair (L,H) is identifiable up to a coordinate transformation.

Firstly, assume the state can be observed, or say $H = I_{2^n}$, then we have

Theorem 13.10. *System (13.1) is identifiable by input-state data, iff the system is controllable.*

Assume we have enough proper input data $\{U(0), U(1), \cdots, U(T)\}$ and the corresponding state data $\{X(0), X(1), \cdots, X(T)\}$ such that (in set sense)

$$\{U_0 \times X_0, U_1 \times X_1, \cdots, U_{T-1} \times X_{T-1}\} = \mathscr{D}^{n+m}. \tag{13.50}$$

Then L can be identified in the following way: In the vector form, if $u(i)x(i) = \delta^j_{2^{m+n}}$, then $\mathrm{Col}_j(L) = x(i+1)$.

Then the key issue is how to design the input sequence. A reasonable method is to choose inputs randomly, but this method can not ensure (13.50) be satisfied. For this, we have the following method.

Theorem 13.11. *If system (13.1) is identifiable, the logical functions f_i can be determined uniquely by the inputs designed as following:*

$$u(t) = \begin{cases} \delta_{2^m}^{i}, & \exists s \text{ s.t. } x(s) = x(t), u(s) = \delta_{2^m}^{i-1}, \\ & \forall t', s < t' < t, x(t') \neq x(t) \\ \delta_{2^m}^{1}, & \text{otherwise}, \end{cases} \tag{13.51}$$

when $x(t)$ enters a cycle, stop the process.

We give an example to illustrate this.

Example 13.12. Consider the following Boolean control network

$$\begin{cases} x_1(t+1) = \neg x_1(t) \vee x_2(t) \\ x_2(t+1) = u(t) \wedge \neg x_1(t) \vee (\neg u(t) \wedge x_1(t) \wedge \neg x_2(t)) \end{cases} \tag{13.52}$$

Setting $x(t) = \ltimes_{i=1}^{2} x_i(t)$, $u = \ltimes_{i=1}^{2} u_i(t)$, we have

$$x(t+1) = Lu(t)x(t) \tag{13.53}$$

where

$$L = \delta_4[2\ 4\ 1\ 1\ 2\ 3\ 2\ 2],$$

Checking the controllability matrix, we have

$$\mathscr{C} = \sum_{s=1}^{2^3}{}_{\mathscr{B}} \sum_{i=1}^{2}{}_{\mathscr{B}} (\mathrm{Blk}_i(\mathscr{J}_0^{(s)})) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} > 0.$$

We conclude that the system is identifiable.

We can choose a sequence of controls and the initial state randomly, then the sequence of states can be determined. First, we choose 20 controls, the input-state data are shown in Table 13.2.

Table 13.2 Input-state data

t	0	1	2	3	4	5	6	7	8	9
$u(t)$	1	1	1	1	1	2	1	1	1	1
$x(t)$	1	2	4	1	2	4	2	4	1	2
t	10	11	12	13	14	15	16	17	18	19
$u(t)$	2	1	2	1	1	2	1	2	2	1
$x(t)$	4	2	4	2	4	1	2	4	2	3

Where the number i in $u(t)$ ($x(t)$) means δ_2^i (δ_4^i). We can get L as

$$L = \delta_4[2\ 4 * 1\ 2\ 3 * 2].$$

Some columns of L are not identified, because not all the input-state $\delta_{2^{m+n}}^{j}$ are reached by the sequence of control chosen randomly.

Then we use (13.51) to obtain the input-state data as Table 13.3.

Table 13.3 input-state data

t	0	1	2	3	4	5	6	7	8	9	10	11
$u(t)$	1	1	1	2	2	1	1	1	2	2	2	1
$x(t)$	1	2	4	1	2	3	1	2	4	2	3	2

Hence, L can be identified as

$$L = \delta_4[2\ 4\ 1\ 1\ 2\ 3\ 2\ 2].$$

When we consider input-output data, the case becomes much more complicated. At present, there is no tricky method to identify the system. One can refer to [5] for an effective algorithm which searches the identification solution one by one in an particular order, but its computation complexity is huge. However, we have a sufficient and necessary condition for identifiability, which is theoretically important.

Theorem 13.12. *The Boolean control network (13.1)-(13.2) is identifiable, iff it is controllable and observable.*

Exercise 10

1. Consider the following system

$$\begin{cases} x_1(t+1) = x_2(t) \leftrightarrow x_3(t) \\ x_2(t+1) = x_3(t) \vee u_1(t) \\ x_3(t+1) = x_1(t) \wedge u_2(t) \end{cases}$$
$$y_1(t) = x_1(t)$$
$$y_2(t) = x_2(t) \vee x_3(t).$$

   Answer the following questions.

   a. Is the system identifiable?
   b. Can the system be stabilized? If so, find a control sequence or a feedback control which stabilizes the system?
   c. How many length-3 cycles does this system have?

2. Theorem 13.1 and 13.2 tell us the number of cycles of Boolean control network (13.1). In fact, by checking $\mathscr{J}^{(s)}$, we can also find what the cycles are. Try to find all the length-3 cycles in Exercise 1.

3. The Boolean control network (13.1) can also be expressed in algebraic form as

$$x(t+1) = \bar{L}x(t)u(t),$$

where $\bar{L} = LW_{[2^n,2^m]}$. Similar to the construction of input-state incidence matrix, we can construct the state-input incidence matrix of (13.1) as

$$\bar{\mathscr{J}} = \mathbf{1}_{2^m} \rtimes \bar{L},$$

where $\rtimes$ is the right semi-tensor product of matrices. Prove,

(a) There exists $\bar{\mathscr{J}}_0^s$ such that

$$\bar{\mathscr{J}}^s = \mathbf{1}_{2^m} \rtimes \bar{\mathscr{J}}_0^s, s \geq 1.$$

(b) $\bar{\mathscr{J}}_0^s = \tilde{\mathscr{J}}_0^s$, where $\tilde{\mathscr{J}}_0^s$ is defined in (13.20).

4. Prove Theorem 13.6
5. Prove Theorem 13.3
6. If $x_i(t), u_j(t), y_\alpha(t) \in \mathscr{D}_k$ in (13.1) and (13.2), we call the system logical control network. In fact, all the properties of Boolean control networks can be generalized to logical control networks.
Consider the following infinitely repeated game. Both of player 1 and player 2 have three actions, $\{$L, M, R$\}$. The payoff bi-matrix is assumed to be the Table 13.4.

Table 13.4 Payoff bi-matrix

$P_1 \backslash P_2$	L	M	R
L	3,3	0,4	9,2
M	4,0	4,4	5,3
R	2,9	3,5	6,6

Assume player 2's strategy is fixed to play R in the first stage, in the t-th stage, if the outcome in the $(t-1)$-th stage is (R, R) then plays R, otherwise, plays M.

a. Represent the game into logical control network.
b. Find a best strategy of player 1 which maximizes his infinite horizon average payoff.

References

1. Akutsu, T., Hayashida, M., Ching, W., Ng, M.: Control of Boolean networks: Hardness results and algorithms for tree structured networks. J. Theoretical Biology **244**(4), 670–679 (2007)
2. Cheng, D., Qi, H.: Controllability and observability of Boolean control networks. Automatica **45**(7), 1659–1667 (2009)

3. Cheng, D., Qi, H., Li, Z.: Analysis and Control of Boolean Networks: A Semi-tensor Product Approach. Springer, London (2011)
4. Cheng, D., Qi, H., Li, Z., Liu, J.B.: Stability and stabilization of Boolean networks. Int. J. Robust Nonlinear Contr. **21**(2), 134–156 (2011). DOI 10.1002/rnc.1581
5. Cheng, D., Zhao, Y.: Identification of Boolean control networks. Automatica To appear
6. Datta, A., Choudhary, A., Bittner, M., Dougherty, E.: External control in Markovian genetic regulatory networks. Machine Learning **52**, 169–191 (2003)
7. Datta, A., Choudhary, A., Bittner, M., Dougherty, E.: External control in Markovian genetic regulatory networks: The imperfect information case. Bioinformatics **20**, 924–930 (2004)
8. Ideker, T., Galitski, T., Hood, L.: A new approach to decoding life: Systems biology. Annu. Rev. Genomics Hum. Genet. **2**, 343–372 (2001)
9. Zhao, Y., Li, Z., Cheng, D.: Optimal control of logical control networks. Preprint (2010)