Chapter 12
Boolean Network

Boolean network was firstly introduced by Kauffman to formulate the cell networks. Then, it has been developed by [1, 2, 16, 11, 3, 15, 8, 13] and many others, becomes a powerful tool in describing, analyzing, and simulating the cell networks, and also be used as models of some complex systems such as neural networks, social and economic networks.

In this chapter, using the logical expression of logic, discussed in Chapter 5, the dynamics of a Boolean network is converted into an equivalent algebraic form as a standard discrete-time linear system. And then the topological structure (fixed points, cycles, transient period, etc.), subspace, input-state description of Boolean networks and higher-order Boolean network are investigated.

12.1 An Introduction

In Boolean networks, gene state is quantized to only two levels: True and False. Then the state of each gene is determined by the states of its neighborhood genes, using logical rules. Precisely speaking, a Boolean network is a directed network graph, $\Sigma = \{\mathscr{N}, \mathscr{E}\}$, consists of a set of nodes, $\mathscr{N} = \{x_i \,|\, i = 1, \cdots, n\}$ and a set of edges, denoted by $\mathscr{E} \subset \{x_1, \cdots, x_n\} \times \{x_1, \cdots, x_n\}$. If $(x_i, x_j) \in \mathscr{E}$, there is an edge from x_i to x_j, which means node x_j is affected by node x_i.

The dynamics of Boolean networks can be expressed as logical dynamic equations. Following is the rigorous definition of dynamics of Boolean networks.

Definition 12.1. [9] A Boolean network is a set of nodes $x_1, x_2, \cdots, x_n$, which interact with each other in a synchronous manner. At each given time $t = 0, 1, 2, \cdots$ a node has only one of two different values: 1 or 0. Thus the network can be described by a set of equations:

$$\begin{cases} x_1(t+1) = f_1(x_1(t), x_2(t), \cdots, x_n(t)) \\ x_2(t+1) = f_2(x_1(t), x_2(t), \cdots, x_n(t)) \\ \vdots \\ x_n(t+1) = f_n(x_1(t), x_2(t), \cdots, x_n(t)), \end{cases} \tag{12.1}$$

where f_i, $i = 1, 2, \cdots, n$ are n-ary logical functions.

We give a simple example to show these.

Example 12.1. Consider a Boolean network, $\Sigma = (\mathscr{N}, \mathscr{E})$ of three nodes as

$$\begin{cases} A(t+1) = B(t) \wedge C(t) \\ B(t+1) = \neg A(t) \\ C(t+1) = B(t) \vee C(t). \end{cases} \tag{12.2}$$

Its set of nodes is $\mathscr{N} = \{x_1 := A,\ x_2 := B,\ x_3 := C\}$, set of edges is $\mathscr{E} = \{(x_1, x_2), (x_2, x_1), (x_2, x_3), (x_3, x_1), (x_3, x_3)\}$. Its network graph is depicted in Fig 12.1.

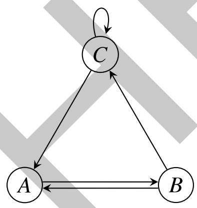

Fig. 12.1 Network Graph of (12.1)

Our first purpose is to convert Boolean network dynamics (12.1) into an algebraic form. Precisely, express it as a conventional discrete-time linear system. Using the technique developed in Chapter 5, we use vector form $x_i(t) \in \Delta$, and define

$$x(t) = x_1(t)x_2(t)\cdots x_n(t) := \ltimes_{i=1}^{n} x_i(t).$$

Using Lemma 5.2, there exist structure matrices, $M_i = M_{f_i}$, $i = 1, \cdots, n$, such that

$$x_i(t+1) = M_i x(t), \quad i = 1, 2, \cdots, n. \tag{12.3}$$

Remark 12.1. Note that usually the right hand side of i-th equation of (12.1) may not have all x_j, $j = 1, 2, \cdots, n$. Say, in the previous example, for node A we have

$$A(t+1) = B(t) \wedge C(t).$$

In matrix form it is

$$A(t+1) = M_{\wedge} B(t) C(t). \tag{12.4}$$

To get the form of (12.3), using dummy matrix $D_{p,q}$ (6.27), we can rewrite (12.4) as

$$A(t+1) = M_{\wedge} D_{2,2} A(t) B(t) C(t) = M_{\wedge} D_{2,2} x(t).$$

Multiplying the equations in (12.3) together yields

$$x(t+1) = M_1 x(t) M_2 x(t) \cdots M_n x(t). \tag{12.5}$$

Using Theorem 6.4, equation (12.5) can be expressed as

$$x(t+1) = L x(t), \tag{12.6}$$

where

$$L = \prod_{i=1}^{n} (I_{2^{i-1}} \otimes M_i) \left(M_r^{2^n} \right)^{n-1}$$

is called the transition matrix. Thus, we have the following result.

Theorem 12.1. *The dynamics of Boolean network (12.1) is uniquely determined by linear dynamic system (12.6).*

Definition 12.2. Equation (12.6) is called the algebraic form of network (12.1). Equation (12.3) is called the component-wise algebraic form of network (12.1).

In fact, a direct computation by using Corollary 6.3 can produce the algebraic form easily. We give a simple example to see how to get algebraic form of the dynamics of Boolean networks.

Example 12.2. Recall the Boolean network in Example 12.1. Its dynamics is described as (12.2). In algebraic form, we have

$$\begin{cases} A(t+1) = M_{\wedge} B(t) C(t) \\ B(t+1) = M_{\neg} A(t) \\ C(t+1) = M_{\vee} B(t) C(t). \end{cases} \tag{12.7}$$

Using dummy matrix $D_{p,q}$, it is easy to convert (12.7) to its component-wise algebraic form

$$\begin{cases} A(t+1) = \delta_2[1\ 2\ 2\ 2\ 1\ 2\ 2\ 2] A(t) B(t) C(t) \\ B(t+1) = \delta_2[2\ 2\ 2\ 2\ 1\ 1\ 1\ 1] A(t) B(t) C(t) \\ C(t+1) = \delta_2[1\ 1\ 1\ 2\ 1\ 1\ 1\ 2] A(t) B(t) C(t). \end{cases} \tag{12.8}$$

Then, setting $x(t) = A(t)B(t)C(t)$, and by Corollary 6.3 we can get the algebraic form of (12.7):

$$x(t+1) = L x(t),$$

where $L = \delta_8[3\ 7\ 7\ 8\ 1\ 5\ 5\ 6]$.

Remark 12.2. Equation (12.6) is a standard linear system with L being a square Boolean matrix. So all classical methods and conclusions for linear systems can be used for analyzing the dynamics of the Boolean network.

12.2 Fixed Points and Cycles

In this section, we consider the structure of a Boolean network, which is described in terms of its cycles and the transient states that lead to them. Two different methods, iteration and scalar form, were developed in [12] to determine cyclic structure and the transient states that lead to them. In [9], a linear reduced scalar equation is derived from a more rudimentary nonlinear scalar equation to get immediate information about both cycle and transient structure of the network. Several useful Boolean networks have been analyzed and their cycles have been revealed (see, e.g., [12, 9] and references therein). It was pointed out in [17] that finding fixed points and cycles of a Boolean network is an NP-complete problem.

Definition 12.3. 1. Given the system (12.6), the pair $(\Delta_n, \mathscr{E})$ where

$$\mathscr{E} = \{(x_i, x_j) | x_j = Lx_i\}$$

forms a directed graph, which is called the state transfer graph (STG).

2. $(x_0, x_1, x_2, \cdots)$ is called a path of system (12.6), if $x_i = Lx_{i-1}, i \geq 1$
3. A cyclic path is called a cycle of system (12.6) with length k, if the smallest period is k, a fixed point is a cycle of length 1.

Remark 12.3. For Boolean network, we can see that the elements in a cycle $(x_0, Lx_0, \cdots, L^{k-1}x_0)$ are pairwise distinct. But this is not true for higher order Boolean network or Boolean control network, which will be considered latter.

The next two theorems are main results of this chapter, which show how many fixed points and cycles of different lengths a Boolean network has.

Theorem 12.2. *Consider Boolean network (12.1). $\delta_{2^n}^i$ is its fixed point, iff in its algebraic form (12.6) the diagonal element ℓ_{ii} of the network transition matrix L equals 1. It follows that the number of fixed points of network (12.1), denoted by N_e, equals the number of i, for which $\ell_{ii} = 1$. Equivalently,*

$$N_e = \mathrm{tr}(L). \tag{12.9}$$

Proof. Assume $\delta_{2^n}^i$ is its fixed point. Note that $L\delta_{2^n}^i = \mathrm{Col}_i(L)$. It is clear that $\delta_{2^n}^i$ is its fixed point, iff $\mathrm{Col}_i(L) = \delta_{2^n}^i$, which completes the proof. □

For statement ease, if $\ell_{ii} = 1$, the $\mathrm{Col}_i(L)$ is called a diagonal nonzero column of L.

Next, we consider the cycles of the Boolean network system (12.1). We need a notation: Let $k \in \mathbb{Z}_+$. A positive integer $s \in \mathbb{Z}_+$ is called a proper factor of k if

$s < k$ and $k/s \in \mathbb{Z}_+$. The set of proper factors of k is denoted by $\mathscr{P}(k)$. For instance, $\mathscr{P}(8) = \{1,2,4\}$, $\mathscr{P}(12) = \{1,2,3,4,6\}$, etc.

Using a similar argument as for Theorem 12.2, we can have the following theorem.

Theorem 12.3. *The number of length s cycles, denoted by N_s, is inductively determined by*

$$\begin{cases} N_1 = N_e, \\ N_s = \frac{\mathrm{tr}(L^s) - \sum_{k \in \mathscr{P}(s)} kN_k}{s}, \quad 2 \le s \le 2^n. \end{cases} \tag{12.10}$$

The proof is left for exercise.

Next, we consider how to find the cycles. If

$$\mathrm{tr}(L^s) - \sum_{k \in \mathscr{P}(s)} kN_k > 0, \tag{12.11}$$

then we call "s" a non-trivial power.

Assume s is a non-trivial power. Denote by ℓ_{ii}^s the (i,i)-th entry of matrix L^s. Then we define

$$C_s = \{i | \ell_{ii}^s = 1\}, \quad s = 1, 2, \cdots, 2^n,$$

and

$$D_s = C_s \bigcap_{i \in \mathscr{P}(s)} C_i^c,$$

where C_i^c is the compliment of C_i.

From the above argument the following result is obvious.

Proposition 12.1. *Let $x_0 = \delta_{2^n}^i$. Then $(x_0, Lx_0, \cdots, L^s x_0)$ is a cycle with length s, iff $i \in D_s$.*

Theorem 12.3 and Proposition 12.1 provide a simple algorithm for constructing cycles. We give an example to show the algorithm.

Example 12.3. Recall Example 12.1. It is easy to check that

$$\mathrm{tr}(L^t) = 0, \quad t \le 3,$$

and

$$\mathrm{tr}(L^t) = 4, \quad t \ge 4.$$

Using Theorem 12.3, we conclude that there is only one cycle of length 4. Moreover, note that

$$L^4 = \delta_8[1\ 3\ 3\ 1\ 5\ 7\ 7\ 3].$$

Then each diagonal nonzero column can generate the cycle. For instance, choosing $Z = \delta_8^1$, then we have

$$LZ = \delta_8^3, \quad L^2 Z = \delta_8^7, \quad L^3 Z = \delta_8^5, \quad L^4 Z = Z.$$

Converting the vector forms back to the scalar form of $A(t)$, $B(t)$, and $C(t)$, we have the cycle as $((1,1,1),(1,0,1),(0,0,1),(0,1,1),(1,1,1))$.

We refer to [5, 7] for the transient period and basins of attractors of Boolean network.

12.3 Invariant Subspace and Input-state Description

12.3.1 State Space and Subspaces

This section present a systematic description of state space and subspaces of Boolean (control) network. This state-space description makes a state-space approach, similar to that of the modern control theory, applicable to the analysis of Boolean networks and the synthesis of Boolean control systems. This section is based on [6]. Unlike the quantity-based dynamic (control) systems, the logic-based dynamic (control) systems do not have a natural vector space structure. To use the state-space approach, we have to define state space and its various subspaces.

Consider a Boolean network

$$\begin{cases} x_1(t+1) = f_1(x_1(t),\cdots,x_n(t)) \\ \vdots \\ x_n(t+1) = f_n(x_1(t),\cdots,x_n(t)), \quad x_i \in \mathscr{D}, \end{cases} \tag{12.12}$$

or a Boolean control network

$$\begin{cases} x_1(t+1) = f_1(x_1(t),\cdots,x_n(t),u_1(t),\cdots,u_m(t)) \\ \vdots \\ x_n(t+1) = f_n(x_1(t),\cdots,x_n(t),u_1(t),\cdots,u_m(t)), \end{cases} \tag{12.13}$$
$$y_j(t) = h_j(x_1(t),\cdots,x_n(t)), \quad x_i,u_i,y_j \in \mathscr{D}.$$

We give the following definitions

Definition 12.4. Consider Boolean network (12.12) or Boolean control network (12.13).

1. The state space $\mathscr{X}$ is defined as the set of all logical functions of $x_1,\cdots,x_n$, denoted by

$$\mathscr{X} = \mathscr{F}_\ell\{x_1,\cdots,x_n\}. \tag{12.14}$$

$\{x_1,\cdots,x_n\}$ is call the basis of $\mathscr{X}$.

2. Let $z_1,\cdots,z_k \in \mathscr{X}$. The subspace $\mathscr{Z}$ generated by $z_1,\cdots,z_k$ is the set of logical functions of $z_1,\cdots,z_k$, denoted by

$$\mathscr{Z} = \mathscr{F}_\ell\{z_1,\cdots,z_k\}. \tag{12.15}$$

$\{z_1,\cdots,z_k\}$ is called the basis of the subspace $\{\mathscr{Z}\}$.

3. Let $Z = \{z_1,\cdots,z_n\} \subset \mathscr{X}$. For notational ease, we also consider $Z = (z_1,\cdots,z_n)^T$ as a column vector. The mapping $G : \mathscr{D}^n \to \mathscr{D}^n$ defined by $X = (x_1,\cdots,x_n)^T \mapsto Z = (z_1,\cdots,z_n)^T$ is called a coordinate transformation, if G is one-to-one and onto.
4. A subspace $\mathscr{Z} = \mathscr{F}_\ell\{z_1,\cdots,z_k\} \subset \mathscr{X}$ is called a regular subspace of dimension k if there are $\{z_{k+1},\cdots,z_n\}$, such that $Z = (z_1,\cdots,z_n)$ is a coordinate frame. Moreover, $\{z_1,\cdots,z_k\}$ is called a regular basis of $\mathscr{Z}$.
5. Consider system (12.12). If it can be expressed (under a suitable coordinate frame) as

$$\begin{cases} z^1(t+1) = F^1(z^1(t)), & z^1 \in \mathscr{D}^s \\ z^2(t+1) = F^2(z(t)), & z^2 \in \mathscr{D}^{n-s}. \end{cases} \tag{12.16}$$

Then $\mathscr{Z} = \mathscr{F}_\ell\{z^1\} = \mathscr{F}_\ell\{z^1_1,\cdots,z^1_s\}$ is called an invariant subspace of (12.12).

Remark 12.4. 1. Let $\xi \in \mathscr{X}$. Then ξ is a logical function of $x_1,\cdots,x_n$. Say,

$$\xi = g(x_1,\cdots,x_n).$$

Then it can be uniquely expressed into an algebraic form as

$$\xi = M_g \ltimes_{i=1}^n x_i,$$

where $M_g \in \mathscr{L}_{2\times 2^n}$. Now M_g can be expressed as

$$\delta_2[i_1\ i_2\ \cdots\ i_{2^n}],$$

where i_s can be either 1 or 2. It follows that there are 2^{2^n} different functions. That is,

$$|\mathscr{X}| = 2^{2^n}.$$

2. Using set of functions to define a (sub) space is reasonable. For instance, in linear space $\mathbb{R}^n$ with the coordinate frame $\{x_1,\cdots,x_n\}$, we consider all the linear functions over $x_{i_1},\cdots,x_{i_k}$, it is

$$L_k = \left\{ \sum_{j=1}^k c_j x_{i_j} \,\middle|\, c_1,\cdots,c_k \in \mathbb{R} \right\}$$

which is obviously a k-dimensional subspace. In fact, we can identify L_k with its domain, which is a k-dimensional subspace in state space $\mathbb{R}^n$, called the dual space of L_k.

The logical space (subspace) defined here is also in dual sense. Precisely, we consider its domain as a subspace of the state space.

3. In vector space, the basis of a subspace can always be expanded to the basis of the full space by adding some vectors in the basis. But this is no longer true in logical space (the example of this is left for exercise).

The following result about coordinate transformation is obvious.

Theorem 12.4. *G is a coordinate transformation, iff its structure matrix T_G is nonsingular.*

Remark 12.5. If a matrix $T \in \mathscr{L}_{s\times s}$ and it is nonsingular, then it is an orthogonal matrix. Hence, if $z = T_G x$ is a coordinate transformation then $x = T_G^T z$.

Next, we consider the logical coordinate transformation of the dynamics of a Boolean network.

Consider a Boolean network in algebraic form as

$$x(t+1) = Lx(t), \quad x \in \Delta_{2^n}. \tag{12.17}$$

Let $z = Tx : \Delta_{2^n} \to \Delta_{2^n}$ be a logical coordinate transformation. Then

$$z(t+1) = Tx(t+1) = TLx(t) = TLT^{-1}z(t).$$

That is, under z coordinate frame Boolean network dynamics (12.17) becomes

$$z(t+1) = \tilde{L}z(t), \tag{12.18}$$

where

$$\tilde{L} = TLT^T. \tag{12.19}$$

Consider a Boolean control system in algebraic form as

$$\begin{cases} x(t+1) = Lu(t)x(t), & x \in \Delta_{2^n},\ u \in \Delta_{2^m} \\ y(t) = Hx(t), & y \in \Delta_{2^p}. \end{cases} \tag{12.20}$$

Let $z = Tx : \Delta_{2^n} \to \Delta_{2^n}$ be a logical coordinate transformation. A straightforward computation shows that (12.20) can be expressed as

$$\begin{array}{ll} z(t+1) = \tilde{L}u(t)z(t), & z \in \Delta_{2^n},\ u \in \Delta_{2^m} \\ y(t) = \tilde{H}z(t), & y \in \Delta_{2^p}, \end{array} \tag{12.21}$$

where

$$\begin{array}{l} \tilde{L} = TL(I_{2^m} \otimes T^T) \\ \tilde{H} = HT^T. \end{array} \tag{12.22}$$

Next, we give criterions of regular subspace and invariant subspace.

Given a set of functions z_i as

$$z_i = g_i(x_1, \cdots, x_n), \quad i = 1, \cdots, k, \tag{12.23}$$

and let $\mathscr{Z} = \mathscr{F}_\ell\{z_1, \cdots, z_k\}$. We would like to know when $\mathscr{Z}$ is a regular subspace with $\{z_1, \cdots, z_k\}$ as its regular sub-basis. Set $z = \ltimes_{i=1}^k z_i$ and $x = \ltimes_{i=1}^n x_i$. We can get its algebraic form as

$$z = Lx := \begin{bmatrix} \ell_{1,1} & \ell_{1,2} & \cdots & \ell_{1,2^n} \\ \vdots & & & \\ \ell_{2^k,1} & \ell_{2^k,2} & \cdots & \ell_{2^k,2^n} \end{bmatrix} x, \tag{12.24}$$

where

$$L = \prod_{i=1}^k (I_{2^{i-1}} \otimes M_i) \left(M_r^{2^k}\right)^{n-1}, \tag{12.25}$$

.

Theorem 12.5. *Assume there is a set of logical variables $z_1, \cdots, z_k$ ($k \le n$) satisfying (12.24). $\mathscr{Z} = \mathscr{F}_\ell\{z_1, \cdots, z_k\}$ is a regular subspace with regular sub-basis $\{z_1, \cdots, z_k\}$, iff the corresponding coefficient matrix L satisfies*

$$\sum_{i=1}^{2^n} \ell_{j,i} = 2^{n-k}, \quad j = 1, 2, \cdots, 2^k, \tag{12.26}$$

where $\ell_{j,i}$ are defined in (12.24).

Proof. (Sufficiency) Note that condition (12.26) means there are 2^{n-k} different x which makes $z = \delta_{2^k}^j$, $j = 1, 2, \cdots, 2^k$. Now we can choose z_{k+1} as follows. Set

$$S_k^j = \{x \,|\, Lx = \delta_{2^k}^j\}, \quad j = 1, 2, \cdots, 2^k.$$

Then the cardinal number $\left|S_k^j\right| = 2^{n-k}$. For half of the elements of S_k^j, define $z_{k+1} = 0$, and for the other half, set $z_{k+1} = 1$. Then it is easy to see that for $\tilde{z} = \ltimes_{i=1}^{k+1} z_i$ the corresponding $\tilde{L}$ satisfies (12.26) with k being replaced by $k+1$.

Continue this process till $k = n$. Then (12.26) becomes

$$\sum_{i=1}^{2^n} \ell_{j,i} = 1, \quad j = 1, 2, \cdots, 2^n. \tag{12.27}$$

(12.27) means the corresponding L contains all the columns of I_{2^n}, *i.e.*, it is obtained from I_{2^n} via a column permutation. It is, hence, a coordinate change.

(Necessity) Note that using the swap matrix, it is easy to see that the order of z_i does not affect the property of (12.26). First, we claim that if $\{z_i \,|\, i = 1, \cdots, k\}$ satisfies (12.26), then any of its subset $\{z_{i_t}\} \subset \{z_i \,|\, i = 1, \cdots, k\}$ also satisfies (12.26) with k be replaced by $|\{z_{i_t}\}|$. Since the order does not affect this property, it is enough to show that a $k-1$ subset $\{z_i \,|\, i = 2, \cdots, k\}$ is a proper sub-basis, because from $k-1$ we can go to $k-2$ and so on. Assume that $z^2 = \ltimes_{i=2}^{k} z_i = Qx$, and $z_1 = Px$. Using Corollary 6.3, we have

$$\mathrm{Col}_i(L) = \mathrm{Col}_i(P)\,\mathrm{Col}_i(Q), \quad i = 1, \cdots, 2^n. \tag{12.28}$$

Next, we split L into two equal-size blocks as

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}.$$

Note that either $\mathrm{Col}_i(P) = \delta_2^1$ or $\mathrm{Col}_i(P) = \delta_2^2$. Using this fact to (12.28), one sees easily that either $\mathrm{Col}_i(L) = \begin{bmatrix} \mathrm{Col}_i(Q) \\ 0 \end{bmatrix}$ (as $\mathrm{Col}_i(P) = \delta_2^1$) or $\mathrm{Col}_i(L) = \begin{bmatrix} 0 \\ \mathrm{Col}_i(Q) \end{bmatrix}$ (as $\mathrm{Col}_i(P) = \delta_2^2$). Hence, $\mathrm{Col}_i(Q) = \mathrm{Col}_i(L_1) + \mathrm{Col}_i(L_2)$. It follows that

$$Q = L_1 + L_2. \tag{12.29}$$

Since L satisfies (12.26), (12.29) ensures that Q satisfies (12.26) too.

Now since $\{z_i \,|\, i = 1, \cdots, k\}$ is a proper sub-basis, so there exists $\{z_i \,|\, i = k+1, \cdots, n\}$ such that $\{z_i \,|\, i = 1, \cdots, n\}$ is a coordinate transformation of x, it satisfies (12.26). (Precisely, it satisfies (12.27) with row sum equal to 1.) According to the claim, the subset $\{z_i \,|\, i = 1, \cdots, k\}$ also satisfies (12.27). $\square$

It is easy to see that (12.26) is equivalent to $|\{i \,|\, \mathrm{Col}_i(L) = \xi\}| = 2^{n-k}$ for any $\xi \in \Delta_{2^k}$.

Example 12.4. Consider the state space $\mathscr{X} = \mathscr{F}_\ell\{x_1, x_2, x_3\}$. And there is a subspace $\mathscr{Z} = \mathscr{F}_\ell\{z_1, z_2\} \subset \mathscr{X}$. Assume

$$\begin{cases} z_1 = x_1 \leftrightarrow x_2 \\ z_2 = x_2 \bar{\vee} x_3. \end{cases} \tag{12.30}$$

Then its algebraic form can be expressed as

$$z_1 z_2 = Mx = \delta_4[2\ 1\ 3\ 4\ 4\ 3\ 1\ 2]x. \tag{12.31}$$

We can see that $\left|\{i \,|\, \mathrm{Col}_i(M) = \delta_4^j\}\right| = 2$ for $j = 1, 2, 3, 4$, thus $\mathscr{Z}$ is a regular subspace.

Then we want to find z_3 such that $\{z_1, z_2, z_3\}$ form a coordinate frame, that is, to make the columns algebraic form T of the coordinate transformation . By Corollary 6.3, we need to construct $M_{z_3} = \delta_2[c_1\ c_2\ c_3\ c_4\ c_5\ c_6\ c_7\ c_8]$ such that

$$\mathrm{Col}_i(M)c_i \neq \mathrm{Col}_j(M)c_j, \quad i \neq j.$$

For this, we only need

$$c_1 \neq c_8;\ c_2 \neq c_7;\ c_3 \neq c_6;\ c_4 \neq c_5.$$

Theorem 12.6. *Consider system (12.12) with its algebraic form (12.17). Assume that a regular subspace $\mathscr{Z} = \mathscr{F}_\ell\{z_1,\cdots,z_s\}$ with $z = \ltimes_{i=1}^s z_i$ has the following algebraic form*

$$z = Qx, \tag{12.32}$$

where $Q \in \mathscr{L}_{2^s\times 2^n}$. Then $\mathscr{Z} = \mathscr{F}_\ell\{z_1,\cdots,z_s\}$ is an invariant subspace of system (12.12)*, iff*

$$\mathrm{Row}(QL) \subset \mathrm{Span}_{\mathscr{B}}\mathrm{Row}(Q), \tag{12.33}$$

where $\mathrm{Span}_{\mathscr{B}}$ *means the coefficients are in $\mathscr{D}$; and L is as in* (12.17), i.e., *it is the transition matrix of the algebraic form of system* (12.12).

We refer to [7] for the proof of this theorem. Note that checking (12.33) is not a straightforward computation. Since $\{\mathscr{Z}\}$ is a regular subspace, Q is of full row rank. Hence, we have

Corollary 12.1. *$\mathscr{Z}$ is an invariant subspace, iff*

$$QL = QLQ^T(QQ^T)^{-1}Q. \tag{12.34}$$

Example 12.5. Consider the following Boolean network

$$\begin{cases} x_1(t+1) = (x_1(t)\wedge x_2(t)\wedge\neg x_4(t))\vee(\neg x_1(t)\wedge x_2(t))\\ x_2(t+1) = x_2(t)\vee(x_3(t)\leftrightarrow x_4(t))\\ x_3(t+1) = (x_1(t)\wedge\neg x_4(t))\vee(\neg x_1(t)\wedge x_2(t))\vee(\neg x_1(t)\wedge\neg x_2(t)\wedge x_4(t))\\ x_4(t+1) = x_1(t)\wedge\neg x_2(t)\wedge x_4(t). \end{cases} \tag{12.35}$$

Let $\mathscr{Z} = \mathscr{F}_\ell\{z_1,z_2,z_3\}$, where

$$\begin{cases} z_1 = x_1\bar{\nabla}x_4\\ z_2 = \neg x_2\\ z_3 = x_3 \leftrightarrow \neg x_4. \end{cases} \tag{12.36}$$

Set $x = \ltimes_{i=1}^4 x_i$, $z = \ltimes_{i=1}^3 z_i$. Then we have

$$z = Qx,$$

where

$$Q = \delta_8[8\ 3\ 7\ 4\ 6\ 1\ 5\ 2\ 4\ 7\ 3\ 8\ 2\ 5\ 1\ 6],$$

and the algebraic form of (12.35) is

$$x(t+1) = Lx(t),$$

where

$$L = \delta_{16}[11\ 1\ 11\ 1\ 11\ 13\ 15\ 9\ 1\ 2\ 1\ 2\ 9\ 15\ 13\ 11].$$

It is easy to calculate that

$$QL = \delta_8[3\ 8\ 3\ 8\ 3\ 2\ 1\ 4\ 8\ 3\ 8\ 3\ 4\ 1\ 2\ 3] = QLQ^T(QQ^T)^{-1}Q.$$

Hence $\mathscr{X}$ is an invariant subspace of (12.35).

In fact we can choose $z_4 = x_4$ such that

$$\begin{cases} z_1 = x_1 \bar{\vee} x_4 \\ z_2 = \neg x_2 \\ z_3 = x_3 \leftrightarrow \neg x_4 \\ z_4 = x_4 \end{cases} \tag{12.37}$$

is a coordinate transformation. Moreover, under coordinate frame z, system (12.35) can be expressed into the cascading form (12.16) as

$$\begin{cases} z_1(t+1) = z_1(t) \to z_2(t) \\ z_2(t+1) = z_2(t) \wedge z_3(t) \\ z_3(t+1) = \neg z_1(t) \\ z_4(t+1) = z_1(t) \vee z_2(t) \vee z_4(t). \end{cases} \tag{12.38}$$

12.3.2 Input-State Description

Consider Boolean control network (12.13). In this chapter we assume the controls are logical variables satisfying certain logical rule, called the input network, described as

$$\begin{cases} u_1(t+1) = g_1(u_1(t), u_2(t), \cdots, u_m(t)) \\ u_2(t+1) = g_2(u_1(t), u_2(t), \cdots, u_m(t)) \\ \vdots \\ u_m(t+1) = g_m(u_1(t), u_2(t), \cdots, u_m(t)). \end{cases} \tag{12.39}$$

Setting $u(t) = \ltimes_{i=1}^m u_i(t)$, its algebraic form is

$$u(t+1) = Gu(t), \quad G \in \mathscr{L}_{2^m \times 2^m}.$$

Then, we consider the cycles of system (12.13) with (12.39) (the case of free inputs will be considered in next chapter). The state space is $\mathscr{X} = \mathscr{F}_\ell\{x_1, \cdots, x_n\}$, input space is $\mathscr{U} = \mathscr{F}_\ell\{u_1, \cdots, u_m\}$, input-state space is $\mathscr{W} = \mathscr{F}_\ell\{u_1, \cdots, u_m, x_1, \cdots, x_n\}$. It has been known that $\mathscr{U}$ is the invariant subspace of $\{\mathscr{W}\}$, we want to investigate the relationship between the cycles in $\mathscr{U}$ and the cycles in $\mathscr{W}$.

Assume there is a cycle of length k in the input-state space $\mathscr{W}$. Say, it is

$$\begin{aligned} C^k_{\mathscr{W}}: \quad & w(0) = w_0 = u_0x_0 \to w(1) = w_1 = u_1x_1 \to \cdots \\ & \to w(k) = w_k = u_kx_k = w_0. \end{aligned}$$

First of all, one sees easily that since $u_0 = u_k$, in the input space $\mathscr{U}$, the sequence $\{u_0, u_1, \cdots, u_k\}$ contains, say, j folds of a cycle of length ℓ, where $j\ell = k$. Hence $u_\ell = u_0$. Now let us see what condition the $\{x_i\}$ in the cycle $C^k_{\mathscr{W}}$ should satisfy. Define a network transition matrix as

$$\Psi := L(u_{\ell-1})L(u_{\ell-2})\cdots L(u_1)L(u_0). \tag{12.40}$$

Starting from $w_0 = u_0x_0$, we have x component of the cycle $C^k_{\mathscr{W}}$ as

$$\begin{aligned} & x_0 \to x_1 = L(u_0)x_0 \to x_2 = L(u_1)L(u_0)x_0 \to \cdots \to x_\ell = \Psi x_0 \to \\ & x_{\ell+1} = L(u_0)\Psi x_0 \to x_{\ell+2} = L(u_1)L(u_0)\Psi x_0 \to \cdots \to x_{2\ell} = \Psi^2 x_0 \to \\ & \vdots \\ & x_{(j-1)\ell+1} = L(u_0)\Psi^{j-1}x_0 \to x_{(j-1)\ell+2} = L(u_1)L(u_0)\Psi^{j-1}x_0 \to \cdots \\ & \to x_{j\ell} = \Psi^j x_0 = x_0. \end{aligned} \tag{12.41}$$

We conclude that $x_0 \in \Delta_{2^n}$ is a fixed point of the equation

$$x(t+1) = \Psi^j x(t). \tag{12.42}$$

For convenience, we assume $j > 0$ is the smallest positive integer, which makes x_0 a fixed point of (12.42).

Conversely, assume $x_0 \in \Delta_{2^n}$ is a fixed point of (12.42) and u_0 is a point on a cycle of control space $C^\ell_{\mathscr{U}}$. Then it is obvious that we have the cycle (12.41).

Summarizing above arguments yields

Theorem 12.7. *Consider Boolean control network (12.13) with (12.39). A set $C^k_{\mathscr{W}} \subset \Delta_{2^{k(n+m)}}$ is a cycle in the input-state space $\mathscr{W}$ with length k, iff for any point $w_0 = u_0x_0 \in C^k_{\mathscr{W}}$, there exists an $\ell \le k$ as a factor of k, such that $u_0, u_1 = Gu_0, u_2 = G^2u_0, \cdots, u_\ell = G^\ell u_0 = u_0$ is a cycle in the control space, and x_0 is a fixed point of equation (12.42) with $j = k/\ell$.*

Theorem 12.7 shows how to find all the cycles in the input-state space. First, we can find cycles in the input space. Pick a cycle in the input space, say $C^\ell_{\mathscr{U}}$, then for each point $u_0 \in C^\ell_{\mathscr{U}}$ we can construct an auxiliary system

$$x(t+1) = \Psi x(t). \tag{12.43}$$

Say, $C^\ell_{\mathscr{U}} = (u_0, u_1, \cdots, u_\ell = u_0)$ is a cycle in $\mathscr{U}$, and $C^j_{\mathscr{X}} = (x_0, x_1, \cdots, x_j = x_0)$ is a cycle of (12.43). Then there is a cycle $C^k_{\mathscr{W}}$, $k = \ell j$, in the input-state STP space, which can be constructed by

$$
\begin{aligned}
w_0 &= u_0x_0 \to w_1 = u_1L(u_0)x_0 \to w_2 = u_2L(u_1)L(u_0)x_0 \to \cdots \to\\
w_\ell &= u_0x_1 \to w_{\ell+1} = u_1L(u_0)x_1 \to w_{\ell+2} = u_2L(u_1)L(u_0)x_1 \to \cdots \to\\
&\vdots\\
w_{(j-1)\ell} &= u_0x_{(j-1)} \to w_{(j-1)\ell+1} = u_1L(u_0)x_{(j-1)} \to\\
&\quad w_{(j-1)\ell+2} = u_2L(u_1)L(u_0)x_{(j-1)} \to \cdots \to\\
w_{j\ell} &= u_0x_j = u_0x_0 = w_0.
\end{aligned}
\tag{12.44}
$$

We call this $C^k_{\mathscr{W}}$ the composed cycle of $C^\ell_{\mathscr{U}}$ and $C^j_{\mathscr{X}}$, denoted by $C^k_{\mathscr{W}} = C^\ell_{\mathscr{U}} \circ C^j_{\mathscr{X}}$.

Note that from a cycle $C^\ell_{\mathscr{U}}$ we can choose any point as the starting point u_0. Then in equation (12.43) we have different Ψ, which produces different $C^j_{\mathscr{X}}$. It is reasonable to guess that the composed cycle $C^k_{\mathscr{W}} = C^\ell_{\mathscr{U}} \circ C^j_{\mathscr{X}}$ is independent of the choice of u_0. In fact, this is true.

Definition 12.5. Let $C^k_{\mathscr{W}} = \{w(t)|t = 0,1,\cdots,k\}$ be a cycle in the input-state space, and $C^\ell_{\mathscr{U}}$ be a cycle in the input space. Splitting $w(t) = u(t)x(t)$, we said that $C^k_{\mathscr{W}}$ is attached to $C^\ell_{\mathscr{U}}$ at u_0, if $w(0) = u_0x_0$, and

1. $u(t) \in C^\ell_{\mathscr{U}}, \quad \text{with} \quad u(0) = u_0;$
2. $x(0) = x_0$ is a fixed point of (12.42) with $j = \frac{k}{\ell} \in \mathbb{Z}_+$.

Proposition 12.2. *The sets of cycles in the input-state space, attached to any point of a given cycle $C^\ell_{\mathscr{U}}$ are the same.*

We refer to [7] or [4] for the proof of Proposition 12.2. Next, we give an illustration.

Example 12.6. Consider system

$$
\begin{cases}
x_1(t+1) = u(t) \to x_2(t)\\
x_2(t+1) = x_1(t) \vee x_3(t)\\
x_3(t+1) = \neg x_1(t),
\end{cases}
$$

the control network is

$$u(t+1) = \neg u(t).$$

We have an obvious kernel cycle: $0 \to 1 \to 0$ in $\mathscr{U}$. Then we can easily calculate that

$$
\begin{aligned}
L(0) &= \delta_8[2\,2\,2\,2\,1\,3\,1\,3],\\
L(1) &= \delta_8[2\,2\,6\,6\,1\,3\,5\,7].
\end{aligned}
$$

Hence we consider an auxiliary system

$$x(t+1) = \Psi x(t),$$

where

$$\Psi = L(1)L(0) = \delta_8[2\,2\,2\,2\,2\,6\,2\,6].$$

A routine calculation shows: (1) non-trivial power of Ψ is 1 and $\mathrm{tr}(\Psi^1) = 2$. So there are two fixed points, which are $\delta_8^1 \sim (1,1,0)$ and $\delta_8^6 \sim (0,1,0)$. The overall composed cycles are depicted in Fig. 12.2, where the dash lines show the duplicated cycles. Overall, we have a cycle in the input space and two product cycles of length 2 in the input-state space.

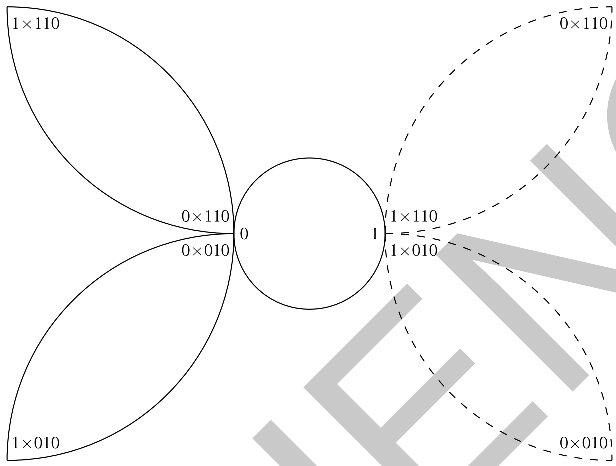

Fig. 12.2 Cycles of a control system

12.4 Higher-order Boolean Networks

In this section we consider higher order Boolean networks. This section is based on [14].

Definition 12.6. A Boolean network is called a μ-th order network if the current states depend on μ length histories. Precisely, its dynamics can be described as

$$\begin{cases} x_1(t+1) = f_1(x_1(t-\mu+1),\cdots,x_n(t-\mu+1),\cdots,x_1(t),\cdots,x_n(t)),\\ x_2(t+1) = f_2(x_1(t-\mu+1),\cdots,x_n(t-\mu+1),\cdots,x_1(t),\cdots,x_n(t)),\\ \vdots \\ x_n(t+1) = f_n(x_1(t-\mu+1),\cdots,x_n(t-\mu+1),\cdots,x_1(t),\cdots,x_n(t)), \quad t \geq \mu-1, \end{cases} \tag{12.45}$$

where $f_i : \mathscr{D}^{\mu n} \to \mathscr{D}$, $i = 1,\cdots,n$ are logical functions.

Note that same as for higher order discrete-time difference equations, to determine the solution (it is also called a trajectory) we need a set of initial conditions as

$$x_i(j) = a_{ij}, \quad i = 1, \cdots, n;\ j = 0, \cdots, \mu - 1. \tag{12.46}$$

We give an example to illustrate this kind of systems. It is a biochemical network of coupled oscillations in the cell cycle [10].

Example 12.7. Consider the following Boolean network

$$\begin{cases} A(t+3) = \neg(A(t) \wedge B(t+1)) \\ B(t+3) = \neg(A(t+1) \wedge B(t)). \end{cases} \tag{12.47}$$

It can be easily converted into the canonical form (12.45) as

$$\begin{cases} A(t+1) = \neg(A(t-2) \wedge B(t-1)) \\ B(t+1) = \neg(A(t-1) \wedge B(t-2)), \quad t \geq 2. \end{cases} \tag{12.48}$$

This is a 3rd order Boolean network.

Now the first natural question is: Can we use the technique developed in the previous sections of this chapter to investigate the structure of higher order Boolean networks? The answer is "Yes". In the following we will discuss two algebraic forms of (12.45).

12.4.1 First Algebraic Form of Higher Order Boolean Networks

Using a vector form, we define

$$\begin{cases} x(t) = \ltimes_{i=1}^{n} x_i(t) \in \Delta_{2^n} \\ z(t) = \ltimes_{i=t}^{t+\mu-1} x(i) \in \Delta_{2^{\mu n}}, \quad t = 0, 1, \cdots. \end{cases}$$

Assume the structure matrix of f_i is $M_i \in \mathscr{L}_{2 \times 2^{\mu n}}$. Then we can express (12.45) into its component-wise algebraic form as

$$x_i(t+1) = M_i z(t - \mu + 1), \quad i = 1, \cdots, n;\ t = \mu - 1, \mu, \mu + 1, \cdots. \tag{12.49}$$

Multiplying the equations in (12.49) together yields

$$x(t+1) = L_0 z(t - \mu + 1), \quad t \geq \mu, \tag{12.50}$$

where

$$L_0 = M_1 \ltimes_{j=2}^{n} \left[(I_{2^{\mu n}} \otimes M_j) M_r^{2^{\mu n}} \right]. \tag{12.51}$$

Note that the L_0 here can be calculated in a standard procedure as we used before. Using some properties of the semi-tensor product of matrix, we have

$$
\begin{aligned}
z(t+1) &= \ltimes_{i=t+1}^{t+\mu} x(i) \\
&= D_n \ltimes_{i=t}^{t+\mu-1} x(i)(L_0 \ltimes_{i=t}^{t+\mu-1} x(i)) \\
&= D_n(I_{2^{\mu n}} \otimes L_0) M_r^{2^{\mu n}} \ltimes_{i=t}^{t+\mu-1} x(i) \\
&:= Lz(t).
\end{aligned}
\tag{12.52}
$$

(12.52) is called the first algebraic form of network (12.45).

In fact, we can prove that the two Boolean networks have the same topological structure, including fixed points, cycles, and transient time, for all points to enter the set of cycles. So the first order Boolean network (12.52) provides all such results for higher order Boolean network (12.50).

Definition 12.7. 1. $(x_0, x_1, x_2, \cdots)$ is called a path of system (12.45), if $x_i = L_0 \ltimes_{j=i-\mu}^{i-1} x_j, i \geq \mu$

2. A cyclic path is called a cycle of system (12.45) with length k, if the smallest period is k, a fixed point is a cycle of length 1.

Theorem 12.8. *There is a one-to-one correspondence between the cycles of (12.50) and the cycles of (12.52).*

We refer to [7] or [14] for the proof.

Example 12.8. Recall Example 12.7. Set $x(t) = A(t)B(t)$. Using vector form, (12.48) can be expressed as

$$
x(t+1) = L_0 x(t-2)x(t-1)x(t), \tag{12.53}
$$

where

$$
\begin{aligned}
L_0 = \delta_4[&4\ 4\ 4\ 4\ 2\ 2\ 2\ 2\ 3\ 3\ 3\ 3\ 1\ 1\ 1\ 1 \\
&3\ 3\ 3\ 3\ 1\ 1\ 1\ 1\ 3\ 3\ 3\ 3\ 1\ 1\ 1\ 1 \\
&2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \\
&1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1].
\end{aligned}
$$

Set $z(t) = x(t)x(t+1)x(t+2)$. Then

$$
z(t+1) = Lz(t), \tag{12.54}
$$

where

$$
\begin{aligned}
L = \delta_{26}[&4\ 8\ 12\ 16\ 18\ 22\ 26\ 30\ 35\ 39\ 43\ 47\ 49\ 53\ 57\ 61 \\
&3\ 7\ 11\ 15\ 17\ 21\ 25\ 29\ 35\ 39\ 43\ 47\ 49\ 53\ 57\ 61 \\
&2\ 6\ 10\ 14\ 18\ 22\ 26\ 30\ 33\ 37\ 41\ 45\ 49\ 53\ 57\ 61 \\
&1\ 5\ 9\ \ 13\ 17\ 21\ 25\ 29\ 33\ 37\ 41\ 45\ 49\ 53\ 57\ 61].
\end{aligned}
$$

To find the cycles of (12.53), it is enough to find all the cycles in system (12.54). We can check $\mathrm{tr}(L^k), k = 1, 2, \cdots, 64$. They can be easily calculated as

$$
\mathrm{tr}(L^2) = 2, \quad \mathrm{tr}(L^5) = 5, \quad \mathrm{tr}(L^{10}) = 17.
$$

Using Theorem 12.3, we conclude that the system does not have fixed point, but it has one cycle of length 2, one cycle of length 5 and one cycle of length 10, which are

$$\begin{array}{l}\delta_{64}^{26} \to \delta_{64}^{29},\\ \delta_{64}^{1} \to \delta_{64}^{4} \to \delta_{64}^{16} \to \delta_{64}^{61} \to \delta_{64}^{49},\\ \delta_{64}^{2} \to \delta_{64}^{8} \to \delta_{64}^{30} \to \delta_{64}^{53} \to \delta_{64}^{17} \to \delta_{64}^{3} \to \delta_{64}^{12} \to \delta_{64}^{47} \to \delta_{64}^{5} \to \delta_{64}^{33}.\end{array}$$

Decompose $z(t)$ into $x(t)\cdots x(t+\mu-1)$, we have the cycles of (12.53):

$$\begin{array}{l}\delta_4^2 \to \delta_4^3,\\ \delta_4^1 \to \delta_4^1 \to \delta_4^1 \to \delta_4^4 \to \delta_4^4,\\ \delta_4^1 \to \delta_4^1 \to \delta_4^2 \to \delta_4^4 \to \delta_4^2 \to \delta_4^1 \to \delta_4^1 \to \delta_4^3 \to \delta_4^4 \to \delta_4^3.\end{array}$$

The result coincides with the one in [12].

12.4.2 Second Algebraic Form of Higher Order Boolean Networks

Define

$$w(\tau) := x(\mu\tau)x(\mu\tau+1)\cdots x(\mu\tau+(\mu-1)) = z(\mu\tau). \tag{12.55}$$

Then we have

$$w(\tau+1) = z(\mu\tau+\mu) = L^{\mu} z(\mu\tau) = L^{\mu} w(\tau),$$

where L is obtained in (12.52). Therefore, we have

$$w(\tau+1) = \Gamma w(\tau), \tag{12.56}$$

where

$$\Gamma = \left[D_n(I_{2^{\mu n}} \otimes L_0)M_r^{2^{\mu n}}\right]^{\mu},$$

with initial value $w(0) = \ltimes_{i=0}^{\mu-1} x(i)$. We call (12.56) the second algebraic form of the μ-th order Boolean network (12.45).

In fact, by re-scheduling the sampling time, the second algebraic form provides state variable, $w(\tau)$, $\tau = 0, 1, \cdots$, as a set of non-overlapped segments of $x(t)$. Hence, there is an obvious one-to-one correspondence between the trajectories of (12.45) and the trajectories of (12.56).

Proposition 12.3. *There is an obvious one-to-one correspondence between the trajectories of (12.45) and the trajectories of its second algebraic form (12.56), by*

$$w(\tau) := x(\mu\tau)x(\mu\tau+1)\cdots x(\mu\tau+(\mu-1)), \quad \tau = 0, 1, \cdots.$$

But there is no one-to-one correspondence between the cycles of (12.45) and the cycles of (12.56). It is easy to give an counterexample, we left this for exercise.

Exercise 9

1. Proof Theorem 12.3.
2. Give an example of 3 of Remark 12.4.
3. Consider the following Boolean control network

$$x(t+1) = (u_1(t) \vee u_2(t)) \wedge x(t),$$

where inputs $u_1(t), u_2(t)$ satisfy

$$\begin{cases} u_1(t+1) = u_1(t) \leftrightarrow u_2(t) \\ u_2(t+1) = \neg u_1(t). \end{cases}$$

Find out the cycles of this system.
4. Consider the following 2-order Boolean network

$$\begin{cases} A(t+1) = C(t-1) \vee (A(t) \wedge B(t)) \\ B(t+1) = \neg(C(t-1) \wedge A(t)) \\ C(t+1) = B(t-1) \wedge B(t). \end{cases}$$

Find out its cycles and its second algebraic form's cycles. Do they have one-to-one correspondence?
5. Consider the following Boolean network

$$\begin{cases} x_1(t+1) = [x_1(t) \wedge (x_2(t) \vee x_3(t))] \vee [\neg(x_1(t) \vee x_2(t))] \\ x_2(t+1) = \neg(x_1(t) \bar{\nabla} x_3(t)) \\ x_3(t+1) = [(x_1(t) \bar{\nabla} x_3(t)) \vee x_2(t)] \wedge x_3(t). \end{cases}$$

Try to find out its invariant subspace and the corresponding coordinate transformation.

References

1. Akutsu, T., Miyano, S., Kuhara, S.: Inferring qualitative relations in genetic networks and metabolic pathways. Bioinformatics **16**, 727–734 (2000)
2. Albert, R., Barabási, A.: Dynamics of complex systems: Scaling laws for the period of Boolean networks. Physical Review Letters **84**(24), 5660–5663 (2000)
3. Aldana, M.: Boolean dynamics of networks with scale-free topology. Physica D: Nonlinear Phenomena **185**(1), 45–66 (2003)
4. Cheng, D.: Input-state approach to Boolean networks. IEEE Trans. Neural Networks **20**(3), 512–521 (2009)
5. Cheng, D., Qi, H.: A lienar representation of dynamics of Boolean networks. IEEE Trans. Aut. Contr. **55**(10), 2251–2258 (2010)
6. Cheng, D., Qi, H.: State-space analysis of Boolean networks. IEEE Trans. Neural Networks **21**(4), 584–594 (2010)

7. Cheng, D., Qi, H., Li, Z.: Analysis and Control of Boolean Networks: A Semi-tensor Product Approach. Springer, London (2011)
8. Drossel, B., Mihaljev, T., Greil, F.: Number and length of attractors in a critical Kauffman model with connectivity one. Physical Review Letters **94**(8), 88,701 (2005)
9. Farrow, C., Heidel, J., Maloney, J., Rogers, J.: Scalar equations for synchronous Boolean networks with biological applications. IEEE Trans. Neural Networks **15**(2), 348–354 (2004)
10. Goodwin, B.: Temporal Organization in Cells. Acad. Press, New York (1963)
11. Harris, S., Sawhill, B., Wuensche, A., Kauffman, S.: A model of transcriptional regulatory networks based on biases in the observed regulation rules. Complexity **7**(4), 23–40 (2002)
12. Heidel, J., Maloney, J., Farrow, C., Rogers, J.: Finding cycles in synchronous Boolean networks with applications to biochemical systems. Int. J. Bifurcation & Chaos **13**(3), 535–552 (2003)
13. Kauffman, S.: The Origins of Order: Self-organization and Selection in Evolution. New York: Oxford University Press (1993)
14. Li, Z., Zhao, Y., Cheng, D.: Structure of higher order Boolean networks. Preprint (2010)
15. Samuelsson, B., Troein, C.: Superpolynomial growth in the number of attractors in Kauffman networks. Physical Review Letters **90**(9), 98,701 (2003)
16. Shmulevich, I., Dougherty, E., Kim, S., Zhang, W.: Probabilistic Boolean networks: A rule-based uncertainty model for gene regulatory networks. Bioinformatics **18**(2), 261–274 (2002)
17. Zhao, Q.: A remark on 'Scalar equations for synchronous Boolean networks with biologicapplications' by C. Farrow, J. Heidel, J. Maloney, and J. Rogers. IEEE Trans. Neural Networks **16**(6), 1715–1716 (2005)