Chapter 9
Fuzzy Relation Equation

Fuzzy relation plays a fundamental rule in the design of fuzzy controllers, fuzzy logical inferences, and the application of fuzzy control to engineering problems, the application of fuzzy inference to medical diagnosis etc. The problem of finding fuzzy relations was firstly investigated by E. Sanchez, who proposed the so called fuzzy relational equation and provided some fundamental principles [4]. Then Y. Tsukamato et.al investigated the solvability of a class of lower dimensional fuzzy relational equations [6].

This chapter ...

9.1 Operators on Boolean Matrices

In the following we use $\wedge$ for "product" and $\vee$ for "plus". That is,

$$a +_{\mathscr{B}} b := a \vee b = \max\{a, b\}, \quad a, b \in [0,1]. \tag{9.1}$$

$$a \times_{\mathscr{B}} b := a \wedge b = \min\{a, b\}, \quad a, b \in [0,1]. \tag{9.2}$$

It follows that let $A = (a_{i,j}) \in \mathscr{M}_{m\times n}$ and $B = (b_{i,j}) \in \mathscr{M}_{n\times s}$, with $a_{i,j}, b_{i,j} \in [0,1]$. Then their Boolean product is defined as

$$C = (c_{i,j}) = A \circ B \in \mathscr{M}_{m\times s}, \tag{9.3}$$

where

$$c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j} = (a_{i,1} \times_{\mathscr{B}} b_{1,j}) +_{\mathscr{B}} (a_{i,2} \times_{\mathscr{B}} b_{2,j}) +_{\mathscr{B}} \cdots +_{\mathscr{B}} (a_{i,n} \times_{\mathscr{B}} b_{n,j}),$$
$$i = 1, \cdots, m; \; j = 1, \cdots, s.$$

The following proposition is an immediate consequence of the definition.

Proposition 9.1. *1. Let $A \in \mathscr{D}^{\infty}_{m\times n}$, $B \in \mathscr{D}^{\infty}_{n\times p}$, and $C \in \mathscr{D}^{\infty}_{p\times q}$. Then (Associativity)*

$$(A \circ B) \circ C = A \circ (B \circ C). \tag{9.4}$$

2. *Let A, B, C be matrices with entries in $\mathscr{D}^\infty$ and of proper dimensions such that the following involved operations are well defined. Then (Distributivity)*

$$\begin{aligned} A \circ (B +_{\mathscr{B}} C) = (A \circ B) +_{\mathscr{B}} (A \circ C); \\ (A +_{\mathscr{B}} B) \circ C = (A \circ C) +_{\mathscr{B}} (B \circ C). \end{aligned} \tag{9.5}$$

Next, we define a partial order $\geq$ on $\mathscr{D}^\infty_{m \times n}$ as follows.

Definition 9.1. 1. Let $A = (a_{i,j}), B = (b_{i,j}) \in \mathscr{D}^\infty_{m \times n}$. We say $A \geq B$ if

$$a_{i,j} \geq b_{i,j}, \quad i = 1, \cdots, m; \; j = 1, \cdots, n.$$

2. If $A \geq B$ and $A \neq B$, then we say $A > B$.
3. Let $\Theta \subset \mathscr{D}^\infty_{m \times n}$. $A \in \Theta$ is called a maximum (minimum) element, if there is no $B \in \Theta$ such that $B > A$ ($B < A$)
4. $A \in \Theta$ is called the largest (smallest) element, if

$$A \geq B, \quad \forall B \in \Theta; \qquad (A \leq B, \quad \forall B \in \Theta.)$$

The following order preserving property is an immediate consequence of the definition of product.

Proposition 9.2. *Let $A,\ B \in \mathscr{D}^\infty_{m \times n}$ and $C,\ D \in \mathscr{D}^\infty_{n \times p}$. Assume $A \geq B$ and $C \geq D$. Then*

$$A \circ C \geq B \circ D. \tag{9.6}$$

Throughout this paper we consider only some universes of discourse, which are finite. Particularly, we set

$$U = \{u_1, \cdots, u_m\}, \quad V = \{v_1, \cdots, v_n\}, \quad W = \{w_1, \cdots, w_s\}.$$

Then we have

Definition 9.2. Let $R \in \mathscr{F}(U \times V)$ be given. The relation matrix of R, denoted by M_R is defined as

$$M_R = \begin{bmatrix} \mu_R(u_1, v_1) & \mu_R(u_1, v_2) & \cdots & \mu_R(u_1, v_n) \\ \mu_R(u_2, v_1) & \mu_R(u_2, v_2) & \cdots & \mu_R(u_2, v_n) \\ \vdots & & & \\ \mu_R(u_m, v_1) & \mu_R(u_m, v_2) & \cdots & \mu_R(u_m, v_n) \end{bmatrix}, \tag{9.7}$$

where μ_R is the membership degree of R on $U \times V$.

For notational ease, conventionally R is also used for M_R. Hereafter, we use this convention.

Usually, two kinds of fuzzy relational equations (FRE) were investigated.

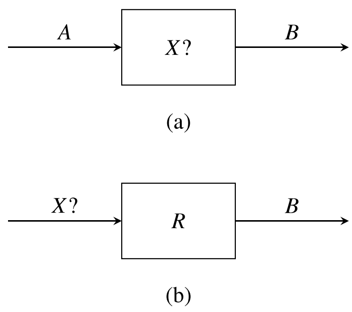

Fig. 9.1 (a): Unknown Fuzzy Relation (b): Unknown Fuzzy Input

- Type 1: Let $A \in \mathscr{F}(U \times V)$, and $B \in \mathscr{F}(U \times W)$. We are looking for a fuzzy relation $X \in \mathscr{F}(V \times W)$, such that
$$A \circ X = B. \tag{9.8}$$
We refer to Fig. 9.1 (a) for this type of FRE's, which is commonly used in the design of fuzzy controllers.
- Type 2: Let $R \in \mathscr{F}(V \times W)$, and $B \in \mathscr{F}(U \times W)$. We are looking for a fuzzy input $X \in \mathscr{F}(U \times V)$, such that
$$X \circ R = B. \tag{9.9}$$
We refer to Fig. 9.1 (b) for this type of FRE's, which may be used for a problem similar to the diagnosing diseases via symptoms, where the fuzzy relation is known [3].

  Taking a transpose on both sides of (9.9), we have
$$R^T \circ X^T = B^T,$$

which has the same form as (9.8). We, therefore, consider the solvability of (9.8) only.

To state the existing result for solving (9.8), we need some preparations. The following statements are copied from [7].

Definition 9.3 ([7]).

(i) Let $a,\ b \in [0,1]$. Then
$$a \oplus b = \begin{cases} b, & a > b \\ 1, & a \le b; \end{cases} \tag{9.10}$$

and

$$a \oslash b = \begin{cases} b, & a \ge b \\ 0, & a < b. \end{cases} \tag{9.11}$$

(ii) Let $A \in \mathscr{F}(U)$ and $B \in \mathscr{F}(V)$. Then $A \oplus B$ is defined by

$$\mu_{A \oplus B}(u,v) := \mu_A(u) \oplus \mu_B(v); \tag{9.12}$$

and $A^T \oslash B$ is defined by

$$\mu_{A^T \oslash B}(u,v) := \mu_A^T(u) \oslash \mu_B(v). \tag{9.13}$$

Then the following result is a commonly used method in fuzzy control design.

Theorem 9.1 ([7]). *In equation (9.8) assume (i) $A \in \mathscr{F}(U)$ and $B \in \mathscr{F}(V)$; (ii) there exists at least one solution, then*

(i)

$$X^* = A^T \oplus B \tag{9.14}$$

is the largest solution;

(ii)

$$X_* = A^T \oslash B \tag{9.15}$$

is also a solution.

It is obvious that the result of Theorem 9.1 is very limited. It provides only some particular solutions under very strong constrains on A and B. The purpose of this paper is to provide a general algorithm, which provides all the solutions of (9.8), without any restrictions on both A and B. This approach is based on the semi-tensor product of matrices and the vector expression of multi-valued logic, which were proposed recently in [1].

9.2 Matrix Expression of k-valued Logical Relations

Let $x \in \mathscr{D}^k$, where $k < \infty$. We identify

$$\frac{i}{k-1} \sim \delta_k^{k-i}, \quad i = 0, 1, \cdots, k-1.$$

Then we have $x \in \Delta_k$, which is called the vector expression of x.

Let $\sigma : \underbrace{\mathscr{D}^k \times \cdots \times \mathscr{D}^k}_{r} \to \mathscr{D}^k$. σ is called an r-ary k-valued logical operator (alternatively, logical function). If the logical variables are expressed in vector form, σ becomes a mapping $\sigma : \underbrace{\Delta_k \times \cdots \times \Delta_k}_{r} \to \Delta_k$.

Theorem 9.2 ([1]). *Let f be an r-ary k-valued logical function. Then there exists a unique logical matrix $M_f \in \mathscr{L}_{k\times k^r}$, such that in vector form*

$$f(x_1,\cdots,x_r) = M_f \ltimes x_1 \ltimes \cdots \ltimes x_r. \tag{9.16}$$

(9.16) is called the algebraic form of f, and M_f is called the structure matrix of f. We refer to [1] for how to calculate M_f from its logical form and how to convert the algebraic form back to its logical form.

Particularly, when σ is a unary operator, we can find its structure matrix M_σ such that

$$\sigma x = M_\sigma x, \quad x \in \Delta_k.$$

when σ is a binary operator, we can find its structure matrix M_σ such that

$$x\sigma y = M_\sigma \ltimes x \ltimes y, \quad x,y \in \Delta_k.$$

In the following example we provide the structure matrices of $\neg$, $\wedge$, and $\vee$ respectively.

Example 9.1. For notational ease, we introduce a set of k-dimensional vectors as:

$$\begin{aligned} U_s &= (1\ 2\ \cdots\ s-1\ \underbrace{s\ \cdots\ s}_{k-s+1}) \\ V_s &= (\underbrace{s\ \cdots\ s}_{s}\ s+1\ s+2\ \cdots\ k), \quad s = 1,2,\cdots,k. \end{aligned}$$

Then we have

1. For k-valued negation ($\neg$), its structure matrix is

$$M_n^k = \delta_k[k\ k-1\ \cdots\ 1]. \tag{9.17}$$

When $k=3$ we have

$$M_n^3 = \delta_3[3\ 2\ 1]. \tag{9.18}$$

When $k=4$ we have

$$M_n^4 = \delta_4[4\ 3\ 2\ 1]. \tag{9.19}$$

2. For k-valued disjunction ($\vee$), its structure matrix is

$$M_d^k = \delta_k[U_1\ U_2\ \cdots\ U_k]. \tag{9.20}$$

When $k=3$ we have

$$M_d^3 = \delta_3[1\ 1\ 1\ 1\ 2\ 2\ 1\ 2\ 3]. \tag{9.21}$$

When $k=4$ we have

$$M_d^4 = \delta_4[1\ 1\ 1\ 1\ 1\ 2\ 2\ 2\ 1\ 2\ 3\ 3\ 1\ 2\ 3\ 4]. \tag{9.22}$$

3. For k-valued conjunction ($\wedge$), its structure matrix is

$$M_c^k = \delta_k[V_1\ V_2\ \cdots\ V_k]. \tag{9.23}$$

When $k = 3$ we have

$$M_c^3 = \delta_3[1\ 2\ 3\ 2\ 2\ 3\ 3\ 3\ 3]. \tag{9.24}$$

When $k = 4$ we have

$$M_c^4 = \delta_4[1\ 2\ 3\ 4\ 2\ 2\ 3\ 4\ 3\ 3\ 3\ 4\ 4\ 4\ 4\ 4]. \tag{9.25}$$

9.3 Structure of the Set of Solutions

Consider equation (9.8). Let $A = (a_{i,j})$, $B = (b_{i,j})$, and $R = (x_{i,j})$, where $x_{i,j}$ are used to emphasize that R is the unknown matrix. We can further convert it into canonical linear algebraic equations as

$$A \circ X_i = B_i, \quad i = 1, \cdots, s, \tag{9.26}$$

where X_i is the i-th column of R and B_i is the i-th column of B.

Collecting different values of the entries of A and B as

$$S = \{a_{i,j}, b_{p,q} | i = 1, \cdots, m; j = 1, \cdots, n; p = 1, \cdots, m; q = 1, \cdots, s\},$$

and adding 1 and/or 0 when they are not in S, we construct an ordered set as

$$\Xi = \{\xi_i | i = 1, \cdots, r; \text{ and } \xi_1 = 0 < \xi_2 < \cdots < \xi_{r-1} < \xi_r = 1\}.$$

Then we have $S \subset \Xi$.

Definition 9.4. Let $x \in [0,1]$. Then we define

(i) $\pi_* : [0,1] \to \Xi$ as

$$\pi_*(x) = \max_i \{\xi_i \in \Xi | \xi_i \le x\}; \tag{9.27}$$

(ii) $\pi^* : [0,1] \to \Xi$ as

$$\pi^*(x) = \min_i \{\xi_i \in \Xi | \xi_i \ge x\}. \tag{9.28}$$

Note that if $x = \xi_i \in \Xi$, then

$$\pi_*(x) = \pi^*(x) = \xi_i.$$

Otherwise, there exists a unique i such that

$$\xi_i < x < \xi_{i+1}.$$

Then we have

$$\pi_*(x) = \xi_i; \quad \pi^*(x) = \xi_{i+1}.$$

For statement ease, we identify ξ_i with $\frac{i-1}{r-1}$, $i = 1, \cdots, r$. It means we identify Ξ with $\mathscr{D}_r$. Then we have the following result, which allows us to search solutions from a finite set.

Lemma 9.1. *Let $R = (x_{i,j}) \in \mathscr{D}^{\infty}_{n\times s}$ be a solution of (9.8). Then $\pi_*(R) := (\pi_*(x_{i,j}))$ is also a solution of (9.8).*

Proof. It suffices to prove it for each equation in (9.26), which is simply denoted by

$$A \circ z = b. \tag{9.29}$$

Assume $z = (z_1, \cdots, z_n)^T$ is a solution of (9.29). Set $Z_0 = \{z_1, \cdots, z_n\}$ and define

$$Z^0 = Z_0 \backslash \Xi.$$

If $Z^0 = \emptyset$, we are done. Otherwise, we can find

$$z^0 = \max_{z_j} \{z_j \in Z^0\}.$$

Then there is an i_0 such that

$$\xi_{i_0} < z^0 < \xi_{i_0+1}. \tag{9.30}$$

Next, we replace all the elements in Z^0, which are greater than ξ_{i_0}, by ξ_{i_0}. Such a replacement converts Z_0 to a new set, called Z_1. We claim that Z_1 is also a solution of (9.29).

Consider a particular equation of (9.29), say, j-th equation, which is

$$[a_{j,1} \wedge z_1] \vee [a_{j,2} \wedge z_2] \vee \cdots [a_{j,n} \wedge z_n] = b_j. \tag{9.31}$$

First, we assume $b_j \ge \xi_{i_0+1}$. Then there must be a term $a_{j,s} \wedge z_s$, which equals b_j. Then replacing any $\xi_{i_0} < z^0 < \xi_{i_0+1}$ by ξ_{i_0} will not affect the equality.

Next, we assume $b_j \le \xi_{i_0}$. Then multiplying both sides of (9.31) by ξ_{i_0} (precisely, operating both sides by $\xi_{i_0} \wedge$). Then the right hand side is still b_j, and on the left hand side, since each term should be less than or equal to b_j, it changes nothing. But if there is a term, say, $a_{j,s} \wedge z_s$, which has z_s satisfying (9.30), then we can replace it by

$$\xi_{i_0} \wedge a_{j,s} \wedge z_s = a_{j,s} \wedge \xi_{i_0}.$$

We conclude that Z_1 is a solution of (9.29). Now for Z_1, we can do the same thing as setting

$$Z^1 = Z_1 \backslash \Xi,$$

defining

$$z^1 = \max_{z_j} \{z_j \in Z^1\},$$

and finding i_1 such that

$$\xi_{i_1} < z^1 < \xi_{i_1+1}. \qquad (9.32)$$

Finally, in the solution $Z_1 = (z_1^1, \cdots, z_1^n)$ all z_1^j, satisfying (9.32), can be replaced by ξ_{i_1} to produce a new solution Z_2. Note that now $\xi_{i_1} < \xi_{i_0}$. Continuing this procedure, finally, we can have $Z^{k^*} = \emptyset$, where $k^* \le r$. The conclusion follows. □

Similar to Lemma 9.1, we can prove the following result.

Lemma 9.2. *Let $R = (x_{i,j}) \in \mathscr{D}^{\infty}_{n\times s}$ be a solution of (9.8). Then $\pi^*(R) := (\pi^*(x_{i,j}))$ is also a solution of (9.8).*

Now we can prove the following result, which shows the structure of the set of solutions.

Theorem 9.3. *$R = (x_{i,j}) \in \mathscr{D}^{\infty}_{n\times s}$ is a solution of (9.8), if and only if both $\pi_*(R)$ and $\pi^*(R)$ are solutions of (9.8).*

Proof. The necessity comes from Lemmas 9.1 and 9.2. We prove the sufficiency. That is, if both $\pi_*(R)$ and $\pi^*(R)$ are solutions of (9.8), then so is R.

If $R = \pi_*(R)$ or $R = \pi^*(R)$, we are done. So we assume $R \ne \pi_*(R)$ and $R \ne \pi^*(R)$. We prove it by contradiction. Assume R is not a solution of (9.8). Since $R \ge \pi_*(R)$, according to Proposition 9.2 we have $A \circ R \ge B$. But since R is not a solution, we have

$$A \circ R > B.$$

Now since $\pi^*(R) \ge R$, we have

$$A \circ \pi^*(R) \ge A \circ R > B.$$

This is absurd. □

Theorem 9.3 gives a complete picture for the set of solutions. It has also clearly demonstrated that the set of PSS's is enough to describe the whole set of solutions.

We have the following useful proposition for the set of solutions. In fact, Theorem 9.1 shows the following result for the particular case, where A and B are restricted.

Proposition 9.3. *If there is a solution of (**??**) in Ξ^n. Then there is a largest solution in Ξ^n.*

Proof. If we can prove the maximum solution is unique, we are done. Now assume both z_1^* and z_2^* are two different maximum solutions. Using (9.5), it is easy to prove that $z_1^* +_{\mathscr{B}} z_2^*$ is also a solution. But $z_1^* +_{\mathscr{B}} z_2^* > z_1^*$, which is a contradiction. □

9.4 Solving Fuzzy Relational Equation

In the following all the matrix products are assumed to be semi-tensor product. For compactness, the symbol $\ltimes$ is omitted unless we want to emphasize it.

To solve the fuzzy relation we have only to solve equations in (9.26). That is, we have only to develop a method to solve (9.29).

Recall that in vector form we have the structure matrices such that all the logical expressions with operators can be expressed as a matrix product. Particularly, in this paper we need the following expressions.

Using algebraic form, we can convert the left hand side (LHS) of equation (9.31) (as the j-th equation of (9.29)), into the following form:

$$
\begin{aligned}
LHS &= (M_d^r)^{n-1}(M_c^r a_{j,1} z_1)\cdots(M_c^r a_{j,n} z_n) \\
&= (M_d^r)^{n-1} M_c^r a_{j,1} [I_r \otimes (M_c^r a_{j,2})][I_{r^2} \otimes (M_c^r a_{j,3})]\cdots[I_{r^{n-1}} \otimes (M_c^r a_{j,n})] \ltimes_{i=1}^n z_i \\
&:= L_j z,
\end{aligned}
\tag{9.33}
$$

where

$$
\begin{aligned}
L_j &= (M_d^r)^{n-1} M_c^r a_{j,1} [I_r \otimes (M_c^r a_{j,2})][I_{r^2} \otimes (M_c^r a_{j,3})]\cdots[I_{r^{n-1}} \otimes (M_c^r a_{j,n})] \in \mathcal{L}_{r\times r^n}; \\
z &= \ltimes_{i=1}^n z_i.
\end{aligned}
$$

Then (9.29) becomes

$$
L_j z = b_j, \quad j = 1, \cdots, m. \tag{9.34}
$$

Multiplying both sides of m equations of (9.34), we can express (9.29) as

$$
Lz = b, \tag{9.35}
$$

where $L = L_1 * L_2 * \cdots * L_m \in \mathcal{L}_{r^m \times r^n}$, and $b = \ltimes_{i=1}^m b_i$. Here " $*$ " is the Khatri-Rao product of matrices [2]. Precisely,

$$
\mathrm{Col}_t(L) = \mathrm{Col}_t(L_1) \ltimes \mathrm{Col}_t(L_2) \ltimes \cdots \ltimes \mathrm{Col}_t(L_s), \quad t = 1, \cdots, r^n.
$$

Next, we show how to solve the equation (9.35). Note that since L is a logical matrix, $b \in \Delta_{r^m}$ and $z \in \Delta_{r^n}$, the following result is obvious.

Theorem 9.4. *Equation (9.35) has solution, if and only if*

$$
b \in \mathrm{Col}(L). \tag{9.36}
$$

Now assume

$$
\Lambda = \{\lambda \,|\, \mathrm{Col}_\lambda(L) = b\}.
$$

Then the solution set is

$$
\left\{ z_\lambda = \delta_{2^n}^{\lambda} \,\middle|\, \lambda \in \Lambda \right\}. \tag{9.37}
$$

Finally, we have to convert z back to $(z_1, \cdots, z_n) \in \Xi^n$.

9.5 Illustrative Examples

This section presents some examples to demonstrate the algorithm for solving the fuzzy relational equations. In fact, the method developed in previous section is applicable to general fuzzy logical equations. First example is a simple one, which is used to show the solving process.

Example 9.2. Consider the following logical equation

$$\begin{cases} x \wedge y = 0.32 \\ (\neg x) \vee y = 0.68. \end{cases} \tag{9.38}$$

First, one sees easily that the logical values can be divided into 4 levels. That is,

$$\Xi = \{1,\ 0.68,\ 0.32,\ 0\}.$$

Then we identify the values with their vector forms as

$$1 \sim \delta_4^1; \quad 0.68 \sim \delta_4^2; \quad 0.32 \sim \delta_4^3; \quad 0 \sim \delta_4^4.$$

Now (9.38) can be converted into its algebraic form as

$$\begin{cases} M_c^4 \ltimes x \ltimes y = \delta_4^3 \\ M_d^4 \ltimes (M_n^4 \ltimes x) \ltimes y = \delta_4^2. \end{cases} \tag{9.39}$$

Setting $z = x \ltimes y$, (9.39) can be converted as

$$\begin{cases} G_1 z = \delta_4^3 \\ G_2 z = \delta_4^2, \end{cases} \tag{9.40}$$

where

$$G_1 = M_c^4 = \delta_4[1\ 2\ 3\ 4\ 2\ 2\ 3\ 4\ 3\ 3\ 3\ 4\ 4\ 4\ 4\ 4],$$
$$G_2 = M_d^4 \ltimes M_n^4 = \delta_4[1\ 2\ 3\ 4\ 1\ 2\ 3\ 3\ 1\ 2\ 2\ 2\ 1\ 1\ 1\ 1].$$

Multiplying two equations together yields

$$Lz = b, \tag{9.41}$$

where

$$L = G_1 * G_2 = \delta_{16}[1\ 6\ 11\ 16\ 5\ 6\ 11\ 15\ 9\ 10\ 10\ 14\ 13\ 13\ 13\ 13],$$

$$b = \delta_4^3 \ltimes \delta_4^2 = \delta_{16}^{10}.$$

Since

$$\mathrm{Col}_{10}(L) = \mathrm{Col}_{11}(L) = \delta_{16}^{10},$$

we have solutions

$$z_1 = \delta_{16}^{10}, \quad z_2 = \delta_{16}^{11}.$$

It follows that

$$\begin{cases} x_1 = \delta_4^3 \\ y_1 = \delta_4^2, \end{cases} \qquad \begin{cases} x_2 = \delta_4^3 \\ y_2 = \delta_4^3. \end{cases}$$

Back to the fuzzy values, we have

$$\begin{cases} x_1 = 0.32 \\ y_1 = 0.68, \end{cases} \qquad \begin{cases} x_2 = 0.32 \\ y_2 = 0.32. \end{cases}$$

The next example is from [5]. It will be used to demonstrate some structure properties of the solution set of fuzzy relational equations.

Example 9.3. Consider the following relational equation [5]

$$Q \circ X = T, \tag{9.42}$$

where

$$Q = \begin{bmatrix} 0.2 & 0 & 0.8 & 1 \\ 0.4 & 0.3 & 0 & 0.7 \\ 0.5 & 0.9 & 0.2 & 0 \end{bmatrix}; \quad T = \begin{bmatrix} 0.7 & 0.3 & 1 \\ 0.6 & 0.4 & 0.7 \\ 0.8 & 0.9 & 0.2 \end{bmatrix}.$$

First, we figure out the levels of the membership degrees and identify them with their vector forms:

$$\begin{array}{l} 1 \sim \delta_{10}^1; \quad 0.9 \sim \delta_{10}^2; \; 0.8 \sim \delta_{10}^3; \; 0.7 \sim \delta_{10}^4; \; 0.6 \sim \delta_{10}^5; \\ 0.5 \sim \delta_{10}^6; \; 0.4 \sim \delta_{10}^7; \; 0.3 \sim \delta_{10}^8; \; 0.2 \sim \delta_{10}^9; \; 0 \sim \delta_{10}^{10}. \end{array}$$

We start by solving the first column of X. Let $X_1 = (x_{11}, x_{21}, x_{31}, x_{41})^T = \mathrm{Col}_1(X)$. Then the algebraic equation for X_1 becomes

$$\begin{cases} (\delta_{10}^9 \wedge x_{11}) \vee (\delta_{10}^{10} \wedge x_{21}) \vee (\delta_{10}^3 \wedge x_{31}) \vee (\delta_{10}^1 \wedge x_{41}) = \delta_{10}^4 \\ (\delta_{10}^7 \wedge x_{11}) \vee (\delta_{10}^8 \wedge x_{21}) \vee (\delta_{10}^{10} \wedge x_{31}) \vee (\delta_{10}^4 \wedge x_{41}) = \delta_{10}^5 \\ (\delta_{10}^6 \wedge x_{11}) \vee (\delta_{10}^2 \wedge x_{21}) \vee (\delta_{10}^9 \wedge x_{31}) \vee (\delta_{10}^{10} \wedge x_{41}) = \delta_{10}^3. \end{cases} \tag{9.43}$$

Let $x_1 = \ltimes_{i=1}^4 x_{i1}$. Then equation (9.43) can be converted into its algebraic form as

$$Lx_1 = b_1, \tag{9.44}$$

where

$$\begin{aligned} L = \delta_{1000}[&32 \;\; 132 \;\; 232 \;\; 232 \;\; 242 \;\; 252 \;\; 262 \;\; 262 \;\; 262 \;\; 262 \cdots \\ &899 \;\; 40 \;\; 140 \;\; 240 \;\; 340 \;\; 450 \;\; 560 \;\; 670 \;\; 780 \;\; 890 \;\; 1000] \in \mathscr{L}_{1000 \times 10000}, \end{aligned}$$

and

$$b_1 = \delta_{10}^4 \ltimes \delta_{10}^5 \ltimes \delta_{10}^3 = \delta_{1000}^{343}.$$

Using the Toolbox[1], we can solve it out as

$$\begin{array}{lll}
X_1^1 = \delta_{10}[1\ 3\ 4\ 5]^T, & X_1^2 = \delta_{10}[2\ 3\ 4\ 5]^T, & X_1^3 = \delta_{10}[3\ 3\ 4\ 5]^T, \\
X_1^4 = \delta_{10}[4\ 3\ 4\ 5]^T, & X_1^5 = \delta_{10}[5\ 3\ 4\ 5]^T, & X_1^6 = \delta_{10}[6\ 3\ 4\ 5]^T, \\
X_1^7 = \delta_{10}[7\ 3\ 4\ 5]^T, & X_1^8 = \delta_{10}[8\ 3\ 4\ 5]^T, & X_1^9 = \delta_{10}[9\ 3\ 4\ 5]^T, \\
X_1^{10} = \delta_{10}[10\ 3\ 4\ 5]^T. & &
\end{array}$$

For the second column, we have

$$Lx_2 = b_2, \tag{9.45}$$

where

$$b_2 = \delta_{10}^8 \ltimes \delta_{10}^7 \ltimes \delta_{10}^2 = \delta_{1000}^{762}.$$

Solving it, we have

$$\begin{array}{lll}
X_2^1 = \delta_{10}[1\ 1\ 8\ 8]^T, & X_2^2 = \delta_{10}[1\ 1\ 8\ 9]^T, & X_2^3 = \delta_{10}[1\ 1\ 8\ 10]^T, \\
X_2^4 = \delta_{10}[1\ 1\ 9\ 8]^T, & X_1^5 = \delta_{10}[1\ 1\ 10\ 8]^T, & X_2^6 = \delta_{10}[1\ 2\ 8\ 8]^T, \\
X_2^7 = \delta_{10}[1\ 2\ 8\ 9]^T, & X_2^8 = \delta_{10}[1\ 2\ 8\ 10]^T, & X_2^9 = \delta_{10}[1\ 2\ 9\ 8]^T, \\
X_2^{10} = \delta_{10}[1\ 2\ 10\ 8]^T, & X_2^{11} = \delta_{10}[2\ 1\ 8\ 8]^T, & X_2^{12} = \delta_{10}[2\ 1\ 8\ 9]^T, \\
X_2^{13} = \delta_{10}[2\ 1\ 8\ 10]^T, & X_2^{14} = \delta_{10}[2\ 1\ 9\ 8]^T, & X_2^{15} = \delta_{10}[2\ 1\ 10\ 8]^T, \\
X_2^{16} = \delta_{10}[2\ 2\ 8\ 8]^T, & X_2^{17} = \delta_{10}[2\ 2\ 8\ 9]^T, & X_2^{18} = \delta_{10}[2\ 2\ 8\ 10]^T, \\
X_2^{19} = \delta_{10}[2\ 2\ 9\ 8]^T, & X_2^{20} = \delta_{10}[2\ 2\ 10\ 8]^T, & X_2^{21} = \delta_{10}[3\ 1\ 8\ 8]^T, \\
X_2^{22} = \delta_{10}[3\ 1\ 8\ 9]^T, & X_2^{23} = \delta_{10}[3\ 1\ 8\ 10]^T, & X_2^{24} = \delta_{10}[3\ 1\ 9\ 8]^T, \\
X_2^{25} = \delta_{10}[3\ 1\ 10\ 8]^T, & X_2^{26} = \delta_{10}[3\ 2\ 8\ 8]^T, & X_2^{27} = \delta_{10}[3\ 2\ 8\ 9]^T, \\
X_2^{28} = \delta_{10}[3\ 2\ 8\ 10]^T, & X_2^{29} = \delta_{10}[3\ 2\ 9\ 8]^T, & X_2^{30} = \delta_{10}[3\ 2\ 10\ 8]^T, \\
X_2^{31} = \delta_{10}[4\ 1\ 8\ 8]^T, & X_2^{32} = \delta_{10}[4\ 1\ 8\ 9]^T, & X_2^{33} = \delta_{10}[4\ 1\ 8\ 10]^T, \\
X_2^{34} = \delta_{10}[4\ 1\ 9\ 8]^T, & X_2^{35} = \delta_{10}[4\ 1\ 10\ 8]^T, & X_2^{36} = \delta_{10}[4\ 2\ 8\ 8]^T, \\
X_2^{37} = \delta_{10}[4\ 2\ 8\ 9]^T, & X_2^{38} = \delta_{10}[4\ 2\ 8\ 10]^T, & X_2^{39} = \delta_{10}[4\ 2\ 9\ 8]^T, \\
X_2^{40} = \delta_{10}[4\ 2\ 10\ 8]^T, & X_2^{41} = \delta_{10}[5\ 1\ 8\ 8]^T, & X_2^{42} = \delta_{10}[5\ 1\ 8\ 9]^T, \\
X_2^{42} = \delta_{10}[5\ 1\ 8\ 10]^T, & X_2^{44} = \delta_{10}[5\ 1\ 9\ 8]^T, & X_2^{45} = \delta_{10}[5\ 1\ 10\ 8]^T, \\
X_2^{46} = \delta_{10}[5\ 2\ 8\ 8]^T, & X_2^{47} = \delta_{10}[5\ 2\ 8\ 9]^T, & X_2^{48} = \delta_{10}[5\ 2\ 8\ 10]^T, \\
X_2^{49} = \delta_{10}[5\ 2\ 9\ 8]^T, & X_2^{50} = \delta_{10}[5\ 2\ 10\ 8]^T, & X_2^{51} = \delta_{10}[6\ 1\ 8\ 8]^T, \\
X_2^{52} = \delta_{10}[6\ 1\ 8\ 9]^T, & X_2^{53} = \delta_{10}[6\ 1\ 8\ 10]^T, & X_2^{54} = \delta_{10}[6\ 1\ 9\ 8]^T, \\
X_2^{55} = \delta_{10}[6\ 1\ 10\ 8]^T, & X_2^{56} = \delta_{10}[6\ 2\ 8\ 8]^T, & X_2^{57} = \delta_{10}[6\ 2\ 8\ 9]^T, \\
X_2^{58} = \delta_{10}[6\ 2\ 8\ 10]^T, & X_2^{59} = \delta_{10}[6\ 2\ 9\ 8]^T, & X_2^{60} = \delta_{10}[6\ 2\ 10\ 8]^T, \\
X_2^{61} = \delta_{10}[7\ 1\ 8\ 8]^T, & X_2^{62} = \delta_{10}[7\ 1\ 8\ 9]^T, & X_2^{63} = \delta_{10}[7\ 1\ 8\ 10]^T, \\
X_2^{64} = \delta_{10}[7\ 1\ 9\ 8]^T, & X_2^{65} = \delta_{10}[7\ 1\ 10\ 8]^T, & X_2^{66} = \delta_{10}[7\ 2\ 8\ 8]^T, \\
X_2^{67} = \delta_{10}[7\ 2\ 8\ 9]^T, & X_2^{68} = \delta_{10}[7\ 2\ 8\ 10]^T, & X_2^{69} = \delta_{10}[7\ 2\ 9\ 8]^T, \\
X_2^{70} = \delta_{10}[7\ 2\ 10\ 8]^T. & &
\end{array}$$

Finally, for the last column, we have

[1] The STP toolbox for Matlab is available at *

$$Lx_3 = b_3, \tag{9.46}$$

where

$$b_3 = \delta_{10}^1 \ltimes \delta_{10}^4 \ltimes \delta_{10}^9 = \delta_{1000}^{39}.$$

Solving it, we have

$$\begin{array}{lll}
X_3^1 = \delta_{10}[9\ 9\ 1\ 1]^T, & X_3^2 = \delta_{10}[9\ 9\ 2\ 1]^T, & X_3^3 = \delta_{10}[9\ 9\ 3\ 1]^T,\\
X_3^4 = \delta_{10}[9\ 9\ 4\ 1]^T, & X_3^5 = \delta_{10}[9\ 9\ 5\ 1]^T, & X_3^6 = \delta_{10}[9\ 9\ 6\ 1]^T,\\
X_3^7 = \delta_{10}[9\ 9\ 7\ 1]^T, & X_3^8 = \delta_{10}[9\ 9\ 8\ 1]^T, & X_3^9 = \delta_{10}[9\ 9\ 9\ 1]^T,\\
X_3^{10} = \delta_{10}[9\ 9\ 10\ 1]^T, & X_3^{11} = \delta_{10}[9\ 10\ 1\ 1]^T, & X_3^{12} = \delta_{10}[9\ 10\ 2\ 1]^T,\\
X_3^{13} = \delta_{10}[9\ 10\ 3\ 1]^T, & X_3^{14} = \delta_{10}[9\ 10\ 4\ 1]^T, & X_3^{15} = \delta_{10}[9\ 10\ 5\ 1]^T,\\
X_3^{16} = \delta_{10}[9\ 10\ 6\ 1]^T, & X_3^{17} = \delta_{10}[9\ 10\ 7\ 1]^T, & X_3^{18} = \delta_{10}[9\ 10\ 8\ 1]^T,\\
X_3^{19} = \delta_{10}[9\ 10\ 9\ 1]^T, & X_3^{20} = \delta_{10}[9\ 10\ 10\ 1]^T, & X_3^{21} = \delta_{10}[10\ 9\ 1\ 1]^T,\\
X_3^{22} = \delta_{10}[10\ 9\ 2\ 1]^T, & X_3^{23} = \delta_{10}[10\ 9\ 3\ 1]^T, & X_3^{24} = \delta_{10}[10\ 9\ 4\ 1]^T,\\
X_3^{25} = \delta_{10}[10\ 9\ 5\ 1]^T, & X_3^{26} = \delta_{10}[10\ 9\ 6\ 1]^T, & X_3^{27} = \delta_{10}[10\ 9\ 7\ 1]^T,\\
X_3^{28} = \delta_{10}[10\ 9\ 8\ 1]^T, & X_3^{29} = \delta_{10}[10\ 9\ 9\ 1]^T, & X_3^{30} = \delta_{10}[10\ 9\ 10\ 1]^T,\\
X_3^{31} = \delta_{10}[10\ 10\ 1\ 1]^T, & X_3^{32} = \delta_{10}[10\ 10\ 2\ 1]^T, & X_3^{33} = \delta_{10}[10\ 10\ 3\ 1]^T,\\
X_3^{34} = \delta_{10}[10\ 10\ 4\ 1]^T, & X_3^{35} = \delta_{10}[10\ 10\ 5\ 1]^T, & X_3^{36} = \delta_{10}[10\ 10\ 6\ 1]^T,\\
X_3^{37} = \delta_{10}[10\ 10\ 7\ 1]^T, & X_3^{38} = \delta_{10}[10\ 10\ 8\ 1]^T, & X_3^{39} = \delta_{10}[10\ 10\ 9\ 1]^T.
\end{array}$$

We conclude the following:

1. We have totally $10 \times 70 \times 39 = 27300$ solutions in Ξ^4.
2. The largest solution, corresponding to the largest solutions of each column, is

$$X^* = [X_1^1\ X_2^1\ X_3^1] \sim \begin{bmatrix} 1 & 1 & 0.2 \\ 0.8 & 1 & 0.2 \\ 0.7 & 0.3 & 1 \\ 0.6 & 0.3 & 1 \end{bmatrix}.$$

3. There is no smallest solution, because for the second column there are two minimum solutions

$$X_2^{68} = \delta_{10}[7\ 2\ 8\ 10]^T \sim \begin{bmatrix} 0.4 \\ 0.9 \\ 0.3 \\ 0 \end{bmatrix}; \quad X_2^{70} = \delta_{10}[7\ 2\ 10\ 8]^T \sim \begin{bmatrix} 0.4 \\ 0.9 \\ 0 \\ 0.3 \end{bmatrix}.$$

Hence, we have also two minimum solutions for X as

$$X_*^1 = [X_1^{10}\ X_2^{68}\ X_3^{39}] \sim \begin{bmatrix} 0 & 0.4 & 0 \\ 0.8 & 0.9 & 0 \\ 0.7 & 0.3 & 0.2 \\ 0.6 & 0 & 1 \end{bmatrix},$$

and

$$X_*^2 = [X_1^{10}\ \ X_2^{70}\ \ X_3^{39}] \sim \begin{bmatrix} 0 & 0.4 & 0 \\ 0.8 & 0.9 & 0 \\ 0.7 & 0 & 0.2 \\ 0.6 & 0.3 & 1 \end{bmatrix}.$$

4. The solution provided in [5] is

$$R = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.8 & 1 & 0 \\ 0.7 & 0 & 0.5 \\ 0.6 & 0.3 & 1 \end{bmatrix}$$

which corresponds to

$$X = [X_1^8\ X_2^{55}\ X_3^{16}].$$

5. Finally, we consider the set of all solutions. Note that for PSS's we have

$$X_1 = \delta_{10}[a\ 3\ 4\ 5]^T, \quad 1 \le a \le 10.$$

It follows from Theorem 9.3 that

$$X_1 = \begin{bmatrix} \alpha \\ 0.8 \\ 0.7 \\ 0.6 \end{bmatrix}, \quad \text{where } 0 \le \alpha \le 1.$$

Similarly, we can calculate that X_2 is either

$$X_2^1 = \begin{bmatrix} \alpha \\ \beta \\ 0.3 \\ \eta \end{bmatrix}, \quad \text{where } 0.4 \le \alpha \le 1,\ 0.9 \le \beta \le 1,\ 0 \le \eta \le 0.3;$$

or

$$X_2^2 = \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ 0.3 \end{bmatrix}, \quad \text{where } 0.4 \le \alpha \le 1,\ 0.9 \le \beta \le 1,\ 0 \le \gamma \le 0.3.$$

X_3 can be expressed as

$$X_3^1 = \begin{bmatrix} 0.2 \\ \beta \\ \gamma \\ 1 \end{bmatrix}, \quad 0 \le \beta \le 0.2;\ 0 \le \gamma \le 1;$$

or

$$X_3^2 = \begin{bmatrix} \alpha \\ 0.2 \\ \gamma \\ 1 \end{bmatrix}, \quad 0 \le \alpha \le 0.2;\ 0 \le \gamma \le 1;$$

or

$$X_3^3 = \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ 1 \end{bmatrix}, \quad 0 \le \alpha \le 0.2;\ 0 \le \beta \le 0.2;\ 0.2 \le \gamma \le 1.$$

Summarizing them, we have 6 groups of solutions (with possible overlaps), which are expressed as

$$R_1 = \begin{bmatrix} 0 \le r_{11} \le 1 & 0.4 \le r_{12} \le 1 & 0.2 \\ 0.8 & 0.9 \le r_{22} \le 1 & 0 \le r_{23} \le 0.2 \\ 0.7 & 0.3 & 0 \le r_{33} \le 1 \\ 0.6 & 0 \le r_{42} \le 0.3 & 1 \end{bmatrix};$$

or

$$R_2 = \begin{bmatrix} 0 \le r_{11} \le 1 & 0.4 \le r_{12} \le 1 & 0 \le r_{13} \le 0.2 \\ 0.8 & 0.9 \le r_{22} \le 1 & 0.2 \\ 0.7 & 0.3 & 0 \le r_{33} \le 1 \\ 0.6 & 0 \le r_{42} \le 0.3 & 1 \end{bmatrix};$$

or

$$R_3 = \begin{bmatrix} 0 \le r_{11} \le 1 & 0.4 \le r_{12} \le 1 & 0 \le r_{13} \le 0.2 \\ 0.8 & 0.9 \le r_{22} \le 1 & 0 \le r_{23} \le 0.2 \\ 0.7 & 0.3 & 0.2 \le r_{33} \le 1 \\ 0.6 & 0 \le r_{42} \le 0.3 & 1 \end{bmatrix};$$

or

$$R_4 = \begin{bmatrix} 0 \le r_{11} \le 1 & 0.4 \le r_{12} \le 1 & 0.2 \\ 0.8 & 0.9 \le r_{22} \le 1 & 0 \le r_{23} \le 0.2 \\ 0.7 & 0 \le r_{32} \le 0.3 & 0 \le r_{33} \le 1 \\ 0.6 & 0.3 & 1 \end{bmatrix};$$

or

$$R_5 = \begin{bmatrix} 0 \le r_{11} \le 1 & 0.4 \le r_{12} \le 1 & 0 \le r_{13} \le 0.2 \\ 0.8 & 0.9 \le r_{22} \le 1 & 0.2 \\ 0.7 & 0 \le r_{32} \le 0.3 & 0 \le r_{33} \le 1 \\ 0.6 & 0.3 & 1 \end{bmatrix};$$

or

$$R_6 = \begin{bmatrix} 0 \le r_{11} \le 1 & 0.4 \le r_{12} \le 1 & 0 \le r_{13} \le 0.2 \\ 0.8 & 0.9 \le r_{22} \le 1 & 0 \le r_{23} \le 0.2 \\ 0.7 & 0 \le r_{32} \le 0.3 & 0.2 \le r_{33} \le 1 \\ 0.6 & 0.3 & 1 \end{bmatrix}.$$

Remark 9.1. Searching all the solutions and providing the overall picture of the set of solutions are significant in applications. For instance, in designing fuzzy controllers it provides a knowledge for finding "best" solutions, or to find where is the problem if there is no solution. Then further improvement can be directed.

Exercise 9

1. (to be completed).

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