Chapter 7
Fuzzy Set and Fuzzy Logic

In1965, L.A. Zadeh firstly proposed the fuzzy set theory to describe fuzzy nature in [1], which created a new area of fuzzy mathematics and applications.

In this chapter we first investigate the matrix expression of general fuzzy sets, their logical operators etc. Then the fuzzy mappings and their expressions are studied. Finally, the fuzzy logic is considered.

7.1 Matrices of General Logical Variables

Definition 7.1. Let $A=(a_{ij})\in M_{m\times n}$. A is called a k-valued Boolean matrix (BM), if its entries $a_{ij}\in \mathscr{D}_k$. Where $2\leq k\leq\infty$. When $k=2$ it is called a Boolean matrix. When $k=\infty$ $\mathscr{D}_\infty:=[0,1]$, and A is called a fuzzy matrix. The set of $m\times n$ k-valued matrices is denoted by $\mathscr{B}^k_{m\times n}$. When $k=2$, the superscript k can be omitted, i.e., $\mathscr{B}_{m\times n}:=\mathscr{B}^2_{m\times n}$.

When $m=1$ ($n=1$) it is called a row (column) k-valued Boolean vector.

Next, we define the logical operators on $\mathscr{B}^k_{m\times n}$.

Definition 7.2. 1. Let $\alpha,\beta\in\mathscr{D}_k$. Then

$$\neg\alpha:=1-\alpha. \tag{7.1}$$
$$\alpha\wedge\beta:=\min\{\alpha,\beta\}. \tag{7.2}$$
$$\alpha\vee\beta:=\max\{\alpha,\beta\}. \tag{7.3}$$

2. Let $A=(a_{ij}),B=(b_{ij})\in\mathscr{B}^k_{m\times n}$. Then

$$\neg A:=(\neg a_{ij}). \tag{7.4}$$
$$A\wedge B:=(a_{ij}\wedge b_{i,j}). \tag{7.5}$$
$$A\vee B:=(a_{ij}\vee b_{i,j}). \tag{7.6}$$

We give some new notations:

(i) $\mathbf{1}_{m\times n}$: $\mathbf{1}_{m\times n} \in \mathscr{B}^k_{m\times n}$ with all entries equal to 1.
(ii) $\mathbf{1}_m := \mathbf{1}_{m\times 1}$. If m is obvious, it can be omitted.
(iii) $\mathbf{0}_{m\times n}$: $\mathbf{0}_{m\times n} \in \mathscr{B}^k_{m\times n}$ with all entries equal to 0.
(iv) $\mathbf{0}_m := \mathbf{0}_{m\times 1}$. If m is obvious, it can be omitted.
(v) Let $A = (a_{ij}), B = (b_{ij}) \in \mathscr{B}^k_{m\times n}$. Then

$$A \le B \;\Leftrightarrow\; a_{ij} \le b_{ij},\; \forall i, j.$$

(vi) Let $\alpha \in \mathscr{D}_k$ and $A = (a_{ij}) \in \mathscr{B}^k_{m\times n}$. Then

$$\alpha A = A\alpha := (\alpha \wedge a_{i,j}) \in \mathscr{B}^k_{m\times n}.$$

The following simple examples are used to depict the operators.

Example 7.1. Let

$$A = \begin{bmatrix} 0.2 & 0.5 \\ 1 & 0.7 \end{bmatrix}; \quad B = \begin{bmatrix} 0.4 & 0.6 \\ 0.8 & 0 \end{bmatrix}.$$

Then

$$\neg A = \begin{bmatrix} 0.8 & 0.5 \\ 0 & 0.3 \end{bmatrix}; \quad A \wedge B = \begin{bmatrix} 0.2 & 0.5 \\ 0.8 & 0 \end{bmatrix}; \quad A \vee B = \begin{bmatrix} 0.4 & 0.6 \\ 1 & 0.7 \end{bmatrix}; \quad (0.5)A = \begin{bmatrix} 0.2 & 0.5 \\ 0.5 & 0.5 \end{bmatrix};$$

$$A \wedge \mathbf{1}_{2\times 2} = A = \begin{bmatrix} 0.2 & 0.5 \\ 1 & 0.7 \end{bmatrix}; \quad B \vee \mathbf{0}_{2\times 2} = B = \begin{bmatrix} 0.4 & 0.6 \\ 0.8 & 0 \end{bmatrix}.$$

The following properties are enhanced from the corresponding properties for scalar logical variables.

Proposition 7.1. *Let $A, B, C, \mathbf{1}, \mathbf{0} \in \mathscr{B}^k_{m\times n}$. Then*

(i) (nilpotent)

$$A \wedge A = A; \quad A \vee A = A. \tag{7.7}$$

(ii)

$$A \wedge \mathbf{0} = \mathbf{0}; \quad A \vee \mathbf{1} = \mathbf{1}. \tag{7.8}$$

(iii)

$$A \vee \mathbf{0} = A; \quad A \wedge \mathbf{1} = A. \tag{7.9}$$

(iv) (commutative)

$$A \wedge B = B \wedge A; \quad A \vee B = B \vee A. \tag{7.10}$$

(v) (associative)

$$(A \wedge B) \wedge C = A \wedge (B \wedge C); \quad (A \vee B) \vee C = A \vee (B \vee C). \tag{7.11}$$

(vi) (distributive)

$$(A \wedge B) \vee C = (A \vee C) \wedge (B \vee C); \quad (A \vee B) \wedge C = (A \wedge C) \vee (B \wedge C). \tag{7.12}$$

(vii) (absorptive)

$$(A \wedge B) \vee A = A; \quad (A \vee B) \wedge A = A. \tag{7.13}$$

(viii)

$$\neg(\neg A) = A. \tag{7.14}$$

(ix) (DeMorgan's Law)

$$\neg(A \wedge B) = (\neg A) \vee (\neg B); \quad \neg(A \vee B) = (\neg A) \wedge (\neg B). \tag{7.15}$$

(x) Assume $A \le B$. Then

$$A \wedge B = B; \quad A \vee B = A. \tag{7.16}$$

(xi) Assume $A \le B$ and $C \le D$. Then

$$A \wedge C \le B \wedge D; \quad A \vee C \le B \vee D. \tag{7.17}$$

(xii) Assume $A \le B$. Then

$$\neg A \ge \neg B. \tag{7.18}$$

7.2 Boolean Operators

Definition 7.3. Consider $\mathscr{D}_k$. We define two Boolean operators as

(i) (Boolean Addition)

$$\alpha +_{\mathscr{B}} \beta := \alpha \vee \beta, \quad \alpha, \beta \in \mathscr{D}_k. \tag{7.19}$$

(i) (Boolean Product)

$$\alpha \times_{\mathscr{B}} \beta := \alpha \wedge \beta, \quad \alpha, \beta \in \mathscr{D}_k. \tag{7.20}$$

Note that we also use the following notations for multi-addition and multi-product.

$$\Sigma_{\mathscr{B}}{}_{i=1}^{n} \alpha_i = \alpha_1 +_{\mathscr{B}} \alpha_2 +_{\mathscr{B}} \cdots +_{\mathscr{B}} \alpha_n;$$
$$\prod_{\mathscr{B}}{}_{i=1}^{n} \alpha_i = \alpha_1 \times_{\mathscr{B}} \alpha_2 \times_{\mathscr{B}} \cdots \times_{\mathscr{B}} \alpha_n.$$

Next, we define the Boolean addition and product for matrices.

Definition 7.4. 1. Let $\alpha \in \mathscr{D}_k$ and $A = (a_{i,j}) \in \mathscr{B}^k_{m\times n}$. Then

$$\alpha \times_{\mathscr{B}} A = A \times_{\mathscr{B}} \alpha := (\alpha \wedge a_{ij}). \tag{7.21}$$

2. Let $A = (a_{i,j}) \in \mathscr{B}^k_{m\times n}$ and $B = (b_{i,j}) \in \mathscr{B}^k_{n\times p}$. Then

$$A \times_{\mathscr{B}} B := C = (c_{ij}) \in \mathscr{B}^k_{m\times p}, \tag{7.22}$$

where

$$c_{ij} = \Sigma_{\mathscr{B}}{}_{k=1}^{n} a_{ik} \times_{\mathscr{B}} b_{kj}, \quad i = 1,\cdots,m;\ j = 1,\cdots p.$$

3. Let $A = (a_{i,j}) \in \mathscr{B}^k_{m\times n}$ and $B = (b_{i,j}) \in \mathscr{B}^k_{p\times q}$. Then

$$A \otimes_{\mathscr{B}} B := \begin{bmatrix} a_{11} \times_{\mathscr{B}} B & a_{12} \times_{\mathscr{B}} B & \cdots & a_{1n} \times_{\mathscr{B}} B \\ a_{21} \times_{\mathscr{B}} B & a_{22} \times_{\mathscr{B}} B & \cdots & a_{2n} \times_{\mathscr{B}} B \\ \vdots & & & \\ a_{m1} \times_{\mathscr{B}} B & a_{m2} \times_{\mathscr{B}} B & \cdots & a_{mn} \times_{\mathscr{B}} B \end{bmatrix} \in \mathscr{B}^k_{mp\times nq}. \tag{7.23}$$

4. Let $A \in \mathscr{B}^k_{n\times n}$. Then

$$A^{(m+1)_{\mathscr{B}}} = A^{m_{\mathscr{B}}} \times_{\mathscr{B}} A \quad m = 1,2,\cdots. \tag{7.24}$$

5. Let $A = (a_{i,j}) \in \mathscr{B}^k_{m\times n}$, $B = (b_{i,j}) \in \mathscr{B}^k_{p\times q}$.

(i) If $n = pt$, then

$$A \ltimes_{\mathscr{B}} B := A \times_{\mathscr{B}} (B \otimes_{\mathscr{B}} I_t). \tag{7.25}$$

(ii)If $nt = p$, then

$$A \ltimes_{\mathscr{B}} B := (A \otimes_{\mathscr{B}} I_t) \times_{\mathscr{B}} B. \tag{7.26}$$

6. Let $A = \in \mathscr{B}^k_{m\times s}$, $B = \in \mathscr{B}^k_{n\times s}$.

$$A *_{\mathscr{B}} B := \left[\mathrm{Col}_1(A) \ltimes_{\mathscr{B}} \mathrm{Col}_1(B)\ \mathrm{Col}_2(A) \ltimes_{\mathscr{B}} \mathrm{Col}_2(B)\ \cdots\ \mathrm{Col}_s(A) \ltimes_{\mathscr{B}} \mathrm{Col}_s(B)\right]. \tag{7.27}$$

The following examples are used to depict these.

Example 7.2. 1. Let

$$A = \begin{bmatrix} 0.2 & 0.4 \\ 0.6 & 0.8 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & 0.2 & 0.4 \\ 0.6 & 0.8 & 1 \end{bmatrix}; \quad C = \begin{bmatrix} 0 & 0.8 \\ 0.2 & 1 \\ 0.4 & 0.7 \\ 0.6 & 0.3 \end{bmatrix}.$$

Then

$$A \times_{\mathscr{B}} B = \begin{bmatrix} 0.4 & 0.4 & 0.4 \\ 0.6 & 0.8 & 0.8 \end{bmatrix};$$

$$A \otimes_{\mathscr{B}} B = \begin{bmatrix} 0 & 0.2 & 0.2 & 0 & 0.2 & 0.4 \\ 0.2 & 0.2 & 0.2 & 0.4 & 0.4 & 0.4 \\ 0 & 0.2 & 0.4 & 0 & 0.2 & 0.4 \\ 0.6 & 0.6 & 0.6 & 0.6 & 0.8 & 0.8 \end{bmatrix};$$

$$A \otimes_{\mathscr{B}} I_2 = \begin{bmatrix} 0.2 & 0 & 0.4 & 0 \\ 0 & 0.2 & 0 & 0.4 \\ 0.6 & 0 & 0.8 & 0 \\ 0 & 0.6 & 0 & 0.8 \end{bmatrix};$$

$$A \ltimes_{\mathscr{B}} C = \begin{bmatrix} 0.4 & 0.4 \\ 0.4 & 0.3 \\ 0.4 & 0.7 \\ 0.6 & 0.6 \end{bmatrix}.$$

$$A^{2_{\mathscr{B}}} = \begin{bmatrix} 0.4 & 0.4 \\ 0.6 & 0.8 \end{bmatrix}; \quad A^{k_{\mathscr{B}}} = \begin{bmatrix} 0.4 & 0.4 \\ 0.6 & 0.8 \end{bmatrix}, \quad k \geq 3.$$

2. Let

$$A = \begin{bmatrix} 0.2 & 0.3 \\ 0.7 & 0.9 \end{bmatrix}; \quad X = \begin{bmatrix} 0.3 \\ 0.5 \end{bmatrix}; \quad Y = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}; \quad \alpha = 0.4.$$

Then

$$\alpha A \ltimes_{\mathscr{B}} (X \wedge Y) = 0.4 \begin{bmatrix} 0.2 & 0.3 \\ 0.7 & 0.9 \end{bmatrix} \ltimes_{\mathscr{B}} \begin{bmatrix} 0.2 \\ 0.5 \end{bmatrix} = 0.4 \begin{bmatrix} 0.3 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.4 \end{bmatrix}.$$

3. Let

$$A = \begin{bmatrix} 0.2 & 0.3 & 0 \\ 0.4 & 0.5 & 0.8 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 0.4 & 0.6 \\ 0.3 & 0.9 & 0.2 \\ 0 & 1 & 0.7 \end{bmatrix}.$$

Then

$$A *_{\mathscr{B}} B = \begin{bmatrix} 0.2 & 0.3 & 0 \\ 0.2 & 0.3 & 0 \\ 0 & 0.3 & 0 \\ 0.4 & 0.4 & 0.6 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0.5 & 0.7 \end{bmatrix}$$

The following properties are easily verifiable.

Proposition 7.2. *Let R, S, T be k-valued BM with proper dimensions such that the following products are well defined. Then*

1.

$$R \times_{\mathscr{B}} I = I \times_{\mathscr{B}} R = R. \tag{7.28}$$

2.

$$R \times_{\mathscr{B}} \mathbf{0} = \mathbf{0} \times_{\mathscr{B}} R = \mathbf{0}. \tag{7.29}$$

3.

$$R^{(m+n)_{\mathscr{B}}} = R^{m_{\mathscr{B}}} \times_{\mathscr{B}} R^{n_{\mathscr{B}}}. \tag{7.30}$$

4. Assume $S \leq T$. Then

$$R \times_{\mathscr{B}} S \leq R \times_{\mathscr{B}} T. \tag{7.31}$$

5.

$$(R \times_{\mathscr{B}} S) \times_{\mathscr{B}} T = R \times_{\mathscr{B}} (S \times_{\mathscr{B}} T). \tag{7.32}$$

6. (Distributive Law)

(i)

$$R \times_{\mathscr{B}} (S \vee T) = (R \times_{\mathscr{B}} S) \vee (R \times_{\mathscr{B}} S). \tag{7.33}$$

(ii)

$$(S \vee T) \times_{\mathscr{B}} R = (S \times_{\mathscr{B}} R) \vee (S \times_{\mathscr{B}} R). \tag{7.34}$$

(iii)

$$R \times_{\mathscr{B}} (S \wedge T) = (R \times_{\mathscr{B}} S) \wedge (R \times_{\mathscr{B}} S). \tag{7.35}$$

(iv)

$$(S \wedge T) \times_{\mathscr{B}} R = (S \times_{\mathscr{B}} R) \wedge (S \times_{\mathscr{B}} R). \tag{7.36}$$

7.

$$(R \times_{\mathscr{B}} S)^T = S^T \times_{\mathscr{B}} R^T. \tag{7.37}$$

7.3 Fuzzy Sets

Definition 7.5. Consider an objective set E, called a universe. A set A is called a fuzzy set over E if for each $e \in E$ there is a membership degree $\mu_A(e) = \alpha_e \in \mathscr{D}_k$. If $E = \{e_1, \cdots, e_n\}$ is a finite set and $\mu_A(e_i) = \alpha_i$, $i = 1, \cdots, n$, then A can be expressed as

$$A = \alpha_1/e_1 + \alpha_2/e_2 + \cdots + \alpha_n/e_n. \tag{7.38}$$

The set of fuzzy sets over universe E is denoted by $\mathscr{F}(E)$.

Consider a set E, its power set, denoted by $\mathscr{P}(E)$, is the set of all subsets of E. That is,

$$\mathscr{P}(E) = \{S | S \subset E\}.$$

Consider $C \in \mathscr{P}(E)$. C can also be considered as a special fuzzy set, with

$$\mu_C(e) = \begin{cases} 1, & e \in C \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, $p \in E$ can be considered as a special power set, which have

$$\mu_p(e) = \begin{cases} 1, & e = p \\ 0, & \text{otherwise.} \end{cases}$$

Under this understanding we have

$$E \subset \mathscr{P}(E) \subset \mathscr{F}(E). \tag{7.39}$$

To distinguish $C \in \mathscr{P}(E)$ with fuzzy sets, we call C a crisp set.

Remark 7.1. 1. Consider $I = [0,1)$, we define a topology as

$$\mathscr{T} := \{[0,r) | 0 \le r \le 1\}.$$

Then it is obvious that $(I, \mathscr{T})$ is a topological space, denote it by $\mathscr{I}$.
2. Consider the universe E and E with discrete topology is denoted by $\mathscr{E}$.
3. Consider the topological product space $\mathscr{E} \times \mathscr{I}$. Then a set $A \in \mathscr{F}(E)$ is a fuzzy set, iff A is an open set in this topological space. (Here we use a trivial identity $\mu_A(e) \sim [0, \mu_A(e))$.)
4. If E has its own topology, then we may consider a fuzzy set as a subset in the topological product space $E \times \mathscr{I}$. Then an open set is a fuzzy set, but not every fuzzy set is an open set.

Definition 7.6. Let A and B be two fuzzy sets on E.

1. $A = \emptyset$, if
$$\mu_A(e) = 0, \quad \forall e \in E.$$
2. $A = E$, if
$$\mu_A(e) = 1, \quad \forall e \in E.$$
3. $A \subset B$, if and only if
$$\mu_A(e) \le \mu_B(e), \quad \forall e \in E.$$
4. $A \cap B$ is defined by
$$\mu_{A\cap B}(e) = \mu_A(e) \wedge \mu_B(e), \forall e \in E.$$

5. $A \cup B$ is defined by

$$\mu_{A\cup B}(e) = \mu_A(e) \vee \mu_B(e), \quad \forall e \in E.$$

6. A^c is defined by

$$\mu_{A^c}(e) = \neg\mu_A(e), \quad \forall e \in E.$$

Definition 7.7. Assume the universe $|E| < \infty$ and a fuzzy set A on E is as in (7.38). Then the vector form of A, denoted by X_A, is defined as

$$X_A = (\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n)^T \in \mathscr{B}^k_{n\times 1}. \tag{7.40}$$

Let $P = (p_{ij}), Q = (q_{ij}) \in M_{m\times n}$. Then $P \le Q$ means

$$p_{i,j} \le q_{i,j}, \quad i = 1, \cdots, m; j = 1, \cdots, n.$$

When $|E| < \infty$, according to the Definition7.6, we have that

Proposition 7.3. *Let A and B be two fuzzy sets on E.*

1. $A = \emptyset$, iff

$$X_A = \mathbf{0}.$$

2. $A = E$, iff

$$X_A = \mathbf{1}.$$

3. $A \subset B$, if and only if

$$X_A \le X_B.$$

4.

$$X_{A\cap B} = X_A \wedge X_B. \tag{7.41}$$

5.

$$X_{A\cup B} = X_A \vee X_B. \tag{7.42}$$

6.

$$X_{A^c} = \neg X_A. \tag{7.43}$$

Remark 7.2. Using Proposition 7.3 and the properties of logical operators, we can calculate the vector of any logical expressions. For instance,

$$X_{(A\cap B)^c} = \neg(X_A \wedge X_B) = (\neg X_A) \vee (\neg X_B) = X_{A^c} \vee X_{B^c}. \tag{7.44}$$

$$X_{(A\cup B)^c} = \neg(X_A \vee X_B) = (\neg X_A) \wedge (\neg X_B) = X_{A^c} \wedge X_{B^c}. \tag{7.45}$$

Example 7.3. 1. Let $E = \{x_1, x_2, x_3, x_4, x_5\}$.

(i) $A = 0.1/x_1 + 0.3/x_2 + 1/x_5 \in \mathscr{F}(E)$. Then

$$X_A = [0.1\ 0.3\ 0\ 0\ 1]^T.$$

(ii)$B = \{x_2, x_4, x_5\} \in \mathscr{P}(E)$. Then

$$X_B = [0\ 1\ 0\ 1\ 1]^T.$$

(iii)$C = x_3 \in E$. Then

$$X_C = [0\ 0\ 1\ 0\ 0]^T.$$

2. Let $E = \{x_1, x_2, x_3\}$ and there are three fuzzy sets

$$\begin{aligned} A_1 &= 0/x_1 + 0.25/x_2 + 0.75/x_3; \\ A_2 &= 0.5/x_1 + 0.75/x_2 + 0/x_3; \\ A_3 &= 1/x_1 + 0.5/x_2 + 0/x_3. \end{aligned}$$

Then we can choose $k = 5$ and find their corresponding vector form as:

$$X_1 = X_{A_1} = \begin{bmatrix} 0 \\ 0.25 \\ 0.75 \end{bmatrix}; \quad X_2 = X_{A_2} = \begin{bmatrix} 0.5 \\ 0.75 \\ 0 \end{bmatrix}; \quad X_3 = X_{A_3} = \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}.$$

Consider $S = A_1 \cup A_2 \cup A_3$. Then

$$X_S = X_1 \vee X_2 \vee X_3 = \begin{bmatrix} 1 \\ 0.75 \\ 0.75 \end{bmatrix}.$$

Equivalently, $S = 1/x_1 + 0.75/x_2 + 0.75/x_3$. Consider $T = A_1 \cap A_2 \cap A_3$. Then

$$X_T = X_1 \wedge X_2 \wedge X_3 = \begin{bmatrix} 0 \\ 0.25 \\ 0 \end{bmatrix}.$$

Equivalently, $T = 0/x_1 + 0.25/x_2 + 0/x_3$.

Throughout this chapter we assume

A1 The fuzzy sets, in which we are interested, is finite.
A2 The number of membership degrees for each fuzzy set is finite.

Theorem 7.1. *Under assumptions* **A1** *and* **A2***, the domain can be equivalent to a finite partition.*

Proof. Assume the domain is E and the fuzzy sets concerned are $A_1, \cdots, A_s$, and the number of membership degrees for A_i is $k_i < \infty$, $i = 1, \cdots, s$. Define

$$E_j^i := \left\{ e \in E \,\middle|\, \mu_{A_i}(e) = \frac{j-1}{k_i - 1} \right\}, \quad j = 1, \cdots, k_i,\ i = 1, \cdots, s.$$

Using E^i_j, we define a set of subsets of E as

$$E_{j_1,\cdots,j_s} := E^1_{j_1} \cap E^2_{j_2} \cap \cdots \cap E^s_{j_s}, \quad j_i = 1,\cdots,k_i;\ i = 1,\cdots,s.$$

Then $\left\{ E_{j_1,\cdots,j_s} \,\middle|\, j_i = 1,\cdots,k_i;\ i = 1,\cdots,s \right\}$ is a partition of E. That is

1.
$$\cup_{j_1=1}^{k_1} \cup_{j_2=1}^{k_2} \cdots \cup_{j_s=1}^{k_s} E_{j_1,\cdots,j_s} = E;$$

2.
$$E_{j_1,\cdots,j_s} \cap E_{j'_1,\cdots,j'_s} = \emptyset, \quad (j_1,\cdots,j_s) \neq (j'_1,\cdots,j'_s).$$

Now it is clear that when considering the fuzzy sets $A_1, \cdots, A_s$, we do not need to distinguish two points within a $E_{j_1,\cdots,j_s}$. Hence we can consider $E_{j_1,\cdots,j_s}$ as "one element" in the domain E provided $E_{j_1,\cdots,j_s} \neq \emptyset$ (If $E_{j_1,\cdots,j_s} = \emptyset$, we can just ignore it.). Then

$$E = \left\{ E_{j_1,\cdots,j_s} \,\middle|\, E_{j_1,\cdots,j_s} \neq \emptyset;\ j_1,\cdots,j_s = 1,\cdots,k \right\}$$

can be treated as a finite set. □

We give an example to depict this.

Example 7.4. Let E be the set of human age. It could be $[0,\infty)$. Say, we are concerning two fuzzy sets: A: One is old. B: One is rational. Assume

$$\mu_A(x) = \begin{cases} 0, & x < 20 \\ \frac{1}{3}, & 20 \le x < 40 \\ \frac{2}{3}, & 40 \le x < 60 \\ 1, & x \ge 60 \end{cases}$$

$$\mu_B(x) = \begin{cases} 0, & x < 10 \\ \frac{1}{4}, & 10 \le x < 14 \\ \frac{1}{2}, & 14 \le x < 18 \\ \frac{3}{4}, & 18 \le x < 25, \text{ or } x \ge 80 \\ 1, & 25 \le x < 80 \end{cases}$$

Then the universe can be partitioned as

$$E = \{E_{11}, E_{12}, E_{13}, E_{14}, E_{24}, E_{25}, E_{35}, E_{45}, E_{44}\},$$

where

$$\begin{aligned} &E_{11} = [0,10); \quad E_{12} = [10,14); E_{13} = [14,18); \\ &E_{14} = [18,20); E_{24} = [20,25); E_{25} = [20,40); \\ &E_{35} = [40,60); E_{45} = [60,80); E_{44} = [80,\infty). \end{aligned}$$

We conclude that when only A, B and the fuzzy sets generated by them are concerned, E can be considered a finite set with 9 elements.

Based on Theorem 7.1 we, hereafter, assume

A3 The domain of any fuzzy set is a finite set, i.e., $|E| < \infty$.

Then for each fuzzy set A we have $X_A \in \mathscr{D}_k^n$.

Note that when $k = 2$ we have a "complement rule" as

$$X \vee \neg X = \mathbf{1}; \quad X \wedge \neg X = \mathbf{0}. \tag{7.46}$$

When $k > 2$ (7.46) is not true.

7.4 Mappings over Fuzzy Sets

First, we consider the decomposition of a fuzzy set.

Definition 7.8. Let $\alpha \in \mathscr{D}_\infty$ and A be a fuzzy set. Then the α-truncated set of A is defined as

$$A_\alpha = \{e | \mu_A(e) \geq \alpha\},$$

which is a crisp set.

Note that let X_A be the vector expression of A. Then the components of X_{A_α} can be determined as

$$x_i^{A_\alpha} = \begin{cases} 0, & x_i^A < \alpha \\ 1, & x_i^A \geq \alpha. \end{cases}$$

The following decomposition theorem can be proved by a straightforward verification.

Theorem 7.2 (Decomposition Theorem). *Let A be a fuzzy set. Then*

$$X_A = \vee_{\alpha \in \mathscr{D}_k} \alpha X_{A_\alpha}. \tag{7.47}$$

Next, we consider how to extend a mapping $f : E \to F$ to the fuzzy sets $f : \mathscr{F}(E) \to \mathscr{F}(F)$. Recall the including relation (7.39). We first extend it to $f : \mathscr{P}(E) \to \mathscr{P}(F)$.

Definition 7.9. Let E and F be two arbitrary sets, and a mapping $f : E \to F$ is given.

1. f can naturally be extended to $f : \mathscr{P}(E) \to \mathscr{P}(F)$ as

$$f(S) = \{f(x) | x \in S\} \in \mathscr{P}(F), \quad S \in \mathscr{P}(E). \tag{7.48}$$

2. $f^{-1} : \mathscr{P}(F) \to \mathscr{P}(E)$ is defined as

$$f^{-1}(T) = \{x | f(x) \in T\}, \quad T \in \mathscr{P}(F). \tag{7.49}$$

Next, we extend f to $\mathscr{F}(E) \to \mathscr{F}(F)$

Definition 7.10. Assume $f : E \to F$ is given.

1. Then f can be extended to $\mathscr{F}(E) \to \mathscr{F}(F)$ as follows:

$$\mu_{f(A)}(y) = \begin{cases} \vee_{x\in f^{-1}(y)} \mu_A(x), & A \in \mathscr{F}(E) \\ \emptyset, & f^{-1}(y) = \emptyset. \end{cases} \tag{7.50}$$

2. The inverse $f^{-1} : \mathscr{F}(F) \to \mathscr{F}(E)$ is defined as

$$\mu_{f^{-1}(B)}(x) = \mu_B(f(x)). \tag{7.51}$$

Assume $E = \{e_1, e_2, \cdots, e_n\}$ and $F = \{d_1, d_2, \cdots, d_m\}$, and $f : E \to F$ is defined by

$$f(e_i) = d_{j_i}, \quad i = 1, \cdots, n;\ 1 \le j_i \le m.$$

Identifying e_i with its vector form as $e_i \sim X_{e_i} = \delta_n^i$, $i = 1, \cdots, n$ and $d_j \sim X_{d_j} = \delta_m^j$, $j = 1, \cdots, m$, we have

$$f(x) = M_f x := \delta_m[j_1\ j_2\ \cdots\ j_n]x, \tag{7.52}$$

where M_f is called the structure matrix of f.

Example 7.5. Let $E = \{1,2,3,4,5\}$, $F = \{0,1,2\}$. and $f : E \to F$ is defined by $f(x) = x^3 (mod\ 3)$. Then it is easy to figure out that

$$M_f = \delta_3[2,3,1,2,3]. \tag{7.53}$$

Using (7.52), let $x = 4 \sim \delta_5^4$. Then

$$f(x) = M_f \delta_5^4 = \delta_5^2 \sim 1.$$

Theorem 7.3. *Assume $|E| = n$ and $|F| = m$ and $f : E \to F$ has its structure matrix $M_f \in \mathscr{L}_{m\times n}$. Then*

1.

$$X_{f(A)} = M_f \times_{\mathscr{B}} X_A, \quad \forall A \in \mathscr{F}(E). \tag{7.54}$$

2.

$$X_{f^{-1}(B)} = M_f^T \times_{\mathscr{B}} X_B, \quad \forall B \in \mathscr{F}(F). \tag{7.55}$$

Proof. 1. Assume $y = \delta_m$. If $f^{-1}(y) = \emptyset$, then $\mathrm{Row}_j \times_{\mathscr{B}} (M_f) = 0$. Then it is not difficult to see that

$$X_{f(A)} = \vee_{x\in E} X_A(x).$$

Then (7.54) follows from the definition of the Boolean product of k-valued matrices.

2. Let $x = \delta_n^i$. Then $f(x) = \mathrm{Row}_i(M_f)$. Say, $f(x) = \delta_m^j$. Then

$$\mu_B(f(x)) = \mathrm{Col}_j(X_B) = f(x)^T X_B.$$

Hence we have (7.55).

□

Example 7.6. Consider the mapping defined in Example 7.5.

1. Let $A = 0.3/1 + 0.8/2 + 1/4 + 0.5/4 \in \mathscr{P}(E)$. Then $X_A = (0.3\ 0.8\ 0\ 1\ 0.5)^T$. Hence,
$$\begin{aligned} X_{f(A)} &= M_f \times_{\mathscr{B}} X_A \\ &= \delta_3[2\ 3\ 1\ 2\ 3] \times_{\mathscr{B}} \begin{bmatrix} 0.3 \\ 0.8 \\ 0 \\ 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0.5 \end{bmatrix}. \end{aligned}$$
It follows that $f(A) = 1/1 + 0.5/2$.
2. Let $B = 0.2/0 + 0.8/1 + 0.4/2 \in \mathscr{P}(F)$. Then $X_B = (0.2\ 0.8\ 0.4)^T$. Hence,
$$\begin{aligned} X_{f^{-1}(B)} &= M_f^T \times_{\mathscr{B}} X_B \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times_{\mathscr{B}} \begin{bmatrix} 0.2 \\ 0.8 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0 \\ 0.2 \\ 0.8 \\ 0.4 \end{bmatrix}. \end{aligned}$$
It follows that $f^{-1}(B) = 0.8/1 + 0.2/3 + 0.8/4 + 0.4/5$.

Corollary 7.1. *Let $f : E \to F$, where $|E| = n$ and $|F| = m$.*

1. *If f is one-to-one, then*
$$X_{f^{-1}(f(A))} = X_A, \quad A \in \mathscr{F}(E). \tag{7.56}$$
2. *If f is one-to-one and onto, then*
$$X_{f(f^{-1}(B))} = X_B, \quad B \in \mathscr{F}(F). \tag{7.57}$$

Proof. 1. Assume the structure matrix of f is $M_f = \delta_n[i_1\ i_2\ \cdots\ i_n]$. Since f is one-to-one, then when $p \neq q$ we have $i_p \neq i_q$. It follows that
$$X_{f^{-1}(f(A))} = M_f^T M_f X_A = I_n X_A = X_A.$$

2. Since f is one-to-one and onto, it is easy to see that $M_f^T = M_f^{-1}$. Hence
$$X_{f(f^{-1}(B))} = M_f M_f^T X_B = X_B.$$

$y = \delta_m$. If $f^{-1}(y) = \emptyset$, then $\mathrm{Row}_j(M_f) = 0$. Then it is not difficult to see that
$$X_{f(A)} = \vee_{x \in E} X_A(x).$$

It is easy to prove the following properties.

Proposition 7.4. *(i)*

$$f(A) = \emptyset \Leftrightarrow A = \emptyset. \tag{7.58}$$

(ii)

$$A \subset B \Rightarrow f(A) \subset f(B). \tag{7.59}$$

If f is one-to-one, Λ is also correct.

(iii)

$$f\left(\cup_{\lambda\in\Lambda} A_\lambda\right) = \cup_{\lambda\in\Lambda} f\left(A_\lambda\right). \tag{7.60}$$

(iv)

$$f\left(\cap_{\lambda\in\Lambda} A_\lambda\right) \subset \cap_{\lambda\in\Lambda} f\left(A_\lambda\right). \tag{7.61}$$

If f is one-to-one, then "$\subset$" can be replaced by "$=$".

(v)

$$f^{-1}(\emptyset) = \emptyset. \tag{7.62}$$

(vi) If f is onto, then

$$f^{-1}(B) = \emptyset \Rightarrow B = \emptyset. \tag{7.63}$$

(vii)

$$B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2). \tag{7.64}$$

(viii)

$$f^{-1}\left(\cup_{\lambda\in\Lambda} B_\lambda\right) = \cup_{\lambda\in\Lambda} f^{-1}\left(B_\lambda\right). \tag{7.65}$$

(ix)

$$f^{-1}\left(\cap_{\lambda\in\Lambda} B_\lambda\right) = \cap_{\lambda\in\Lambda} f^{-1}\left(B_\lambda\right). \tag{7.66}$$

(x)

$$f^{-1}(B^c) = \left[f^{-1}(B)\right]^c. \tag{7.67}$$

7.5 Fuzzy Logic

Throughout this section we assume the universe $E = \{e_1, e_2, \cdots, e_n\}$ is unique for all fuzzy objects.

Definition 7.11. 1. A fuzzy proposition a is a fuzzy set. Precisely, $a \in \mathscr{F}(E)$ is an element.
2. A fuzzy logical variable x is a variable which takes values from $\mathscr{F}(E)$.
3. A fuzzy logical function is an expression with some fuzzy propositions and fuzzy variables connected by (fuzzy) logical operators.

Remark 7.3. 1. Traditionally, the logical operators allowed in a fuzzy logical function are $\{\neg, \wedge, \vee\}$. But we assume **A2** (refer to Section 3), then any logical operators are allowed.
2. In this section we consider only the k-valued (fuzzy) logic. The results obtained can easily be extended to mix-valued (fuzzy) logic.

Assume $a, x, x_1, \cdots, x_m$, and $f(x_1, \cdots, x_m)$ are fuzzy proposition, fuzzy variables, and fuzzy logical function respectively. Moreover, assume for any $\xi \in \mathscr{F}(E)$

$$\mu_\xi(e) \in \mathscr{D}_k, \quad e \in E.$$

Then for a fixed $e_0 \in E$ the $\mu_a(e_0)$, $\mu_x(e_0)$ $\mu_{x_i}(e_0)$ $i = 1, \cdots, m$, and $\mu_f(e_0)$ are simply the k-valued proposition (or constant), k-valued logical variables, and k-valued logical functions. So as $|E| = n$ is assumed, they are n-dimensional k-valued proposition, n-dimensional k-valued logical variables, and n-dimensional k-valued logical function.

Then we have the following result.

Proposition 7.5. *Let $X^1, \cdots, X^m$ be a set of fuzzy logical variables, and a fuzzy logical function $f(X^1, \cdots, X^m)$ has its structure matrix as $M_f \in \mathscr{L}_{k \times k^n}$. Denote the vector form of the j th component of X^i by x^i_j, and in vector form $x^i_j \in \Delta_k$. Then*

$$f_j = M_f x^1_j x^2_j \cdots x^n_j, \quad j = 1, \cdots, n. \tag{7.68}$$

We give an example to depict this.

Example 7.7. Assume the universe is $E = \{e_1, e_2, e_3, e_4\}$ and $k = 3$. $X, Y, Z \in \mathscr{P}(E)$. A fuzzy logical function f is defined as

$$f(X, Y, Z) = (X \wedge Y) \leftrightarrow Z. \tag{7.69}$$

Then we have the algebraic form of (A.2) as

$$f(X, Y, Z) = M_f XYZ, \tag{7.70}$$

where the structure matrix of f is

$$M_f = M_{e,3}M_{c,3} = \delta_3[1\ 2\ 3\ 2\ 2\ 2\ 3\ 2\ 1\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 3\ 2\ 1\ 3\ 2\ 1\ 3\ 2\ 1\ 3\ 2\ 1]$$

Precisely speaking, (7.70) means

$$f(X,Y,Z)(e) = M_f X(e)Y(e)Z(e), \quad e \in E.$$

Now assume the vector form of X, Y, and Z are respectively as

$$X_X = \begin{bmatrix} 0.5 \\ 0 \\ 0.5 \\ 1 \end{bmatrix}; \quad X_Y = \begin{bmatrix} 0 \\ 0.5 \\ 1 \\ 0.5 \end{bmatrix}; \quad X_Z = \begin{bmatrix} 0.5 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Then

$$\begin{aligned} f(e_1) &= M_f x_1 y_1 z_1 = M_f \delta_3^2 \delta_3^3 \delta_3^2 = \delta_3^1; \\ f(e_2) &= M_f x_2 y_2 z_2 = M_f \delta_3^3 \delta_3^2 \delta_3^1 = \delta_3^3; \\ f(e_3) &= M_f x_3 y_3 z_3 = M_f \delta_3^2 \delta_3^1 \delta_3^1 = \delta_3^2; \\ f(e_4) &= M_f x_4 y_4 z_4 = M_f \delta_3^1 \delta_3^2 \delta_3^3 = \delta_3^2. \end{aligned}$$

We conclude that

$$f(X,Y,Z) = \begin{bmatrix} 1 \\ 0 \\ 0.5 \\ 0.5 \end{bmatrix}.$$

Exercise 7

1. (to be completed).

References

1. Zadeh, L.: Fuzzy sets. Information and Control **8**, 338–353 (1965)