Chapter 1
Multi-dimensional Data

Abstract Roughly speaking, the classical matrix theory can mainly deal with one- or two-dimensional data. The main purpose of semi-tensor product of matrices is to use matrix tools to deal with higher-dimensional data. Hence the multi-dimensional data become the main objective of this book. This chapter considers how to arrange a set of higher-dimensional data into a vector or a matrix. First, the ordered multi-index is introduced to arrange a set of data into a properly ordered form. Then we briefly introduce some other matrix products, including the Kronecker Product (also called the tensor product) of matrices, which is a fundamental tool in this book; the Hadamard product, which is also used in the sequel; etc. Tensor and Nash equilibrium are two useful examples for multi-dimensional data. They are introduced in this chapter and will be used in the sequel. Symmetric group is another useful tool and it is also introduced here. Finally, we propose a special matrix, called the swap matrix. Its certain properties are discussed. It will be used largely to overcome the non-commutativity of matrix product.

1.1 Multi-dimensional Data

In scientific research we have to deal with various kinds of data. First, we would like to clarify what do we mean the dimension of data. A set of data may depend on $k$ factors. Assume each factor can have $n_j$ levels, $j = 1, \cdots, k$. Then to label a data, we may need $k$ indices $i_1, i_2, \cdots, i_k$, and allow $i_j$ runs from 1 to $n_j$. Then we can have a finite set of data as:

$$D := \{d_{i_1, \cdots, i_k} \mid 1 \leq i_j \leq n_j, j = 1, \cdots, k\}. \quad (1.1)$$

Throughout this book only finite sets of data are considered, unless elsewhere stated.

We say that the dimension of $D$ is $k$, i.e., $\dim(D) = k$. Roughly speaking, the dimension of a set of data is the number of indices of the data.
Example 1.1. 1. Consider a vector $X \in \mathbb{R}^n$. It can be expressed as $X = (x_1, x_2, \cdots, x_n)^T$. Hence a vector can be considered as a set of one-dimensional data.

2. Consider a matrix $A \in \mathbb{M}_{m \times n}$. It can be expressed as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$  

Hence in a natural way, a matrix can be considered as a set of two-dimensional data.

3. Let $y = f(x_1, x_2, \cdots, x_n)$ be a function of $n$ variables. To get numerical expression of $f$ we let $x_i$ take values $x_1^j, x_2^j, \cdots, x_n^j$. Then we have a set of $n$-dimensional data as

$$Y := \{ y_{i_1, \cdots, i_n} = f(x_1^{i_1}, \cdots, x_n^{i_n}) | 1 \leq i_j \leq n_j; \ j = 1, \cdots, n \}.$$

To arrange a multi-dimensional data, or equivalently, to decide the order of a data, we introduce an ordered multi-index, briefly, multi-index.

Definition 1.1. A set of $k$-dimensional data (1.1) is said to be arranged by the (ordered) multi-index $i(i_1, \cdots, i_k; n_1, \cdots, n_k)$, if the data are labeled by indices $i_1, \cdots, i_k$ and arranged in the order that $a_{i_1, \cdots, i_k} < a_{i_1, \cdots, i_k}$, iff there exists a $1 \leq r \leq k$, such that $p_i = q_i, i < r$ and $p_r < q_r$.

Example 1.2. Let

$$D = \{ x_{i,j,k} | 1 \leq i \leq 2; \ 1 \leq j \leq 3; \ 1 \leq k \leq 4 \}.$$  

(i) If we arrange it by the multi-index $i(i, j, k; 2, 3, 4)$, we have

$$[x_{111}, x_{112}, x_{113}, x_{114}, x_{211}, x_{122}, x_{123}, x_{124}, x_{131}, x_{132}, x_{133}, x_{134}, x_{211}, x_{212}, x_{213}, x_{214}, x_{221}, x_{222}, x_{223}, x_{224}, x_{231}, x_{232}, x_{233}, x_{234}].$$  

(ii) If we arrange it by the multi-index $i(i, j, k; 3, 2, 4)$, we have

$$[x_{111}, x_{112}, x_{113}, x_{114}, x_{211}, x_{212}, x_{213}, x_{214}, x_{121}, x_{122}, x_{123}, x_{124}, x_{221}, x_{222}, x_{223}, x_{224}, x_{131}, x_{132}, x_{133}, x_{134}, x_{231}, x_{232}, x_{233}, x_{234}].$$  

(iii) If we arrange it by the multi-index $i(k, j, i; 3, 2, 4)$, we have

$$[x_{111}, x_{211}, x_{121}, x_{131}, x_{131}, x_{121}, x_{122}, x_{122}, x_{123}, x_{123}, x_{124}, x_{124}, x_{221}, x_{222}, x_{223}, x_{224}, x_{231}, x_{232}, x_{233}, x_{234}].$$  

Assume we have a set of $n$ data, where $n = \prod_{i=1}^{k} n_i$. Then we can either use a single index to label the data as

$$D = [x_1, x_2, \cdots, x_n],$$  

or use multi-index $(i_1, \cdots, i_k; n_1, \cdots, n_k)$ to arrange the data as

$$D = \{ x_{i_1, i_2, \cdots, i_k} | 1 \leq i_1 \leq n_1; \ 1 \leq i_2 \leq n_2; \ \cdots; \ 1 \leq i_k \leq n_k \}.$$
Then we need to find formulas to convert the single index to multi-index and vice versa. In the following we deduce the formulas. For notational ease, we introduce some notations in the following:

- Let $a \in \mathbb{Z}$, $b \in \mathbb{Z}_+$. As in C-language, the $a\%b$ is used for the remaining of $a/b$.
- Denote by $[t]$ the largest integer, which is less than or equal to $t$.

For instance,

$$100\% 3 = 1, \quad 100\% 7 = 2, \quad (-7)\% 3 = 2; \quad \left\lfloor \frac{7}{3} \right\rfloor = 2, \quad [-1.25] = -2.$$  

It is easy to see that

$$a = \left\lfloor \frac{a}{b} \right\rfloor b + a\% b. \quad (1.4)$$

Next, we give the converting formulas between single index and multi-index of a set of data. We leave the proves to the reader.

**Proposition 1.1.** Let $D$ be a set of $n = \prod_{j=1}^{k} n_j$ data. It has been labeled by single index as in (1.2) and by multi-index as in (1.3). An element $x \in D$ is labeled by single index $p$ and multi-index $\mu_1 \cdots \mu_k$. That is, $x \in D$ is expressed as

$$x = x_p = x_{\mu_1 \cdots \mu_k}. \quad (1.5)$$

Then we have the following converting formulas:

1. Set $p_k := p - 1$, then $(\mu_1, \cdots, \mu_k)$ can be calculated inductively by

$$\begin{cases}
\mu_k = p_k \% n_k + 1, \\
p_j = \left\lfloor \frac{p_{j+1}}{n_{j+1}} \right\rfloor, \quad \mu_j = p_j \% n_j + 1, \\
j = k - 1, \cdots, 1.
\end{cases}$$  

2. Conversely, from single index to multi-index we have:

$$p = \sum_{j=1}^{k-1} (\mu_j - 1) n_{j+1} n_{j+2} \cdots n_k + \mu_k. \quad (1.6)$$

The following example shows the conversions.

**Example 1.3.** A set of data

$$D = \{d_1, d_2, \cdots, d_{100}\}.$$

1. Given a number $p = 35$, what is the label of $d_p$ under multi-index id$(i_1, i_2, i_3; 4, 5, 5)$?

Using (1.5), we have
\[ p_3 = p - 1 = 34, \]
\[ \mu_3 = p_3 \% n_3 + 1 = 34 \% 5 + 1 = 4 + 1 = 5, \]
\[ p_2 = [p_3/n_3] = [34/5] = 6, \]
\[ \mu_2 = p_2 \% n_2 + 1 = 6 \% 5 + 1 = 1 + 1 = 2, \]
\[ p_1 = [p_2/n_2] = [6/4] = 1, \]
\[ \mu_1 = p_1 \% n_1 + 1 = 1 \% 4 + 1 = 1 + 1 = 2. \]

Thus the label of \( p \) is \((2, 2, 5)\).

2. Assume the label of \( d_q \in D \) under multi-index \( \text{id}(i_1, i_2, i_3; 5, 2, 10) \) is \((3, 2, 8)\).

What is the single index \( q \)?

Using (1.6), we have

\[ q = (\mu_1 - 1)n_2n_3 + (\mu_2 - 1)n_3 + \mu_3 = 2 \cdot 2 \cdot 10 + 10 + 8 = 58. \]

1.2 Arrangement of Data

Let \( D \) be a \( k \)-dimensional data as in (1.1) with \( n = \prod_{i=1}^{k} n_i \). When \( k = 1, n = n_1 \) and the data can be arranged into a vector as

\[ V_D = (d_1, d_2, \cdots, d_n)^T. \] (1.7)

When \( k = 2 \), the data can naturally be arranged into a matrix as

\[ M_D = \begin{bmatrix}
  d_{11} & d_{12} & \cdots & d_{1n_2} \\
  d_{21} & d_{22} & \cdots & d_{2n_2} \\
  \vdots & \vdots & \ddots & \vdots \\
  d_{n_11} & d_{n_12} & \cdots & d_{n_1n_2}
\end{bmatrix}. \] (1.8)

Vector and matrix are two major objects for matrix theory or linear algebra. Roughly speaking, matrix theory is a theory for one- or two-dimensional data. Now when \( k \geq 3 \), what can we do? Let us first consider the case of \( k = 3 \). Say, we have a set of data \( D \) as

\[ D = \{d_{ijk} | i = 1, \cdots, p; j = 1, \cdots, m; k = 1, \cdots, n\}. \] (1.9)

Then how can we arrange the data? It was proposed by some researchers that the data are arranged into a cube, which consists of \( p \) layers and each layer is an \( m \times n \) matrix. Such a compounded matrix is called a cubic matrix (Fig. 1.1). Now a natural question is: when \( k > 3 \) what can we do? It seems that this is not a proper way to arrange the higher-dimensional data.

As we know that a set of higher-dimensional data can easily be stored in a computer memory, where they are not arranged into a cubic or even higher-dimensional cube. In fact, the data are arranged into a line regardless the dimension of the data.
1.2 Arrangement of Data

Then how do the computers find the hierarchy structure of the data? They use some marks. Say, in C-language the pointer, pointer to pointer, pointer to pointer to pointer etc. are used to indicate the data structure.

Motivated by the computer technology, we propose to arrange a set of data into either a vector or a matrix. One may ask that why not just use vector only. It might be more convenient. In fact, to use tools developed in matrix theory, both vector and matrix forms are necessary.

To arrange a set of multi-dimensional data into a vector is rather easy. Formula (1.6) provides the single index label for each data. We are particularly interested in arranging the entries of a matrix into a vector.

**Definition 1.2.** Consider the matrix \( M_D \) in (1.8).

1. Its row stacking form, denoted by \( V_r(M_D) \), is defined as

\[
V_r(M_D) = (d_{11}, d_{12}, \cdots , d_{1n_1}, d_{21}, d_{22}, \cdots , d_{2n_2}, \cdots , d_{m_1, 1}, d_{m_1, 2}, \cdots , d_{m_1 n_2})^T.
\]

(1.10)

2. Its column stacking form, denoted by \( V_c(M_D) \), is defined as

\[
V_c(M_D) = (d_{11}, d_{21}, \cdots , d_{n_1, 1}, d_{12}, d_{22}, \cdots , d_{n_2, 1}, \cdots , d_{m_1 n_2}, d_{m_2 n_2}, \cdots , d_{m_1 n_2})^T.
\]

(1.11)

By definition, it is obvious that for any matrix \( A \)

\[
V_r(A) = V_c(A^T); \quad V_c(A) = V_r(A^T).
\]

(1.12)
Next, we consider how to arrange a set of multi-dimensional data into a matrix form. Let \( \text{id}(i_1, \ldots, i_k; n_1, \ldots, n_k) \) be a multi-index.

\[
\{i_{j_1}, \ldots, i_{j_p}\} \subset \{i_1, \ldots, i_k\}.
\]

Then \( \text{id}(i_{j_1}, \ldots, i_{j_p}; n_{j_1}, \ldots, n_{j_p}) \) is called a sub-index of \( \text{id}(i_1, \ldots, i_k; n_1, \ldots, n_k) \).

**Definition 1.3.** Let \( D \) be a \( k \)-dimensional data as in (1.1). Assume there is a partition of the indices as

\[
\{i_1, i_2, \ldots, i_k\} = \{i_{r_1}, i_{r_2}, \ldots, i_{r_p}\} \cup \{i_{c_1}, i_{c_2}, \ldots, i_{c_q}\},
\]

where \( p \geq 1, q \geq 1 \) and \( p + q = k \). Then the multi-indexed matrix \( M_D \), corresponding to this index partition, is defined as follows:

(i) \( M_D \in \mathcal{M}_{n_r \times n_c} \), where \( n_r = \prod_{r=1}^{p} n_{r_i} \) and \( n_c = \prod_{c=1}^{q} n_{c_j} \).

(ii) The rows of \( M_D \) is labeled by multi-index id \( \{i_{r_1}, i_{r_2}, \ldots, i_{r_p}; n_{r_1}, n_{r_2}, \ldots, n_{r_p}\} \) and its columns is labeled by multi-index id \( \{i_{c_1}, i_{c_2}, \ldots, i_{c_q}; n_{c_1}, n_{c_2}, \ldots, n_{c_q}\} \).

(iii) the \( ((\alpha_1, \ldots, \alpha_p), (\beta_1, \ldots, \beta_q)) \)th element of \( M_D \) is \( d_{i_r,i_c} \), where \( i_r = \alpha_r, s = 1, \ldots, p \), and \( i_c = \beta_s, s = 1, \ldots, q \).

The set of multi-indexed matrices, which have row and column multi-indexes as \( \text{id}(i_1, i_2, \ldots, i_p; m_1, m_2, \ldots, m_p) \) and \( \text{id}(i_1, i_2, \ldots, i_q; n_1, n_2, \ldots, n_q) \) respectively, is in \( \mathcal{M}_{\prod_{p=1}^{P} m_{p} \times \prod_{q=1}^{Q} n_{q}} \). To emphasize its multi-index structure, it can alternatively be denoted by \( \mathcal{M}_(m_1, \ldots, m_p) \times (n_1, \ldots, n_q) \).

**Example 1.4.** Given a 4-dimensional data

\[
D = \{d_{i,j,k,r} \mid i = 1, 2; j = 1, 2, 3; k = 1, 2, 3, 4; r = 1, 2\}.
\]

1. Partition the index set as \( \{i, j, k, r\} = \{i, j\} \cup \{k, r\} \), and let the corresponding matrix \( M_D \) be row-indexed by id \( (i, j, 2, 3) \) and the column-indexed by id \( (k, r, 4, 2) \). Then we have the matrix as

\[
M_D = \begin{bmatrix}
    d_{1111} & d_{1112} & d_{1121} & d_{1122} & d_{1131} & d_{1132} & d_{1141} & d_{1142} \\
    d_{1211} & d_{1212} & d_{1221} & d_{1222} & d_{1231} & d_{1232} & d_{1241} & d_{1242} \\
    d_{1311} & d_{1312} & d_{1321} & d_{1322} & d_{1331} & d_{1332} & d_{1341} & d_{1342} \\
    d_{1411} & d_{1412} & d_{1421} & d_{1422} & d_{1431} & d_{1432} & d_{1441} & d_{1442} \\
    d_{2111} & d_{2112} & d_{2121} & d_{2122} & d_{2131} & d_{2132} & d_{2141} & d_{2142} \\
    d_{2211} & d_{2212} & d_{2221} & d_{2222} & d_{2231} & d_{2232} & d_{2241} & d_{2242} \\
    d_{2311} & d_{2312} & d_{2321} & d_{2322} & d_{2331} & d_{2332} & d_{2341} & d_{2342} \\
    d_{2411} & d_{2412} & d_{2421} & d_{2422} & d_{2431} & d_{2432} & d_{2441} & d_{2442}
\end{bmatrix} \in \mathcal{M}_{(2,3) \times (4,2)}.
\]

2. Partition the index set as \( \{i, j, k, r\} = \{i, k\} \cup \{j, r\} \), and let the corresponding matrix \( M_D \) be row-indexed by id \( (i, k, 2, 4) \) and the column-indexed by id \( (j, r, 3, 2) \). Then we have the matrix as
\[ M_D = \begin{bmatrix} d_{1111} & d_{1112} & d_{1121} & d_{1212} & d_{1311} & d_{1312} \\ d_{1121} & d_{1122} & d_{1123} & d_{1222} & d_{1321} & d_{1322} \\ d_{1131} & d_{1132} & d_{1133} & d_{1232} & d_{1331} & d_{1332} \\ d_{2111} & d_{2112} & d_{2121} & d_{2212} & d_{2311} & d_{2312} \\ d_{2121} & d_{2122} & d_{2123} & d_{2222} & d_{2321} & d_{2322} \\ d_{2131} & d_{2132} & d_{2133} & d_{2232} & d_{2331} & d_{2332} \\ d_{2141} & d_{2142} & d_{2241} & d_{2242} & d_{2341} & d_{2342} \end{bmatrix} \in \mathcal{M}_{(2,4) \times (3,2)} \]

### 1.3 Matrix Products

In addition to conventional matrix product, there are some other matrix products. This section gives a brief survey on their definitions and basic properties without proving. We refer to many standard references of matrix theory for details.

#### 1.3.1 Kronecker Product of Matrices

The Kronecker product of matrices is also called the tensor product of matrices. This product is applicable to any two matrices. It will be used from time to time throughout this book. We refer to [4] for a complete discussion.

**Definition 1.4.** Let \( A = [a_{ij}] \in \mathcal{M}_{m \times n} \) and \( B = [b_{ij}] \in \mathcal{M}_{p \times q} \). The Kronecker product of \( A \) and \( B \) is defined as

\[
A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \in \mathcal{M}_{mp \times nq}.
\] (1.13)

Next, we introduce some basic properties of Kronecker product:

**Proposition 1.2.**

1. (Associative Law)

\[ A \otimes (B \otimes C) = (A \otimes B) \otimes C. \] (1.14)

2. (Distributive Law)

\[
(\alpha A + \beta B) \otimes C = \alpha (A \otimes C) + \beta (B \otimes C),
\] (1.15)

\[
A \otimes (\alpha B + \beta C) = \alpha (A \otimes B) + \beta (A \otimes C), \quad \alpha, \beta \in \mathbb{R}.
\] (1.16)

**Proposition 1.3.**

1. \[
(A \otimes B)^T = A^T \otimes B^T.
\] (1.17)
2. Assume A and B are invertible. Then
\[ (A \otimes B)^{-1} = A^{-1} \otimes B^{-1}. \] (1.18)

3. 
\[ \text{rank}(A \otimes B) = \text{rank}(A) \text{rank}(B). \] (1.19)

4. Let \( A \in \mathcal{M}_{m \times m} \) and \( B \in \mathcal{M}_{n \times n} \). Then
\[ \det(A \otimes B) = (\det(A))^m (\det(B))^n. \] (1.20)
\[ \text{tr}(A \otimes B) = \text{tr}(A) \text{tr}(B). \] (1.21)

The next proposition is very useful.

**Proposition 1.4.** Let \( A \in \mathcal{M}_{m \times n}, B \in \mathcal{M}_{p \times q}, C \in \mathcal{M}_{q \times r} \), and \( D \in \mathcal{M}_{r \times n} \). Then
\[ (A \otimes B)(C \otimes D) = (AC) \otimes (BD). \] (1.22)

Particularly, we have
\[ A \otimes B = (A \otimes I_p) (I_n \otimes B). \] (1.23)

The next proposition is about the vector form of matrices.

**Proposition 1.5.** 1. Let \( X \in \mathbb{R}^n \) and \( Y \in \mathbb{R}^n \) be two column vectors. Then
\[ V_c(XY^T) = Y \otimes X. \] (1.24)

2. Let \( A \in \mathcal{M}_{m \times p}, B \in \mathcal{M}_{p \times q}, \) and \( C \in \mathcal{M}_{q \times n} \). Then
\[ V_c(ABC) = (C^T \otimes A)V_c(B). \] (1.25)

### 1.3.2 Hadamard Product

Hadamard product of matrices is another useful product in certain problems. It will also used in the sequel. The reader is referred to [4, 8] for more about it.

**Definition 1.5.** Let \( A = [a_{i,j}], B = [b_{i,j}] \in \mathcal{M}_{m \times n} \). The Hadamard product of \( A \) and \( B \) is defined as
\[ A \odot B = [a_{i,j} b_{i,j}] \in \mathcal{M}_{m \times n}. \] (1.26)

Hadamard product has some important properties as

**Proposition 1.6.** 1. (Commutativity) For any two matrices \( A, B \in \mathcal{M}_{m \times n} \),
1.3 Matrix Products

\[ A \odot B = B \odot A. \]  
(1.27)

2. (Associative Law) Let \( A, B, C \in \mathcal{M}_{m \times n} \). Then

\[ (A \odot B) \odot C = A \odot (B \odot C). \]  
(1.28)

3. (Distributive Law) Let \( A, B, C \in \mathcal{M}_{m \times n} \). Then

\[ (\alpha A + \beta B) \odot C = \alpha (A \odot C) + \beta (B \odot C), \quad \alpha, \beta \in \mathbb{R}. \]  
(1.29)

**Proposition 1.7.** 1.

\[ (A \odot B)^T = A^T \odot B^T. \]  
(1.30)

2. Let \( A \in \mathcal{M}_n \) and \( E = 1_n \), i.e., \( E = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}^T \). Then

\[ A \odot (EE^T) = A = (EE^T) \odot A. \]  
(1.31)

3. Let \( X, Y \in \mathbb{R}^n \) be two column vectors. Then

\[ (XX^T) \odot (YY^T) = (X \odot Y)(X \odot Y)^T. \]  
(1.32)

Define

\[ H_n = \text{diag}(\delta^1_n, \ldots, \delta^n_n). \]

Then we have

**Proposition 1.8.** Let \( A, B \in \mathcal{M}_{m \times n} \). Then

\[ A \odot B = H_n^T (A \odot B) H_n. \]  
(1.33)

**Proposition 1.9** (Schur’s Theorem). Let \( A, B \in \mathcal{M}_n \) be symmetric.

(i) If \( A \succeq 0 \) and \( B \succeq 0 \), then \( A \odot B \succeq 0 \);

(ii) If \( A \succeq 0 \) and \( B \succeq 0 \), then \( A \odot B \succeq 0 \).

**Proposition 1.10** (Oppenbein’s Theorem). Let \( A, B \in \mathcal{M}_n \) be symmetric. If \( A \succeq 0 \) and \( B \succeq 0 \), then

\[ \det(A \odot B) \geq \det(A) \det(B). \]  
(1.34)

1.3.3 Khatri-Rao Product

Definition 1.6. Let $A \in \mathcal{M}_{m \times r}$ and $B \in \mathcal{M}_{n \times r}$. The Khatri-Rao product of $A$ and $B$ is defined as

$$A \ast B = \left[ \text{Col}_1(A) \otimes \text{Col}_1(B), \text{Col}_2(A) \otimes \text{Col}_2(B), \ldots, \text{Col}_r(A) \otimes \text{Col}_r(B) \right].$$  \hfill (1.35)

Proposition 1.11. 1. (Associative Law) Let $A \in \mathcal{M}_{m \times r}$, $B \in \mathcal{M}_{n \times r}$, and $C \in \mathcal{M}_{p \times r}$. Then

$$ (A \ast B) \ast C = A \ast (B \ast C). \hfill (1.36) $$

2. (Distributive Law) Let $A, B \in \mathcal{M}_{m \times r}$ and $C \in \mathcal{M}_{n \times r}$. Then

$$ (aA + bB) \ast C = a(A \ast C) + b(B \ast C), \quad a, b \in \mathbb{R}. \hfill (1.37) $$

$$ C \ast (aA + bB) = a(C \ast A) + b(C \ast B), \quad a, b \in \mathbb{R}. \hfill (1.38) $$

The following example is useful in the sequel.

Example 1.5. A matrix $A \in \mathcal{M}_{m \times r}$ is called a logical matrix if all its columns are of the form $\delta^i_m$, $1 \leq i \leq m$. The set of $m \times r$ logical matrices is denoted by $\mathcal{L}_{m \times r}$.

Assume $A \in \mathcal{L}_{m \times r}$ and $B \in \mathcal{L}_{n \times r}$. Then

$$A \ast B \in \mathcal{L}_{m \times r}.$$

Remark 1.1. In addition to conventional matrix product, we have introduced Kronecker product, Hadamard product, and Khatri-Rao product of matrices. One sees easily that the associativity and distributivity are two common properties. These two properties may be considered as two fundamental requirements for any matrix products.

1.4 Tensor

Tensor is a typical multi-linear mapping. This section is a brief introduction. We refer to [2] for details.

Let $V$ be an $n$-dimensional vector space with a basis $\{d_1, \ldots, d_n\}$. Denote by $V^*$ the dual space of $V$, that is $V^*$ is the set of linear functions on $V$. Let $\{e_1, \ldots, e_n\} \subset V^*$ be a basis of $V^*$, dual to $\{d_1, \ldots, d_n\}$. That is,

$$d_i(d_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Then $X = \sum_{i=1}^n x_i d_i \in V$ can be expressed as a column vector $X = (x_1, \ldots, x_n)^T$, and $\omega = \sum_{i=1}^n \omega_i e_i \in V^*$ as a row vector $\omega = (\omega_1, \ldots, \omega_n)$. 

Definition 1.7. 1. Let \( f : V^s \to \mathbb{R} \) be an \( s \)-linear mapping, and
\[
f(d_{i_1}, \cdots, d_{i_s}) = \mu_{i_1 \cdots i_s}, \quad 1 \leq i_p \leq n, \ p = 1, \cdots, s.
\]
Arrange \( \{ \mu_{i_1 \cdots i_s} \mid 1 \leq i_j \leq n, \ j = 1, \cdots, s \} \) into a row by using multi-index \( \text{id}(i_1 \cdots i_s; n, \cdots, n) \) as
\[
M_f = [\mu_{1 \cdots 1} \cdots \mu_{1 \cdots n} \cdots \mu_{n \cdots n}] . \tag{1.39}
\]
\( M_f \) is called the structure matrix of \( f \). By the linearity, it is easy to check that for \( X_1, \cdots, X_s \in V \)
\[
f(X_1, \cdots, X_s) = M_f (X_1 \otimes \cdots \otimes X_s) . \tag{1.40}
\]
\( f \) is called a tensor of covariant order \( s \). The set of tensors on \( V \) of covariant order \( s \) is denoted by \( \mathcal{T}^s \).

2. Let \( f : (V^*)^t \to \mathbb{R} \) be a \( t \)-linear mapping, and
\[
f(e_{j_1}, \cdots, e_{j_t}) = \mu^{j_1 \cdots j_t}, \quad 1 \leq j_q \leq n, \ q = 1, \cdots, t.
\]
Arrange \( \{ \mu^{j_1 \cdots j_t} \mid 1 \leq j_q \leq n, \ q = 1, \cdots, t \} \) into a column by using multi-index \( \text{id}(j_1 j_2 \cdots j_t; n, \cdots, n) \) as
\[
M_f = [\mu^{1 \cdots 1} \cdots \mu^{1 \cdots n} \cdots \mu^{n \cdots n}]^T . \tag{1.41}
\]
\( M_f \) is called the structure matrix of \( f \). By the linearity, it is easy to check that for \( \omega_1, \cdots, \omega_t \in V^* \)
\[
f(\omega_1, \cdots, \omega_t) = (\omega_1 \otimes \cdots \otimes \omega_t) M_f . \tag{1.42}
\]
\( f \) is called a tensor of contravariant order \( t \). The set of tensors on \( V \) of contravariant order \( t \) is denoted by \( \mathcal{T}_t \).

3. Let \( f : V^s \times (V^*)^t \to \mathbb{R} \) be an \( s + t \)-linear mapping, and
\[
f(d_{i_1}, \cdots, d_{i_s}, e_{j_1}, \cdots, e_{j_t}) = \mu^{i_1 j_1 \cdots i_s j_t},
\]
\[
1 \leq i_p \leq n, \ p = 1, \cdots, s; \ 1 \leq j_q \leq n, \ q = 1, \cdots, t.
\]
Arrange \( \{ \mu^{i_1 j_1 \cdots i_s j_t} \mid 1 \leq i_p \leq n, \ p = 1, \cdots, s; \ 1 \leq j_q \leq n, \ q = 1, \cdots, t \} \) into a matrix, whose columns are labeled by multi-index \( \text{id}(i_1 \cdots i_s; n, \cdots, n) \) and rows are labeled by multi-index \( \text{id}(j_1 j_2 \cdots j_t; n, \cdots, n) \). Then we have
\[ M_f = \begin{bmatrix}
\mu^{11-1} & \mu^{11-2} & \cdots & \mu^{11-n} & \mu^{11-1-1} \\
\mu^{11-2} & \mu^{11-3} & \cdots & \mu^{11-n-2} & \mu^{11-2-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\mu^{11-n} & \mu^{11-n-1} & \cdots & \mu^{11-n-n} & \mu^{11-n-1-1} \\
\mu^{11-n-1} & \mu^{11-n-2} & \cdots & \mu^{11-n-n-2} & \mu^{11-n-n-1}
\end{bmatrix} \]  

(1.43)

By the linearity, it is also easy to check that for \( X_1, \ldots, X_e \in V \) and \( \omega_1, \ldots, \omega_k \in V^* \)

\[ f(X_1, \ldots, X_e, \omega_1, \ldots, \omega_k) = (\omega_1 \otimes \cdots \otimes \omega_k)M_f(X_1 \otimes \cdots \otimes X_e). \]  

(1.44)

\( f \) is called a tensor of covariant order \( s \) and contravariant order \( t \). The set of tensors on \( V \) of covariant order \( s \) and contravariant \( t \) is denoted by \( \mathcal{T}^{s,t} \).

In the following we assume contravariant order \( t = 0 \).

**Definition 1.8.** A tensor \( f \in \mathcal{T}^s \) is symmetric if for any \( i \neq j \)

\[ f(X_1, \ldots, X_i, \ldots, X_j, \ldots, X_e) = f(X_1, \ldots, X_j, \ldots, X_i, \ldots, X_e), \quad X_1, \ldots, X_e \in V. \]  

(1.45)

A tensor \( f \in \mathcal{T}^s \) is sky-symmetric if for any \( i \neq j \)

\[ f(X_1, \ldots, X_i, \ldots, X_j, \ldots, X_e) = -f(X_1, \ldots, X_j, \ldots, X_i, \ldots, X_e), \quad X_1, \ldots, X_e \in V. \]  

(1.46)

From high school algebra we know that in \( \mathbb{R}^3 \) two products were defined: (i) inner product; (ii) cross product. Fix a basis \( \{i, j, k\} \) as \( i = (1, 0, 0)^T \), \( j = (0, 1, 0)^T \), and \( k = (0, 0, 1)^T \). Let \( X = (x_1, x_2, x_3)^T \) and \( Y = (y_1, y_2, y_3)^T \). The inner product is defined as

\[ \langle X, Y \rangle := x_1y_1 + x_2y_2 + x_3y_3. \]  

(1.47)

The cross product, denoted by \( \times \), is defined as

\[ X \times Y = \det \begin{bmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}. \]

**Example 1.6.** 1. The inner product on \( \mathbb{R}^3 \) is a tensor of covariant order \( r = 2 \). It is symmetric. (This result is also true for \( \mathbb{R}^n \).)  
2. Let \( X, Y, Z \in \mathbb{R}^3 \).

\[ v := \langle X, Y \times Z \rangle. \]
We leave to the reader to check that \(\psi\) is a tensor of covariant order \(r = 3\). It is skew-symmetric. In fact, \(\psi\) is the volume of the parallelepiped (as \(X, Y, Z\) satisfy the right hand rule, otherwise, it is the negative volume).

### 1.5 Nash Equilibrium

This section gives a very limited introduction to game theory, including strategic form and Nash equilibrium. We refer to [3] for details.

Assume a game has \(n\) players, denoted by \(P_1, \ldots, P_n\), and each player \(P_i\) can have \(n_i > 0\) possible actions, called his strategies, denoted by \(\{s^1_i, s^2_i, \ldots, s^n_i\}\). Let

\[
f^i(s^1_k, s^2_k, \ldots, s^n_k), \quad i = 1, \cdots, n
\]

be the payoff of the \(i\)th player, which means what the player \(i\) obtains from the game when \(P_j\) takes his strategy \(s^j_k\), \(j = 1, \cdots, n\). For compactness, denote by

\[
\mu^i_{k_1, k_2, \ldots, k_n} := f^i(s^1_{k_1}, s^2_{k_2}, \ldots, s^n_{k_n}), \quad i = 1, \cdots, n; \quad k_j = 1, \cdots, n_j, \quad j = 1, \cdots, n.
\]

(1.49)

It is reasonable to assume that each player is pursuing his maximum payoff.

**Definition 1.9.** A set of strategies \(\{s^1_{k_1}, s^2_{k_2}, \ldots, s^n_{k_n}\}\) is called a Nash equilibrium, if

\[
f^i(s^1_{k_1}, s^2_{k_2}, \ldots, s^n_{k_n}) \geq f^i(s^1_{k_{i+1}}, s^2_{k_{i+2}}, \ldots, s^n_{k_n}), \quad \forall s^i, \quad i = 1, \cdots, n.
\]

(1.50)

Nash equilibrium is extremely important because once it is reached, each player intends to stick on this strategy forever.

Now let us see how to find the Nash equilibrium. The following procedure comes from definition directly: For each \(i\) we can put the data

\[
D_i = \{\mu^i_{k_1, k_2, \ldots, k_n} | k_j = 1, \cdots, n_j; \quad j = 1, \cdots, n\}
\]

into a matrix \(M_i\), which has \(id(k_1; n_i)\) as its row index and use all other indexes to label columns, that is, \(id(k_1, \cdots, k_{i-1}, k_{i+1}, \cdots, k_n, n_1, \cdots, n_{i-1}, n_{i+1}, \cdots, n_n)\) is used as its column index. Then for each column of \(M_i\) we can find at least one \(n\)-index \((k^*_1, \cdots, k^*_n)\), which corresponds to the largest value of \(\mu^i\). Denote by \(K_i\) the set of \(n\)-indexed found from each columns of \(M_i\). Then

\[
E_N := \cap_{i=1}^n K_i
\]

is the set of Nash equilibriums.

We give some examples to depict this.

**Example 1.7.** Two suspects are arrested by the police. The police have insufficient evidence for a conviction, and, having separated the prisoners, visit each of them
to offer the same deal. If one testifies for the prosecution against the other (defects) and the other remains silent (cooperates), the defector goes free and the silent accomplice receives the full 10-year sentence. If both remain silent, both prisoners are sentenced to only 1 year in jail for a minor charge. If each betrays the other, each receives a 5-year sentence. Each prisoner must choose to betray the other or to remain silent. Each one is assured that the other would not know about the betrayal before the end of the investigation. How should the prisoners act?

The payoff bi-matrix is given in Table 1.1.

<table>
<thead>
<tr>
<th>Table 1.1 Payoff of Prisoner’s Dilemma</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1 \setminus P_2$</td>
</tr>
<tr>
<td>----------------------</td>
</tr>
<tr>
<td>C</td>
</tr>
<tr>
<td>D</td>
</tr>
</tbody>
</table>

Now we have $M_1$ and $M_2$ as

$$M_1 = \begin{bmatrix} -1 & -10 \\ 0 & -5 \end{bmatrix}; \quad M_2 = \begin{bmatrix} -1 & -10 \\ 0 & -5 \end{bmatrix};$$

where the underline elements are the column maximal elements.

Note that in $M_1$ the row index is $k_1$ and the column index is $k_2$ while in $M_2$ the row index is $k_2$ and the column index is $k_1$, hence we have the $K_i$ set as

$$K_1 = \{(1, 2), (2, 2)\}; \quad K_2 = \{(1, 2), (2, 2)\}.$$

It follows that the set of Nash equilibrium(s) is

$$E_N = K_1 \cap K_2 = \{(2, 2)\},$$

which means $(D, D)$ is the only Nash equilibrium.

**Example 1.8.** Assume a game has three players. Their strategies are: $S_1 = \{s_1, s_2, s_3\}$, $S_2 = \{s_1^2, s_2^2\}$, $S_3 = \{s_1^3, s_2^3\}$. And the payoffs are shown in Table 1.2.

<table>
<thead>
<tr>
<th>Table 1.2 Payoffs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1 \setminus P_2$</td>
</tr>
<tr>
<td>---------------------</td>
</tr>
<tr>
<td>$s_1$</td>
</tr>
<tr>
<td>$s_2$</td>
</tr>
<tr>
<td>$s_3$</td>
</tr>
</tbody>
</table>

Then we have $M_i, i = 1, 2, 3$ as
1.6 Symmetric Group

\[ M_1 = \begin{bmatrix} 3 & -1 & 2 & 2 \\ 2 & 1 & -1 & 0 \\ 1 & 2 & 0 & 2 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 2 & 2 & -1 & 3 & 1 \\ -1 & 0 & 1 & 1 & 2 & 4 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 2 & 3 & 0 & 2 & -1 & 1 \\ 1 & -1 & 2 & 2 & 1 & 3 \end{bmatrix} \]

Note the in \( M_1 \) the row index is \( k_1 \) and the column index is \( Id(k_2, k_3; 2, 2) \), it follows that
\[ K_1 = \{(1, 1, 1), (2, 1, 2), (1, 2, 1), (1, 2, 2), (3, 2, 2)\}. \]

In \( M_2 \) the row index is \( k_2 \) and the column index is \( Id(k_1, k_3; 3, 2) \), it follows that
\[ K_2 = \{(1, 1, 1), (1, 1, 2), (2, 1, 1), (2, 2, 2), (3, 1, 1), (3, 2, 2)\}. \]

In \( M_3 \) the row index is \( k_3 \) and the column index is \( Id(k_1, k_2; 3, 2) \), hence
\[ K_3 = \{(1, 1, 1), (1, 2, 1), (2, 1, 2), (2, 2, 2), (3, 1, 2), (3, 2, 2)\}. \]

Therefore,
\[ E_N = K_1 \cap K_2 \cap K_3 = \{(1, 1, 1), (3, 2, 2)\}. \]

That is, \((x^1_1, x^2_2, x^3_3)\) and \((x^3_3, x^2_2, x^3_3)\) are two Nash equilibriums.

1.6 Symmetric Group

Let \( S = \{1, 2, \cdots, k\} \). A permutation \( \sigma \) on \( S \) is a one-to-one mapping from \( S \) onto \( S \). All the permutations on \( S \) with the product as the combination of two permutations as their product form a group called the symmetric group on \( k \) letters, or \( k \)th order symmetric group, denoted by \( S_k \) [5].

We use some numerical examples to depict it. For instance, let \( k = 5 \). A \( \sigma \in S_5 \) may be expressed as
\[ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 5 & 1 & 4 & 3 \end{pmatrix}. \]

Let another permutation \( \tau \in S_5 \) be expressed as
\[ \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 5 & 3 & 4 & 1 & 2 \end{pmatrix}. \]

The product on \( S_5 \) is defined as
\[
\tau \sigma = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
2 & 5 & 1 & 4 & 3 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
3 & 2 & 5 & 1 & 4
\end{pmatrix} = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
2 & 1 & 4 & 5 & 6 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
3 & 2 & 5 & 1 & 4
\end{pmatrix}.
\]

An alternative expression of an element in \( S_k \) is expressing it as a product of cycles. For instance, we can express \( \sigma = (1 \ 2 \ 3 \ 5 \ 3) \), \( \tau = (1 \ 5 \ 2 \ 3 \ 4) \), and \( \tau \sigma = (1 \ 3 \ 5 \ 4) \). Let
\[
\mu = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
2 & 1 & 4 & 5 & 6 & 3
\end{pmatrix} \in S_6.
\]

Then it can be expressed as \( \mu = (1\ 2)(3\ 4\ 5\ 6) \).

It is easy to check that the cardinal number \(|S_k| = k!\).

A cycle of two elements, such as \((a, b) \in S_k\), is called a transposition.

**Proposition 1.12.** Every permutation can be expressed as a product of transpositions.

**Proof.** We have only to prove that each cycle can be expressed as a product of transpositions. Assume the length of a cycle is 1; we have \((r_1) = (r_1 \ r_2)(r_2 \ r_1)\). Assume the length of a cycle is greater than 1, then we have \((r_1 \ r_2 \ \cdots \ r_k) = (r_1 \ r_k)(r_1 \ r_{k-1}) \cdots (r_1 \ r_2)\). \(\square\)

Note that a permutation \( \sigma \) can have different products of transpositions, but the number of transpositions can either be even or odd, but not both \([5]\). When the number of the transpositions is even we say \(\text{sgn}(\sigma) = 1\), otherwise, \(\text{sgn}(\sigma) = -1\).

For a \( \sigma \in S_k \) define a matrix \( M_\sigma \) as
\[
M_\sigma = \delta_{ij} [\sigma(1) \ \sigma(2) \ \cdots \ \sigma(k)].
\]

Then \( M_\sigma \) can realize the permutation as
\[
(\sigma(1) \ \sigma(2) \ \cdots \ \sigma(k)) = (1 \ 2 \ \cdots \ k)M_\sigma.
\]

Moreover,
\[
\text{sgn}(\sigma) = \det(M_\sigma).
\] (1.51)

Note that sometimes to label a set of data the index order and the index arrange order do not coincide. Say the data are labeled by multi-index \((i_1, \cdots, i_k)\) and the multi-index may be ordered by \(\text{id}(i_{\sigma(1)}, \cdots, i_{\sigma(k)}; n_{\sigma(1)}, \cdots, n_{\sigma(k)})\). See the following example.

**Example 1.9.** Consider a set of data
\[
D = \{d_{i_1, i_2, i_3} \mid i_1 = 1, 2; i_2 = 1, 2, 3; i_3 = 1, 2, 3\}.
\]
1.7 Swap Matrix

Let $\sigma = (1,3,2) \in S_3$. Assume $D$ is required to be arranged in the order of $\text{id}(\sigma(1), \sigma(2), \sigma(3); n_{\sigma(1)}, n_{\sigma(2)}, n_{\sigma(3)})$. Since $\sigma(1) = 3$, $\sigma(2) = 1$, and $\sigma(3) = 2$, the data are arranged in the order of $\text{id}(i_3, i_1, i_2; 3, 2, 1)$. Hence we have

$$
\begin{align*}
&d_{111} \quad d_{121} \quad d_{131} \quad d_{211} \quad d_{221} \quad d_{231} \\
&d_{112} \quad d_{122} \quad d_{132} \quad d_{212} \quad d_{222} \quad d_{232} \\
&d_{113} \quad d_{123} \quad d_{133} \quad d_{213} \quad d_{223} \quad d_{233}
\end{align*}
$$

1.7 Swap Matrix

In this section we define a special matrix, called the swap matrix. It is very useful in overcoming the non-commutativity of the matrix product. Swap matrix was firstly introduced in [4], where it is called commutation matrix.

**Definition 1.10.** A swap matrix $W_{mn} \in \mathcal{M}_{mn \times mn}$ is constructed in the following way:

Step 1. Label its columns by index $(i,j)$ in the order of $\text{id}(i; n)$ and its rows by index $(I,J)$ in the order of id$(J, n; n)$.

Step 2. The entry at row $(I,J)$ and column $(i,j)$, denoted by $w_{(I,J),(i,j)}$, is assigned as

$$
w_{(I,J),(i,j)} = \begin{cases} 
1, & I = i \text{ and } J = j \\
0, & \text{otherwise.}
\end{cases} \quad (1.52)
$$

We give some examples to depict swap matrices.

**Example 1.10.** 1. Consider $W_{2,3}$. Labeling its columns by $(i,j)$ in the order of $\text{id}(i; 2, 3)$ and its rows by $(I,J)$ in the order of $\text{id}(J, 3, 2)$. Then the swap matrix can be constructed as

$$
W_{2,3} = \begin{bmatrix}
11 & 12 & 13 & 21 & 22 & 23 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

2. Consider $W_{3,2}$. Labeling its columns by $(i,j)$ in the order of $\text{id}(i; 3, 2)$ and its rows by $(I,J)$ in the order of $\text{id}(J, 2, 3)$. Then the swap matrix can be constructed as
According to the construction of the swap matrix, the following two propositions are immediate consequences.

**Proposition 1.13.** Let $A \in \mathbb{M}_{m \times n}$. Then

$$W_{[m,n]}V_r(A) = V_r(A); \quad W_{[n,m]}V_r(A) = V_r(A). \quad (1.53)$$

**Proposition 1.14.**

1. Let $X \in \mathbb{R}^m$ and $Y \in \mathbb{R}^n$ be two column vectors. Then

$$W_{[m,n]}(X \otimes Y) = Y \otimes X. \quad (1.54)$$

2. Let $\omega \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^n$ be two row vectors. Then

$$(\omega \otimes \mu)W_{[n,m]} = \mu \otimes \omega. \quad (1.55)$$

This proposition has an equivalent statement.

**Corollary 1.1.** Let $D = \{x_{ij} | i = 1, \ldots, m; j = 1, \ldots, n\}$ be a set of data. $X$ is a column vector of the elements of $D$, labeled by $(i,j)$ and arranged in the order of $\text{id}(i,j;m,n)$, and $Y$ is a column vector of the elements of $D$, labeled by $(i,j)$ and arranged in the order of $\text{id}(j,i;n,m)$. Then

$$W_{[m,n]}X = Y; \quad W_{[n,m]}Y = X. \quad (1.56)$$

Swap matrix has some special properties, which follow from its definition immediately.

**Proposition 1.15.**

1. A swap matrix is an orthogonal matrix. It satisfies

$$W_{[m,n]}^T = W_{[m,n]}^{-1} = W_{[n,m]}. \quad (1.57)$$

2. When $m = n$, (1.55) becomes

$$W_{[n,n]} = W_{[n,n]}^T = W_{[n,n]}^{-1}. \quad (1.58)$$

3.

$$W_{[1,n]} = W_{[n,1]} = I_n. \quad (1.59)$$

$m = n$ is a particularly useful in the sequel. To simplify the notation, we define

$$W_{[n]} := W_{[n,n]}.$$
1.7 Swap Matrix

**Exercise 1**

1. Prove the formulas (1.5) and (1.6).
2. A multidimensional data $D$ as in (1.1). Assume there is a partition of the indices as

$$\{i_1, i_2, \ldots, i_k\} = \{i_{e_1}, i_{e_2}, \ldots, i_{e_q}\} \cup \{i_{c_1}, i_{c_2}, \ldots, i_{c_q}\},$$

such that $M_D$ is row-indexed by $\{i_{e_1}, i_{e_2}, \ldots, i_{e_q}; n_1, n_2, \ldots, n_p\}$ and column-indexed by $\{i_{c_1}, i_{c_2}, \ldots, i_{c_q}; n_1, n_2, \ldots, n_q\}$. Find the $p$th element in $V_r(M_D)$ and the $q$th element in $V_c(M_D)$.
3. Let $Z \in \mathcal{L}_{m \times n \times k}$. Show that there exist unique $X \in \mathcal{L}_{m \times k}$ and $Y \in \mathcal{L}_{n \times k}$ such that $Z$ is the Khatri-Rao product of $X$ and $Y$. That is,

$$Z = X \ast Y.$$

4. Let $\xi$ be an eigenvector of $A$ with respect to the eigenvalue $\lambda \in \sigma(A)$ and $\eta$ be an eigenvector of $B$ with respect to the eigenvalue $\mu \in \sigma(B)$. Prove that $\xi \ast \eta$ is an eigenvector of $AB$ with respect to the eigenvalue $\lambda \mu \in \sigma(AB)$.
5. Check the $v$ defined in Example 1.6 is a skew-symmetric tensor of covariant order $r = 3$.
6. Consider the game illustrated in Table 1.3.

<table>
<thead>
<tr>
<th>$P_1$</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_2$</td>
<td>L</td>
<td>R</td>
</tr>
<tr>
<td>$U$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$D$</td>
<td>5</td>
<td>-5</td>
</tr>
</tbody>
</table>

- Find Nash equilibrium of this game.
- If Player 1 and Player 2 form a coalition, the coalition’s payoff is the sum of their payoffs, then do the Nash equilibriums found in (a) remain equilibriums?

7. Prove $S_k$ is a group.
8. A set of data $D$ is arranged by $\text{id}(i_1, \ldots, i_k; n_1, \ldots, n_k)$, and under this order a data $d_p \in D$ is the $p$th element. $\sigma \in S_k$ is a known permutation. Find the multi-index of $d_p$ in the order of $\text{id}(i_{\sigma(1)}, \ldots, i_{\sigma(k)}; n_1, \ldots, n_k)$.
9. Let $V$ be an $n$-dimensional vector space with a basis $\{e_1, \ldots, e_n\}$, which has dual basis $\{e_1, \ldots, e_n\}$. Consider a tensor $\omega \in T^1_1(V)$. The structure matrix of $\omega$ under these bases is $M_\omega$. Let $\{\tilde{d}_1, \ldots, \tilde{d}_n\}$ be another basis of $V$, with dual bases $\{\hat{e}_1, \ldots, \hat{e}_n\}$. Moreover,

$$\begin{bmatrix} \hat{d}_1 \\ \vdots \\ \hat{d}_n \end{bmatrix} = A \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}.$$
Find the structure matrix of ω under new bases.

10. Proof the following two alternative expressions of the swap matrix.

\[
W_{[m,n]} = \begin{bmatrix}
    I_m \otimes (\delta^1_m)^T \\
    I_m \otimes (\delta^2_m)^T \\
    \vdots \\
    I_m \otimes (\delta^m_m)^T
\end{bmatrix}.
\] (1.60)

\[
W_{[m,n]} = \left[ I_n \otimes \delta^1_m, I_n \otimes \delta^2_m, \ldots, I_n \otimes \delta^m_m \right].
\] (1.61)

11. Let \( X \in \mathbb{R}^m, Y \in \mathbb{R}^n, Z \in \mathbb{R}^p \) be columns. Prove the following equations:

\[
Y \otimes X \otimes Z = (W_{[m,n]} \otimes I_p) X \otimes Y \otimes Z.
\] (1.62)

\[
X \otimes Z \otimes Y = (I_m \otimes W_{[n,p]}) X \otimes Y \otimes Z.
\] (1.63)

\[
Z \otimes Y \otimes X = (W_{[n,n]} \otimes I_n) \left( I_n \otimes W_{[m,p]} \right) (W_{[m,n]} \otimes I_p) X \otimes Y \otimes Z.
\] (1.64)

12. Let \( A \in \mathbb{M}_{m \times n} \). Prove the following equations:

\[
V_c(A^T) = W_{[n,m]} V_c(A),
\]

\[
V_r(A^T) = W_{[m,m]} V_r(A).
\]

References