Final Examination

Semi-tensor Product of Matrices and Its Applications
Graduate Course, Shandong University, 2010-2011
Lecturer: Daizhan Cheng

1. (20%) Let $\omega = (\omega_x, \omega_y, \omega_z)^T \in \mathbb{R}^3$ be vector of angular velocities of a solid body. Then the matrix

$$\omega^\times := \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

plays an important role in investigating the rotation of a solid body.

   (i) Find a matrix M_ω such that
$$\omega^\times = M_\omega \omega.$$

   (ii) Find a matrix M_{ω^2} such that
$$\left[\omega^\times\right]^2 = M_{\omega^2}\omega^2.$$

2. (20%) Given a logical matrix $Z \in \mathcal{L}_{mn\times k}$. Show that there exist unique $X \in \mathcal{L}_{m\times k}$ and unique $X \in \mathcal{L}_{n\times k}$ such that $Z = X * Y$, (where $*$ is the Khatri-Rao Product).

3. (20%) Similar to cross product on $\mathbb{R}^3$, we intend to define a cross product on $\mathbb{R}^4$, satisfying

   (i)
$$\delta_4^1 \times_c \delta_4^2 = \delta_4^3; \tag{1}$$

   (ii)
$$\delta_4^2 \times_c \delta_4^3 = \delta_4^4; \tag{2}$$

   (iii)
$$\delta_4^3 \times_c \delta_4^4 = \delta_4^1; \tag{3}$$

   (iv)
$$\delta_4^4 \times_c \delta_4^1 = \delta_4^2. \tag{4}$$

   (a) Is there a skew-symmetric algebra, which satisfies (1)-(4)? If so give an example. If not explain why?

   (b) Is there a Lie algebra, which satisfies (1)-(4)? If so give an example. If not explain why?

4. (20%) Consider a Boolean network

$$\begin{cases} A(t+1) = B(t) \vee C(t) \\ B(t+1) = C(t) \\ C(t+1) = A(t) \leftrightarrow B(t). \end{cases}$$

(a) Find its fixed point(s).

(b) Find its cycle(s).

5. (20%) Consider a Boolean control network

$$\begin{cases} A(t+1) = \neg B(t) \wedge u(t) \\ B(t+1) = C(t) \vee A(t) \\ C(t+1) = u(t) \rightarrow A(t). \end{cases}$$

(a) Find the reachable set of point $(1,1,1)$.

(b) Is this system controllable? Is it controllable at any point?