W. D. Zhu
Assistant Professor,
Department of Mechanical and Automation Engineering,
The Chinese University of Hong Kong,
Shatin, N.T., Hong Kong
Assoc. Mem. ASME

C. D. Mote, Jr.
FANUC Chair in Mechanical Systems and Vice Chancellor,
Department of Mechanical Engineering,
University of California,
Berkeley, CA 94720

B. Z. Guo
Department of Applied Mathematics,
Beijing Institute of Technology,
Beijing, China

# Asymptotic Distribution of Eigenvalues of a Constrained Translating String

*A new spectral analysis for the asymptotic locations of eigenvalues of a constrained translating string is presented. The constraint modeled by a spring-mass-dashpot is located at any position along the string. Asymptotic solutions for the eigenvalues are determined from the characteristic equation of the coupled system of contraint and string for all constraint parameters. Damping in the constraint dissipates vibration energy in all modes whenever its dimensionless location along the string is an irrational number. It is shown that although all eigenvalues have strictly negative real parts, an infinite number of them approach the imaginary axis. The analytical predictions for the distribution of eigenvalues are validated by numerical analyses.*

## 1 Introduction

A class of flexible translating elements including textile fibers, magnetic tapes, transmission belts, band saws, and tramway cables is commonly modeled as an axially moving string (Wickert and Mote, 1988). The model of a constrained translating string can also describe a bandsaw passing over a guide bearing and a magnetic tape traveling over a read-write head. Perkins (1990) analyzed the natural frequencies and modes of a string translating across a discrete, and uniform, elastic foundation. By transfer function formulation, Yang (1992) presented an eigenvalue inclusion principle for the translating string under nondissipative, pointwise constraints. Characterized by multiple wave scattering, the transient response of constrained translating strings under arbitrary disturbances was determined by Zhu and Mote (1995).

Control of vibration of the translating string by a point force applied in the domain requires the dimensionless location of it to be an irrational number (Yang and Mote, 1991b). A criterion for design of a stabilizing controller that ensures that all closed-loop eigenvalues lie in the left half-plane was given by Yang and Mote (1991a). The distances of the eigenvalues of the controlled continuous system from the imaginary axis, especially the infinite number of high modes, have not been investigated.

In the present study, a new spectral analysis for the constrained translating string is developed. The constraint, represented by mass $m$, stiffness $k$, and damping $c$, is located at an arbitrary position $d$ along the span. The asymptotic locations of all eigenvalues are determined from the characteristic Eq. (23) through the use of the Rouché's Theorem. When $m \neq 0$ all eigenvalues of large modulus approach the imaginary axis. When $m = 0$ and $c \neq 0$, all eigenvalues remain in the left half-plane if $d$ is irrational. However, an infinite number of eigenvalues approach the imaginary axis. Hence the system is not exponentially stable in any case. The methodology is applicable to predicting the closed-loop eigenvalues for the controller designs in Yang and Mote (1991a).

Contributed by the Applied Mechanics Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS for publication in the ASME JOURNAL OF APPLIED MECHANICS.

Discussion on the paper should be addressed to the Technical Editor, Professor Lewis T. Wheeler, Department of Mechanical Engineering, University of Houston, Houston, TX 77204-4792, and will be accepted until four months after final publication of the paper itself in the ASME JOURNAL OF APPLIED MECHANICS.

Manuscript received by the ASME Applied Mechanics Division, Nov. 20, 1995; final revision, Sept. 9, 1996. Associate Technical Editor: N. C. Perkins.

## 2 Model and Eigenvalue Problem

As shown schematically in Fig. 1, a string of tension $P$ and mass per unit length $\rho$ is traveling at a subcritical speed $V$ ($V < \sqrt{P/\rho}$) between two supports separated by $L$. A flexible constraint with mass $M$, stiffness $K$, and damping constant $C$ is located at a distance $D$ ($0 < D < L$) from the left end. The interaction force between the string and constraint is $R(T)$. The string is subjected to gravitational force $\rho g$ and a distributed external force $F(X, T)$. The transverse displacements of the string and the mass $M$, relative to the horizontal $X$-axis, are $U(X, T)$ and $Z(T)$, respectively.

The string transverse displacement $U(X, T)$ is small and planar. The friction force between the string and the constraint is negligible compared to the tension. Introduce the following dimensionless variables:

$$x = X/L \quad u = U/L \quad z = Z/L \quad d = D/L$$
$$v = V(\rho/P)^{1/2} \quad t = T(P/\rho L^2)^{1/2}$$
$$m = M/\rho L \quad k = KL/P \quad c = C/(P\rho)^{1/2}$$
$$w = \rho g L/P \quad f = FL/P \quad r = R/P. \tag{1}$$

The equation governing transverse motion of the translating string is

$$u_{tt}(x, t) + 2v u_{xt}(x, t) + (v^2 - 1)u_{xx}(x, t) = r(t)\delta(x - d) + f(x, t) - w \tag{2}$$

with the boundary conditions

$$u(0, t) = u(1, t) = 0. \tag{3}$$

The equation of motion for the flexible constraint is

$$-r(t) + k[z_0 + z^* - z(t)] - c\dot{z}(t) - mw = m\ddot{z}(t) \tag{4}$$

where $z(t) = u(d, t)$, $z^*$ is the equilibrium displacement of the constraint mass to be determined, and $z_0$ is the compression of the spring at equilibrium.

The equilibrium displacement of the string $u^*(x)$ and static preload $r^*$ are derived from the equilibrium balance following (2)-(4):

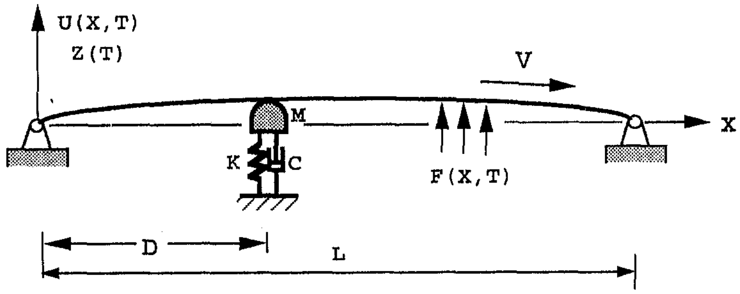


**Fig. 1 Schematic of a constrained translating string**

$$(v^2 - 1)u^*_{xx}(x) = r^*\delta(x - d) + f^*(x) - w \quad (5)$$

$$u^*(0) = u^*(1) = 0 \quad (6)$$

$$r^* = kz_0 - mw \quad z^* = u^*(d) \quad (7)$$

where $f^*(x)$ is the equilibrium component of $f(x, t)$. Substitution of $u(x, t) = u^*(x) + \tilde{u}(x, t)$, $z(t) = z^* + \tilde{z}(t)$, $r(t) = r^* + \tilde{r}(t)$, and $f(x, t) = f^*(x) + \tilde{f}(x, t)$ into (2)–(4), and use of (5)–(6), yield the equations describing small amplitude motions of the string and constraint around the equilibrium:

$$\tilde{u}_{tt}(x, t) + 2v\tilde{u}_{xt}(x, t) + (v^2 - 1)\tilde{u}_{xx}(x, t) = \tilde{r}(t)\delta(x - d) + \tilde{f}(x, t) \quad (8)$$

$$\tilde{u}(0, t) = \tilde{u}(1, t) = 0 \quad (9)$$

$$\tilde{r}(t) = -k\tilde{z}(t) - c\dot{\tilde{z}}(t) - m\ddot{\tilde{z}}(t), \quad \tilde{z}(t) = \tilde{u}(d, t). \quad (10)$$

The natural frequencies and vibration modes of the constrained translating string around its equilibrium are derived from (8)–(10). By setting $\tilde{f}(x, t) = 0$ in (8), assuming a separable solution

$$\tilde{u}(x, t) = U(x)e^{\lambda t} = \begin{cases} U_1(x)e^{\lambda t}, & 0 < x < d \\ U_2(x)e^{\lambda t}, & d < x < 1 \end{cases} \quad (11)$$

where $U(x)$ and $\lambda$ are, in general, complex, and substituting (11) into (8)–(10), yields

$$\lambda^2 U_1(x) + 2v\lambda U_1'(x) + (v^2 - 1)U_1''(x) = 0, \quad 0 < x < d \quad (12a)$$

$$\lambda^2 U_2(x) + 2v\lambda U_2'(x) + (v^2 - 1)U_2''(x) = 0, \quad d < x < 1 \quad (12b)$$

$$U_1(0) = 0 \quad U_2(1) = 0. \quad (13)$$

Because $u(x, t)$ is continuous at $x = d$, we have

$$U_1(d) = U_2(d) \quad (14)$$

By (11) $u_t(x, t)$ and $u_{tt}(x, t)$ are also continuous at $x = d$. Integration of (8) from $x = d^-$ to $x = d^+$ and use of (10)–(11) gives

$$(v^2 - 1)[U_2'(d) - U_1'(d)] + [m\lambda^2 + c\lambda + k]U_1(d) = 0. \quad (15)$$

The eigenvalue problem (12a)–(15) leads to the characteristic equation

$$(m\lambda^2 + c\lambda + k)\sinh\frac{\lambda d}{1 - v^2}\sinh\frac{\lambda(1 - d)}{1 - v^2} + \lambda\sinh\frac{\lambda}{1 - v^2} = 0 \quad (16)$$

whose roots are the complex eigenvalues $\lambda = \mu + \omega i$, where $\mu$ and $\omega$ are real and $i = \sqrt{-1}$. They appear in complex conjugate pairs, $\lambda_{\pm n} = \mu_n \pm \omega_n i$ ($n = 1, 2, 3, \ldots$), where the positive $\omega_n$, arranged in ascending order of magnitude, gives the sequence of the dimensionless natural frequencies of the system. Temporal variation of the vibration amplitude for mode $n$ is described by $\mu_n$, whose positive and negative values indicate the rate of amplitude growth and decay, respectively. For $c = 0$, eigenvalues are imaginary and (16) reduces to

$$(k - m\omega^2)\sin\frac{\omega d}{1 - v^2}\sin\frac{\omega(1 - d)}{1 - v^2} + \omega\sin\frac{\omega}{1 - v^2} = 0. \quad (17)$$

The special case of $m = 0$ in (17) returns the characteristic equation for a translating string guided by a single spring (Perkins, 1990). If in addition, $k = 0$ in (17), the positive roots of (17) recover the natural frequencies of the classical moving threadline, $\omega_n = n\pi(1 - v^2)$ ($n = 1, 2, 3, \ldots$) (Sack, 1954).

The complex eigenfunction $U_n(x)$ corresponding to the complex eigenvalue $\lambda_n$ is obtained from (12a)–(15) as follows:

$$U_n(x) = e^{\lambda_n x/(1-v)} - e^{-\lambda_n/(1+v)}, \quad 0 < x < d \quad (18a)$$

$$U_n(x) = -e^{-\lambda_n/(1-v^2)}\frac{\sinh\dfrac{\lambda_n d}{1 - v^2}}{\sinh\dfrac{\lambda_n(1 - d)}{1 - v^2}} \times (e^{\lambda_n x/(1-v)} - e^{2\lambda_n/(1-v^2)}e^{-\lambda_n x/(1+v)}), \quad d < x < 1. \quad (18b)$$

Hence, the general solution $\tilde{u}(x, t)$ describing free response can be obtained by superposition of the separable form (11) for each eigensolution $\{\lambda_n, U_n(x)\}$ so determined:

$$\tilde{u}(x, t) = \sum_{n=1}^{\infty}(A_n U_n(x)e^{\lambda_n t} + A_{-n}U_{-n}(x)e^{\lambda_{-n}t}) \quad (19)$$

where the eigenfunction associated with the eigenvalue $\lambda_{-n} = \bar{\lambda}_n$ is $U_{-n}(x) = \bar{U}_n(x)$ by (18a, b), with the overbar denoting complex conjugation. Because $\tilde{u}(x, t)$ is real, $A_{-n} = \bar{A}_n$ with $A_n$ determined from initial conditions.

## 3 Spectral Analysis

The solutions to (16) are symmetric with respect to the center of the string $d = \frac{1}{2}$. For $c \neq 0$ and subcritical transport speed $v < 1$, Re $\lambda = \mu \leq 0$. The system is asymptotically stable, i.e., $\mu_n < 0$ for all $n \in \mathbf{N}$, only when $d$ is irrational. This can be shown by substituting $\lambda = \omega i$ into (16) to give

$$(k - m\omega^2)\sin\frac{\omega d}{1 - v^2}\sin\frac{\omega(1 - d)}{1 - v^2} + \omega\sin\frac{\omega}{1 - v^2} + ic\,\omega\sin\frac{\omega d}{1 - v^2}\sin\frac{\omega(1 - d)}{1 - v^2} = 0. \quad (20)$$

Separating the real and imaginary parts, we obtain for $c \neq 0$:

$$\sin \frac{\omega}{1-v^2} = \sin \frac{\omega d}{1-v^2} = 0 \quad \text{or}$$

$$\sin \frac{\omega}{1-v^2} = \sin \frac{\omega(1-d)}{1-v^2} = 0. \qquad (21)$$

Hence $\mu = 0$ if and only if $d = p/q$, where $p, q \in \mathbf{N}$ and $0 \le p \le q$. For $p = 0$ and $p = q$ the constraint location coincides with one or the other support. The other discrete locations $d = p/q$ with $p = 1, 2, \ldots, q-1$ give the $q-1$ nodal points of mode $q$ of the classical moving threadline. To provide damping to all the vibration modes, $d \ne p/q$ for all $q$ and $0 < p < q$ in agreement with Yang and Mote (1991b). For a rational $d = p/q$ with $p$ and $q$ co-prime, the imaginary eigenvalues are $nq\pi(1-v^2)i$ ($n = 1, 2, 3, \ldots$) by (21).

Introduction of the new variables

$$\lambda^* = \frac{\lambda}{1-v^2} \quad m^* = m\frac{1-v^2}{2} \quad c^* = \frac{c}{2}$$

$$k^* = \frac{k}{2(1-v^2)} \qquad (22)$$

into (16) and deletion of the asterisks in the notation yields

$$(m\lambda^2 + c\lambda + k)[e^{\lambda} - e^{(1-2d)\lambda} - e^{-(1-2d)\lambda} + e^{-\lambda}] + \lambda[e^{\lambda} - e^{-\lambda}] = 0. \qquad (23)$$

**3.1 Spectrum for $d = \frac{1}{2}$.** When $d = \frac{1}{2}$ two branches of the solution to (23) result:

$$e^{\lambda} = 1; \quad e^{\lambda} = \frac{m\lambda^2 + c\lambda + k - \lambda}{m\lambda^2 + c\lambda + k + \lambda}. \qquad (24a, b)$$

The eigenvalues of the first branch, $\lambda_n = 2n(1-v^2)i$ ($n = 1, 2, 3, \ldots$) by (22) and (24$a$), are the even-numbered modes of a classical moving threadline. The eigenvalues of the second branch are obtained from (24$b$) for the following cases:

*Case I:* $m \ne 0$. Equation (24$b$) is written in the form

$$e^{\lambda} - 1 + \frac{2}{m}\frac{1}{\lambda} + O(|\lambda|^{-2}) = 0. \qquad (25)$$

The zeros of $e^{\lambda} = 1$ are $\sigma_n = 2n\pi i$ ($n = 1, 2, 3, \ldots$). The zeros of (25) for $\lambda$ of large modulus are asymptotic to $\sigma_n$ following the theorem of Rouché (Carrier et al., 1983):

*Rouché's Theorem:* Let $f(z)$ and $g(z)$ be analytic inside and on $C$, with $|g(z)| < |f(z)|$ on $C$. Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside $C$.

In the present case, let $f(\lambda) = e^{\lambda} - 1$ and $g(\lambda) = (2/m)(1/\lambda) + O(|\lambda|^{-2})$. A disk $C_n$ centered at $\sigma_n$ is defined: $\lambda = \sigma_n + (\epsilon/|\sigma_n|)e^{i\theta}$, where $0 \le \theta \le 2\pi$. On $C_n$, we have

$$|f(\lambda)| = |e^{(\epsilon/|\sigma_n|)e^{i\theta}} - 1| = \left|\frac{\epsilon}{|\sigma_n|}e^{i\theta} + O(|\sigma_n|^{-2})\right|$$

$$= \frac{\epsilon}{|\sigma_n|} + O(|\sigma_n|^{-2}) \qquad (26)$$

where the Taylor expansion has been used in (26). Because $|\lambda| \ge |\sigma_n| - (\epsilon/|\sigma_n|)$, we have

$$\frac{1}{|\lambda|} \le \frac{1}{|\sigma_n| - \dfrac{\epsilon}{|\sigma_n|}}. \qquad (27)$$

Take $\epsilon > 0$, such that

$$\frac{2}{m}\frac{1}{|\sigma_n| - \dfrac{\epsilon}{|\sigma_n|}} < \frac{\epsilon}{|\sigma_n|}. \qquad (28)$$

That is, $(2/m)|\sigma_n| < \epsilon|\sigma_n| - \epsilon^2/|\sigma_n|$. So if $\epsilon > 2/m$, there exists $N > 0$, such that for $n \ge N$, (28) is satisfied. Hence by (27), $(2/m)/(1/|\lambda|) < \epsilon/|\sigma_n|$. Take $N_0 \ge N$, such that for $n \ge N_0$,

$$|g(\lambda)| = \left|\frac{2}{m}\frac{1}{\lambda} + O(|\lambda|^{-2})\right|$$

$$= \frac{2}{m}\frac{1}{|\lambda|} + O(|\lambda|^{-2}) < |f(\lambda)| \qquad (29)$$

on $C_n$. By Rouché's Theorem, there exists one solution $\lambda_n$ to (25) inside $C_n$ for $n \ge N_0$, i.e., $|\lambda_n - \sigma_n| < \epsilon/|\sigma_n|$. Hence by returning to the former variables in (22), the eigenvalues of the second branch are

$$\lambda_n = (1-v^2)\sigma_n + O\left(\frac{1}{n}\right) = 2n\pi(1-v^2)i + O\left(\frac{1}{n}\right). \qquad (30)$$

Each eigenvalue $\lambda_n$ of the second branch in (30) is asymptotic to one on the first branch, $2n\pi(1-v^2)i$. Hence eigenvalues of high modes exist in closely located pairs near the imaginary axis. They are independent of the constraint parameters, $m$, $c$, and $k$, to the first order.

*Case II: $m = 0$ and $c \ne 1$.* Equation (24$b$) becomes

$$e^{\lambda} = \frac{c-1}{c+1} + \frac{2k}{1+c}\frac{1}{(1+c)\lambda + k}. \qquad (31)$$

The solutions to $e^{\lambda} = (c-1)/(c+1)$ are

$$\sigma_n = \ln\frac{c-1}{c+1} + 2n\pi i, \quad \text{for} \quad c > 1 \qquad (32a)$$

$$\sigma_n = \ln\frac{1-c}{1+c} + (2n-1)\pi i, \quad \text{for} \quad c < 1 \qquad (32b)$$

for $n = 1, 2, 3, \ldots$. Hence by (22) the exact eigenvalues of the second branch for $k = 0$ are $\lambda_n = (1-v^2)\sigma_n$. By use of the Rouché's Theorem and (22) similar to Case I, the eigenvalues of the second branch for $k \ne 0$ are

$$\lambda_n = (1-v^2)\ln\frac{c-2}{c+2} + 2n\pi(1-v^2)i + O\left(\frac{1}{n}\right),$$

$$\text{for} \quad c > 2, \qquad (33a)$$

$$\lambda_n = (1-v^2)\ln\frac{2-c}{2+c} + (2n-1)\pi(1-v^2)i + O\left(\frac{1}{n}\right),$$

$$\text{for} \quad c < 2. \qquad (33b)$$

They are independent of $k$ to the first order. In either case $\mu_n = \text{Re}\lambda_n = (1-v^2)\ln|(2-c)/(2+c)| + O(1/n)$. Hence the decay rates for high modes are nearly constant.

*Case III: $m = 0$ and $c = 1$.* Equation (24$b$) reduces to

$$(2\lambda + k)e^{\lambda} = k \qquad (34)$$

and $k \ne 0$, $e^{\text{Re}\lambda} = k/|2\lambda + k|$. Hence $\text{Re}\lambda = \ln(k/|2\lambda + k|)$, and $\mu = \text{Re}\lambda \to -\infty$ as $|\lambda| \to \infty$. Reducing $k$ increases the damping rates for all the modes on the second branch. $\text{Re}\lambda \to -\infty$ in (34) yields $2\lambda e^{\lambda} = k$. Hence (34) is asymptotic to

$$\mathrm{Re}\lambda + \ln|\lambda| = \ln\frac{k}{2}. \quad (35)$$

The asymptotic locations of the eigenvalues of (35) can be obtained. Let $\lambda = |\lambda| e^{i\theta}$,

$$\cos\theta = \frac{\mathrm{Re}\lambda}{|\lambda|} = \frac{1}{|\lambda|}\ln\frac{k}{2} - \frac{\ln|\lambda|}{|\lambda|} \to 0, \quad \text{as} \quad |\lambda| \to \infty. \quad (36)$$

Hence $\theta \to (\pi/2)$ as $|\lambda| \to \infty$. Also,

$$e^{i\,\mathrm{Im}\lambda} = \frac{k}{2\lambda} e^{-\mathrm{Re}\lambda} = \frac{|\lambda|}{\lambda} = e^{-i\theta}. \quad (37)$$

Hence by (22), (35), and (37) the asymptotic eigenvalues of the second branch are given by

$$\omega_n = \mathrm{Im}\lambda_n = (1 - v^2)\left(2n\pi - \frac{\pi}{2}\right) \quad (38a)$$

$$\mu_n = \mathrm{Re}\lambda_n = (1 - v^2)\ln\frac{k}{4(1 - v^2)} - (1 - v^2)\ln\left(2n\pi - \frac{\pi}{2}\right). \quad (38b)$$

By (38$a$) $\omega_n$ is independent of $k$ to the first order.

For $k = 0$ there are no finite solutions to $e^{\lambda} = 0$. Hence there are no eigenvalues corresponding to (34). $\mathrm{Re}\lambda = -\infty$ in this case implies that all the modes of the second branch are completely dissipated by damping after a finite time.

**3.2 Spectrum for $m \neq 0$ and arbitrary $d$.** Equation (23) is written in the form

$$e^{2\lambda} - e^{2d\lambda} - e^{2(1-d)\lambda} + 1 = -\frac{\lambda(e^{2\lambda} - 1)}{m\lambda^2 + c\lambda + k}. \quad (39)$$

If $\mathrm{Re}\lambda \to -\infty$ as $|\lambda| \to \infty$, (39) yields a contradiction, $1 = 0$. Hence there exists $A > 0$, such that $-A < \mathrm{Re}\lambda \le 0$. Therefore $e^{\lambda}$ and $e^{-\lambda}$ are bounded. As $|\lambda| \to \infty$, we have from (23):

$$(e^{d\lambda} - e^{-d\lambda})(e^{(1-d)\lambda} - e^{-(1-d)\lambda}) = -\frac{\lambda(e^{\lambda} - e^{-\lambda})}{m\lambda^2 + c\lambda + k} \to 0. \quad (40)$$

Hence either $e^{2d\lambda} \to 1$ or $e^{2(1-d)\lambda} \to 1$ as $|\lambda| \to \infty$. In either case $\mathrm{Re}\lambda \to 0$ as $|\lambda| \to \infty$. For irrational $d$, though $\mathrm{Re}\lambda_n < 0$ for all $n$, all eigenvalues approach the imaginary axis as $|\lambda| \to \infty$. Hence the system is asymptotically, but not exponentially, stable.

*Determination of Eigenvalues.* By (23) we have

$$(e^{2d\lambda} - 1)(e^{2(1-d)\lambda} - 1) + \frac{1}{m\lambda}(e^{2\lambda} - 1) + O(|\lambda|^{-2}) = 0. \quad (41)$$

Let $f(\lambda) = (e^{2d\lambda} - 1)(e^{2(1-d)\lambda} - 1)$ and $g(\lambda) = (1/m\lambda)(e^{2\lambda} - 1) + O(|\lambda|^{-2})$. The roots of $e^{2d\lambda} = 1$ are $\sigma_n = n\pi i/d$. Define $C_n$ around $\sigma_n$: $\lambda = \sigma_n + (\epsilon/|\sigma_n|)e^{i\theta}$, where $0 \le \theta \le 2\pi$. For any $\lambda$ on $C_n$, using the Taylor expansion we have

$$|e^{2d\lambda} - 1| = |e^{2d(\epsilon/|\sigma_n|)e^{i\theta}} - 1| = 2d\frac{\epsilon}{|\sigma_n|} + O(|\sigma_n|^{-2}) \quad (42)$$

$$|e^{2(1-d)\lambda} - 1| = |e^{(2n\pi/d)i}e^{2(1-d)(\epsilon/|\sigma_n|)e^{i\theta}} - 1| = |e^{(2n\pi/d)i} - 1| + O(|\sigma_n|^{-1})$$

$$|e^{2\lambda} - 1| = |e^{(2n\pi/d)i} - 1| + O(|\sigma_n|^{-1}). \quad (43)$$

Hence on $C_n$,

$$|f(\lambda)| = |e^{2d\lambda} - 1||e^{2(1-d)\lambda} - 1| = 2d\frac{\epsilon}{|\sigma_n|}|e^{(2n\pi/d)i} - 1| + O(|\sigma_n|^{-2})$$

$$|g(\lambda)| = \frac{1}{m}\frac{1}{|\sigma_n|}|e^{(2n\pi/d)i} - 1| + O(|\sigma_n|^{-2}). \quad (44)$$

For irrational $d$, $|e^{(2n\pi/d)i} - 1| \neq 0$. For rational $d = p/q$, with $p$ and $q$ co-prime, $|e^{(2n\pi/d)i} - 1| = 0$ only when $p$ divides $n$. In this case (43) and (44) become

$$|e^{2(1-d)\lambda} - 1| = \frac{2(1-d)\epsilon}{|\sigma_n|} + O(|\sigma_n|^{-2})$$

$$|e^{2\lambda} - 1| = \frac{2\epsilon}{|\sigma_n|} + O(|\sigma_n|^{-2}) \quad (45)$$

$$|f(\lambda)| = \frac{4d(1-d)\epsilon^2}{|\sigma_n|^2} + O(|\sigma_n|^{-3})$$

$$|g(\lambda)| = \frac{2\epsilon}{m|\sigma_n|^2} + O(|\sigma_n|^{-3}). \quad (46)$$

Choose $\epsilon > 1/[2d(1-d)m]$, then for either (44) or (46) there exists $N > 0$ such that when $n \ge N$, $|g(\lambda)| < |f(\lambda)|$ on $C_n$. By Rouché's Theorem, there is one solution $\lambda_n$ to (41) inside $C_n$ for $n \ge N$, i.e., $|\lambda_n - \sigma_n| < \epsilon/|\sigma_n|$. Hence by (22) one branch of eigenvalues is

$$\lambda_n = \frac{n\pi}{d}(1 - v^2)i + O\left(\frac{1}{n}\right). \quad (47a)$$

Similarly the roots of $e^{2(1-d)\lambda} - 1 = 0$ are $\sigma_n = n\pi i/(1 - d)$. Following the same analysis, the other branch of eigenvalues is

$$\lambda_n = \frac{n\pi}{1-d}(1 - v^2)i + O\left(\frac{1}{n}\right). \quad (47b)$$

For $d = \frac{1}{2}$ the two branches of eigenvalues are both of the form $2n\pi(1 - v^2)i + O(1/n)$, consistent with Section 3.1.

**3.3 Spectrum for $m = 0$ and Arbitrary $d$.** Because of symmetry of the spectrum with respect to $d = \frac{1}{2}$, we consider $d < \frac{1}{2}$. Because $\lambda = 0$ is not an eigenvalue, (23) becomes

$$\left(c + 1 + \frac{k}{\lambda}\right)e^{2\lambda} - \left(c + \frac{k}{\lambda}\right)e^{2(1-d)\lambda} - \left(c + \frac{k}{\lambda}\right)e^{2d\lambda} + c - 1 + \frac{k}{\lambda} = 0. \quad (48)$$

For $c \neq 1$ the roots of (48) of large modulus are asymptotic to those of the characteristic equation corresponding to $k = 0$ (Bellman and Cooke, 1963):

$$(c + 1)e^{2\lambda} - ce^{2(1-d)\lambda} - ce^{2d\lambda} + c - 1 = 0. \quad (49)$$

For $c = 1$ and $k \neq 0$ (48) is written as

$$2e^{2(1-d)\lambda} - e^{2(1-2d)\lambda} - 1 + \frac{k}{\lambda}[e^{-2d\lambda} + e^{2(1-d)\lambda} - e^{2(1-2d)\lambda} - 1] = 0. \quad (50)$$

For $c = 1$ and $k = 0$ (48) becomes

$$2e^{2\lambda} = e^{2(1-d)\lambda} + e^{2d\lambda}. \tag{51}$$

If $\text{Re}\lambda \to -\infty$ as $|\lambda| \to \infty$, (48) implies $c = 1$. For $c = 1$ and $k = 0$, we have by (51)

$$2 = e^{2(d-1)\lambda}(1 + e^{2(1-2d)\lambda}) \tag{52}$$

$\text{Re}\lambda \to -\infty$ in (52) leads to a contradiction, $2 = \infty$. Hence there exists $A > 0$, such that $-A < \text{Re}\lambda \le 0$. For $c = 1$ and $k \ne 0$, $\text{Re}\lambda \to -\infty$ in (50) yields $1 = ke^{-2d\lambda}/\lambda$. Hence, $e^{-2d\text{Re}\lambda} = |\lambda|/k$. A branch of eigenvalues of (50) of large modulus is asymptotic to

$$\text{Re}\lambda + \frac{1}{2d}\ln|\lambda| = \frac{1}{2d}\ln k, \tag{53}$$

which is similar in form to (35) for the case $d = \frac{1}{2}$. Let $\lambda = |\lambda|e^{i\theta}$, we have

$$\cos\theta = \frac{\text{Re}\lambda}{|\lambda|} = -\frac{1}{2d}\frac{\ln|\lambda|}{|\lambda|} + \frac{1}{2d}\frac{\ln k}{|\lambda|} \to 0, \quad \text{as } |\lambda| \to \infty. \tag{54}$$

Hence $\theta \to \pi/2$ as $|\lambda| \to \infty$. Further,

$$e^{-i2d\text{Im}\lambda} = \frac{\lambda}{k}e^{2d\text{Re}\lambda} = \frac{\lambda}{|\lambda|} = e^{i\theta}. \tag{55}$$

Hence by (22) and (53)–(55) the asymptotic locations of the eigenvalues on (53) are

$$\omega_n = \text{Im}\lambda_n = \frac{1 - v^2}{2d}\left(2n\pi - \frac{\pi}{2}\right) \tag{56a}$$

$$\mu_n = \text{Re}\lambda_n = \frac{1 - v^2}{2d}\left[\ln\frac{k}{2(1 - v^2)} - \ln\frac{1}{2d}\left|2n\pi - \frac{\pi}{2}\right|\right]. \tag{56b}$$

All other branches of eigenvalues of (50) must satisfy $-A < \text{Re}\lambda \le 0$ for some constant $A > 0$. They are determined next.

*Case I: Rational d.* Let $d = p/q$ with $p$ and $q$ co-prime, and $2p < q$. Equations (49) and (51) reduce, respectively, to the polynomial equations

$$(c + 1)z^q - cz^{q-p} - cz^p + c - 1 = 0 \tag{57}$$

$$2z^q - z^{q-p} - z^p = 0 \tag{58}$$

where $z = e^{(2/q)\lambda}$. Because there are no finite solutions for $\lambda$ corresponding to the root $z = 0$, (58) reduces to

$$2z^{q-p} - z^{q-2p} - 1 = 0. \tag{59}$$

There are at most $q$ and $q - p$ distinct roots for (57) and (59) respectively. By (22) the branch of eigenvalues corresponding to the root $z_l$ of (57) or (59) is

$$\lambda_n = \frac{q}{2}(1 - v^2)[\ln|z_l| + i(\arg z_l + 2\pi n)], \quad n = 1, 2, 3, \ldots. \tag{60}$$

Each branch of eigenvalues in (60) lies on a straight line parallel to the imaginary axis and hence represents a constant rate of damping. Because $z = 1$ is a root of (57) or (59), the corresponding branch of eigenvalues is imaginary: $nq\pi(1 - v^2)i$ ($n = 1, 2, 3, \ldots$) by (22), in agreement with (21). Note that (60) is the exact solution to (48) when $k = 0$.

In addition to the branch of eigenvalues given by (56$a$, $b$), we show that all other eigenvalues of (50) are asymptotic to those of (51) determined by (59) and (60). Let

$$f(\lambda) = 2e^{2(1-d)\lambda} - e^{2(1-2d)\lambda} - 1$$

$$g(\lambda) = \frac{k}{\lambda}[e^{-2d\lambda} + e^{-2(1-d)\lambda} - e^{2(1-2d)\lambda} - 1]. \tag{61}$$

Equation (50) becomes $f(\lambda) + g(\lambda) = 0$. Each branch of zeros, $\sigma_n$, of $f(\lambda) = 0$ satisfies

$$e^{2\sigma_n/q} = z_n \quad 2z_n^{q-p} = z_n^{q-2p} + 1. \tag{62}$$

Define $C_n$ around $\sigma_n$ by $\lambda = \sigma_n + (1/\sqrt{|\sigma_n|})e^{i\theta}$, where $0 \le \theta \le 2\pi$. For any $\lambda$ on $C_n$, by using the Taylor expansion and (62) we obtain

$$|f(\lambda)| = \left|2z_n^{q-p}\left[1 + \frac{2(1-d)}{\sqrt{|\sigma_n|}}e^{i\theta}\right] - z_n^{q-2p}\left[1 + \frac{2(1-2d)}{\sqrt{|\sigma_n|}}e^{i\theta}\right] - 1 + O(|\sigma_n|)^{-1})\right|$$

$$= \frac{1}{\sqrt{|\sigma_n|}}|2dz_n^{q-2p} + 2(1-d)| + O(|\sigma_n|^{-1}) \tag{63}$$

$$|g(\lambda)| = \left|\frac{k}{\lambda}\left[z_n^{-p} + z_n^{q-p} - z_n^{q-2p} - 1 + O(|\sigma_n|^{-1/2})\right]\right|$$

$$= \frac{k}{|\sigma_n|}|z_n^{-p} - z_n^{q-p}| + O(|\sigma_n|^{-3/2}). \tag{64}$$

Because there are only a finite number of zeros $z_n$, $|2dz_n^{q-2p} + 2(1-d)|$ and $|z_n^{-p} - z_n^{q-p}|$ are bounded. It can be further shown that $|2dz_n^{q-2p} + 2(1-d)| \ne 0$ in (63). Hence $|g(\lambda)| < |f(\lambda)|$ on $C_n$ for sufficiently large $|\sigma_n|$. By Rouché's Theorem there is one solution $\lambda_n$ to (50) inside $C_n$ such that $|\lambda_n - \sigma_n| < 1/\sqrt{|\sigma_n|}$. On the other hand, if (50) has a branch of zeros $\lambda_l$ other than those of $\lambda_n$, let $F(\lambda) = f(\lambda) + g(\lambda)$ and $G(\lambda) = -g(\lambda)$. Because $-A < \text{Re}\lambda \le 0$, $e^{-2d\text{Re}\lambda}$ is bounded. Following the same approach we can show that $|G(\lambda)| < |F(\lambda)|$ on a disk $C_l$ around $\lambda_l$ for sufficiently large $|\lambda_l|$. Hence $F(\lambda) + G(\lambda) = f(\lambda)$ also has another branch of zeros around $\lambda_l$, which is impossible. Therefore eigenvalues of (50) of large modulus are asymptotic to those given by (59), (60) and (56$a$, $b$). Note that (56$a$, $b$) apply for both rational and irrational $d$.

*Case II: Irrational d.* For irrational $d$ all eigenvalues lie strictly within the left half-plane and the system is asymptotically stable. We will show that there are an infinite number of eigenvalues arbitrarily close to the imaginary axis, hence the system is not exponentially stable.

By Theorem 185 of Hardy and Wright (1979), every irrational number $0 < d < 1$ can be approximated by an infinite number of rational fractions $p/q$ such that $|p/q - d| < 1/q^2$. Hence we assume $d = p_n/q_n + b_n/q_n^2$ ($n = 1, 2, 3, \ldots$), where $p_n$ and $q_n$ are co-prime positive integers arranged in the ascending order of magnitude of $q_n$, and $|b_n| < 1$. Let

$$f(\lambda) = (c + 1)e^{2\lambda} - ce^{2[1-(p_n/q_n)\lambda]} - ce^{2(p_n/q_n)\lambda} + c - 1 \tag{65}$$

$$g(\lambda) = \frac{k}{\lambda}e^{2\lambda} - \frac{k}{\lambda}e^{2(1-d)\lambda} - \frac{k}{\lambda}e^{2d\lambda} + \frac{k}{\lambda} + ce^{2(1-p_n/q_n)\lambda}(1 - e^{-2(b_n/q_n^2)\lambda}) + ce^{2(p_n/q_n)\lambda}(1 - e^{2(b_n/q_n^2)\lambda}). \tag{66}$$

Equation (48) becomes $f(\lambda) + g(\lambda) = 0$. Because $f(\lambda) = 0$ has zeros $\sigma_n = lq_n\pi i$, where $l$ is any positive integer, we define $C_n$ around $\sigma_n$ by $\lambda = \sigma_n + (1/|\sigma_n|)e^{i\theta}$, where $0 \le \theta \le 2\pi$. For any $\lambda$ on $C_n$, using the Taylor expansion we have

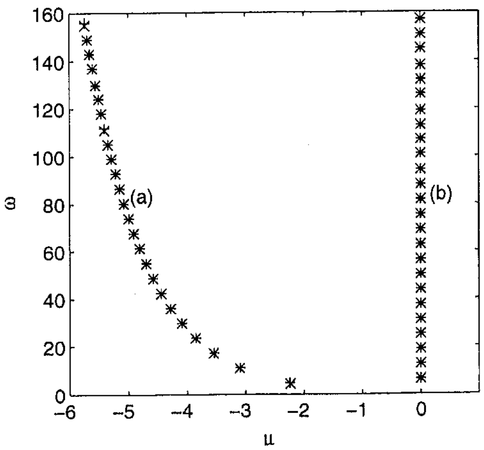


**Fig. 2 Distribution of the first 50 eigenvalues for $m = 0$, $d = \frac{1}{2}$, $v = 0.1$, and $c = 2$. (*a*) Numerical ("+") and asymptotic ("×") solutions for $k = 2$; (*b*) numerical solutions for $k = 2$ ("+") and $k = 0$ ("×").**

$$
\begin{aligned}
|f(\lambda)| &= |(c+1)e^{(2/|\sigma_n|)e^{i\theta}} - ce^{(1-p_n/q_n)(2/|\sigma_n|)e^{i\theta}} \\
&\quad - ce^{(p_n/q_n)(2/|\sigma_n|)e^{i\theta}} + c - 1| \\
&= \left|\frac{2}{|\sigma_n|}e^{i\theta} + O(|\sigma_n|^{-2})\right| \\
&= \frac{2}{|\sigma_n|} + O(|\sigma_n|^{-2}) \qquad (67)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{k}{\lambda}[e^{2\lambda} - e^{2(1-d)\lambda} - e^{2d\lambda} + 1] \\
&= \frac{k}{\lambda}[e^{(2/|\sigma_n|)e^{i\theta}} - e^{[1-(p_n/q_n)](2/|\sigma_n|)e^{i\theta}} \times e^{-2(b_n/q_n^2)[\sigma_n + (1/|\sigma_n|)e^{i\theta}]} \\
&\quad - e^{(p_n/q_n)(2/|\sigma_n|)e^{i\theta}} e^{2(b_n/q_n^2)[\sigma_n + (1/|\sigma_n|)e^{i\theta}]} + 1] \\
&= \frac{k}{\lambda}\left[\frac{4b_n\pi^2 l^2 i}{|\sigma_n|^2}\left(1 - 2\frac{p_n}{q_n}\right)e^{i\theta} + \frac{4}{|\sigma_n|^2}e^{2i\theta} + O(|\sigma_n|^{-3})\right] \\
&= O(|\sigma_n|^{-3}) \qquad (68)
\end{aligned}
$$

$$
\begin{aligned}
&ce^{2[1-(p_n/q_n)]\lambda}(1 - e^{-2(b_n/q_n^2)\lambda}) + ce^{2(p_n/q_n)\lambda}(1 - e^{2(b_n/q_n^2)\lambda}) \\
&= c\left[-2\pi^2\frac{b_n l^2}{\sigma_n} + O(|\sigma_n|^{-2})\right] \\
&\quad + c\left[2\pi^2\frac{b_n l^2}{\sigma_n} + O(|\sigma_n|^{-2})\right] = O(|\sigma_n|^{-2}). \qquad (69)
\end{aligned}
$$

Hence $|g(\lambda)| = O(|\sigma_n|^{-2}) < |f(\lambda)|$ on $C_n$. By Rouché's Theorem, there exists one solution $\lambda_n$ to (48) inside $C_n$, that is $|\lambda_n - \sigma_n| < 1/|\sigma_n| = 1/lq_n\pi$. Because there are an infinite number of distinct $q_n$, $q_n \to \infty$ as $n \to \infty$. Therefore $\text{Re}\lambda_n \to \text{Re}\sigma_n = 0$ as $n \to \infty$. As noted, $\sigma_n = lq_n\pi i$ ($l = 1, 2, 3, \ldots$) correspond to the infinite number of eigenvalues on the imaginary axis when $d$ is approximated by the rational fraction $p_n/q_n$. As $n \to \infty$, $p_n/q_n \to d$ and $\sigma_n$ ($l = 1, 2, 3, \ldots$) approach those corresponding to the irrational $d$.

## 4 Examples and Discussions

Eigenvalues for different cases are calculated numerically from (16) and compared with the analytical solutions in Section 3.

In the first example $m = 0$, $d = \frac{1}{2}$, $v = 0.1$, and $c = 1$. When $k = 0$ the locations of the first 50 eigenvalues agree with the exact solutions in (24*a*) and (32*b*). One branch of eigenvalues lies on the imaginary axis: $2n\pi(1 - v^2)i = 6.22ni$ ($n = 1, 2, 3, \ldots$). They correspond to the even-numbered modes with nodal points at $d = \frac{1}{2}$. The other branch of eigenvalues given by $(1 - v^2)[\ln(2 - c)/(2 + c) + (2n - 1)\pi i] = -1.09 + 3.11(2n - 1)i$ ($n = 1, 2, 3, \ldots$) corresponds to the odd-numbered modes. These eigenvalues have a constant real part $\mu = -1.09$. When $k = 2$ the branch of eigenvalues on the imaginary axis remains unchanged, while the other branch is quickly asymptotic to that for $k = 0$ (not shown here), in agreement with (33*b*).

In the example shown in Fig. 2, $c = 2$ and all other parameters remain the same as those in the previous example. The first 50 eigenvalues for $k = 2$ separate along two branches. The first branch lies on the imaginary axis as in the previous case. Locations of the eigenvalues on the second branch agree with the asymptotic solution in (38*a*, *b*). Unlike any other damping constant $c$ which would yield a nearly constant decay rate, rates of decay of the eigenvalues on the second branch increase monotonically. Hence $c = 2$ is optimal damping in this sense. When $k = 0$ as shown in Fig. 2, the second branch disappears and all eigenvalues lie on the imaginary axis, as predicted in Case III of Section 3.1.

The distribution of the first 50 eigenvalues for $m = v = d = 0.1$, $k = 2$, and $c = 1$ is shown in Fig. 3. The eigenvalues on the imaginary axis are $10n\pi(1 - v^2)i = 31.1ni$ ($n = 1, 2, \ldots, 5$), as predicted by (21). In addition there is a sequence of eigenvalues asymptotic to each eigenvalue on the imaginary axis, in agreement with (47*a*). Hence introduction of a small inertia $m$ alters the behavior of the spectrum significantly. For $m = 0.5$ and other parameters unchanged (not shown here), the eigenvalues approach the imaginary axis faster and all $\lambda_n$ for $n > 10$ are located close to the imaginary axis.

For $m = k = 0$, $d = \frac{1}{4}$, $v = 0.1$, and $c = 1$, the first 50 eigenvalues shown in Fig. 4 are in agreement with the exact solutions in (57) and (60). The roots of (57) are 1, −5.523, and $-0.0572 \pm 0.7748i$. By (60) the resulting four branches of eigenvalues are: $12.44ni$, $-1.18 + 6.22(2n + 1)i$, $-0.50 + (3.26 + 12.44n)i$, and $-0.50 + (9.18 + 12.44n)i$ ($n = 1, 2, \ldots$). They are distributed along three $\mu$ = constant lines because the last two branches are both located on $\mu = -0.50$. When $k = 2$ with other parameters unchanged, eigenvalues of high modes are asymptotic to those corresponding to $k = 0$, as predicted by (49).

In Fig. 5 are shown the first 50 eigenvalues for $c = 2$ and other parameters same as those in Fig. 4. The roots of (58) are 1 and $-0.25 \pm 0.6614i$. The corresponding branches of eigenvalues for $k = 0$: $12.44ni$, $-0.686 + (3.83 + 12.44n)i$, and $-0.686 + (8.61 + 12.44n)i$ ($n = 1, 2, 3, \ldots$) by (60) are distributed along two lines $\mu = 0$ and $\mu = -0.686$, as shown

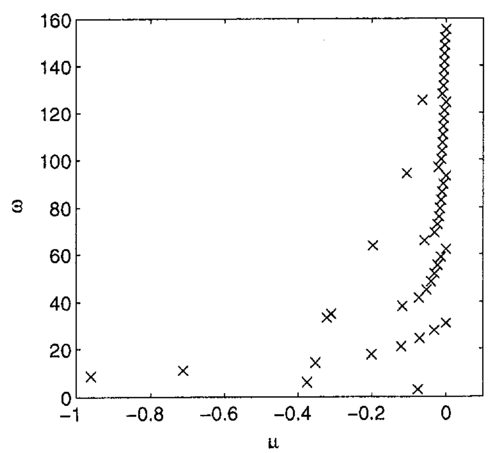


**Fig. 3 Distribution of the first 50 eigenvalues for $m = v = d = 0.1$, $k = 2$, and $c = 1$**

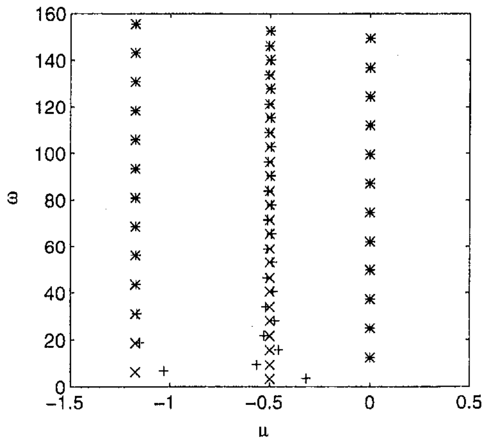


**Fig. 4 Distribution of the first 50 eigenvalues for $m = 0$, $d = \frac{1}{4}$, $v = 0.1$, and $c = 1$. (1) $k = 0$ ("×"); (2) $k = 2$ ("+").**

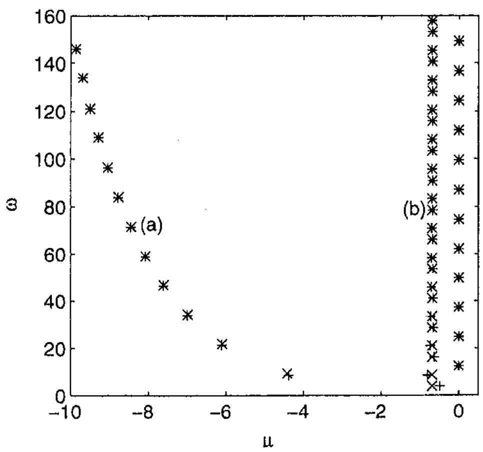


**Fig. 5 Distribution of the first 50 eigenvalues for $c = 2$ and other parameters same as those in Fig. 4. (*a*) Numerical ("+") and asymptotic ("×") solutions for $k = 2$; (*b*) numerical solutions for $k = 2$ ("+") and $k = 0$ ("×").**

in Fig. 5. When $k = 2$, the branch of eigenvalues on the imaginary axis is unchanged, and the other branch is asymptotic to that corresponding to $k = 0$, as expected. In addition to those two branches, there is a branch of eigenvalues shown in Fig. 5($a$) with increasing rates of decay $|\mu|$, as predicted by (56$a$, $b$).

## 5 Conclusions

1 When $c \neq 0$ the constrained translating string is asymptotically stable if and only if $d$ is irrational. However, even for irrational $d$, there are an infinite number of eigenvalues approaching the imaginary axis. Hence the system is not exponentially stable. If $d = p/q$ is rational, where $p$ and $q$ are co-prime, the branch of eigenvalues on the imaginary axis is given by $nq\pi(1 - v^2)i$ ($n = 1, 2, \ldots$).

2 When $m \neq 0$ and $c$ and $k$ are arbitrary, eigenvalues of the high modes are asymptotic to $(n\pi/d)(1 - v^2)i$ and $[n\pi/(1 - d)](1 - v^2)i$ ($n = 1, 2, \ldots$). The asymptotic behavior of the eigenvalues for sufficiently high modes is independent of $m$, $c$, and $k$.

3 For $d = p/q$, $m = k = 0$, and $c \neq 2$, the exact solutions for the eigenvalues are given by (57) and (60). All eigenvalues are distributed along the imaginary axis and along at most $q - 1$ lines of constant $\mu = \text{Re}\lambda$ in the left half-plane. The distribution of the eigenvalues for nonzero $k$ is asymptotic to that corresponding to $k = 0$. Hence the asymptotic locations of the eigenvalues are independent of $k$.

4 $c = 2$ is a special damping constant when $m = 0$. If $d$ is rational, the exact eigenvalues for $k = 0$ are given by (58) and (60). They are distributed along the imaginary axis and along a maximum number of $q - p - 1$ lines of constant $\mu = \text{Re}\lambda$ in the left half-plane. The vibration corresponding to the other $p$ branches of eigenvalues is dissipated by damping in finite time. When $k \neq 0$, in addition to the branch of eigenvalues in (56$a$, $b$) which has monotonically increasing decay rates, all other eigenvalues are asymptotic to those corresponding to $k = 0$.

## Acknowledgments

This work was supported by the Engineering Direct Grant and the Summer Research Grant of the Chinese University of Hong Kong.

## References


Bellman, R., and Cooke, K., 1963, *Differential-Difference Equations*, Academic Press, New York.

Carrier, G. F., Krook, M., and Pearson, C. E., 1983, *Functions of a Complex Variable: Theory and Technique*, Hod Books, Ithaca, NY.

Hardy, G. H., and Wright, E. M., 1979, *An Introduction to the Theory of Numbers*, 5th ed, Oxford University Press, Oxford, UK.

Perkins, N. C., 1990, "Linear Dynamics of a Translating String on an Elastic Foundation," *Journal of Vibration and Acoustics*, Vol. 112, No. 5, pp. 2–7.

Sack, R. A., 1954, "Transverse Oscillations in Travelling Strings," *British Journal of Applied Physics*, Vol. 5, pp. 224–226.

Wickert, J. A., and Mote, C. D. Jr., 1988, "Current Research on the Vibration and Stability of Axially Moving Materials," *Shock and Vibration Digest*, Vol. 20, No. 5, pp. 3–13.

Yang, B., 1992, "Eigenvalue Inclusion Principles for Distributed Gyroscopic Systems," ASME Journal of Applied Mechanics, Vol. 59, pp. 425–443.

Yang, B., and Mote, C. D., Jr., 1991a, "Active Vibration Control of the Axially Moving String in the *s* Domain," ASME Journal of Applied Mechanics, Vol. 58, pp. 189–196.

Yang, B., and Mote, C. D., Jr., 1991b, "Controllability and Observability of Distributed Gyroscopic Systems," ASME *Journal of Dynamic Systems, Measurement and Control*, Vol. 113, pp. 11–17.

Zhu, W. D., and Mote, C. D., Jr., 1995, "Propagation of Boundary Disturbances in an Axially Moving Strip in Contact with Rigid and Flexible Constraints," ASME Journal of Applied Mechanics, Vol. 62, pp. 873–879.