Local null controllability for a chemotaxis system of parabolic–elliptic type

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\section*{ARTICLE INFO}

Article history:
Received 18 April 2013
Received in revised form
17 October 2013
Accepted 28 October 2013

Keywords:
Local null controllability
Chemotaxis system
Parabolic–elliptic type
Kakutani’s fixed-point theorem

\section*{ABSTRACT}

We consider the controllability of a chemotaxis system of parabolic–elliptic type. By linearizing the nonlinear system into two separate linear equations, we can bypass the obstacle caused by the nonlinear drift term and establish local null controllability of the original nonlinear system. This approach is different from the usual method for dealing with coupled parabolic systems.

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1. Introduction and main result

We consider null controllability for the following controlled system of parabolic–elliptic type with initial boundary values:

\begin{equation}
\begin{aligned}
\partial_t u &= \nabla \cdot (\nabla u - \chi u \nabla v) + 1_{\omega} f &\text{in } \Omega \times (0,T), \\
\partial_t v &= \Delta v - \gamma v + \delta u &\text{in } \Omega \times (0,T), \\
\partial_n u &= 0, \quad \partial_n v &= 0 &\text{on } \partial \Omega \times (0,T), \\
u(x,0) &= u_0(x) &\text{for } x \in \Omega,
\end{aligned}
\end{equation}

where \( u \) and \( v \) denote states \( u(x,t) \) and \( v(x,t) \) at spatial position \( x \in \Omega \) and time \( t \geq 0 \), \( \partial_n \) denotes the outward normal derivative along the outward unit normal vector \( \nu \) on \( \partial \Omega \), \( 1_{\omega} \) denotes the characteristic function of \( \omega \), \( f \) is the control function, \( u_0 \) is the initial value, and \( \chi, \gamma, \) and \( \delta \) are given positive constants.

System (1.1) without control (i.e., \( f \equiv 0 \)) is a simple chemotaxis system used by Keller and Segel as a model to describe the aggregation process in slime mold morphogenesis, assuming that the cells directly emit a chemoattractant that immediately diffuses [1]. The unknown function \( u \) represents the cell density and \( v \) is the chemoattractant concentration. The Keller–Segel chemotaxis model was validated by experiments on \textit{Escherichia coli} bacteria and many other interesting physical phenomena. It has also been extensively applied in many medical and biological applications and relevant areas such as ecology and environmental sciences. Because the model has a rich structure from a mathematical perspective, many challenging problems can be addressed. Some interesting cases have been studied, such as aggregation, blow-up of solutions, and chemotactic collapse. Some significant results have been achieved from different perspectives, as reviewed elsewhere [2,3].

Here we study the Keller–Segel system from a control point of view. We say that system (1.1) is \textit{locally null controllable at time} \( T \) if there exists a neighborhood of the origin such that for any initial data \( u_0 \) belonging to this neighborhood, there exists a control \( f \) such that the solution \((u, v)\) of (1.1) satisfies \( u(x, T) = 0 \) for almost all \( x \in \Omega \). Physically, null controllability means that the cell density can be driven to zero in finite time. Here we consider the local rather than the global exact null controllability. The reason for this is that solutions of the Keller–Segel system may blow up in either finite or infinite time, as previously shown for the 3D case [4]. The 2D case is even more interesting and attractive. For the latter case, it has been found that a global solution exists in finite time when...
the initial mass is less than a threshold value, while the solution blows up in either finite or infinite time when the initial mass is greater than the threshold value [5].

The study of controllability for parabolic equations is an area of active research [6,7]. There is special interest in the controllability of coupled parabolic systems with control imposed on one equation. This is the most significant case in engineering applications and has attracted wide attention, as reviewed by Ammar-Khodja et al. [8].

However, to the best of our knowledge, very few results are available on control problems for Keller–Segel system (1.1) for which a parabolic equation is coupled to an elliptic equation through a drift term. Null controllability for a nonlinear parabolic–elliptic system of the following form was recently considered [9]:

\[
\begin{align*}
\partial_t y - \Delta y &= F(y,z) + 1_y f, & \text{in } \Omega \times (0,T), \\
-\Delta z &= f(y,z), & \text{in } \Omega \times (0,T),
\end{align*}
\]

where \( F(y,z) \) and \( f(y,z) \) are nonlinear terms. The authors considered controllability of the system as a limit of the following parabolic system as \( \varepsilon \to 0 \):

\[
\begin{align*}
\partial_t y - \Delta y &= \varepsilon F(y,z) + \varepsilon_1 f, & \text{in } \Omega \times (0,T), \\
\varepsilon \partial_t z - \Delta z &= \varepsilon f(y,z), & \text{in } \Omega \times (0,T).
\end{align*}
\]

Ryu and Yagi considered an optimal control problem for system (1.1) with control imposed on the second equation [10]. Their previous study was the first work to consider local exact controllability of the following parabolic–parabolic type of Keller–Segel system [11]:

\[
\begin{align*}
\partial_t u &= \nabla \cdot (\nabla u - \chi u \nabla v) + 1_y f, & \text{in } \Omega \times (0,T), \\
\partial_t v &= \Delta v - \gamma v + \delta u, & \text{in } \Omega \times (0,T).
\end{align*}
\]

In the present study, we attempt to establish controllability of system (1.1), which is probably the first work for this type of Keller–Segel system. Since the drift term \( -\chi \nabla \cdot (u \nabla v) \) in (1.1) destroys some good properties of the diffusion operator that usually ensure regularity of the solution, some mathematical techniques that are much more difficult than the aforementioned coupled parabolic systems occur. These include the regularity of the solution and estimation of an observability inequality, among many others. The usual way to establish controllability of a nonlinear system is to linearize the nonlinear system into some coupled linear systems. Then, for the controllability result established for the linearized system, some fixed-point results can be applied to establish controllability for the nonlinear system. The approach for establishing controllability of systems (1.2) and (1.3) is along these lines and has been described elsewhere [9,11]. However, here we investigate the controllability of system (1.1) in a very different way inspired intuitively by its special mathematical structure. We decompose this nonlinear system into two separate linear ones: one is a controlled parabolic system and the other is an irrelevant elliptic equation. In this way we can bypass the obstacle caused by the nonlinear drift term. The reason behind this is that (1.1) can be considered as a scalar parabolic equation with a nonlinear and nonlocal first-order term as follows:

\[
\begin{align*}
\partial_t u &= \Delta u - \nabla \cdot (u \nabla (-\Delta u + \gamma u)^{-1} u_1) + 1_y f, & \text{in } \Omega \times (0,T), \\
\partial_t u &= 0 & \text{on } \partial \Omega \times (0,T).
\end{align*}
\]

This point of view was also adopted for the controllability of a scalar conservation law with nonlocal velocity [12]. The techniques we apply here would be useful for other coupled systems such as drift–diffusion equations for semiconductor devices.

In this section, we present some regularity results for linear equations of both parabolic and elliptic types. We first consider the well-posedness of the following linear elliptic equation:

\[
\begin{align*}
\partial_t u &= \Delta u - \gamma v + \delta \eta, & \text{in } \Omega, \\
\partial_t u &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

where \( \gamma \) and \( \delta \) are positive constants. The following proposition is from Agmon et al. [14].

**Proposition 2.1.** For any \( \eta \in L^p(\Omega), p > 1, (2.1) \) admits a unique solution \( v \in W^{2,p}(\Omega) \) with

\[
\| v \|_{W^{2,p}(\Omega)} \leq C |\eta|_p.
\]

Next we consider the parabolic equation

\[
\begin{align*}
\partial_t u &= \Delta u - \nabla \cdot (Bu) + F, & \text{in } Q, \\
\partial_t u &= 0 & \text{on } \Sigma, \\
u(x,0) &= u_0(x) & x \in \Omega.
\end{align*}
\]
Proposition 2.2. Let $B \in L^\infty(Q)^N$ with $B \cdot v = 0$ on $\Sigma$, $F \in L^\infty(Q)$, and $u_0 \in L^\infty(\Omega)$. Then (2.2) admits a (weak) solution $u \in W(0, T) \cap L^\infty(Q)$ with
\[ \|u\|_\infty \leq e^{\varrho_0} (\|u_0\|_\infty + \|F\|_\infty), \]
where $\varrho = C(\Omega)$ is a positive constant depending on $\Omega$, and $\varrho_0$ is given by
\[ \varrho_0 = (1 + \|B\|_\infty^2) (1 + T). \]

A similar inequality to (2.3) was proposed by Ladyzhenskaya et al. [13, p. 181], but here we improve the estimate so that it depends on time $T$ explicitly. To prove Proposition 2.2, we need the following lemma [13, Lemma 5.6, p. 95].

Lemma 2.1. Suppose that a sequence $Y_s, s = 0, 1, 2, \ldots$ of non-negative numbers satisfies the recursion relation
\[ Y_{s+1} \leq cB^s Y_s^{1+s}, \quad s = 0, 1, 2, \ldots \]
with some positive constants $c, \epsilon$, and $b > 1$. If $Y_0 \leq c^{-1} b^{-1 - \epsilon}$, then $Y_s \to 0$, $s \to \infty$.

Proof of Proposition 2.2. Let $(u - k)_+ = \max(u - k, 0)$ and $A_\kappa(t) = \max\{x \in \Omega : u(x, t) > k\}$ for $t \in [0, T]$, where
\[ k \geq K_0 = \|u\|_\infty + \|u_0\|_\infty. \]

Multiplying both sides of (2.2) by $(u - k)_+$, integrating by parts, and applying the Hölder inequality, we obtain
\[ \frac{d}{dt} \int_\Omega |(u - k)_+|^2 dx + \int_\Omega |\nabla (u - k)_+|^2 dx \leq \|B\|_\infty^2 \int_{A_\kappa(t)} u^2 dx + \int_\Omega |(u - k)_+|^2 dx + \int_{A_\kappa(t)} F^2 dx \]
\[ \leq (2 \|B\|_\infty^2 + 1) \int_{A_\kappa(t)} |(u - k)_+|^2 dx + \int_{A_\kappa(t)} k^2 dx \]
From Gronwall’s inequality, it follows that
\[ \int_\Omega |(u - k)_+|^2 dx + \int_0^t \int_\Omega |\nabla (u - k)_+|^2 dx dt \leq e^{\varrho_0} \int_0^T \int_{A_\kappa(t)} k^2 dx dt \]
for all $t \in [0, T]$, where $\varrho_0$ is the constant defined by (2.4). However, by Proposition I.3.2 of DiBenedetto [15],
\[ \|u\|_{2N+2} \leq C(\Omega)(1 + T) \frac{N}{2N+2} \|v\|_{V_2(\Omega)}, \quad \forall v \in V_2(Q), \]
(2.7)
where $V_2(Q) = L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ is endowed with its graph norm. By (2.6) and (2.7), we have
\[ \|u - k\|_{2N+2} \leq e^{\varrho_0} \int_0^T \int_{A_\kappa(t)} k^2 dx dt. \]
Let $\varphi(k) = \max\{x \in \Omega : u(x, t) > k\}$. For any $h > k$, it follows from (2.8) that
\[ (h - k)^2 \varphi(h) \frac{h^2}{N} \leq \|u - k\|_{2N+2} \leq e^{\varrho_0} \varphi(k) k^2. \]
This gives
\[ \varphi(h) \leq e^{\varrho_0} \left( \frac{k}{h - k} \right)^{\frac{2N+2}{h^2/N}} \varphi(k) k^s. \]
Next we set $Y_s = \varphi(k_s), k_s = M(2 - 1/\nu)$ and substitute $h = k_{s+1}$ and $k = k_s$ in (2.9) to obtain
\[ Y_{s+1} \leq \tilde{c} T^s \cdot Y_s^{1+s}, \]
where $r = 2(N + 2)/\nu, \nu = 2/N$, and $\tilde{c} = e^{\varrho_0}$. By Lemma 2.1, $\varphi(2M) = 0$ provided that
\[ Y_0 = \varphi(k_0) = \varphi(M) \leq \left( \tilde{c} T^s \right)^{-1} (2^s)^{1/s} \]
(2.10)
for some positive real number $M$. To determine the value of $M$, let $m > 1$ be an integer and $M = mk_0$. Then $k_0$ is given by (2.5). Let $h = M = mk_0$ and $k = k_0$ in (2.9) to obtain
\[ \varphi(M) \leq \tilde{c} \left( \frac{1}{m - 1} \right)^T \varphi(k_0)^{1+\epsilon} \]
\[ \leq \tilde{c} \left( \frac{1}{m - 1} \right)^T (1 + \epsilon) \left( \text{meas}(\Omega) \right)^{1+\epsilon}. \]
(2.11)
Now we choose a suitable $m$ such that (2.10) holds. By (2.10) and (2.11), we can choose an integer $m$ so that
\[ m \geq 1 + \tilde{c} \frac{T^{1+\epsilon}}{T^{1+\epsilon}} \left( \text{meas}(\Omega) \right)^{1+\epsilon} 2^{s+1/\nu}. \]
In this way, $\varphi(2M) = \varphi(2mk_0) = 0$. This gives
\[ u \leq 2mk_0 \leq e^{\varrho_0} (\|u\|_\infty + \|u_0\|_\infty). \]
Applying a similar argument, we can also obtain the other half of (2.3) for $-u$. This completes the proof. □

3. Null controllability for a linear parabolic equation

In this section, we consider null controllability for the following linear parabolic equation:
\[ \begin{align*}
\partial_t u - \Delta u - \nabla \cdot (Bu) + \chi u &= f(x) \quad \text{in } Q, \\
\partial_n u &= 0 \quad \text{on } \Sigma, \\
u(x, 0) &= u_0(x) \quad \text{for } x \in \Omega.
\end{align*} \]
(3.1)

Theorem 3.1. Let $T > 0$ and $B \in L^\infty(Q)^N$ with $B \cdot v = 0$ on $\Sigma$. For any $u_0 \in L^2(\Omega)$, there exists a control $f \in L^\infty(Q_0)$ such that the solution $u$ of system (3.1) corresponding to $f$ satisfies $u \in W(0, T)$ and $u(x, T) = 0$ for $x \in \Omega$ almost everywhere. Moreover, the control $f$ satisfies
\[ \|\chi u\|_\infty \leq e^\kappa \|u_0\|_2, \]
(3.2)
where
\[ \kappa = (1 + \|B\|_\infty^2) (1 + T) + \frac{1}{T}. \]
(3.3)

To prove Theorem 3.1, we need to establish a type of observability inequality for the following system, which is the dual of system (3.1):
\[ \begin{align*}
-\partial_t \phi &= \Delta \phi + B \cdot \nabla \phi \quad \text{in } Q, \\
\partial_n \phi &= 0 \quad \text{on } \Sigma, \\
\phi(x, T) &= \phi^T(x) \quad \text{in } \Omega,
\end{align*} \]
(3.4)
where $\phi^T \in L^2(\Omega)$. By Lemma 1.1 of Fursikov and Imanuvilov [7], there is a function $\beta \in C^2(\Omega)$ such that $\beta(x) > 0$ for all $x \in \Omega$, $\beta|_{\partial \Omega} = 0$, and $|\nabla \beta(x)| > 0$ for all $x \in \Omega \setminus \omega$. For $\lambda > 0$, we set
\[ \varphi = e^{\lambda \beta}, \quad \alpha = e^{\lambda \beta} - e^{2\lambda \beta|_{\partial \Omega}}. \]
(3.5)
We then have a Carleman inequality.
Lemma 3.1. There exists a constant \( \lambda_0 = \lambda_0(\Omega, \omega) > 1 \) such that for all \( \lambda \geq \lambda_0 \) and \( s \geq \gamma(\lambda)(T + T^2) \),
\[
\int_Q [s\phi|\nabla \phi|^2 + (s\phi)^3|\phi|^2] e^{2su} \, dx \, dt \\
\leq C \int_Q e^{2su}|\partial_t \phi \pm \Delta \phi|^2 \, dx \, dt + C \int_Q (s\phi)^3 e^{2su}|\phi|^2 \, dx \, dt,
\]
\( \forall \, y \in X = \{ x \in \mathbb{C}^2{:} (Q_1)|\partial_0 \phi = 0 \text{ on } \Sigma \} \),
where \( C = C(\Omega, \omega) \) and \( \gamma(\lambda) \) is given by
\[
\gamma(\lambda) = e^{2\kappa ||\phi||_{L^\infty}}. \tag{3.7}
\]

A similar result for Lemma 3.1 was described by Fursikov and Imanuvilov [7], but the explicit dependence of \( C \) on \( \Omega \) and \( \omega \) can be specified according to a previous argument [16] in obtaining a similar inequality for a heat equation with the Dirichlet boundary condition. The following proposition is an observability inequality for the adjoint equation (3.4).

Proposition 3.1. Let \( \delta_0 \in (1, 2) \). There exist positive constants \( \lambda \) and \( s \) such that for all \( T > 0 \) and \( \phi^T \in L^2(\Omega) \), the solution \( \phi \) to (3.4) satisfies
\[
|\phi(\cdot, 0)_T|^2 \leq e^{\kappa} \int_{Q_0} e^{2su}|\phi|^2 \, dx \, dt,
\]
where \( \kappa \) is given by (3.3).

Proof. First, by Lemma 3.1, there exists a positive constant \( \lambda_1 = \lambda_1(\Omega, \omega) \) such that for any \( \lambda \geq \lambda_1, s \geq \gamma(\lambda)(T + T^2) \), and \( \phi^1 \in L^2(\Omega) \), the solution \( \phi \) to (3.4) satisfies
\[
\int_Q [s\phi|\nabla \phi|^2 + (s\phi)^3|\phi|^2] e^{2su} \, dx \, dt \\
\leq C||B||_\infty \int_Q e^{2su}|\nabla \phi|^2 \, dx \, dt + C \int_Q (s\phi)^3|\phi|^2 e^{2su} \, dx \, dt,
\]
where \( \gamma(\lambda_1) \) and \( \gamma(\lambda) \) are given by (3.7). Here and in what follows, \( C = C(\Omega, \omega) \) denotes a positive constant depending only on \( \Omega \) and \( \omega \) whose value may change from one position to another. We choose \( \lambda_2 = C(1 + ||B||_\infty^2) \) so that for any \( \lambda > \lambda_2 > \lambda_1 \) and \( s > \gamma(\lambda)(T + T^2) \), \( C||B||_\infty^2 \leq s\phi \). Hence,
\[
\int_Q [s\phi|\nabla \phi|^2 + (s\phi)^3|\phi|^2] e^{2su} \, dx \, dt \\
\leq C \int_Q (s\phi)^3|\phi|^2 e^{2su} \, dx \, dt. \tag{3.9}
\]

By (3.4),
\[
\frac{d}{dt} (e^{s|\phi|^T} |\phi|^2) \geq 0.
\]
This gives
\[
|\phi(\cdot, 0)_T|^2 \leq e^{s|\phi|^T} \int_{Q_0} |\phi|^2 \, dx \, dt \quad \forall \, t \in [0, T]. \tag{3.10}
\]
Integrating on both sides of (3.10) over [T/4, 3T/4], we obtain
\[
|\phi(\cdot, 0)_T|^2 \leq \frac{2}{T} \int_{T/4}^{3T/4} \int_Q |\phi|^2 \, dx \, dt. \tag{3.11}
\]
Since \((s\phi)^{-1} e^{-2su} \leq e^{C/T^2} \in \Omega \times \lbrack T/4, 3T/4\], inequality (3.8) then follows from (3.9) and (3.11) if \( \lambda \) and \( s \) are large enough. □

Proof of Theorem 3.1. Let \( \varepsilon > 0 \). We set
\[
J_\varepsilon(u, f) = \frac{1}{2} \int_{Q_0} |T|^2 e^{-\varepsilon u} \, dx \, dt + \frac{1}{2\varepsilon} \int_{\Omega} |u(x, T)|^2 \, dx
\]
and consider the extremal problem for
\[
\inf_{(u_\varepsilon, f_\varepsilon)} J_\varepsilon(u, f). \tag{3.12}
\]
where \( u \) is the totality of \( (u, f) \in W(0, T) \times L^2(\Omega) \) solving (3.1). The existence of an optimal pair \((u_\varepsilon, u_\varepsilon)\) for this extremal problem follows from a standard argument. By the maximum principle [17], we obtain the optimality system for this problem as follows:
\[
\begin{align*}
-\partial_t \phi_\varepsilon &= \Delta \phi_\varepsilon + B \cdot \nabla \phi_\varepsilon \quad \text{in } Q, \\
\partial_\varepsilon \phi_\varepsilon &= 0 \quad \text{on } \Sigma, \\
\phi_\varepsilon(x, T) &= -\frac{1}{\epsilon} u_\varepsilon(x, T) \quad \text{for } x \in \Omega, \\
\partial_\varepsilon u_\varepsilon &= \Delta u_\varepsilon - \nabla \cdot (Bu_\varepsilon) + 1_{u_\varepsilon} \, f_\varepsilon \quad \text{in } Q, \\
\partial_\varepsilon u_\varepsilon &= 0 \quad \text{on } \Sigma, \\
u_\varepsilon(x, 0) &= u_\varepsilon(x, 0) = 0 \quad \text{on } Q \setminus \Sigma, \\
f_\varepsilon &= 1_{u_\varepsilon} \phi^\varepsilon \quad \text{on } \Sigma.
\end{align*}
\]

Note that we can choose \( s \) and \( \lambda \) such that (3.8) holds and
\[
\omega(\lambda) = e^{-\kappa ||\phi||_{L^\infty}} < \delta_0 - 1. \tag{3.16}
\]
Furthermore, by (3.13), (3.14), (3.15), and Proposition 3.1, we obtain
\[
\int_{Q_0} |\phi_\varepsilon|^2 e^{2su} \, dx \, dt + \frac{1}{\varepsilon} \int_{\Omega} |u_\varepsilon(x, T)|^2 \, dx \leq e^{\kappa} |u_\varepsilon_0|^2. \tag{3.17}
\]
This inequality together with (3.15) leads to \( ||1_{u_\varepsilon}f_\varepsilon||_2 \leq e^{\kappa} |u_\varepsilon_0|_2 \). That is, the control \( f_\varepsilon \) can be taken in \( L^2 \) space.

Now we show that \( f_\varepsilon \) can actually be taken in \( L^\infty \) space. To this end, we apply a bootstrap method [6, 18]. We first set \( \alpha_0 = \min_{\Omega} \alpha \). The following inequalities can be easily verified:
\[
\alpha_0 \leq \alpha \leq \alpha_0 + \omega(\lambda) < 0, \tag{3.18}
\]
where \( \omega(\lambda) \) is defined by (3.16). Second, we let \( r \) be a sufficiently small positive constant and let \( \{ r_i \}_{i=0}^M \) be a finite increasing sequence so that \( 0 < r_j < r, j = 0, 1, \ldots, M, \tau_M = r \). Let \( \{ p_i \}_{i=0}^M \) be another finite increasing sequence so that \( p_0 = 2, p_M = \infty \) and
\[
\left( \frac{N}{2} + 1 \right) \left( \frac{1}{p_i} - \frac{1}{p_{i+1}} \right) + 1 > \frac{1}{2}, \quad i = 0, 1, \ldots, M - 1. \tag{3.19}
\]
For each \( i = 0, 1, \ldots, M, \) we define
\[
z_i(x, t) = e^{(s + r_i)u_\varepsilon} \phi_\varepsilon(x, T - t), \\
f_i(x, t) = \left( \frac{1}{p_{i+1}} \right) \phi_\varepsilon(x, T - t), \\
\hat{B}(x, t) = B(x, T - t).
\]
Then the adjoint equation (3.13) can be transformed into the following initial–boundary problem for every \( i = 0, 1, \ldots, M \):
\[
\begin{align*}
\partial_t z_i - \Delta z_i &= \hat{B} \cdot \nabla z_i + f_i \quad \text{in } Q, \\
\partial_\varepsilon z_i &= 0 \quad \text{on } \Sigma, \\
z_i(x, 0) &= 0 \quad \text{on } Q, \\
z_i(x, 0) &= 0 \quad \text{on } \Sigma, \\
z_i(x, 0) &= 0 \quad \text{on } \Sigma.
\end{align*}
\]
Let \( \{ S(t) \}_{t \geq 0} \) be the semigroup generated by the Laplace operator with a Neumann boundary condition. It follows that [19]
\[
|S(t)|_{\mathcal{L}^p_\Omega} \leq C_m(t) \frac{1}{T^2} \left( \frac{1}{T^2} \right)^{1/2} |u|_{L^p}. \tag{3.21}
\]
for all $u \in L^2(\Omega)$, $t > 0$, and $1 < p \leq q \leq \infty$, where $m(t) = \min\{1, t\}$. Note that the solution $z_t$ of (3.20) can be represented as
\[ z_t(\cdot, t) = \int_0^t S(t - s) \left( B \cdot \nabla z_t + F_t \right)(\cdot, s) \, ds, \]
i = 0, 1, \ldots, M. \tag{3.22}

Applying (3.21) to (3.22) gives
\[ |z_t(\cdot, t)|_{p_i} \leq C \int_0^t m(t - s) \frac{\delta}{(p_i - 1)} \left( B \cdot \nabla z_t + F_t \right)(\cdot, s) \, ds, \quad i = 0, 1, \ldots, M. \tag{3.23}

With (3.19), we apply Young’s convolution inequality [19] to the right-hand side of (3.23) to obtain
\[ |z_t(\cdot, t)|_{p_i} \leq e^{\delta(1 + t)} \left( ||B||_\infty ||\nabla z_t||_{p_{i-1}} + ||F_t||_{p_{i-1}} \right). \tag{3.24}

Applying a standard energy estimate to (3.20), we can obtain the following $L^p$-estimate for $z_t$:
\[ |z_t||_{p_{i-1}} + ||\nabla z_t||_{p_{i-1}} \leq e^{\delta t} ||F||_{p_{i-1}}. \tag{3.25}

By the definition of $F_t$ and (3.18),
\[ ||F||_{p_{i-1}} \leq CT |\nabla z_t|_{p_{i-1}}. \tag{3.26}

Combining (3.24), (3.25), and (3.26) gives
\[ |z_t||_{p_{i-1}} \leq e^{\delta t} ||\nabla z_t||_{p_{i-1}}. \tag{3.27}

This iteration inequality from 0 to $M$ yields
\[ |z_t||_{p_M} \leq e^{\delta(1 + t)} \left( ||B||_\infty ||\nabla z_t||_{p_0} + ||F_t||_{p_0} \right). \tag{3.28}

Thus this shows that $z_t$ can be applied in $L^\infty$ space.

Finally, by (3.29) we can extract a subsequence of $\{f_t\}_{t \geq 0}$, still denoted by itself, such that $1_{\Omega}f_t \to 1_{\Omega}f$ weakly in $L^2(\Omega)$ as $\varepsilon \to 0$. We denote by $u_\varepsilon$ the solution to system (3.14) corresponding to $f_t$. By Proposition 2.2, $\{u_\varepsilon\}_{\varepsilon > 0}$ is uniformly bounded in $W(0, T)$. Thus, we can extract a subsequence of $\{u_\varepsilon\}_{\varepsilon > 0}$, still denoted by itself, such that $u_\varepsilon \to u$ weakly in $W(0, T)$ for all $u \in W(0, T) \subset C([0, T]; L^2(\Omega))$. Such a $u$ is the weak solution of (3.1) corresponding to $f$. In addition, by (3.17), $u(x, T) = 0$ in $\Omega$ almost everywhere. This completes the proof. \qed

4. Proof of Theorem 1.1

Let $K = \{\xi \in L^\infty(\Omega) \mid ||\xi||_\infty \leq 1 \} \cap L^\infty(0, T; L^p(\Omega)) \subset L^2(\Omega)$, $p > \max\{N, 2\}$. For every $\xi \in K$, consider the following two linear equations:
\[ \begin{align*}
0 &= \Delta v(\cdot, t) - \nu v(\cdot, t) + \delta \xi(\cdot, t) & \text{in } \Omega, \\
\partial_t v(\cdot, t) &= 0 & \text{on } \partial \Omega \tag{4.1}
\end{align*} \]
and
\[ \begin{align*}
\partial_t u &= \Delta u - \nabla \cdot (B u) + 1_\Omega f & \text{in } Q, \\
\partial_t u &= 0 & \text{on } \Sigma, \\
u(x, 0) &= \nu_0(x) & x \in \Omega \tag{4.2}
\end{align*} \]
for almost every $t \in [0, T]$, where $B = \chi \nabla \xi$. In what follows, we denote by $v_t = v(\cdot, t)$ the unique solution to equation (4.1) corresponding to $\xi(\cdot, t)$ for $t \in [0, T]$. First, by Proposition 2.1, $v_t \in L^\infty(0, T; W^{1, p}(\Omega))$ for $p > \max\{N, 2\}$ provided that $\xi \in K$. Hence, embedding theory between Sobolev spaces for $p > N [15]$ implies that
\[ B = \chi \nabla v_t \in L^\infty(0, T; W^{1, p}(\Omega))^N \subset L^\infty(0, T; C(\Omega))^N \]
with $B \cdot v = 0$ on $\Sigma$

and hence
\[ ||B||_\infty = \chi ||\nabla v_t||_{L^\infty(0, T; W^{1, p}(\Omega))} \leq C ||\xi||_\infty \leq C. \tag{4.3} \]

We can then define a linear continuous operator $\Phi$ from $K$ to $L^\infty(0, T; C(\Omega))^N \subset L^\infty(\Omega)^N$ as
\[ \Phi(\xi) = B = \chi \nabla v_t, \quad \forall \xi \in K. \]

Second, by Theorem 3.1, for any $B \in L^\infty(\Omega)^N$ with $B \cdot v = 0$ on $\Sigma$, there exists a pair $(u, f) \in L^2(\Omega) \times L^2(\Omega)$ that solves (4.2) with $u(x, T) = 0$ for almost all $x \in Q$. Moreover, the control $f$ satisfies (3.2). By (4.3),
\[ ||1_{\Omega}f||_\infty \leq e^{\delta(x)} ||u_0||_2. \tag{4.4} \]

where $\delta(x) = 1 + T + \frac{1}{T}$ and $C > 0$ is a positive constant independent of $T$. By abuse of the notation, we still use $C$ in what follows to denote such a type of constant. By (2.5) in Proposition 2.2 and (4.4), we have the following estimate:
\[ ||u||_{L^\infty(0, T)} + ||u||_\infty \leq e^{\delta(x)} ||u_0||_\infty. \tag{4.5} \]

We then define a multi-valued mapping $\Psi : L^\infty(Q)^N \to 2^L(\Omega)$ by
\[ \Psi(B) = \left\{ u \in L^2(Q) \mid 3f \in L^\infty(Q), \text{satisfying (4.4) such that } u \text{ is the solution of (4.2) corresponding to } f \right\} \]
where $L^2(\Omega)$ denotes all subsets of $L^2(\Omega)$. Since both $\Phi$ and $\Psi$ are well defined, which is guaranteed by Proposition 2.1 and Theorem 3.1, we can set
\[ \Lambda = \Psi \circ \Phi : K \subset L^2(\Omega) \to 2^L(\Omega). \tag{4.6} \]

Now we apply Kakutani’s fixed-point theorem [17, p. 7] to the map $\Lambda$ to prove Theorem 1.1. In fact, it is clear that $K$ is a convex subset of $L^2(\Omega)$. By Proposition 2.1 and Theorem 3.1 again, for any $\xi \in K$, $\Lambda(\xi)$ is nonempty and convex owing to the linear nature of the equations. Moreover, from (4.6) it follows that for each $\xi \in K$, $\Lambda(\xi)$ is bounded in $L^2(\Omega)$, $\Lambda(\xi)$ is bounded in $L^2(\Omega)$, $\Lambda(\xi)$ is compact in $L^2(\Omega)$.

We claim that $\Lambda$ is upper semi-continuous. Let $\{\xi_n\}_{n=1}^\infty$ be a sequence of functions in $K$ such that
\[ \xi_n \to \xi \text{ strongly in } L^2(\Omega) \text{ as } n \to \infty. \tag{4.7} \]

For every $n$, let $B_n = \Phi(\xi_n) = \chi \nabla v_{\xi_n}$ and take $u_n \in \Lambda(\xi_n) = \Psi(B_n)$, where $v_{\xi_n}$ solves
\[ \begin{align*}
0 &= \Delta u_n(\cdot, t) - \nu v_{\xi_n}(\cdot, t) + \delta \xi_n(\cdot, t) & \text{in } \Omega, \\
\partial_t u_n(\cdot, t) &= 0 & \text{on } \Sigma \tag{4.9}
\end{align*} \]
for almost all $t \in [0, T]$ and $u_n$ solves
\[ \begin{align*}
\partial_t u_n &= \Delta u_n - \nabla \cdot (B_n u_n) + 1_{\Omega} f_n & \text{in } Q, \\
\partial_t u_n &= 0 & \text{on } \Sigma, \\
u_n(x, 0) &= \nu_0(x) & x \in \Omega \tag{4.10}
\end{align*} \]
for almost every $t \in [0, T]$. Hence, embedding theory between Sobolev spaces for $p > N$ [15]
with \( u_n(x, T) = 0 \) for almost all \( x \in \Omega \). Moreover, the control \( f_n \) satisfies
\[
\| u_n \|_{L^\infty} \leq e^{\varepsilon \lambda_0} | u_0 |_2.
\]
(4.11)
To show that \( \Lambda \) is upper semi-continuous, it suffices to prove that there exists a subsequence of \( \{ u_n \}_{n=1}^{\infty} \) that converges strongly to an element of \( \Lambda(\varepsilon) \) in \( L^2(\Omega) \) topology.

In what follows, we do not distinguish the sequence and its subsequence by abuse of the notation. First, the estimate (4.11) enables us to obtain a function \( f \in L^\infty(\Omega) \) and a subsequence of \( \{ u_n \}_{n=1}^{\infty} \) such that
\[
\| u_n \|_{L^\infty} \to 0\quad \text{weakly in } L^2(\Omega);\quad \text{weak* in } L^\infty(\Omega) \quad \text{as } n \to \infty.
\]
(4.12)
By (4.11) and Proposition 2.2, \( u_n \) satisfies (4.6); that is,
\[
\| u_n \|_{W^{2,1}(\Omega)} + \| u_n \|_{L^p(\Omega)} \leq e^{\varepsilon \lambda_0} | u_0 |_2.
\]
(4.13)
Applying the Aubin–Lions lemma again, we obtain \( u \in W(0, T) \cap L^\infty(\Omega) \) and a subsequence of \( \{ u_n \}_{n=1}^{\infty} \) such that
\[
u_n \to u \quad \text{weakly in } W(0, T);\quad \text{strongly in } L^2(\Omega), \quad \text{as } n \to \infty.
\]
(4.14)
Furthermore, by the strong convergence of \( \{ u_n \}_{n=1}^{\infty} \) in \( L^2(\Omega) \), we can extract a subsequence of \( \{ u_n \}_{n=1}^{\infty} \) such that
\[
u_n \to u \quad \text{almost everywhere in } \Omega \quad \text{as } n \to \infty.
\]
(4.15)
By Proposition 2.1, for any \( n \) and \( p > 1 \),
\[
\| v_n \|_{L^2(\Omega)} \leq C \| \xi_n \|, \quad \forall \xi \in [0, T] \quad \text{a.e.}
\]
(4.16)
Thus, \( u_n \) satisfies (4.3); that is,
\[
\| u_n \|_{L^\infty} \leq C \| \nabla v_n \|_{L^\infty(\Omega \times (0, T))} \leq C \| \xi_n \| \leq C.
\]
(4.17)
Let \( v \) be the unique solution of (4.1) corresponding to \( \xi \). By the linear nature of (4.1) and (4.16),
\[
\| v_n \|_{L^2(\Omega)} \leq C \| \xi_n \| \quad \forall \xi \in [0, T] \quad \text{a.e.}
\]
Since by (4.8), \( \xi_n \to \eta \) strongly in \( L^2(\Omega) \), it follows that
\[
\| v \|_{L^2(\Omega \times (0, T))} \leq C \| \xi \|_2 \to 0.
\]
(4.18)
This shows that
\[
\| u_n \|_{L^2(\Omega \times (0, T))} \leq \| v \|_{L^2(\Omega \times (0, T))} \leq C \| \xi \|_2 \to 0.
\]
(4.19)
It then follows from Grönwall’s lemma that
\[
\| u_n \|_{L^2(\Omega \times (0, T))} \leq e^{\varepsilon |u_0|_2, T} \left( C \| u \|_{L^p} \| B_n - B \|_2^2 + C \int_\Omega |v| \right).
\]
(4.21)
By (4.12), (4.17), and (4.18), we see that the right-hand side of (4.21) tends to 0 as \( n \to \infty \), and hence \( \| u \|_{L^2(\Omega \times (0, T))} \to 0 \) as \( u \to u \) for almost all \( x \in \Omega \). This shows that \( u \in \Psi(\Lambda) = \Lambda(\xi) \). Therefore, \( \Lambda \) is upper semi-continuous.

It remains to show that \( \Lambda(\Lambda(\xi)) \subset K \). By the standard energy estimate, we can show that for any \( \xi \in K \), every element of \( \Lambda(\xi) \) satisfies
\[
\| u \|_{L^2(\Omega \times (0, T))} \leq e^{\varepsilon |u_0|_2, T} \left( C \| u \|_{L^p} \| B_n - B \|_2^2 + C \int_\Omega |v| \right).
\]
This, together with (2.5) in Proposition 2.2 and (4.4), leads to \( u \in L^\infty(\Omega \times (0, T); L^p(\Omega)) \) and
\[
\| u \|_{L^2(\Omega \times (0, T))} + \| u \|_{L^\infty(\Omega \times (0, T))} \leq e^{\varepsilon |u_0|_2} | u_0 |_p.
\]
where \( c_1 \) is a positive constant independent of \( T \) and \( k \) is given by (4.5). If \( | u_0 |_p \leq e^{-kT} \), which is (4.1), then \( \| u \|_{L^\infty} \leq 1 \) and hence \( \Lambda(\Lambda(\xi)) \subset K \). We apply Kakutani’s fixed-point theorem to obtain at least one fixed point \( u \) of \( \Lambda \), that is, \( u = \Lambda(u) \). This \( u \), together with \( v = \sqrt{\mu} \) and the solution of (4.1) with \( \xi = w \), gives the solution \( u \) of (1.1) to satisfy \( u(x, T) \equiv 0 \) with some control \( f \). This completes the proof.

References