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The Existence of Optimal Solution for a Shape Optimization Problem on Starlike Domain

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Abstract In this paper, a shape optimization problem over a multi-dimensional starlike domain with boundary payoff is considered. The function, which characterizes the boundary of the domain with respect to some ball contained inside domain, is shown to be Lipschitz continuous. The existence of an optimal solution is proved.

Keywords Shape optimization · Starlike domain · Existence

1 Introduction

The existence theory for shape optimization problems has been studied extensively by many researchers. There are several types of results: using regularity assumptions for the boundary of the unknown domains [1–7], using certain capacitary constraints [1, 8, 9] or using the notion of a generalized perimeter and constraints or penalty terms constructed with it [4, 10–12]. In the second case, conditions on the dimension of the...
underlying Euclidean space have to be imposed in order to obtain the compactness of certain families of open sets with respect to the Hausdorff distance.

Motivated by [5–7], we study, in this paper, a shape optimization problem, with constraints by some penalty functions, and obtain the existence of the shape optimization.

2 Preliminaries

Let $D$ be an open and bounded set with Lipschitz boundary in $\mathbb{R}^N$, where $N$ is a given positive integer, and $B$ the $N$-dimensional ball, $D \subset B$. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set. $\Omega$ is said to be a starlike domain with respect to a point $x_0 \in \Omega$, if the straight line $\{x_0 + te: t \in \mathbb{R}\}$ intersects the boundary $\partial \Omega$ of $\Omega$ at exactly two points for any point $e \in \mathbb{R}^N$. Furthermore, $\Omega$ is said to be starlike with respect to an open ball $U(x_0, r_0)$ centered at $x_0$ with radius $r_0$, if $U(x_0, r_0) \subset \Omega$, and $\Omega$ is starlike with respect to every point of $U(x_0, r_0)$. Define

$$\mathcal{O}_s := \{ \Omega \subset D | \Omega \text{ is open and starlike with respect to some open ball } U(x_\Omega, r_\Omega), r_\Omega \geq r_0 \},$$

(1)

where $r_0$ is a given positive constant.

Many compact classes of open sets under Hausdorff distance have been found, see for instance, the examples in [2, 7–12], and [1, 3, 4].

Since for every $\Omega \in \mathcal{O}_s$, $\Omega$ is starlike with respect to some open ball $U(x_\Omega, r_0)$, then there exists a function $f_\Omega : \frac{r_0}{2} \mathbb{S}^{N-1} \rightarrow \mathbb{R}^+$ such that

$$\partial \Omega = \left\{ x_\Omega + f_\Omega(\omega)\omega | \omega \in \frac{r_0}{2} \mathbb{S}^{N-1} \right\},$$

(3)

where $\mathbb{S}^{N-1}$ is the unit sphere of $\mathbb{R}^N$, and $\frac{r_0}{2} \mathbb{S}^{N-1} = \{ \frac{r_0}{2} \omega | \omega \in \mathbb{S}^{N-1} \}$.

Let $A \in M_{N \times N}(C^1(B))$ be a given smooth symmetric matrix, and $\langle A\xi, \xi \rangle \geq \alpha \|\xi\|^2$, $\forall \xi \in \mathbb{R}^N$

for some constant $\alpha > 0$. Define the associated operator $\mathcal{A} : H_0^1(B) \rightarrow H^{-1}(B)$ by

$$\mathcal{A} := \text{div}(A \nabla),$$

(5)
where $H^1_0(B)$ is the first order Sobolev space with usual inner product induced norm
\[
\|u\|_{H^1_0(B)} = (\int_B |\nabla u|^2 \, dx)^{1/2}.
\]

Let $f \in H^{-1}(B)$ be a given function. For any $\Omega \in \mathcal{O}_s$, consider the Dirichlet problem in $\Omega$:
\[
-Au_\Omega = f \text{ in } \Omega, \quad u_\Omega \in H^1_0(\Omega),
\]
whose solution is understood in the variational sense following:
\[
\int_\Omega A\nabla u_\Omega \cdot \nabla \phi \, dx = \langle f, \phi \rangle_{H^{-1}(\Omega) \times H^1_0(\Omega)} \quad \forall \phi \in C_\infty^0(\Omega).
\]

Since $H^1_0(\Omega) = C_\infty^0(\Omega)$, it is well known that (7) admits a unique solution $u_\Omega \in H^1_0(\Omega)$, which can be extended as zero on $B \setminus \Omega$. Denote this extension by $u^0_\Omega$. Then
\[
u_\Omega^0 \in H^1_0(B), \quad \|u^0_\Omega\|_{H^1_0(B)} = \|u_\Omega\|_{H^1_0(\Omega)}.
\]

When we consider the solution of (6), we do not distinguish $u_\Omega$ with its extension $u^0_\Omega$, unless it is necessary. The shape optimization problem, we are concerned with in this paper, is described by
\[
\inf_{\Omega \in \mathcal{O}_s} J(\Omega) := \inf_{\Omega \in \mathcal{O}_s} \left\{ \frac{1}{2} \int_B |u_\Omega - g|^2 \, dx + \int_{\mathbb{R}^N} f_\Omega(\omega) \, d\omega \right\}.
\]

We note that the similar shape optimization problem has been studied in [1] on pages 37–47.

In Sect. 3, we first prove that the function $f_\Omega$ defined in (3) is locally strongly Lipschitz continuous for every $\Omega \in \mathcal{O}_s$. Then it is shown that the convergence of $f_\Omega$ implies the convergence of $\Omega$. Finally, the existence of the optimal solution for problem (8) is proved.

3 Main Results

Before proving the main results, we introduce some notation. For any compacts $K_1, K_2$ of $\mathbb{R}^N$, let
\[
\delta(K_1, K_2) := \max\left\{ \sup_{x \in K_1} d(x, K_2), \sup_{y \in K_2} d(K_1, y) \right\}.
\]

It is seen from (2) that $\rho(\Omega_1, \Omega_2) = \delta(\overline{D} \setminus \Omega_1, \overline{D} \setminus \Omega_2)$.

We denote by $d_0$ the diameter of $B$: $U(x, r)$ the open ball centered at $x$ with radius $r$ in $\mathbb{R}^N$; $L(x, y)$ the line passing through $x$ and $y$, where $x, y \in \mathbb{R}^N, x \neq y$; $C(\alpha, y, z)$ the open cone in $\mathbb{R}^N$ with vertex $z$, direction $y$ and angle $\alpha$, where $\alpha \in \mathbb{R}, y, z \in \mathbb{R}^N$; $\overrightarrow{yz}$ the ray in $\mathbb{R}^N$ starting from $y$ with direction $z$, where $y, z \in \mathbb{R}^N, z \neq 0$; $[x, y]$ the segment connected by $x, y \in \mathbb{R}^N, [x, y] := \{tx + (1-t)y; t \in [0, 1]\}$; $(x, y) := \{tx + (1-t)y; t \in (0, 1]\}; d(A, B) := \inf_{x \in A, y \in B} \|x - y\|$ the usual distance between two sets $A, B \subset \mathbb{R}^N$.

The following preliminary results Lemmas 3.1–3.3 are available in literature.
Lemma 3.1 [5] For any sequence \( \{ \Omega_n \}_{n=1}^{\infty} \subset \mathcal{O}_s \), there is a subsequence \( \{ \Omega_{n_k} \}_{k=1}^{\infty} \) of \( \{ \Omega_n \}_{n=1}^{\infty} \) and \( \Omega^* \in \mathcal{O}_s \) such that \( \text{Hlim} \, \Omega_{n_k} = \Omega^* \).

Lemma 3.2 [1, 5] [\( \Gamma \)-property for \( \mathcal{O}_s \)] Assume that \( \{ \Omega_n \}_{n=1}^{\infty} \subset \mathcal{O}_s \), \( \Omega_0 \in \mathcal{O}_s \) and \( \Omega_0 = \text{Hlim} \, \Omega_n \). Then for any open subset \( K \), \( \overline{K} \subset \Omega_0 \), there exists a positive integer \( n_K \) depending on \( K \), such that \( \overline{K} \subset \Omega_n \) for all \( n \geq n_K \).

Lemma 3.3 [5] [\( \hat{\Gamma} \)-property for \( \mathcal{O}_s \)] Assume that \( \{ \Omega_n \}_{n=1}^{\infty} \subset \mathcal{O}_s \), \( \Omega_0 \in \mathcal{O}_s \) and \( \Omega_0 = \text{Hlim} \, \Omega_n \). Then for any open subset \( K \), \( \overline{K} \subset B \setminus \overline{\Omega}_0 \), there exists a positive integer \( n_K \) depending on \( K \), such that \( \overline{K} \subset B \setminus \overline{\Omega}_n \) for all \( n \geq n_K \).

Theorem 3.1 below is one of main results in this paper.

**Theorem 3.1** Let \( f_\Omega \) be defined in (3). Then \( f_\Omega \) is locally strongly Lipschitz continuous for every \( \Omega \in \mathcal{O}_s \). That is, for every \( \omega \in \frac{\partial}{2} S^{N-1} \), there exists a \( \delta > 0 \), independent of \( f_\Omega \) and \( \omega \), such that for all \( \omega_1, \omega_2 \in U(\omega, \delta) \cap \frac{\partial}{2} S^{N-1} \),

\[
|f_\Omega(\omega_1) - f_\Omega(\omega_2)| \leq M_0|\omega_1 - \omega_2|,
\]

where \( M_0 \) is a constant independent of \( f_\Omega \) and \( \omega \).

**Proof** The proof will be split into several steps.

**Claim 1.** For any \( \Omega \in \mathcal{O}_s \) and all \( p \in \partial \Omega \), we claim that there exists a cone \( C := U(p, \frac{\partial}{2}) \cap C(\theta_0, \overrightarrow{p \Omega\Omega}, p) \) such that \( C \subset \Omega \), where \( x_\Omega \) is defined in (1), and \( 0 < \theta_0 \leq \frac{\pi}{2} \) is a constant depending only on \( d_0 \) and \( r_0 \). In other words, any \( \Omega \in \mathcal{O}_s \) satisfies the uniform cone condition.

Indeed, for any \( y \in U(x_\Omega, r_\Omega) \subset \Omega \), since \( \Omega \) is starlike with respect to the ball \( U(x_\Omega, r_\Omega) \), \( (p, y) \subset \Omega \). Hence \( \{tp + (1-t)y; \, y \in U(x_\Omega, r_\Omega), \, t \in (0, 1]\} \subset \Omega \), and \( C_1 := \{tp + (1-t)y; \, y \in U(x_\Omega, r_\Omega), \, t \in (0, 1]\} \cap U(p, \frac{\partial}{2}) \) is a cone contained in \( \Omega \). Suppose that the angle of \( C_1 \) is \( \theta \). Then \( \sin \theta = r_\Omega/|x_\Omega - p| \). Since \( r_\Omega \geq r_0 \), and \( |x_\Omega - p| \leq d_0 \), we obtain that \( \sin \theta \geq r_0/d_0 \), i.e., \( \arcsin \frac{r_0}{d_0} \leq \theta \leq \pi - \arcsin \frac{r_0}{d_0} \). It is seen that \( C := U(p, \frac{\partial}{2}) \cap C(\theta_0, \overrightarrow{p \Omega\Omega}, p) \subset C_1 \subset \Omega \) for \( \theta_0 = \arcsin \frac{r_0}{d_0} \).

**Claim 2.** For any \( \Omega \in \mathcal{O}_s \) and all \( p \in \partial \Omega \), there exists a cone \( C := C(\theta_0, \overrightarrow{p \Omega\Omega}, p) \) such that \( C \cap \overline{\Omega} = \emptyset \), where \( x_\Omega, \theta_0 \) are given in Claim 1.

This is because otherwise, there exists a point \( z \in \overline{\Omega} \cap C \), and so \( L(z, p) \cap U(x_\Omega, r_\Omega) \neq \emptyset \). Taking \( y \in L(z, p) \cap U(x_\Omega, r_\Omega) \) yields that \( L(y, p) \cap \partial \Omega = L(z, p) \cap \partial \Omega \) has at least three points, which contradicts the definition of \( \Omega \).

We are now in a position to show the required result that for any given \( \omega \in \frac{\partial}{2} S^{N-1} \), and all \( \omega_1, \omega_2 \in U(\omega, \delta) \cap \frac{\partial}{2} S^{N-1} \), where \( \delta = \frac{r_0}{2} \tan \frac{\theta_0}{2} \), it has

\[
|f_\Omega(\omega_1) - f_\Omega(\omega_2)| \leq M_0|\omega_1 - \omega_2|,
\]

where \( M_0 = \frac{4d_0}{r_0^2 \sin \frac{\theta_0}{2}} \).
First, for any \( \Omega \in \mathcal{O} \), i.e., \( \Omega \) is starlike with respect to some open ball \( U(x_\Omega, r_0) \), and for any \( x_\Omega + f_\Omega(\omega)\omega \in \partial \Omega \), where \( \omega \in \frac{r_0}{2} S^{N-1} \), we have
\[
 r_0 \leq |x_\Omega + f_\Omega(\omega)\omega - x_\Omega| = f_\Omega(\omega)\frac{r_0}{2} \leq d_0.
\]
i.e.,
\[
 2 \leq f_\Omega(\omega) \leq \frac{2d_0}{r_0}. \tag{12}
\]

Next, we notice that for any \( \omega_1, \omega_2 \in U(\omega, \delta) \cap \frac{r_0}{2} S^{N-1}, \omega_1 \neq \omega_2 \), there exists a circle \( \mathcal{C} \) centered at \( x_\Omega \) with radius \( \frac{1}{2} f_\Omega(\omega_1) r_0 \sin \theta_0 \) such that two points \( A' = x_\Omega + (f_\Omega(\omega_1) \sin \theta_0)\omega_1, B' = x_\Omega + (f_\Omega(\omega_1) \sin \theta_0)\omega_2 \) are in \( \mathcal{C} \). Take \( A = x_\Omega + f_\Omega(\omega_1)\omega_1, B = x_\Omega + f_\Omega(\omega_2)\omega_2 \in \partial \Omega \). We denote the plane which is formed by the three points \( O, A, B \) by \( \mathcal{P} \). Then \( \mathcal{P} \cap \partial U(x_\Omega, \frac{1}{2} f_\Omega(\omega_1) r_0 \sin \theta_0) = \mathcal{C} \). Take \( L(O', A) \) be the tangent line of \( \mathcal{C} \), and \( O' \) the tangent point such that \( L(O', A) \cap L(O, B) = D \) (see e.g. Fig. 1). By Claims 1 and 2 we get
\[
x_\Omega + f_\Omega(\omega_2)\omega_2 \notin \{t(x_\Omega + f_\Omega(\omega_1)\omega_1) + (1 - t)y; t \in (0, 1], y \in U(x_\Omega, r_0)\}
\]
and
\[
x_\Omega + f_\Omega(\omega_2)\omega_2 \notin \mathcal{C}(\theta_0, x_\Omega(x_\Omega + f_\Omega(\omega_1)\omega_1), x_\Omega + f_\Omega(\omega_1)\omega_1).
\]

From above, if we take \( \angle BAD = \varphi \), then we have
\[
0 \leq \varphi \leq \pi - 2\theta_0. \tag{13}
\]

Take \( \angle AOB = \theta \). By the law of sine functions, we have
\[
\frac{|OA|}{\sin \angle OBA} = \frac{|OB|}{\sin \angle OAB},
\]
i.e.,
\[
\frac{|f_\Omega(\omega_1)\omega_1|}{\sin(\varphi + \theta_0 - \theta)} = \frac{|f_\Omega(\omega_2)\omega_2|}{\sin(\pi - \varphi - \theta_0)}.
\]

Hence one has
\[
|f_\Omega(\omega_2) - f_\Omega(\omega_1)| = f_\Omega(\omega_1)\left|1 - \frac{\sin(\pi - \varphi - \theta_0)}{\sin(\varphi + \theta_0 - \theta)}\right| = \frac{f_\Omega(\omega_1)}{\sin(\varphi + \theta_0 - \theta)}|\sin(\varphi + \theta_0) - \sin(\varphi + \theta_0 - \theta)| = \frac{f_\Omega(\omega_1)}{\sin(\varphi + \theta_0 - \theta)}\frac{\theta}{2} \sin \left(\frac{\varphi + \theta_0 - \theta}{2}\right).
\]
Note that \( \omega_1, \omega_2 \in U(\omega, \delta) \cap \frac{r_0}{2} S^{N-1} \) and \( \delta = \frac{r_0}{2} \tan \frac{\theta_0}{2} \). We then have \( 0 \leq \theta \leq \frac{\theta_0}{2} \). This together with (13) gives \( \frac{\theta_0}{2} \leq \varphi + \theta_0 - \theta \leq \pi - \theta_0 \). Furthermore, by (12) we have

\[
|\omega_1 - \omega_2| = r_0 \sin \frac{\theta}{2}.
\]  

(15)

By (14) and (15), it follows that

\[
|f_{\Omega}(\omega_2) - f_{\Omega}(\omega_1)| \leq \frac{4d_0}{r_0 \sin \frac{\theta_0}{2}} |\omega_1 - \omega_2|.
\]  

(16)

The proof is complete.

\[\square\]

**Remark 3.1** Since \( \frac{r_0}{2} S^{N-1} \) is compact in \( \mathbb{R}^N \), we know that \( C(\frac{r_0}{2} S^{N-1}) \), the space of continuous functions on \( \frac{r_0}{2} S^{N-1} \), is a complete metric space. By Theorem 3.1, the family of functions \( \{f_{\Omega}; \Omega \in \mathcal{O}_s\} \subset C(\frac{r_0}{2} S^{N-1}) \), and \( \{f_{\Omega}; \Omega \in \mathcal{O}_s\} \) is equicontinuous. On the other hand, since \( f_{\Omega}(\omega) \leq \frac{d_0}{r_0} \) for any \( \omega \in \frac{r_0}{2} S^{N-1} \), i.e., \( \{f_{\Omega}; \Omega \in \mathcal{O}_s\} \) is piecewisely uniformly bounded, it follows from the Arzela–Ascoli theorem that \( \{f_{\Omega}; \Omega \in \mathcal{O}_s\} \) is relatively compact in \( C(\frac{r_0}{2} S^{N-1}) \). Therefore, any sequence \( \{f_{\Omega_n}\}_{n=1}^{\infty} \) contains a convergent subsequence.

**Theorem 3.2** Use the notation of (3). Let \( \{f_{\Omega_n}\}_{n=1}^{\infty}, \Omega_n \in \mathcal{O}_s, n = 1, 2, \ldots \), be a sequence in \( C(\frac{r_0}{2} S^{N-1}) \) and suppose that \( f_{\Omega_n} \) converges uniformly to a function \( f \). Then there exists a domain \( \Omega \in \mathcal{O}_s \) and \( x_{\Omega} + f(\omega) \omega; \omega \in \frac{r_0}{2} S^{N-1} \).
Proof Firstly, since \( \{ \Omega_n \}_{n=1}^{\infty} \subset \mathcal{O}_s \), by Lemma 3.1, there exists a subsequence of \( \{ \Omega_n \}_{n=1}^{\infty} \), still denoted by itself without confusion, and an \( \Omega \in \mathcal{O}_s \) such that \( \text{Hlim} \; \Omega_n = \Omega \). On the other hand, since \( \partial \Omega_n = \{ \chi_{\Omega_n} + f_{\Omega_n}(\omega) \omega \mid \omega \in \frac{r_0}{2} S^{N-1} \} \) and \( \{ \chi_{\Omega_n} \}_{n=1}^{\infty} \subset B \), there exists a subsequence of \( \{ \chi_{\Omega_n} \}_{n=1}^{\infty} \), still denoted by itself without confusion, and \( \chi_{\Omega} \in \overline{B} \) such that \( \chi_{\Omega_n} \to \chi_{\Omega} \) as \( n \to \infty \).

Secondly, we show that \( \partial \Omega = \{ \chi_{\Omega} + f(\omega) \omega \mid \omega \in \frac{r_0}{2} S^{N-1} \} \). Actually, if there is some \( \omega \in \frac{r_0}{2} S^{N-1} \) such that \( \chi_{\Omega} + f(\omega) \omega \notin \partial \Omega \), then there are two cases.

**Case 1:** \( \chi_{\Omega} + f(\omega) \omega \in \Omega \). In this case, there exists a \( \gamma > 0 \) such that \( U(\chi_{\Omega} + f(\omega) \omega, \gamma) \subset \Omega \). By Lemma 3.1, there exists a \( n_\gamma > 0 \) such that for any \( n \geq n_\gamma \), \( U(\chi_{\Omega} + f(\omega) \omega, \gamma) \subset \Omega_n \). On the other hand, since \( f_{\Omega_n} \) converges uniformly to \( f \) and \( \chi_{\Omega_n} \to \chi_{\Omega} \), we can take \( \chi_{\Omega_m} + f_{\Omega_m}(\omega) \omega \in U(f(\omega) \omega, \gamma) \) for sufficiently large \( m \geq n_\gamma \), that is, \( \chi_{\Omega_m} + f_{\Omega_m}(\omega) \omega \in \Omega_m \) for sufficiently large \( m \), which contradicts the fact \( \chi_{\Omega_m} + f_{\Omega_m}(\omega) \omega \notin \partial \Omega_m \).

**Case 2:** \( \chi_{\Omega} + f(\omega) \omega \in B \setminus \Omega \). For this case, along the same way as proof of Case 1, we can also get a contradiction by Lemma 3.3. Therefore, \( \{ \chi_{\Omega} + f(\omega) \omega \mid \omega \in \frac{r_0}{2} S^{N-1} \} \subset \partial \Omega \). The inverse part \( \partial \Omega \subset \{ \chi_{\Omega} + f(\omega) \omega \mid \omega \in \frac{r_0}{2} S^{N-1} \} \) is trivial since \( \Omega \in \mathcal{O}_s \). This completes the proof.

The following Theorem 3.3 is about the existence of problem (8).

**Theorem 3.3** The shape optimization problem (8) admits at least one solution.

**Proof** In what follows, we use \( \text{spt}(u) \) to denote the support of \( u \).

Let \( h_0 = \inf_{\Omega \in \mathcal{O}_s} J(\Omega) \). Then there exists a sequence \( \{ \Omega_m \}_{m=1}^{\infty} \subset \mathcal{O}_s \) such that

\[
h_0 = \lim_{m \to \infty} \left\{ \frac{1}{2} \int_B |u_m - g|^2 \, dx + \int_{\frac{r_0}{2} S^{N-1}} f_m(\omega) d\omega \right\} \geq 0,
\]

where \( u_m := u_{\Omega_m} \) is the weak solution of (6), and \( f_m := f_{\Omega_m} \). By Lemma 3.1, there exist a subsequence of \( \{ \Omega_m \}_{m=1}^{\infty} \), still denoted by itself without confusion, and \( \Omega^* \in \mathcal{O}_s \) such that \( \text{Hlim} \; \Omega_m = \Omega^* \). Take \( u = u_m, \Omega = \Omega_m \) in (7), to get

\[
\int_{\Omega_m} A \nabla u_m \cdot \nabla \phi \, dx = \langle f|\Omega_m, \phi \rangle_{H^{-1}(\Omega_m) \times H^1_0(\Omega_m)}, \quad \forall \phi \in C_0^\infty(\Omega_m).
\]

Since \( u_m \in H^1_0(\Omega_m) \), it has

\[
\alpha \int_{\Omega_m} |\nabla u_m|^2 \, dx \leq \int_{\Omega_m} A \nabla u_m \cdot \nabla u_m \, dx
\]

\[
= \langle f|\Omega_m, u_m \rangle_{H^{-1}(\Omega_m) \times H^1_0(\Omega_m)} \leq \| f \|_{H^{-1}(B)} \cdot \| u_m \|_{H^1_0(\Omega_m)},
\]

which, by (4), implies that

\[
\int_{\Omega_m} |\nabla u_m|^2 \, dx \leq C_0;
\]
here and in what follows, $C_0 > 0$ denotes constant independent of $m$ although it may be different in different contexts. Let

$$
\hat{u}_m(x) = \begin{cases} u_m(x) & \text{in } \Omega_m, \\ 0 & \text{in } B \setminus \Omega_m. \end{cases}
$$

(18)

Then $\{\hat{u}_m\}_{m=1}^\infty$ is bounded in $H^1_0(B)$. Hence there exists a subsequence of $\{\hat{u}_m\}$, still denoted by itself without confusion, such that

$$
\hat{u}_m \to \hat{u} \text{ weakly in } H^1_0(B) \text{ and strongly in } L^2(B)
$$

(19)

for some $\hat{u} \in H^1_0(B)$. We claim that

$$
\hat{u}(x) \in H^1_0(\Omega^*).
$$

(20)

To this purpose, it suffices to show that

$$
\hat{u}(x) = 0 \text{ a.e. in } B \setminus \overline{\Omega^*}.
$$

(21)

By Lemma 3.3, we have $K \subset B \setminus \overline{\Omega^*}$, $\forall m \geq m_0$. Therefore

$$
\int_K |\hat{u}(x)|^2 \, dx = \lim_{m \to \infty} \int_K |\hat{u}_m(x)|^2 \, dx \leq \lim_{m \to \infty} \int_{B \setminus \overline{\Omega^*}} |\hat{u}_m(x)|^2 \, dx = 0,
$$

which implies that $\hat{u}(x) = 0$ almost everywhere in $K$. Since $K \subset K \subset B \setminus \overline{\Omega^*}$ is arbitrary, (21) and consequently (20) hold true.

Next we show that

$$
\int_{\Omega^*} A \nabla \hat{u} \cdot \nabla \phi \, dx = \langle f |_{\Omega^*}, \phi \rangle_{H^{-1}(\Omega^*) \times H^1_0(\Omega^*)}, \quad \forall \phi \in C_0^\infty(\Omega^*)
$$

(22)

or

$$
\int_{spt(\phi)} A \nabla \hat{u} \cdot \nabla \phi \, dx = \langle f |_{spt(\phi)}, \phi \rangle_{H^{-1}(\Omega^*) \times H^1_0(\Omega^*)}, \quad \forall \phi \in C_0^\infty(\Omega^*).
$$

Let

$$
\hat{\phi} = \begin{cases} \phi & \text{in } \Omega^*, \\ 0 & \text{in } B \setminus \overline{\Omega^*}. \end{cases}
$$

(23)

By Lemma 3.3, there exists a positive integer $m_1(\phi)$ such that

$$
spt(\hat{\phi}) = spt(\phi) \subset \Omega_m \quad \text{for all } m \geq m_1(\phi),
$$

and hence $\hat{\phi} \in C_0^\infty(\Omega_m)$ for all $m \geq m_1(\phi)$. By (7), we have

$$
\int_{\Omega_m} A \nabla \hat{u}_m \cdot \nabla \hat{\phi} \, dx = \langle f |_{\Omega_m}, \hat{\phi} \rangle_{H^{-1}(\Omega_m) \times H^1_0(\Omega_m)}
$$
which, together with (23) and \( \text{spt}(\hat{\phi}) = \text{spt}(\phi) \subseteq \Omega_m \) for all \( m \geq m_1(\phi) \), implies that
\[
\int_{\Omega_m} A \nabla \hat{u}_m \cdot \nabla \phi \, dx = (f|_{\Omega_m}, \phi)_{H^{-1}(\Omega_m) \times H^1_0(\Omega_m)}.
\]
Passing to the limit as \( m \to \infty \) and using (19) give (22).

Thirdly, since
\[
\hat{u}_m \to \hat{u} \text{ strongly in } L^2(B),
\]
we have
\[
\frac{1}{2} \int_B |\hat{u}_m - g|^2 \to \frac{1}{2} \int_B |\hat{u} - g|^2.
\]
(24)

Finally, since \( \{\Omega_m\}_{m=1}^{\infty} \subset \partial \) by Remark 3.1, there exists a subsequence of \( \{f_m\}_{m=1}^{\infty} \), still denoted by itself without confusion, such that \( f_m \) converges uniformly to a function \( h \). On the other hand, by Theorem 3.2 and \( \Omega^* = \text{Hlim} \Omega_m \), \( \partial \Omega^* = \{x_{\Omega^*} + h(\omega)\omega; \omega \in \frac{\partial S}{2} S^{N-1}\} \). Hence
\[
f_{\Omega^*} = h
\]
(25)
and
\[
\int_{\frac{\partial S}{2} S^{N-1}} f_m(\omega) \, d\omega \to \int_{\frac{\partial S}{2} S^{N-1}} h(\omega) \, d\omega.
\]
(26)
Combining (24) and (26) gives
\[
h_0 = \frac{1}{2} \int_B |\hat{u} - g|^2 + \int_{\frac{\partial S}{2} S^{N-1}} h(\omega) \, d\omega.
\]
That is, \( \Omega^* \) is a solution of problem (8) by virtue of (25). This completes the proof. \( \square \)

4 Concluding Remarks

This paper discusses a shape optimization problem of the second-order elliptic equation with Dirichlet boundary condition. The Hausdorff distance is used for the topology of the open sets. The geometrical properties of starlike domains are investigated. It is shown that the function, which characterizes the boundary of the domain with respect to some ball contained inside domain, is Lipschitz continuous. Based on these results, we show the existence of the proposed optimization problem.

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