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BOUNDARY STABILIZATION OF A FLEXIBLE MANIPULATOR WITH ROTATIONAL INERTIA

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Abstract. We design a stabilizing linear boundary feedback control for a one-link flexible manipulator with rotational inertia. The system is modelled as a Rayleigh beam rotating around one endpoint, with the torque at this endpoint as the control input. The closed-loop system is nondissipative, so that its well posedness is not easy to establish. We study the asymptotic properties of the eigenvalues and eigenvectors of the corresponding operator $\mathcal{A}$ and establish that the generalized eigenvectors form a Riesz basis for the energy state space. It follows that $\mathcal{A}$ generates a C_0-semigroup that satisfies the spectrum-determined growth assumption. This semigroup is exponentially stable under certain conditions on the feedback gains. If the higher-order feedback gain is set to zero, then we obtain a polynomial decay rate for the semigroup.

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1. Introduction

Motivated by increasing demand for the high speed performance, reduced weight, and low energy consumption of robot systems, much effort has been concentrated on the modelling and control of flexible manipulators in the past two decades ([2]). Many technical methods such as the HUM method in [6], the Lyapunov techniques in [3], or the frequency domain multiplier method in [1], to name just a few, have been developed to suppress the vibration of flexible structures.

It is well known from beam theory that when the transverse section cannot be neglected, the rotational inertia of the manipulator has the significant effects on its vibration behaviors, especially for the "higher order" modal frequencies of beam-like structure ([15]). On the other hand, in order to achieve high speed and precision end-point positioning of the flexible manipulator, the boundary control is one of the practical useful designs for suppression of vibration of the beam in productions and space applications.

In this paper, a one-link flexible manipulator with rotational inertia and transverse section are considered, which is usually referred to as a Rayleigh beam. The beam is rotated by a motor in a horizontal plane at the hinged end. It is assumed that the beam is a uniform rectangular transverse section fixed on a hub with rotational inertia $\widetilde{I}_H$ of the motor which is rotated in the horizontal plane (no gravity effects) as shown in Figure 1 in ([15]), where "OR" is the fixed reference line, and "OX" is the tangent line attached to the hub. It is also assumed that the initial neutral longitudinal axis of the arm coincides with the $x-$axis.

Let the beam be of length ℓ with a transverse section of inertia moment I and area A and let EI be the Young's modulus, ρ_m the uniform mass density per unit length. Let $\theta(t)$ be the angular rotation of the motor at time t and $w(x,t)$ be the flexible displacement of the longitudinal axis of the arm at position x and time t. Suppose the deformation $w(x,t)$ is small and the pure bending rotation of the beam is neglected. Moreover, the elongation of the longitudinal axis is assumed to be negligible.

Define a variable $v(x,t)$ as

$$v(x,t) := w(x,t) + x\theta(t). \tag{1.1}$$

Then the dynamic motion equation for the one-link flexible manipulator with rotational inertia can be modelled by the following Rayleigh beam equation

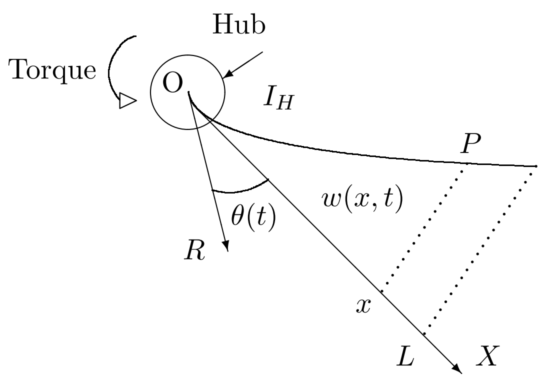

FIGURE 1. The vibration control of a one-link flexible manipulator

([15]):

$$\begin{cases} \rho_m v_{tt}(x,t) - \rho_m \widetilde{S} v_{ttxx}(x,t) \\ \qquad\qquad + EI v_{xxxx}(x,t) = 0, 0 < x < \ell,\ t > 0, \\ v(0,t) = 0, \quad EI v_{xx}(0,t) - \widetilde{I}_H v_{xtt}(0,t) + \widetilde{u}(t) = 0, \\ v_{xx}(\ell,t) = 0, \quad EI v_{xxx}(\ell,t) - \rho_m \widetilde{S} v_{xtt}(\ell,t) = 0, \end{cases} \tag{1.2}$$

where $\widetilde{u}(t)$, applied at the root of the beam, is the torque developed by the motor and $\widetilde{S} = I/A > 0$ is the parameter that characterizes the effect of rotational inertia.

For brevity in notation, we make the following transformation:

$$\begin{cases} y(x,t) = v\Big(\ell x, \sqrt{\dfrac{\rho_m \ell^4}{EI}} t\Big),\ I_H = \dfrac{\widetilde{I}_H}{\rho_m \ell^3}, \\ u(t) = \dfrac{\ell^2}{EI} \widetilde{u}\Big(\sqrt{\dfrac{\rho_m \ell^4}{EI}} t\Big),\ S = \dfrac{\widetilde{S}}{\ell^2}. \end{cases} \tag{1.3}$$

Then y satisfies

$$\begin{cases} y_{tt}(x,t) - S y_{ttxx}(x,t) + y_{xxxx}(x,t) = 0, \quad 0 < x < 1,\ t > 0, \\ y(0,t) = 0, \quad y_{xx}(0,t) - I_H y_{xtt}(0,t) + u(t) = 0, \\ y_{xx}(1,t) = 0, \quad y_{xxx}(1,t) - S y_{xtt}(1,t) = 0. \end{cases} \tag{1.4}$$

Design a feedback controller:

$$u(t) = k y_{xxt}(0,t) - \alpha y_{xt}(0,t) - \beta y_x(0,t), \tag{1.5}$$

where k, α are nonnegative feedback gains, $\beta > 0$. All of them can be tuned in practise. For instance, in [10], the simple direct strain feedback control is adopted to control the motion of the motor so that $\ddot{\theta}(t) = w_{xxt}(0,t)$. Such an objective is easily achieved by direct feedback of the bending moment $w_{xx}(0,t)$ which can be measured by cementing strain gauge foils at the root end of the arm. For interested readers, we refer to [10] for the practical implementability of the feedback law (1.5). By this controller, the closed-loop system becomes

$$\begin{cases} y_{tt}(x,t) - Sy_{ttxx}(x,t) + y_{xxxx}(x,t) = 0,\ 0 < x < 1,\ t > 0, \\ y(0,t) = 0, \\ y_{xx}(0,t) - I_H y_{xtt}(0,t) + ky_{xxt}(0,t) - \alpha y_{xt}(0,t) - \beta y_x(0,t) = 0, \\ y_{xx}(1,t) = 0, \quad y_{xxx}(1,t) - Sy_{xtt}(1,t) = 0. \end{cases} \tag{1.6}$$

Integrating the first equation of system (1.6) over $[x, 1]$ with respect to the spatial variable yields its weak form:

$$\begin{cases} \displaystyle\int_x^1 y_{tt}(\xi,t)d\xi + Sy_{ttx}(x,t) - y_{xxx}(x,t) = 0, \\ y(0,t) = 0, \\ y_{xx}(0,t) - I_H y_{xtt}(0,t) + ky_{xxt}(0,t) - \alpha y_{xt}(0,t) - \beta y_x(0,t) = 0, \\ y_{xx}(1,t) = 0. \end{cases} \tag{1.7}$$

The problem formulated above gives rise to two questions: a) how to show the well posedness or C_0-semigroup generation for the system (1.7) due to the failure of dissipativity? We know that, in this case, it is almost impossible to apply the traditional Hille-Yosida theorem ([12]) to check the C_0-semigroup generation because it is hard to find the n-th power of the resolvent operator; b) does the spectrum of the system determine the stability of the system? In this article, we shall answer these hard questions by virtue of the Riesz basis approach. In Section 2, the system (1.7) is formulated as an evolution equation in the energy state Hilbert space and then the asymptotics of the eigenvalues are developed. Section 3 is devoted to the estimate of the resolvent which will lead to the completeness of the root subspace. The approach used in this part is the Green's function approach which avoids the estimate for the eigenfunctions as was done in previous works (see, e.g., [5]). The Riesz basis property, C_0-semigroup generation, and stability are presented in Section 4.

2. Well posedness and asymptotics of eigenvalues

We begin by expressing the system (1.7) as an evolution equation in the energy Hilbert space. Define an unbounded linear operator A in $L^2(0,1)$ by

$$\begin{cases} Af(x) = Sf'(x) + \displaystyle\int_x^1 f(\tau)d\tau, \ \forall\, f \in D(A), \\ D(A) = \{f \in H^1(0,1) : f(0) = 0\}. \end{cases} \tag{2.1}$$

Denote two Hilbert spaces with inner product induced norms by

$$V := \{f \in H^1(0,1) : f(0) = 0\},\ \|f\|_V^2 := \int_0^1 \left(|f(x)|^2 + S|f'(x)|^2\right) dx$$

and

$$W := \{f \in H^2(0,1) : f(0) = 0\},\ \|f\|_W^2 := \beta|f'(0)|^2 + \int_0^1 |f''(x)|^2 dx.$$

Lemma 2.1. *A has a continuous inverse in $L^2(0,1)$ and is given by*

$$\begin{aligned} A^{-1}g(x) = \ & -\frac{\displaystyle\int_0^1 g(\tau)\sinh\sqrt{\frac{1}{S}}(1-\tau)d\tau}{S\cosh\sqrt{\frac{1}{S}}}\sinh\sqrt{\frac{1}{S}}x \\ & + \frac{1}{S}\int_0^x g(\tau)\cosh\sqrt{\frac{1}{S}}(x-\tau)d\tau,\ \ \forall\, g \in L^2(0,1). \end{aligned} \tag{2.2}$$

Proof. Consider first that $g \in H^1(0,1)$, and solve for $Af = g$, which gives

$$Sf'(x) + \int_x^1 f(\tau)d\tau = g(x), \quad f(0) = 0.$$

Since $g \in H^1(0,1)$, this equation is equivalent to

$$f''(x) - \frac{1}{S}f(x) = \frac{1}{S}g'(x), \quad f(0) = 0,\ \ Sf'(1) = g(1),$$

and solving this yields

$$\begin{aligned} f(x) \ &= c\sinh\sqrt{\frac{1}{S}}x + \frac{1}{\sqrt{S}}\int_0^x g'(\tau)\sinh\sqrt{\frac{1}{S}}(x-\tau)d\tau \\ &= \left[c - \frac{1}{\sqrt{S}}g(0)\right]\sinh\sqrt{\frac{1}{S}}x + \frac{1}{S}\int_0^x g(\tau)\cosh\sqrt{\frac{1}{S}}(x-\tau)d\tau, \end{aligned} \tag{2.3}$$

where c is a constant determined by the condition $Sf'(1) = g(1)$. To find it, we differentiate f and let $x = 1$ to yield

$$Sf'(1) = \sqrt{S}\Big[c - \frac{1}{\sqrt{S}} g(0)\Big] \cosh\sqrt{\frac{1}{S}} + \frac{1}{\sqrt{S}} \int_0^1 g(\tau) \sinh\sqrt{\frac{1}{S}}(1-\tau)d\tau + g(1)$$

and so

$$c - \frac{1}{\sqrt{S}} g(0) = -\frac{\displaystyle\int_0^1 g(\tau) \sinh\sqrt{\frac{1}{S}}(1-\tau)d\tau}{S\cosh\sqrt{\frac{1}{S}}}.$$

Substituting it back into (2.3) gives (2.2), and the general result on $L^2(0,1)$ follows from a density argument. □

Lemma 2.2. *For any $\varphi \in L^2(0,1)$ and $g \in V$, we have*

$$\langle A^{-1}\varphi, g\rangle_V = \int_0^1 \varphi(x)\overline{g'(x)}dx.$$

Proof. Denote $\psi := A^{-1}\varphi$. Then

$$\begin{aligned}
\langle A^{-1}\varphi, g\rangle_V = \langle \psi, g\rangle_V &= \int_0^1 \psi(x)\overline{g(x)}dx + S\int_0^1 \psi'(x)\overline{g'(x)}dx \\
&= \int_0^1 \psi(x)\overline{g(x)}dx + \int_0^1 (A\psi)(x)\overline{g'(x)}dx - \int_0^1\int_x^1 \psi(\tau)d\tau\overline{g'(x)}dx \\
&= \int_0^1 \psi(x)\overline{g(x)}dx + \int_0^1 \varphi(x)\overline{g'(x)}dx - \int_0^1\int_x^1 \psi(\tau)d\tau\overline{g'(x)}dx \\
&= \int_0^1 \varphi(x)\overline{g'(x)}dx.
\end{aligned}$$

This gives the desired result. □

With the operator A at hand, the closed-loop system (1.7), and equivalently (1.6), can be written

$$\begin{cases}
y_{tt} = A^{-1}y_{xxx}, \\
y(0,t) = 0, \\
y_{xx}(0,t) - I_H y_{xtt}(0,t) + k y_{xxt}(0,t) - \alpha y_{xt}(0,t) - \beta y_x(0,t) = 0, \\
y_{xx}(1,t) = 0.
\end{cases} \tag{2.4}$$

We can further pose the system (2.4) in the energy state space $\mathcal{H} := W \times V \times \mathbb{C}$ with the inner product induced norm:

$$\|(\phi,\psi,\eta)\|_{\mathcal{H}}^2 := \|\phi\|_W^2 + \|\psi\|_V^2 + \frac{1}{I_H}|\eta|^2, \ \ \forall\, (\phi,\psi,\eta) \in \mathcal{H},$$

by defining another unbounded linear operator $\mathcal{A} : D(\mathcal{A})\ (\subset \mathcal{H}) \to \mathcal{H}$:

$$\mathcal{A}\begin{pmatrix}\phi\\ \psi\\ \eta\end{pmatrix}^{\top} := \begin{pmatrix}\psi\\ A^{-1}\phi'''\\ \phi''(0)-\beta\phi'(0)\end{pmatrix}^{\top}, \ \forall\ \begin{pmatrix}\phi\\ \psi\\ \eta\end{pmatrix}^{\top} \in \mathcal{D}(\mathcal{A}) \tag{2.5}$$

with

$$\begin{aligned}\mathcal{D}(\mathcal{A}) :=& \Big\{(\phi,\psi,\eta) \in (H^3 \times H^2 \times \mathbb{C}) \cap \mathcal{H} :\\ &\phi''(1) = 0, \eta = I_H\psi'(0) - k\phi''(0) + \alpha\phi'(0)\Big\}.\end{aligned} \tag{2.6}$$

Then system (2.4) is equivalent to an evolution equation in $\mathcal{H}$,

$$\frac{dY(t)}{dt} = \mathcal{A}Y(t), \qquad Y(0) = Y_0 \in \mathcal{H} \tag{2.7}$$

with $Y(t) := \big(y(\cdot,t),\ y_t(\cdot,t),\ I_H y_{xt}(0,t) - k y_{xx}(0,t) + \alpha y_x(0,t)\big)$.

Lemma 2.3. *Let $\mathcal{A}$ be defined by (2.5) and (2.6). Then $\mathcal{A}$ is densely defined and $\mathcal{A}^{-1}$ exists and is compact on $\mathcal{H}$. Hence, the spectrum $\sigma(\mathcal{A})$ consists entirely of isolated eigenvalues only.*

Proof. For any $(f,g,c) \in \mathcal{H}$, we solve the equation

$$\mathcal{A}\begin{pmatrix}\phi\\ \psi\\ \eta\end{pmatrix}^{\top} = \begin{pmatrix}f\\ g\\ c\end{pmatrix}^{\top}$$

and come up with $\psi(x) = f(x)$, $A^{-1}\phi'''(x) = g(x)$, $\phi(0) = 0$, $\phi''(1) = 1$, and $\phi''(0) - \beta\phi'(0) = c$. Hence,

$$\phi(x) = \int_0^x \int_0^\tau \int_0^\xi (Ag)(\zeta)d\zeta d\xi d\tau + \frac{1}{2}\phi''(0)x^2 + \phi'(0)x$$

with

$$\phi''(0) = 1 - \int_0^1 (Ag)(\zeta)d\zeta, \ \phi'(0) = \frac{1}{\beta}\left(\phi''(0) - c\right).$$

Therefore, $\mathcal{A}^{-1}$ exists and is bounded. In light of the Sobolev embedding theorem, $\mathcal{A}^{-1}$ is compact and the proof is completed. □

Since every $\lambda \in \sigma(\mathcal{A})$ is an eigenvalue, $\lambda \in \sigma(\mathcal{A})$ if and only if there exists a nontrivial function ϕ satisfying

$$\begin{cases} \lambda^2\phi(x) - S\lambda^2\phi''(x) + \phi^{(4)}(x) = 0, \;\; 0 < x < 1, \\ \phi(0) = 0, \;\; (1+k\lambda)\phi''(0) - I_H\lambda^2\phi'(0) - \alpha\lambda\phi'(0) - \beta\phi'(0) = 0, \\ \phi''(1) = 0, \quad\; \phi'''(1) - S\lambda^2\phi'(1) = 0 \end{cases} \tag{2.8}$$

where and henceforth primes above symbols representing functions denotes differentiation with respect to the spatial variable x.

Lemma 2.4. *If*

$$\alpha - \beta k > 0, \tag{2.9}$$

then $\mathrm{Re}\lambda < 0$ *for all* $\lambda \in \sigma(\mathcal{A})$.

Proof. Without loss of generality, we may consider the case that $1+k\lambda \neq 0$. Multiply the first equation of (2.8) by $\overline{\phi}$, the conjugate of ϕ, and integrate over $[0,1]$ with respect to x, to give

$$\lambda^2 \int_0^1 \Big[|\phi(x)|^2 + S|\phi'(x)|^2\Big]dx + \int_0^1 |\phi''(x)|^2 dx + \frac{I_H\lambda^2 + \alpha\lambda + \beta}{1+k\lambda}|\phi'(0)|^2 = 0. \tag{2.10}$$

Set $\lambda = \mathrm{Re}\lambda + i\mathrm{Im}\lambda$. Then we have

$$[(\mathrm{Re}\lambda)^2 - (\mathrm{Im}\lambda)^2]\int_0^1 [|\phi(x)|^2 + S|\phi'(x)|^2]dx + \int_0^1 |\phi''(x)|^2 dx \tag{2.11}$$

$$+\frac{[(\mathrm{Re}\lambda)^2 - (\mathrm{Im}\lambda)^2]I_H + \beta + \alpha k|\lambda|^2 + (\mathrm{Re}\lambda)(kI_H|\lambda|^2 + \alpha + \beta k)}{|1+k\lambda|^2}|\phi'(0)|^2 = 0$$

and

$$\begin{aligned} &2(\mathrm{Re}\lambda)(\mathrm{Im}\lambda)\int_0^1 \Big[|\phi(x)|^2 + S|\phi'(x)|^2\Big]dx \\ &+\frac{2(\mathrm{Re}\lambda)(\mathrm{Im}\lambda)I_H + (\mathrm{Im}\lambda)(kI_H|\lambda|^2 + \alpha - \beta k)}{|1+k\lambda|^2}|\phi'(0)|^2 = 0. \end{aligned} \tag{2.12}$$

If $\mathrm{Im}\lambda = 0$, $\mathrm{Re}\lambda < 0$ follows from (2.11). Otherwise when $\mathrm{Im}\lambda \neq 0$, $\mathrm{Re}\lambda < 0$ follows from (2.12) and the assumption that $\alpha - \beta k > 0$. $\square$

Throughout this article, we adopt the convention that when we mention the feedback gains, we always mean that (2.9) holds true. For convenience, let

$$\gamma := \frac{1}{\sqrt{S}}, \;\; \rho := \sqrt{S}\lambda, \tag{2.13}$$

and change (2.8) into

$$\begin{cases} \phi^{(4)}(x) = \rho^2 \left[\phi''(x) - \gamma^2\phi(x)\right], \quad 0 < x < 1, \\ U_4(\phi) = \phi(0) = 0, \\ U_3(\phi) = (1 + k\gamma\rho)\phi''(0) - I_H\gamma^2\rho^2\phi'(0) - \alpha\gamma\rho\phi'(0) - \beta\phi'(0) = 0, \\ U_2(\phi) = \phi''(1) = 0, \qquad U_1(\phi) = \phi'''(1) - \rho^2\phi'(1) = 0. \end{cases} \tag{2.14}$$

Now we are in a position to find asymptotics of "higher" frequencies of the system (2.14). Due to Lemma 2.4 and the fact that the eigenvalues are symmetric about the real axis, we consider only those λ which are in the left-half complex plane:

$$\mathcal{S} := \left\{z \in \mathbb{C} : \frac{\pi}{2} \le \arg z \le \pi\right\}. \tag{2.15}$$

For $\mathcal{S}$, define square roots of -1

$$\omega_1 := e^{i\frac{\pi}{2}} = i, \;\; \omega_2 := e^{i\frac{3}{2}\pi} = -i, \tag{2.16}$$

so that

$$\mathrm{Re}(\rho\omega_1) \le \mathrm{Re}(\rho\omega_2), \; \forall \; \rho \in \mathcal{S}. \tag{2.17}$$

Lemma 2.5. *For $\rho \in \mathcal{S}$ with $|\rho|$ sufficiently large, the equation*

$$\phi^{(4)}(x) = \rho^2 \left[\phi''(x) - \gamma^2\phi(x)\right] \tag{2.18}$$

has four linearly independent fundamental solutions $\phi_s(x,\rho)$ $(s = 1, 2, 3, 4)$

$$\phi_s(x,\rho) = h_s(x) + h_{s1}(x)\rho^{-2} + \mathcal{O}(\rho^{-4}), \; s = 1, 2 \tag{2.19}$$

and

$$\phi_3(x,\rho) = e^{\rho x}\big[1 - \frac{1}{2}\gamma^2 x\rho^{-1} + \frac{1}{8}\gamma^4 x^2\rho^{-2} + \mathcal{O}(\rho^{-3})\big], \tag{2.20}$$

$$\phi_4(x,\rho) = e^{-\rho x}\big[1 + \frac{1}{2}\gamma^2 x\rho^{-1} + \frac{1}{8}\gamma^4 x^2\rho^{-2} + \mathcal{O}(\rho^{-3})\big], \tag{2.21}$$

where and henceforth $\mathcal{O}(\rho^{-m})$ denotes the term satisfying

$$\lim_{|\rho|\to\infty} |\rho^m \mathcal{O}(\rho^{-m})| < \infty.$$

Here

$$\begin{cases} h_1(x) = e^{\gamma x}, \; h_{11}(x) = \frac{1}{2}\gamma^3 x e^{\gamma x}, \\ h_2(x) = e^{-\gamma x}, \; h_{21}(x) = -\frac{1}{2}\gamma^3 x e^{-\gamma x}, \end{cases} \tag{2.22}$$

where $h_s(x)$ and $h_{s1}(x)$ have the following properties $(s = 1, 2)$

$$h''_{s1}(x) - r^2 h_{s1}(x) = r^4 h_s(x).$$

Moreover,

$$\begin{cases} D_1 := h_1'(1)h_2'(0) - h_2'(1)h_1'(0) = -\gamma^2(e^{\gamma} - e^{-\gamma}) = -2\gamma^2 \sinh \gamma, \\ D_2 := h_1'(1)h_2''(1) - h_2'(1)h_1''(1) = 2\gamma^3, \\ D_3 := h_1'(1) - h_2'(1) = \gamma e^{\gamma} + \gamma e^{-\gamma} = 2\gamma \cosh \gamma. \end{cases} \tag{2.23}$$

Proof. This follows directly from Theorem 3 of [14] (or see [16]). $\square$

Set $[a]_3 := a + \mathcal{O}(\rho^{-3})$. Substitution of (2.19)-(2.21) into the boundary conditions in (2.14), we obtain immediately the following Lemma 2.6.

Lemma 2.6. *Let $U_i, i = 1, 2, 3, 4$ be defined as in* (2.14). *Then*

$$\begin{aligned} U_1(\phi_s) &= \phi_s'''(1,\rho) - \rho^2 \phi_s'(1,\rho) \\ &= \begin{cases} -\rho^2 \left[h_s'(1) + (h_{s1}'(1) - h_s'''(1))\rho^{-2} + \mathcal{O}(\rho^{-4})\right], \ s = 1, 2, \\ \rho^3 e^{\rho} \left[-\gamma^2 \rho^{-2} + \mathcal{O}(\rho^{-3})\right], \ s = 3, \\ \rho^3 e^{-\rho} \left[\gamma^2 \rho^{-2} + \mathcal{O}(\rho^{-3})\right], \ s = 4, \end{cases} \\ &:= \begin{cases} -\rho^2 \Big[h_s'(1) + (h_{s1}'(1) - h_s'''(1))\rho^{-2}\Big]_3, \ s = 1, 2, \\ \rho^3 e^{\rho} \left[-\gamma^2 \rho^{-2}\right]_3, \ s = 3, \\ \rho^3 e^{-\rho} \left[\gamma^2 \rho^{-2}\right]_3, \ s = 4; \end{cases} \end{aligned} \tag{2.24}$$

$$\begin{aligned} U_2(\phi_s) = \phi_s''(1,\rho) &= \begin{cases} h_s''(1) + h_{s1}''(1)\rho^{-2} + \mathcal{O}(\rho^{-4}), \ s = 1, 2, \\ \rho^2 e^{\rho}[1 - \frac{1}{2}\gamma^2 \rho^{-1} + (\frac{1}{8}\gamma^4 - \gamma^2)\rho^{-2} + \mathcal{O}(\rho^{-3})], \ s = 3, \\ \rho^2 e^{-\rho}[1 + \frac{1}{2}\gamma^2 \rho^{-1} + (\frac{1}{8}\gamma^4 - \gamma^2)\rho^{-2} + \mathcal{O}(\rho^{-3})], \ s = 4, \end{cases} \\ &:= \begin{cases} \left[h_s''(1) + h_{s1}''(1)\rho^{-2}\right]_3, \ s = 1, 2, \\ \rho^2 e^{\rho} \left[1 - \frac{1}{2}\gamma^2 \rho^{-1} + E_0 \rho^{-2}\right]_3, \ s = 3, \\ \rho^2 e^{-\rho} \left[1 + \frac{1}{2}\gamma^2 \rho^{-1} + E_0 \rho^{-2}\right]_3, \ s = 4; \end{cases} \end{aligned} \tag{2.25}$$

$$\begin{aligned} U_3(\phi_s) &= (1 + k\gamma\rho)\phi_s''(0,\rho) - (I_H \gamma^2 \rho^2 + \alpha\gamma\rho + \beta)\phi_s'(0,\rho) \\ &= \begin{cases} \rho^2 \left[-I_H \gamma^2 h_s'(0) + (k\gamma h_s''(0) - \alpha\gamma h_s'(0))\,\rho^{-1}\right. \\ \left. + (h_s''(0) - I_H\gamma^2 h_{s1}'(0) - \beta h_s'(0))\rho^{-2} + \mathcal{O}(\rho^{-3})\right], \ s = 1, 2, \\ \rho^3 \left[k\gamma + (-1)^s I_H \gamma^2 + (1 + (-1)^s \alpha\gamma)\rho^{-1}\right. \\ \left. + \left((-1)^{s-1}\frac{1}{2} I_H \gamma^4 - k\gamma^3\right)\rho^{-2} + \mathcal{O}(\rho^{-3})\right], \ s = 3, 4, \end{cases} \\ &:= \begin{cases} \rho^2 \Big[- I_H \gamma^2 h_s'(0) + \gamma D_{s+3} \rho^{-1} + E_s \rho^{-2}\Big]_3, \ s = 1, 2, \\ \rho^3 \left[E_s + E_{s+2}\rho^{-1} + E_{s+4}\rho^{-2}\right]_3, \ s = 3, 4; \end{cases} \end{aligned} \tag{2.26}$$

$$U_4(\phi_s) = \phi_s(0,\rho) = 1 + \mathcal{O}(\rho^{-3}) := [1]_3,\ s = 1,2,3,4, \tag{2.27}$$

where

$$\begin{aligned}
&D_4 := kh_1''(0) - \alpha h_1'(0) = k\gamma^2 - \alpha\gamma,\ D_5 := kh_2''(0) - \alpha h_2'(0) = k\gamma^2 + \alpha\gamma,\\
&E_0 := (\frac{1}{8}\gamma^4 - \gamma^2),\ E_1 := h_1''(0) - I_H\gamma^2 h_{11}'(0) - \beta h_1'(0) = \gamma^2 - \frac{1}{2}I_H\gamma^5 - \beta\gamma,\\
&E_2 := h_2''(0) - I_H\gamma^2 h_{21}'(0) - \beta h_2'(0) = \gamma^2 + \frac{1}{2}I_H\gamma^5 + \beta\gamma,\ E_3 := k\gamma - I_H\gamma^2,\\
&E_4 := k\gamma + I_H\gamma^2,\ E_5 := 1 - \alpha\gamma,\ E_6 := 1 + \alpha\gamma,\\
&E_7 := \frac{1}{2}I_H\gamma^4 - k\gamma^3,\ E_8 := -\frac{1}{2}I_H\gamma^4 - k\gamma^3.
\end{aligned} \tag{2.28}$$

Obviously, $0 \neq \lambda \in \sigma(\mathcal{A})$ if and only if the characteristic determinant $\Delta(\rho) = 0$, where

$$\Delta(\rho) := \begin{vmatrix} U_4(\phi_1) & U_4(\phi_2) & U_4(\phi_3) & U_4(\phi_4) \\ U_3(\phi_1) & U_3(\phi_2) & U_3(\phi_3) & U_3(\phi_4) \\ U_2(\phi_1) & U_2(\phi_2) & U_2(\phi_3) & U_2(\phi_4) \\ U_1(\phi_1) & U_1(\phi_2) & U_1(\phi_3) & U_1(\phi_4) \end{vmatrix}. \tag{2.29}$$

Now substitute (2.24)-(2.27) into the characteristic determinant, to obtain

$$\Delta(\rho) = \left| \begin{array}{c} [1]_3 \\ \rho^2[-I_H\gamma^2 h_1'(0) + \gamma D_4\rho^{-1} + E_1\rho^{-2}]_3 \\ [h_1''(1) + h_{11}''(1)\rho^{-2}]_3 \\ -\rho^2[h_1'(1) + (h_{11}'(1) - h_1'''(1))\rho^{-2}]_3 \end{array} \right.$$

$$\begin{array}{c} [1]_3 \\ \rho^2[-I_H\gamma^2 h_2'(0) + \gamma D_5\rho^{-1} + E_2\rho^{-2}]_3 \\ [h_2''(1) + h_{21}''(1)\rho^{-2}]_3 \\ -\rho^2[h_2'(1) + (h_{21}'(1) - h_2'''(1))\rho^{-2}]_3 \end{array}$$

$$\begin{array}{c} [1]_3 \\ \rho^3\left[E_3 + E_5\rho^{-1} + E_7\rho^{-2}\right]_3 \\ \rho^2 e^{\rho}[1 - \frac{1}{2}\gamma^2\rho^{-1} + E_0\rho^{-2}]_3 \\ \rho^3 e^{\rho}[-\gamma^2\rho^{-2}]_3 \end{array}$$

$$
\begin{matrix} [1]_3 \\ \rho^3\left[E_4+E_6\rho^{-1}+E_8\rho^{-2}\right]_3 \\ \rho^2 e^{-\rho}[1+\frac{1}{2}\gamma^2\rho^{-1}+E_0\rho^{-2}]_3 \\ \rho^3 e^{\rho}[-\gamma^2\rho^{-2}]_3 \end{matrix} \Bigg|
$$

$$
\begin{aligned}
&= \rho^7\Big\{D_3\big[E_3e^{-\rho}-E_4e^{\rho}\big]+\rho^{-1}\big[D_6e^{-\rho}-D_7e^{\rho}\big] \\
&\qquad +\rho^{-2}\big[E_9e^{-\rho}+E_{10}+E_{11}e^{\rho}\big]+\mathcal{O}(\rho^{-3})\Big\},
\end{aligned}
$$

where

$$
\begin{aligned}
&D_6 := D_3\Big(\frac{1}{2}\gamma^3(k-I_H\gamma)+(1-\alpha\gamma)\Big)+I_H\gamma^2D_1,\\
&D_7 := D_3\Big((1+\alpha\gamma)-\frac{1}{2}\gamma^3(k+I_H\gamma)\Big)+I_H\gamma^2D_1,\\
&E_9 := D_3(E_3E_0+E_7+\frac{1}{2}\gamma^2E_5)-\frac{1}{2}\gamma^2(D_1+D_3)E_3+\frac{1}{2}I_HD_1\gamma^4-k\gamma^3D_3-\alpha D_1\gamma,\\
&E_{10} := 4I_H\gamma^3,\\
&E_{11} := D_3(-E_3E_0-E_7+\frac{1}{2}\gamma^2E_5+I_H\gamma^4-2I_H\gamma^2E_0+\alpha\gamma^3)\\
&\qquad +\frac{1}{2}\gamma^2(D_1+D_3)E_4+\frac{1}{2}I_HD_1\gamma^4+k\gamma^3D_3+\alpha D_1\gamma. \qquad (2.30)
\end{aligned}
$$

With these preparations, we come to the proof of the asymptotic behavior of the eigenvalues.

Theorem 2.1. *In sector $\mathcal{S}$, the characteristic determinant $\Delta(\rho)$ of the eigenvalue problem* (2.14) *has an asymptotic expansion*

$$
\begin{aligned}
\Delta(\rho) = \rho^7\Big\{&2\gamma^2(\cosh\gamma)\big[(k-\gamma I_H)e^{-\rho}-(k+\gamma I_H)e^{\rho}\big]+\rho^{-1}\big[D_6e^{-\rho}-D_7e^{\rho}\big]\\
&+\rho^{-2}\big[E_9e^{-\rho}+E_{10}+E_{11}e^{\rho}\big]+\mathcal{O}(\rho^{-3})\Big\}, \qquad (2.31)
\end{aligned}
$$

where D_6, D_7, E_9, E_{10} and E_{11} are given by (2.30) *respectively. Moreover, if $k \neq \gamma I_H$, then the eigenvalues $\{\lambda_n, \overline{\lambda_n}\}$ of the eigenvalue problem* (2.8) *have the following asymptotic expansion*

$$
\lambda_n = \frac{1}{2}\gamma\xi+n\gamma\pi i+\frac{\gamma D_8}{\frac{1}{2}\xi+n\pi i}+\gamma\frac{D_9+D_{10}+D_{11}e^{\frac{1}{2}\xi+n\pi i}}{(\frac{1}{2}\xi+n\pi i)^2}+\mathcal{O}(n^{-3}) \quad (2.32)
$$

as $n \to \infty$. *Here,* $n \in \mathbb{N}$, $\gamma := \frac{1}{\sqrt{S}}$ *is given in (2.13), and*

$$\xi := \begin{cases} \ln \dfrac{k-\gamma I_H}{k+\gamma I_H}, & k > \gamma I_H, \\ \ln \dfrac{\gamma I_H - k}{k+\gamma I_H} + \pi i, & k < \gamma I_H, \end{cases} \tag{2.33}$$

$$\begin{aligned} D_8 &:= \frac{1}{2}\gamma^2 + \frac{I_H - k\alpha}{k^2 - \gamma^2 I_H^2} - \frac{\gamma^3 I_H^2 \sinh\gamma}{\cosh\gamma(k^2-\gamma^2 I_H^2)}, \\ D_9 &:= -\frac{1}{4\gamma^2 D_3^2}\Big(\frac{D_6^2}{(k-\gamma I_H)^2} - \frac{D_7^2}{(k+\gamma I_H)^2}\Big), \\ D_{10} &:= \frac{1}{2\gamma D_3}\Big(\frac{E_9}{k-\gamma I_H} + \frac{E_{11}}{k+\gamma I_H}\Big), \\ D_{11} &:= \frac{1}{2\gamma D_3}\frac{E_{10}}{k-\gamma I_H} \end{aligned} \tag{2.34}$$

with $D_6, D_7, E_9, E_{10}, E_{11}$ *being given in* (2.30).

It is clear from Theorem 2.1 that when $k \neq \gamma I_H$,

$$\begin{aligned} \mathrm{Re}\{\lambda_n, \overline{\lambda_n}\} &= \frac{1}{2}\gamma \mathrm{Re}\xi + \frac{\gamma D_8}{|\frac{1}{2}\xi + n\pi i|^2}\frac{1}{2}\mathrm{Re}\xi \\ &\quad + \gamma \frac{\big(D_9 + D_{10} + D_{11}e^{(1/2)\xi}(-1)^n\big)\big(\frac{1}{4}(\mathrm{Re}\xi)^2 - n^2\pi^2\big)}{|\frac{1}{2}\xi + n\pi i|^4} + \mathcal{O}(n^{-3}) \end{aligned} \tag{2.35}$$

and hence

$$\mathrm{Re}\{\lambda_n, \overline{\lambda_n}\} \to \frac{1}{2}\gamma \mathrm{Re}\xi = \frac{1}{2}\gamma \ln\Big|\frac{k-\gamma I_H}{k+\gamma I_H}\Big| \quad \text{as} \quad n \to \infty. \tag{2.36}$$

For the case that $k = \gamma I_H$, it follows from (2.31) that there are at most finitely many eigenvalues because in this case $\Delta(\rho)$ is an analytic function and $\Delta(\rho) \neq 0$ for $|\rho|$ sufficiently large.

Proof of Theorem 2.1. It follows from (2.31) that ρ satisfies (in sector $\mathcal{S}$)

$$\begin{aligned} &\gamma D_3\Big[(k-\gamma I_H)e^{-\rho} - (k+\gamma I_H)e^{\rho}\Big] + \rho^{-1}\Big[D_6 e^{-\rho} - D_7 e^{\rho}\Big] \\ &\qquad + \rho^{-2}\Big[E_9 e^{-\rho} + E_{10} + E_{11}e^{\rho}\Big] + \mathcal{O}(\rho^{-3}) = 0, \end{aligned} \tag{2.37}$$

which leads to

$$\big[(k-\gamma I_H)e^{-\rho} - (k+\gamma I_H)e^{\rho}\big] + \mathcal{O}(\rho^{-1}) = 0. \tag{2.38}$$

Since the equation

$$(k-\gamma I_H)e^{-\rho}-(k+\gamma I_H)e^{\rho}=0$$

has solutions

$$\tilde{\rho}_n=\frac{1}{2}\xi+n\pi i,\quad n\in\mathbb{N} \tag{2.39}$$

with ξ defined in (2.33), we can apply Rouché's theorem to (2.38) to conclude that

$$\rho_n=\tilde{\rho}_n+\alpha_n=\frac{1}{2}\xi+n\pi i+\mathcal{O}(n^{-1}),\ \alpha_n=\mathcal{O}(n^{-1}) \tag{2.40}$$

for sufficiently large positive integers n. Substituting $\rho=\rho_n$ into (2.37) and using the fact that $(k-\gamma I_H)e^{-\tilde{\rho}_n}=(k+\gamma I_H)e^{\tilde{\rho}_n}$, we get

$$\begin{aligned}&\gamma D_3\left[e^{-\alpha_n}-e^{\alpha_n}\right]+\rho_n^{-1}\Big[\frac{D_6}{k-\gamma I_H}e^{-\alpha_n}-\frac{D_7}{k+\gamma I_H}e^{\alpha_n}\Big]\\&\quad+\rho_n^{-2}\Big[\frac{E_9}{k-\gamma I_H}e^{-\alpha_n}+\frac{E_{10}}{k-\gamma I_H}e^{\tilde{\rho}_n}+\frac{E_{11}}{k+\gamma I_H}e^{\alpha_n}\Big]+\mathcal{O}(\rho_n^{-3})=0.\end{aligned}$$

Expanding the exponential functions above in terms of Taylor series, we obtain

$$\begin{aligned}\alpha_n=&\frac{1}{2\tilde{\rho}_n\gamma D_3}\Big(\frac{D_6}{k-\gamma I_H}-\frac{D_7}{k+\gamma I_H}\Big)-\frac{1}{4\tilde{\rho}_n^2\gamma^2D_3^2}\Big(\frac{D_6^2}{(k-\gamma I_H)^2}-\frac{D_7^2}{(k+\gamma I_H)^2}\Big)\\&+\frac{1}{2\tilde{\rho}_n^2\gamma D_3}\Big(\frac{E_9}{k-\gamma I_H}+\frac{E_{10}}{k-\gamma I_H}e^{\tilde{\rho}_n}+\frac{E_{11}}{k+\gamma I_H}\Big)+\mathcal{O}(n^{-3}).\end{aligned}$$

Since

$$\frac{1}{2\gamma D_3}\Big(\frac{D_7}{k-\gamma I_H}-\frac{D_8}{k+\gamma I_H}\Big)=\frac{1}{2}\gamma^2+\frac{I_H-k\alpha}{k^2-\gamma^2I_H^2}-\frac{\gamma^3I_H^2\sinh\gamma}{\cosh\gamma(k^2-\gamma^2I_H^2)}=D_8,$$

we have

$$\rho_n=\frac{1}{2}\xi+n\pi i+\frac{D_8}{\frac{1}{2}\xi+n\pi i}+\frac{D_9+D_{10}+D_{11}e^{\frac{1}{2}\xi+n\pi i}}{(\frac{1}{2}\xi+n\pi i)^2}+\mathcal{O}(n^{-3}). \tag{2.41}$$

This proves the required result because (2.13) says that $\lambda_n=\gamma\rho_n$. $\square$

Theorem 2.2. *Assume $k\neq\gamma I_H$. Let $\sigma(\mathcal{A})=\{\lambda_n,\overline{\lambda}_n\}$ be the eigenvalues of $\mathcal{A}$ and let $\lambda_n=\gamma\rho_n$ with λ_n and ρ_n being given by* (2.32) *and* (2.41) *respectively. Then the corresponding eigenfunctions* $\{(\phi_n,\lambda_n\phi_n,\eta_n),(\overline{\phi}_n,\overline{\lambda}_n\overline{\phi}_n,$

$\overline{\eta}_n)\}$ *have the following asymptotics:*

$$\begin{cases} \lambda_n\phi_n'(x) & = \gamma(1+e^{2\gamma})e^{\rho_n x} + \gamma(1+e^{2\gamma})e^{\rho_n}e^{\rho_n(1-x)} + \mathcal{O}(n^{-1}), \\ \phi_n''(x) & = (1+e^{2\gamma})e^{\rho_n x} - (1+e^{2\gamma})e^{\rho_n}e^{\rho_n(1-x)} + \mathcal{O}(n^{-1}), \\ \eta_n & = \mathcal{O}(n^{-1}) \end{cases} \tag{2.42}$$

for sufficiently large positive integers n. *Moreover,* $(\phi_n, \lambda_n\phi_n, \eta_n)$ *are approximately normalized in* $\mathcal{H}$ *in the sense that there exist positive constants* c_1, c_2 *independent of* n *such that*

$$c_1 \le \|\phi_n''\|_{L^2(0,1)},\ \|\lambda_n\phi_n'\|_{L^2(0,1)}, |\eta_n| \le c_2 \tag{2.43}$$

for all integers n.

Proof. We only need show the first two equalities of (2.42) because if they are valid, then

$$\begin{aligned}\eta_n &= I_H\lambda_n\phi_n'(0) - k\phi_n''(0) + \mathcal{O}(n^{-1}) \\ &= (1+e^{2\gamma})e^{\rho_n}\Big[(k+I_H\gamma)e^{\rho_n} - (k-\gamma I_H)e^{-\rho_n}\Big] + \mathcal{O}(n^{-1}) = \mathcal{O}(n^{-1}).\end{aligned}$$

In the last step, we used (2.38).

From (2.14), Lemma 2.6, and linear algebra theory, we have the eigenfunction ϕ corresponding to the eigenvalue $\lambda = \gamma\rho$ given by

$$\begin{aligned}\phi(x,\rho) &= e^{\gamma}\rho^{-4}\begin{vmatrix} U_4(\phi_1) & U_4(\phi_2) & U_4(\phi_3) & U_4(\phi_4)e^{\rho} \\ U_2(\phi_1) & U_2(\phi_2) & U_2(\phi_3) & U_2(\phi_4)e^{\rho} \\ U_1(\phi_1) & U_1(\phi_2) & U_1(\phi_3) & U_1(\phi_4)e^{\rho} \\ \phi_1(x,\rho) & \phi_2(x,\rho) & \phi_3(x,\rho) & \phi_4(x,\rho)e^{\rho}\end{vmatrix} \\ &= e^{\gamma}\begin{vmatrix} 1 & 1 & 1 & e^{\rho} \\ 0 & 0 & e^{\rho} & 1 \\ -\gamma e^{\gamma} & \gamma e^{-\gamma} & 0 & 0 \\ e^{\gamma x} & e^{-\gamma x} & e^{\rho x} & e^{\rho(1-x)}\end{vmatrix} + \mathcal{O}(\rho^{-1}) \\ &= \gamma\begin{vmatrix} 1 & e^{\gamma} & 1 & e^{\rho} \\ 0 & 0 & e^{\rho} & 1 \\ -e^{\gamma} & 1 & 0 & 0 \\ e^{\gamma x} & e^{\gamma(1-x)} & e^{\rho x} & e^{\rho(1-x)}\end{vmatrix} + \mathcal{O}(\rho^{-1}).\end{aligned}$$

By (2.38), it follows that

$$\gamma^{-1}\phi(x,\rho) \quad = \quad -\Big[1 - \frac{k-\gamma I_H}{k+\gamma I_H}\Big]e^{\gamma x} - \Big[1 - \frac{k-\gamma I_H}{k+\gamma I_H}\Big]e^{\gamma}e^{\gamma(1-x)}$$

$$+(1+e^{2\gamma})e^{\rho x}-(1+e^{2\gamma})e^{\rho}e^{\rho(1-x)}+\mathcal{O}(\rho^{-1}). \quad (2.44)$$

Similarly, by

$$\rho^{-1}\phi'(x,\rho)=e^{\gamma+\rho}\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & e^{\rho} & e^{-\rho} \\ -\gamma e^{\gamma} & \gamma e^{-\gamma} & 0 & 0 \\ 0 & 0 & e^{\rho x} & -e^{-\rho x} \end{vmatrix}+\mathcal{O}(\rho^{-1})$$

and

$$\rho^{-2}\phi''(x,\rho)=e^{\gamma+\rho}\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & e^{\rho} & e^{-\rho} \\ -\gamma e^{\gamma} & \gamma e^{-\gamma} & 0 & 0 \\ 0 & 0 & e^{\rho x} & e^{-\rho x} \end{vmatrix}+\mathcal{O}(\rho^{-1}),$$

we can obtain

$$\rho^{-1}(1+e^{2\gamma})^{-1}\gamma^{-1}\phi'(x,\rho)=e^{\rho x}+e^{\rho}e^{\rho(1-x)}+\mathcal{O}(\rho^{-1}) \quad (2.45)$$

and

$$\rho^{-2}(1+e^{2\gamma})^{-1}\gamma^{-1}\phi''(x,\rho)=e^{\rho x}-e^{\rho}e^{\rho(1-x)}+\mathcal{O}(\rho^{-1}). \quad (2.46)$$

Expression (2.42) then follows from (2.44)-(2.46) by setting

$$\phi_n(x)=\rho_n^{-2}\gamma^{-1}\phi(x,\rho_n)$$

in (2.44), (2.45), and (2.46), respectively. Finally, in order to prove (2.43), we notice (2.41), to obtain

$$\|e^{\rho_n x}\|^2_{L^2(0,1)}=\frac{1}{|\xi|}(e^{|\xi|}-1)+\mathcal{O}(n^{-1}),\ \|e^{\rho_n(1-x)}\|^2_{L^2(0,1)}=\frac{1}{|\xi|}(e^{|\xi|}-1)+\mathcal{O}(n^{-1}).$$

These together with (2.42) yield (2.43). $\square$

To end this section, we remark that the same process can be carried over to $\mathcal{A}^*$, the adjoint operator of $\mathcal{A}$, which is given by

$$\begin{cases} \mathcal{A}^*\begin{pmatrix} f \\ g \\ z \end{pmatrix}^{\top}=\begin{pmatrix} -g+\frac{1}{I_H}[k-\frac{\alpha}{\beta}][\beta f'(0)-f''(0)]x \\ -A^{-1}f''' \\ \beta f'(0)-f''(0) \end{pmatrix}^{\top},\ \forall\ \begin{pmatrix} f \\ g \\ z \end{pmatrix}^{\top}\in\mathcal{D}(\mathcal{A}^*), \\ \mathcal{D}(\mathcal{A}^*):=\Big\{(f,g,z)\in(H^3\times H^2\times\mathbb{C})\cap\mathcal{H}: f''(1)=0, \\ \qquad z=I_Hg'(0)-k[\beta f'(0)-f''(0)]\Big\}. \end{cases} \quad (2.47)$$

First, since $\mathcal{A}$ is a discrete operator, so is $\mathcal{A}^*$ ([4], page 2354). Second, since the eigenvalues of $\mathcal{A}$ are symmetric about the real axis, $\mathcal{A}^*$ will have the same eigenvalues as $\mathcal{A}$ ([9], page 26) with the same algebraic multiplicity ([4], page 2354). Finally, the exact same proof as in Theorem 2.2 will yield the counterpart of Theorem 2.2 for $\mathcal{A}^*$.

Theorem 2.3. *Assume $k \neq \gamma I_H$. Let $\sigma(\mathcal{A}^*) = \{\lambda_n, \overline{\lambda}_n\}$ be the eigenvalues of $\mathcal{A}^*$, let $\lambda_n = \gamma\rho_n$ with λ_n and ρ_n being given by* (2.32) *and* (2.41) *respectively. Then the corresponding eigenfunctions $\{(\psi_n, \lambda_n\psi_n, \xi_n), (\overline{\psi_n}, \overline{\lambda}_n\overline{\psi}_n, \overline{\xi}_n)\}$ have the following asymptotics:*

$$\begin{cases} \lambda_n\psi_n'(x) & = \gamma(1+e^{2\gamma})e^{\rho_n x} + \gamma(1+e^{2\gamma})e^{\rho_n}e^{\rho_n(1-x)} + \mathcal{O}(1), \\ \psi_n''(x) & = (1+e^{2\gamma})e^{\rho_n x} - (1+e^{2\gamma})e^{\rho_n}e^{\rho_n(1-x)} + \mathcal{O}(n^{-1}), \\ \xi_n & = \mathcal{O}(n^{-1}) \end{cases} \tag{2.48}$$

for sufficiently large positive integers n. Moreover, $(\psi_n, \lambda_n\psi_n, \xi_n)$ are approximately normalized in $\mathcal{H}$. □

3. Completeness of the root subspace

Theorem 3.1. *Suppose $k \neq \gamma I_H$. Let $\mathcal{A}$ be defined as in* (2.5) *and* (2.6) *and $\{\lambda_n, n \in \mathbb{J}\}$ be a numeration of all eigenvalues of $\mathcal{A}$, where $\mathbb{J}$ is a subset of all integers. Let $\delta > 0$. Then there exists a constant $M > 0$ such that for any $\lambda \in \rho(\mathcal{A})$ with $|\lambda - \lambda_n| > \delta$ for all $n \in \mathbb{J}$, it holds that*

$$\|R(\lambda, \mathcal{A})\| \leq M\Big(1 + |\lambda|^3\Big) \tag{3.1}$$

where M is independent of λ.

Proof. Let $\lambda \in \rho(\mathcal{A})$ and $(f, g, c) \in \mathcal{H}$. We solve the resolvent equation

$$(\lambda I - \mathcal{A})\begin{pmatrix} \phi \\ \psi \\ \eta \end{pmatrix}^\top = \begin{pmatrix} f \\ g \\ c \end{pmatrix}^\top,$$

which is the same as

$$\begin{cases} \lambda\phi - \psi = f, \ \lambda\psi - A^{-1}\phi''' = g, \ \eta = I_H\psi'(0) - k\phi''(0) + \alpha\phi'(0), \\ \lambda\eta - \big[\phi''(0) - \beta\phi'(0)\big] = c, \end{cases}$$

to obtain $\psi = \lambda\phi - f$ with ϕ satisfying

$$\begin{cases} A^{-1}\phi''' = \lambda^2\phi - \lambda f - g, \\ \phi''(0) - \lambda^2 I_H\phi'(0) + \lambda k\phi''(0) - \lambda\alpha\phi'(0) - \beta\phi'(0) + \lambda I_H f'(0) + c = 0 \end{cases}$$

or

$$\begin{cases} \phi^{(4)}(x) - S\lambda^2\phi''(x) + \lambda^2\phi(x) = F(x,\lambda), \\ \phi(0) = 0, \\ \phi''(0) - \lambda^2 I_H\phi'(0) + \lambda k\phi''(0) - \lambda\alpha\phi'(0) - \beta\phi'(0) = F_1(\lambda), \\ \phi''(1) = 0, \quad \phi'''(1) - S\lambda^2\phi'(1) = F_2(\lambda) \end{cases} \tag{3.2}$$

where

$$\begin{cases} F(x,\lambda) := -S\lambda f''(x) - Sg''(x) + \lambda f(x) + g(x), \\ F_1(\lambda) := -\lambda I_H f'(0) - c, \\ F_2(\lambda) := -S\lambda f'(1) - Sg'(1). \end{cases} \tag{3.3}$$

Set

$$\Phi(x,\lambda) := \phi(x) + \Psi(x,\lambda) \tag{3.4}$$

where

$$\Psi(x,\lambda) := \frac{C_1^*(x,\lambda)}{C^*(\lambda)}F_1(\lambda) + \frac{C_2^*(x,\lambda)}{C^*(\lambda)}F_2(\lambda), \tag{3.5}$$

$$\begin{cases} C^*(\lambda) := (S\lambda^2+2)(\lambda^2 I_H + \lambda\alpha + \beta) + S\lambda^2(2+2\lambda k), \\ C_1^*(x,\lambda) := (S\lambda^2+2)x - S\lambda^2x^2(1-\frac{1}{3}x), \\ C_2^*(x,\lambda) := (2+2\lambda k)x + (\lambda^2 I_H + \lambda\alpha + \beta)x^2(1-\frac{1}{3}x). \end{cases} \tag{3.6}$$

We may assume without loss of generality that $C^*(\lambda) \neq 0$. Then, $\Phi(x,\lambda)$ satisfies

$$\begin{cases} \Phi^{(4)}(x,\lambda) - S\lambda^2\Phi''(x,\lambda) + \lambda^2\Phi(x,\lambda) = F(x,\lambda) - S\lambda^2\Psi''(x,\lambda) + \lambda^2\Psi(x,\lambda), \\ \Phi(0,\lambda) = 0, \\ \Phi''(0,\lambda) - \lambda^2 I_H\Phi'(0,\lambda) + \lambda k\Phi''(0,\lambda) - \lambda\alpha\Phi'(0,\lambda) - \beta\Phi'(0,\lambda) = 0, \\ \Phi''(1,\lambda) = 0, \quad \Phi'''(1,\lambda) - S\lambda^2\Phi'(1,\lambda) = 0. \end{cases} \tag{3.7}$$

Instead of $\phi_j(x)$, we use $\phi_j(x,\lambda), j = 1,2,3,4$, to denote the fundamental solutions of the first equation of (2.8) relative to λ. Then every solution $\Phi(x,\lambda)$ of (3.7) can be represented as (see, e.g., Theorem 2 of [11], page 31)

$$\Phi(x,\lambda) = \int_0^1 G(x,\xi,\lambda)\left[F(\xi,\lambda) - S\lambda^2\Psi''(\xi,\lambda) + \lambda^2\Psi(\xi,\lambda)\right]d\xi. \tag{3.8}$$

Hence in light of (3.4) and (3.8), the solution of (3.2) can be represented as

$$\phi(x) = \int_0^1 G(x,\xi,\lambda)\Big[F(\xi,\lambda) - S\lambda^2\Psi''(\xi,\lambda) + \lambda^2\Psi(\xi,\lambda)\Big]d\xi - \Psi(x,\lambda) \tag{3.9}$$

where $G(x,\xi,\lambda)$ is the Green's function:

$$G(x,\xi,\lambda) := \frac{1}{\Delta(\lambda)} H(x,\xi,\lambda)$$

with

$$H(x,\xi,\lambda) := \begin{vmatrix} \phi_1(x,\lambda) & \phi_2(x,\lambda) & \phi_3(x,\lambda) & \phi_4(x,\lambda) & \eta(x,\xi,\lambda) \\ U_1(\phi_1) & U_1(\phi_2) & U_1(\phi_3) & U_1(\phi_4) & U_1(\eta) \\ U_2(\phi_1) & U_2(\phi_2) & U_2(\phi_3) & U_2(\phi_4) & U_2(\eta) \\ U_3(\phi_1) & U_3(\phi_2) & U_3(\phi_3) & U_3(\phi_4) & U_3(\eta) \\ U_4(\phi_1) & U_4(\phi_2) & U_4(\phi_3) & U_4(\phi_4) & U_4(\eta) \end{vmatrix}, \quad (3.10)$$

$$\eta(x,\xi,\lambda) := \frac{1}{2}\mathrm{sign}(\mathrm{x}-\xi)\sum_{j=1}^{4}\phi_j(x,\lambda)\psi_j(\xi,\lambda) \quad (3.11)$$

where $\psi_j(x,\lambda) := \frac{W_j(x,\lambda)}{W(x,\lambda)}$, $W(x,\lambda)$ is the Wronskian determinant determined by ϕ_i $(i=1,2,3,4)$, and $W_j(x,\lambda)$ is the cofactor determinant of ϕ_j in $W(x,\lambda)$.

We may assume without loss of generality that $\lambda = \gamma\rho$ with $\rho \in \mathcal{S}$. Then substituting (2.19)-(2.21) and (2.24)-(2.27) into (3.10) and (3.11), respectively, we have, for $\lambda \in \rho(\mathcal{A})$ with $|\lambda|$ large enough, that there exists a constant M independent of $x,\xi \in [0,1]$ so that

$$\begin{aligned} &|H(x,\xi,\lambda)| \le M|\lambda|^7 e^{|\rho|}, \qquad \left|\frac{\partial}{\partial x}H(x,\xi,\lambda)\right| \le M|\lambda|^8 e^{|\rho|}, \\ &\left|\frac{\partial^2}{\partial x^2}H(x,\xi,\lambda)\right| \le M|\lambda|^9 e^{|\rho|}. \end{aligned} \quad (3.12)$$

Since $k \neq \gamma I_H$, this together with the assumption that $|\lambda - \lambda_n| \ge \delta$ for all $n \in \mathbb{J}$ and (2.31) gives

$$|G(x,\xi,\lambda)| \le M_1, \left|\frac{\partial}{\partial x}G(x,\xi,\lambda)\right| \le M_1|\lambda|, \left|\frac{\partial^2}{\partial x^2}G(x,\xi,\lambda)\right| \le M_1|\lambda^2|, \quad (3.13)$$

where M_1 is some constant independent of $x,\xi \in [0.1]$. These will in turn yield estimates for $\phi(x)$ and its derivatives

$$\begin{aligned} &|\phi^{(j)}(x)| \\ &\le \int_0^1 \left|\frac{\partial^j}{\partial x^j}G(x,\xi.\lambda)\left(F(\xi,\lambda) - S\lambda^2\Psi''(\xi,\lambda) + \lambda^2\Psi(\xi,\lambda)\right)\right| d\xi + |\Psi^{(j)}(x,\lambda)| \\ &\le M_1|\lambda^j| \int_0^1 \left|F(\xi,\lambda) - S\lambda^2\Psi''(\xi,\lambda) + \lambda^2\Psi(\xi,\lambda)\right| d\xi + |\Psi^{(j)}(x,\lambda)|, \end{aligned}$$

$j = 0, 1, 2, 3$. Combining all these estimates, we obtain eventually that

$$\begin{aligned}
\|(\phi, \psi, \eta)\|^2 =& \beta|\phi'(0)|^2 + \int_0^1 |\phi''(x)|^2 dx + \int_0^1 |\psi(x)|^2 + S|\psi'(x)|^2 dx + \frac{1}{I_H}|\eta|^2 \\
=& \beta|\phi'(0)|^2 + \int_0^1 |\phi''(x)|^2 dx + \int_0^1 \Big[|\lambda\phi(x) - f(x)|^2 \\
&+ S|\lambda\phi'(x) - f'(x)|^2\Big] dx + \frac{1}{I_H}|\lambda I_H \phi'(0) - f'(0) - k\phi''(0)|^2 \\
\le& M_2^2|\lambda^6|\Big[\|f\|_W^2 + \|g\|_V^2 + \frac{1}{I_H}|c|^2\Big], \qquad (3.14)
\end{aligned}$$

where M_2 is some constant independent of λ. Therefore, $\|R(\lambda, \mathcal{A})\| \le M_2(1 + |\lambda|^3)$. $\square$

Recall that a nonzero $Y \in \mathcal{H}$ is called a generalized eigenvector of $\mathcal{A}$, corresponding to an eigenvalue λ (with finite algebraic multiplicity) of $\mathcal{A}$, if there is a positive integer n such that $(\lambda - \mathcal{A})^n Y = 0$. Let $\mathrm{Sp}(\mathcal{A})$ be the root subspace of $\mathcal{A}$ that is a closed subspace spanned by all generalized eigenfunctions of $\mathcal{A}$. A sequence in $\mathcal{H}$ is said to be complete in $\mathcal{H}$, if its linear span is dense in $\mathcal{H}$.

Corollary 3.1. *Under the hypotheses of Theorem* 3.1, *for sufficiently large n, each eigenvalue λ_n of $\mathcal{A}$ is algebraically simple.*

Proof. From (3.9), the multiplicity of each $\lambda \in \sigma(\mathcal{A})$ with sufficiently large modulus, as a pole of $R(\lambda, \mathcal{A})$, is less than or equal to the multiplicity of λ as a zero of the entire function $\Delta(\rho)$ with respect to ρ. On the other hand, it is a routine exercise to verify that λ is geometrically simple. Since, from (2.38), all zeros of $\Delta(\rho) = 0$ with large moduli are simple, the result then follows from the general formula: $m_a \le p \cdot m_g$ (see e.g. [7], page 148), where p denotes the order of the pole of the resolvent operator and m_a, m_g denote the algebraic and geometric multiplicities respectively. $\square$

Theorem 3.2. *Suppose $k \ne \gamma I_H$. Let $\mathcal{A}$ be defined as in* (2.5) *and* (2.6). *Then the root subspaces of both $\mathcal{A}$ and $\mathcal{A}^*$ are complete in $\mathcal{H}$; that is,* $\mathrm{Sp}(\mathcal{A}) = \mathrm{Sp}(\mathcal{A}^*) = \mathcal{H}$.

Proof. We prove $\mathrm{Sp}(\mathcal{A}) = \mathcal{H}$ only because the proof for the other part is similar. From Lemma 5 on page 2355 of [4], the following orthogonal decomposition holds:

$$\mathcal{H} = \sigma_\infty(\mathcal{A}^*) \oplus \mathrm{Sp}(\mathcal{A})$$

where $\sigma_\infty(\mathcal{A}^*)$ consists of those $Y \in \mathcal{H}$ such that $R(\lambda, \mathcal{A}^*)Y$ is an analytic function of λ in the whole complex plane. Hence $\mathrm{Sp}(\mathcal{A}) = \mathcal{H}$ if and only if $\sigma_\infty(\mathcal{A}^*) = \{0\}$. Now suppose that $Y \in \sigma_\infty(\mathcal{A}^*)$. Since $R(\lambda, \mathcal{A}^*)Y$ is an analytic function of λ, it is analytic as a function of ρ. By the maximum modulus principle (or the Phragmén-Lindelöf theorem) of analytic functions and the fact that $\|R(\lambda, \mathcal{A}^*)\| = \|R(\overline{\lambda}, \mathcal{A})\|$, it follows from Theorem 3.1 that

$$\|R(\lambda, \mathcal{A}^*)Y\| \le M(1 + |\lambda|^3)\|Y\|, \ \forall\, \lambda \in \mathbb{C},$$

for some constant $M > 0$. By Theorem 1 on page 3 of [8], we conclude that $R(\lambda, \mathcal{A}^*)Y$ is a polynomial in λ of degree less or equal to 3; i.e.,

$$R(\lambda, \mathcal{A}^*)Y = Y_0 + \lambda Y_1 + \lambda^2 Y_2 + \lambda^3 Y_3 \ \text{ for some } Y_0, Y_1, Y_2, Y_3 \in \mathcal{H}.$$

Thus, for all $\lambda \in \mathbb{C}$,

$$\begin{aligned} Y &= (\lambda - \mathcal{A}^*)(Y_0 + \lambda Y_1 + \lambda^2 Y_2 + \lambda^3 Y_3) \\ &= -\mathcal{A}^* Y_0 + \lambda(Y_0 - A^* Y_1) + \lambda^2(Y_1 - A^* Y_2) + \lambda^3(Y_2 - A^* Y_3) + \lambda^4 Y_3. \end{aligned}$$

Comparing the coefficients of λ^j, we see that $Y_0 = Y_1 = Y_2 = Y_3 = 0$, proving the result. □

4. Riesz basis generation and stability

Let us recall that a sequence in a Hilbert space H is called *minimal* if each element of this sequence lies outside the closed linear span of the remaining elements. Two sequences $\{e_i\}$ and $\{e_i^*\}$ are said to be *biorthogonal* in H if

$$\langle e_i, e_j^* \rangle = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \ne j \end{cases}$$

for every i and j. It is well known that for a given sequence $\{e_i\}$ a biorthogonal sequence $\{e_i^*\}$ exists if and only if $\{e_i\}$ is minimal, and $\{e_i^*\}$ is uniquely determined if and only if $\{e_i\}$ is complete. A sequence $\{e_i\}_{i=1}^\infty$ is called a Bessel sequence in H if for any $x \in H$, the series $\{\langle x, e_i \rangle\}_{i=1}^\infty \in \ell^2$.

A sequence $\{e_i\}_{i=1}^\infty$ is called a basis for H if any element $x \in H$ has a unique representation

$$x = \sum_{i=1}^{\infty} a_i e_i \tag{4.1}$$

and the convergence of the series is in the norm of H. A sequence $\{e_i\}_{i=1}^\infty$ with biorthogonal sequence $\{e_i^*\}_{i=1}^\infty$ is called a Riesz basis for H if $\{e_i\}_{i=1}^\infty$ is an approximately normalized basis of H and the series in (4.1) converges unconditionally in the norm of H. An equivalent definition of Riesz basis is that $\{e_i\}_{i=1}^\infty$ satisfies the following two conditions ([17], page 27):

a) both $\{e_i\}_{i=1}^\infty$ and $\{e_i^*\}_{i=1}^\infty$ are complete in H; and

b) both $\{e_i\}_{i=1}^\infty$ and $\{e_i^*\}_{i=1}^\infty$ are Bessel sequences in H.

It is also well known that $\{e_i\}_{i=1}^\infty$ is a Riesz basis for H if and only if its biorthogonal sequence $\{e_i^*\}_{i=1}^\infty$ is a Riesz basis for H.

In a Hilbert space, the most important bases are orthonormal bases. The second most important ones are Riesz bases, which are bases that are equivalent to some orthonormal bases ([17]).

In order to establish the Riesz basis property of the root subspace of $\mathcal{A}$, we need the following Lemma 3.2 of [13].

Lemma 4.1. *Suppose that a sequence $\{\mu_n\}$ has asymptotics*

$$\mu_n = \alpha_0(n + i\beta_0 \ln n) + \mathcal{O}(1), \quad \alpha_0 \neq 0,\ n = 1, 2, 3, \cdots \tag{4.2}$$

where β_0 is a real number and $\sup_{n\geq 1} Re\mu_n < \infty$. Then the sequence $\{e^{\mu_n x}\}_{n=1}^\infty$ is a Bessel sequence in $L^2(0,1)$.

Lemma 4.2. *Let ρ_n be given by* (2.40). *Then $\{e^{\rho_n x}\}_{n=1}^\infty$ is a Bessel sequence in $L^2(0,1)$.*

Proof. If we set $\mu_n = \rho_n$, then we can take $\beta_0 = 0$ and $\alpha_0 = \pi i$. The result then follows from (2.33) and Lemma 4.1 directly. □

By Theorem 2.1 and the assumption that $k \neq \gamma I_H$, we may assume without loss of generality that $\sigma(\mathcal{A}) = \sigma(\mathcal{A}^*) = \{\lambda_n, \overline{\lambda}_n\}_{n=1}^\infty$. Corollary 3.1 and Theorem 2.1 tell us that there exists an integer $N > 0$ such that all $\lambda_n, \overline{\lambda}_n,\ n \geq N$, are algebraically simple. Furthermore, for $n \leq N$ assume that the algebraic multiplicity of each λ_n is m_n. We say that $\Phi_{n,1}$ is the highest-order generalized eigenvector of $\mathcal{A}$ if

$$(\mathcal{A} - \lambda_n)^{m_n}\Phi_{n,1} = 0 \qquad \text{but} \qquad (\mathcal{A} - \lambda_n)^{m_n - 1}\Phi_{n,1} \neq 0.$$

Then the other lower-order linearly independent generalized eigenvectors associated with λ_n can be found through $\Phi_{n,j} = (\mathcal{A} - \lambda_n)^{j-1}\Phi_{n,1},\ j = 2, 3, \cdots, m_n$. Assume Φ_n is an eigenfunction of $\mathcal{A}$ corresponding to λ_n with $n \geq N$. Then $\big\{\{\{\Phi_{n,j}\}_{j=1}^{m_n}\}_{n<N} \cup \{\Phi_n\}_{n\geq N}\big\} \cup \{\text{their conjugates}\}$ are all linearly independent generalized eigenfunctions of $\mathcal{A}$. Let $\big\{\{\Psi_{n,j}\}_{j=1}^{m_n}\big\}_{n<N} \cup \{\Psi_n\}_{n\geq N}$ be the bi-orthogonal sequence of $\big\{\{\Phi_{n,j}\}_{j=1}^{m_n}\big\}_{n<N} \cup \{\Phi_n\}_{n\geq N}$. Then $\big\{\{\{\Psi_{n,j}\}_{j=1}^{m_n}\}_{n<N} \cup \{\Psi_n\}_{n\geq N}\big\} \cup \{\text{their conjugates}\}$ are all linearly independent generalized eigenfunctions of $\mathcal{A}^*$. It is well known that these two

sequences are minimal in $\mathcal{H}$ and, from Theorem 3.2, they are also complete in $\mathcal{H}$. We now come to the main result of this paper.

Theorem 4.1. *Assume $k \neq \gamma I_H$. Then the generalized eigenfunctions of $\mathcal{A}$ form a Riesz basis for $\mathcal{H}$.*

Proof. From the discussion above and the definition of a Riesz basis, it suffices to show that both $\{\Phi_n\}_{n\geq N}$ and $\{\Psi_n\}_{n\geq N}$ are Bessel sequences in $\mathcal{H}$. Since $1 \leq \|\Phi_n\|\|\Psi_n\| \leq M$ for some constant M independent of n ([17], page 19), we may assume without loss of generality that $\Phi_n = (\phi_n, \lambda_n\phi_n, \eta_n)$ given by (2.42) and $\Psi_n = (\psi_n, \lambda_n\psi_n, \xi_n)$ given by (2.48) for all $n \geq N$. Then it follows from Lemma 4.2 and the expansions of (2.42) and (2.48) that all sequences $\{\phi_n''\}_{n=N}^{\infty}, \{\lambda_n\phi_n'\}_{n=N}^{\infty}$, and $\{\psi_n''\}_{n=N}^{\infty}, \{\lambda_n\psi_n'\}_{n=N}^{\infty}$ are Bessel sequences in $L^2(0,1)$, and $\{\eta_n\}_{n=N}^{\infty}$ and $\{\xi_n\}_{n=N}^{\infty}$ are Bessel sequences in $\mathbb{C}$. So $\{\Phi_n\}_{n\geq N}$ and $\{\Psi_n\}_{n\geq N}$ are also Bessel sequences in $\mathcal{H}$ and the result follows. $\square$

Corollary 4.1. *Suppose $k \neq \gamma I_H$. Let $\mathcal{A}$ be defined as in* (2.5) *and* (2.6)*. Then $\mathcal{A}$ generates a C_0-semigroup $e^{\mathcal{A}t}$ on $\mathcal{H}$ and the spectrum-determined growth condition holds true for the semigroup $e^{\mathcal{A}t}$; that is to say, $s(\mathcal{A}) = \omega(\mathcal{A})$, where $s(\mathcal{A})$ is the spectral bound of $\mathcal{A}$ and $\omega(\mathcal{A})$ is the growth order of $e^{\mathcal{A}t} : \omega(\mathcal{A}) = \inf\big\{\omega :$ there exists $M > 1$ such that $\|e^{\mathcal{A}t}\| \leq Me^{\omega t}$ for all $t \geq 0\big\}$.*

Proof. This is the direct consequence of Theorem 4.1 and Corollary 3.1. $\square$

Corollary 4.2. *Assume that $\alpha > \beta k > 0$, $k \neq \gamma I_H$. Let $\mathcal{A}$ be defined as in (2.5) and (2.6). Then the system (2.7) is exponentially stable; i.e., there exist constants $M, \omega > 0$ such that any mild solution (*[12]*) $Y(t)$ to the equation (2.7) with initial value $Y_0 \in \mathcal{H}$ satisfies*

$$\|Y(t)\| \leq Me^{-\omega t}\|Y_0\|.$$

Proof. From Lemma 2.4, $Re\lambda < 0$ for all $\lambda \in \sigma(\mathcal{A})$. This together with the assumption and Theorem 2.1 gives that $s(\mathcal{A}) < 0$. The result then follows from the spectrum-determined growth condition claimed by Corollary 4.1. $\square$

Corollary 4.3. *Assume that $k = 0, \alpha > 0$, and $\beta > 0$. Let $\mathcal{A}$ be defined as in* (2.5) *and* (2.6)*. Then the system* (2.7) *is not exponentially stable but asymptotically stable and for any given integer $m \geq 1$ the following polynomial decay holds true*

$$\|e^{\mathcal{A}t}Y_0\| \leq C\frac{\|\mathcal{A}^{2m}Y_0\|}{t^m}, \ \ \forall\, t > 0, Y_0 \in D\big(\mathcal{A}^{2m}\big) \tag{4.3}$$

for some constant $C > 0$ depending on m.

Proof. By (2.35), it follows that $\xi = \pi i$ and

$$\mathrm{Re}\{\lambda_n, \overline{\lambda_n}\} = -\gamma \frac{D_9 + D_{10}}{(n+\frac{1}{2})^2\pi^2} + \mathcal{O}(n^{-3}). \tag{4.4}$$

So the system is not exponentially stable. A straightforward computation shows that $D_9 + D_{10} > 0$. On the other hand, Theorem 2.1 tells us that

$$\lambda_n = (n + \frac{1}{2})\gamma\pi i + \mathcal{O}(n^{-1}). \tag{4.5}$$

By Corollary 3.1, we may suppose without loss of generality that $\sigma(\mathcal{A}) = \{\lambda_n, \overline{\lambda}_n\}_{n=1}^{\infty}$. Corollary 3.1 and Theorem 2.1 tell us that there exists an integer $N > 0$ such that all $\lambda_n, \overline{\lambda}_n,\ n > N$, are algebraically simple. Furthermore, for $n \le N$ assume that the algebraic multiplicity of each λ_n is m_n. Let $\Phi_{n,1}$ be the highest-order generalized eigenvector of $\mathcal{A}$ and other lower-order linearly independent generalized eigenvectors associated with λ_n can be found through $\Phi_{n,j} = (\mathcal{A} - \lambda_n)^{j-1}\Phi_{n,1},\ j = 2, 3, \cdots, m_n$. Assume Φ_n is a normalized eigenfunction of $\mathcal{A}$ corresponding to λ_n with $n > N$ (i.e., $\|\Phi_n\| = 1$). Denote by $\overline{\Phi_{n,j}}, j = 1, 2, \cdots, m_n$ the generalized eigenfunctions relative to $\overline{\lambda_n}$ with $n \le N$ and $\overline{\Phi_n}$ the normalized generalized eigenfunction relative to $\overline{\lambda_n}$ with $n > N$. Then by Theorem 4.1, $\big\{\{\{\Phi_{n,j}\}_{j=1}^{m_n}\}_{n\le N} \cup \{\Phi_n\}_{n>N}\big\} \cup \big\{\{\{\overline{\Phi_{n,j}}\}_{j=1}^{m_n}\}_{n\le N} \cup \{\overline{\Phi_n}\}_{n>N}\big\}$ forms a Riesz basis for $\mathcal{H}$. Hence for any initial value Y_0 of equation (2.7), we can expand Y_0 in terms of eigenpairs of $\mathcal{A}$ as follows

$$Y_0 = \sum_{n=1}^{N}\sum_{j=1}^{m_n} a_{n,j}\Phi_{n,j} + \sum_{n=N+1}^{\infty} a_n\Phi_n + \sum_{n=1}^{N}\sum_{j=1}^{m_n} b_{n,j}\overline{\Phi_{n,j}} + \sum_{n=N+1}^{\infty} b_n\overline{\Phi_n} \tag{4.6}$$

where $a_n, a_{n,j}, b_n, b_{n,j}$ are constants. Therefore,

$$\begin{aligned} Y(t) = e^{\mathcal{A}t}Y_0 &= \sum_{n=1}^{N} e^{\lambda_n t}\sum_{j=1}^{m_n} a_{n,j}\sum_{i=1}^{m_n}\frac{(\mathcal{A}-\lambda_n)^{i-1}t^{i-1}}{(i-1)!}\Phi_{n,j} + \sum_{n=N+1}^{\infty} a_n e^{\lambda_n t}\Phi_n \\ &+ \sum_{n=1}^{N} e^{\overline{\lambda_n}t}\sum_{j=1}^{m_n} b_{n,j}\sum_{i=1}^{m_n}\frac{(\mathcal{A}-\overline{\lambda_n})^{i-1}t^{i-1}}{(i-1)!}\overline{\Phi_{n,j}} + \sum_{n=N+1}^{\infty} b_n e^{\overline{\lambda_n}t}\overline{\Phi_n}. \end{aligned} \tag{4.7}$$

Let

$$f_n(t) = t^{2m} e^{-2\gamma\dfrac{D_9 + D_{10} + (-1)^{n+1}D_{11}}{(n+\frac{1}{2})^2\pi^2}t + \mathcal{O}(n^{-3})t}$$

where $\mathcal{O}(n^{-3})$ is the same as in (4.4). Then it is easily seen that $f_n(0) = f_n(+\infty) = 0$ and $f_n(t)$ attains its unique maximum at

$$t = \frac{m(n+\frac{1}{2})^2\pi^2}{\gamma(D_9 + D_{10} + (-1)^{n+1}D_{11})} + \mathcal{O}(n^{-1});$$

that is to say,

$$\sup_{t\geq 0} f_n(t) \leq \Big[\frac{m(n+\frac{1}{2})^2\pi^2}{\gamma(D_9 + D_{10} + (-1)^{n+1}D_{11})} + \mathcal{O}(n^{-1})\Big]^{2m}.$$

By (2.32), it follows that

$$\sup_{t\geq 0} f_n(t) \leq C_1|\lambda_n|^{4m} \tag{4.8}$$

for some constant $C_1 > 0$ independent of n. By (4.4)-(4.8), there exist positive constants ω, C_2, C_3 such that for all $t > 0$

$$\begin{aligned}\|Y(t)\|^2 &\leq C_2 e^{-\omega t}\sum_{n=1}^{N}\sum_{j=1}^{m_n}[|a_{n,j}|^2 + |b_{n,j}|^2] + C_2\sum_{n=N+1}^{\infty}[|a_n|^2 + |b_n|^2]\frac{f_n(t)}{t^{2m}} \\ &\leq C_2 e^{-\omega t}\sum_{n=1}^{N}\sum_{j=1}^{m_n}[|a_{n,j}|^2 + |b_{n,j}|^2] + C_1C_2\sum_{n=N+1}^{\infty}[|a_n|^2 + |b_n|^2]\frac{|\lambda_n|^{4m}}{t^{2m}} \\ &\leq C_3^2\frac{\|\mathcal{A}^{2m}Y_0\|^2}{t^{2m}}. \end{aligned} \tag{4.9}$$

Therefore,

$$\|e^{\mathcal{A}t}Y_0\| \leq C_3\frac{\|\mathcal{A}^{2m}Y_0\|}{t^m}, \;\; \forall\, t > 0, Y_0 \in D(\mathcal{A}^{2m}).$$

This is (4.3). □

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