On exponential stability of a semilinear wave equation with variable coefficients under the nonlinear boundary feedback

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\textbf{ABSTRACT}

The uniform stabilization of an originally regarded nondissipative system described by a semilinear wave equation with variable coefficients under the nonlinear boundary feedback is considered. The existence of both weak and strong solutions to the system is proven by the Galerkin method. The exponential stability of the system is obtained by introducing an equivalent energy function and using the energy multiplier method on the Riemannian manifold. This equivalent energy function shows particularly that the system is essentially a dissipative system. This result not only generalizes the result from constant coefficients to variable coefficients for these kinds of semilinear wave equations but also simplifies significantly the proof for constant coefficients case considered in [A. Guesmia, A new approach of stabilization of nondissipative distributed systems, SIAM J. Control Optim. 42 (2003) 24–52] where the system is claimed to be nondissipative.

\section{1. Introduction}

Many results concerning the boundary stabilization of classical wave equations are available in literature. We refer the reader to [1–4] for linear cases and [5–12] for nonlinear ones. The paper [5] is of special interest since the boundary feedback comprises both the nonlinear component and the memory source term that brings much difficulty to the problem.

The earlier attempt using the classical analysis method for the wave equation with variable coefficients is probably [13]. The recent efforts on nonlinear boundary stabilization can be found in [14–16]. In [14], the stabilization of transmission problem for the wave equation with variable coefficients was investigated. [17] considered the stabilization of an Euler–Bernoulli plate equation with variable coefficients subject to nonlinear boundary feedback. The uniform stabilization of the damped Cauchy–Ventcel problem with variable coefficients and dynamic boundary conditions was obtained in [18]. In all these recent works mentioned above, the Riemannian geometry method is adopted in investigations. This method was first introduced in [19] to obtain the observability inequality, in terms of certain geometry conditions, for the wave equation with variable coefficients. The method was then applied to establish the exact controllability for the second-order
hyperbolic equations with variable coefficients and first-order terms [20], and the observability estimates for the second-order hyperbolic equations with variable coefficients [21].

In [22], the following classical semilinear wave equation was considered:

\[
\begin{cases}
  u'' - \Delta u + h(\nabla u) + f(u) = 0 & \text{in } \Omega \times (0, \infty), \\
  u = 0 & \text{on } \Gamma_0 \times (0, \infty), \\
  \frac{\partial u}{\partial v} + l(u') = 0 & \text{on } \Gamma_1 \times (0, \infty), \\
  u(x, 0) = u_0(x), \quad u'(x, 0) = u_1(x) & \text{in } \Omega.
\end{cases}
\]  

(1.1)

The system (1.1) was claimed to be nondissipative in [22] and a special inequality was introduced to prove the exponential stability of the system. The system (1.1) was also considered later in [23] where both the existence of strong (or weak) solution and the uniform decay rate of the solution were established. Motivated from these works, we consider, in this paper, an equation of the same form as (1.1) by replacing \( \Delta \) in (1.1) with \( \Delta_g \) for a Riemannian metric \( g = \langle \cdot, \cdot \rangle_g \). The system can be considered as a semilinear wave equation with variable coefficients, which covers the system (1.1) as a special case.

The stability of a nondissipative system described by partial differential equations (PDEs) represents most often an extremely mathematical challenge. In recent years, the nondissipative systems have attracted much attention in PDEs. [24] developed the exponential stability for an abstract nondissipative linear system, and in [25], the Riesz basis property was developed for a beam equation with nondissipativity. A new inequality criterion was introduced in [22] to achieve the uniform stabilization for the system (1.1) which is regarded as a nondissipative system. However, in this paper, using a completely different approach and the Riemannian geometry method, we show that the system (1.1) is essentially a dissipative system by introducing an equivalent energy function of the system. The uniform stabilization of such a semilinear wave equation with variable coefficients is thus obtained. This approach not only generalizes the result from constant coefficients to variable coefficients but also simplifies significantly the proof for constant coefficients case considered in [22].

The main contributions of this paper are: (a) a detailed proof for the existence of the solution of the system is given by the Faedo–Galerkin method as in [23] and [5]; (b) the Riemannian geometry approach seems necessary to solve the problem, which is different from the classical analysis; (c) the generalization from the case of constant coefficients to variable ones is not direct, and it needs some additional geometry constraints on the Riemannian manifold formed by variable coefficients that is not needed for constant coefficients case.

We proceed as follows. In Section 2, some necessary notations are introduced. The main results are presented in Section 3 and some preliminary results are proven. Section 4 is devoted to the proofs of the main results.

2. Some notation

Let \( \Omega \subseteq \mathbb{R}^n (n \geq 2) \) be an open bounded domain with \( C^2 \)-boundary \( \partial \Omega = \Gamma_0 \cup \Gamma_1 \), where \( \Gamma_0 \) and \( \Gamma_1 \) are nonempty closed and disjoint, and \( v := \frac{\partial}{\partial v} = \nu_i \frac{\partial}{\partial x_i} \) be the unit normal outer vector field on \( \partial \Omega \). Here and in the rest of the paper, the symbol of Einstein summation is used.

Let \( \{a_{ij}(x), 1 \leq i, j \leq n\} \) be smooth functions in \( \mathbb{R}^n \), satisfying

\[
a_{ij}(x) = a_{ji}(x), \quad \lambda |\xi|^2 \leq a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2, \quad \forall x \in \mathbb{R}^n, \quad \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n,
\]  

(2.1)

where \( \lambda, \Lambda \) are positive constants.

Let \( A(x) = (a_{ij}(x)) \) be an \( n \times n \) matrix for \( x \in \mathbb{R}^n \) and \( G(x) = (g_{ij}(x)) = A^{-1}(x) \) be its inverse matrix. For each \( x \in \mathbb{R}^n \) define the inner product and norm on the tangent space \( \mathbb{R}^n_x = \mathbb{R}^n \) by

\[
g(X, Y)_x = g_{ij}(x) \alpha_i \beta_j, \quad |X|_x = (X, X)_x^{\frac{1}{2}}, \quad \forall X = \alpha_i \frac{\partial}{\partial x_i}, \quad Y = \beta_i \frac{\partial}{\partial x_i} \in \mathbb{R}^n_x.
\]  

(2.2)

Then \( (\mathbb{R}^n, g) \) is a Riemannian manifold with the Riemannian metric \( g \) [19]. Denote by \( D, \nabla_g, \text{div}_g, \) and \( \Delta_g \) the Levi-Civita connection, the gradient operator, the divergence operator and the Beltrami–Laplace operator in terms of the Riemannian metric \( g \), respectively. It can be easily shown that under the Euclidean coordinate,

\[
\nabla_g u(x) = \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) \frac{\partial}{\partial x_i}, \quad |\nabla_g u(x)|^2_g = a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i},
\]  

(2.3)

and

\[
\Delta_g u = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} \left( \sqrt{g} a_{ij} \frac{\partial u}{\partial x_j} \right) = Au - (\nabla_g \varphi, \nabla_g u)_g = Au - a_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial u}{\partial x_j},
\]  

(2.4)

where \( g = \det(G), \quad Au = \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial u}{\partial x_j} \right) \) and \( \varphi = \frac{1}{2} \log(\det(A)) \).
Let $v_A = a_i v_i \frac{\partial}{\partial x_i}$ be the co-normal vector field on $\partial \Omega$. If we define
\begin{equation}
\mu := \frac{\partial}{\partial \mu} = \frac{v_A}{|v_A|^2} = \frac{a_i v_i}{\sqrt{a_i v_j v_j}} \frac{\partial}{\partial x_k},
\end{equation}
then $\mu$ is the unit normal vector field on $\partial \Omega$ in terms of the metric $g$.

Let $H$ be a vector field on $(\mathbb{R}^n, g)$. Then for each $x \in \mathbb{R}^n$, the covariant differential $DH$ of $H$ determines a bilinear form on $\mathbb{R}^n$:
\begin{equation}
DH(X, Y) = \langle D_{\nu}H, X \rangle_g \quad \forall \ X, Y \in \mathbb{R}^n,
\end{equation}
where $D_{\nu}H$ stands for the covariant derivative of the vector field $H$ with respect to $Y$.

### 3. The main results

We consider the semilinear wave equation with variable coefficients under the nonlinear boundary feedback, which is of the same form of (1.1) by replacing $\Delta, \nu$ with $\Delta_g, \mu$ produced by the Riemannian metric $g$ introduced in Section 2:
\begin{equation}
\begin{cases}
u'' - \Delta_g u + h(\nabla u) + f(u) = 0 & \text{in } \Omega \times (0, \infty), \\
u = 0 & \text{on } \Gamma_0 \times (0, \infty), \\
\frac{\partial u}{\partial \mu} + l(u') = 0 & \text{on } \Gamma_1 \times (0, \infty), \\
u(x, 0) = u_0(x), & u'(x, 0) = u_1(x) \text{ in } \Omega,
\end{cases}
\end{equation}
where $f, l : \mathbb{R} \rightarrow \mathbb{R}$ and $h : \mathbb{R}^n \rightarrow \mathbb{R}$ are continuous nonlinear functions, and $\Delta_g$ and $\mu$ are defined in (2.4) and (2.5), respectively. The operator $\nabla$ stands for the gradient operator in the Euclidean space $\mathbb{R}^n$.

The following assumptions are needed.

**Assumption 3.1 (Assumption on $f$).** $f : \mathbb{R} \rightarrow \mathbb{R}$ is a $C^1$-function deriving from a potential $F$:
\begin{equation}
F(s) = \int_0^s f(\tau)d\tau \geq 0 \quad \forall \ s \in \mathbb{R},
\end{equation}
and satisfies
\begin{equation}
|f(s)| \leq b_1 |s|^\rho + b_2, \quad |f'(s)| \leq b_1 |s|^{\rho - 1} + b_2,
\end{equation}
where $b_1, b_2$ are positive constants, and the parameter $\rho$ satisfies
\begin{equation}
1 \leq \rho \leq \begin{cases} 2, & n \leq 3, \\
\frac{n}{n - 2}, & n \geq 4. 
\end{cases}
\end{equation}

**Assumption 3.2 (Assumption on $h$).** $h : \mathbb{R}^n \rightarrow \mathbb{R}$ is a $C^1$-function, and there exist constants $\beta > 0$ and $L > 0$ such that
\begin{align}
|h(\xi)| & \leq \beta |\xi| \quad \forall \ \xi \in \mathbb{R}^n, \\
|\nabla h(\xi)| & \leq L \quad \forall \ \xi \in \mathbb{R}^n.
\end{align}

**Assumption 3.3 (Assumption on $l$).** $l : \mathbb{R} \rightarrow \mathbb{R}$ is a $C^1$-nondecreasing function, and there exist two positive constants $c_1$ and $c_2$ such that
\begin{equation}
c_1 |s|^2 \leq l(s) s \leq c_2 |s|^2 \quad \forall \ s \in \mathbb{R}.
\end{equation}

**Remark 3.1.** An example of the function $f$ satisfying **Assumption 3.1** and (3.11) in **Theorem 3.2** later is given by
\begin{equation}
f(s) = \gamma |s|^{\rho - 1} \quad \text{for some constants } \gamma > 0 \quad \text{and} \quad 1 < \rho \leq \begin{cases} 2, & n \leq 3, \\
\frac{n}{n - 2}, & n \geq 4. 
\end{cases}
\end{equation}

**Assumption 3.4.** There exists a vector field on the Riemannian manifold $(\mathbb{R}^n, g)$ such that
\begin{equation}
DH(X, X) = c(x) |X|^2_g \quad \forall \ x \in \overline{\Omega}, \ X \in \mathbb{R}^n,
\end{equation}
where $b = \min_{\overline{\Omega}} c(x) > 0$ and $B = \max_{\overline{\Omega}} c(x)$
\begin{equation}
B < \min \left\{ b + \frac{2b - 3 \varepsilon_0}{n}, \ r \left( b - \frac{\varepsilon_0}{n} \right) \right\} \quad \text{for some } \varepsilon_0 \in (0, b) \text{ and } r > 1.
\end{equation}
Moreover,
\[ H \cdot v \leq 0 \quad \text{on } \Gamma_0 \quad \text{and} \quad H \cdot v \geq \delta > 0 \quad \text{on } \Gamma_1 \]  
for some constant \( \delta \).

**Remark 3.2.** For the constant coefficients case, \( a_0 = \delta_0 \), the condition (3.8) is automatically satisfied by choosing \( H(x) = x - x_0 \) for fixed \( x_0 \) and \( c(x) \equiv 1 \). For the variable coefficients case, [19] presents some nontrivial examples for which the condition (3.8) is satisfied without constraint on \( B \). Due to the continuity of the function \( c(x) \), there always exists a geodesic ball in \((\mathbb{R}^n, g)\) with small radius centered at the minimum point of \( c(x) \). Hence the condition (3.8) is satisfied for any \( \Omega \) located in the ball. An example in \( \mathbb{R}^2 \) will be given at the end of this section to meet Assumption 3.4.

Let
\[ V = \{ v \in H^1(\Omega) \mid v = 0 \quad \text{on } \Gamma_0 \} \quad \text{and} \quad W = H^2(\Omega) \cap V. \]

**Definition 3.1.** The function \( u \) is said to be a weak solution of system (3.1) if \( u \in L^\infty(0, T; V) \), \( u' \in L^\infty(0, T; L^2(\Omega)) \) and satisfies
\[ \int_0^T \int_\Omega [u\phi'' + (\nabla_g u, \nabla_g \phi)_g + h(\nabla u)\phi + f_u(\phi)] dx dt = \int_\Omega u_0 \phi(0) dx - \int_\Omega u_0 \phi'(0) dx - \int_0^T \int_{\Gamma_1} (u') \phi \ d\Gamma dt \]
for any \( \phi \in C^2(0, T; V) \) with \( \phi(T) = \phi'(T) = 0 \).

Now we state our main results.

**Theorem 3.1.** Under Assumptions 3.1–3.3, for any initial data \( (u_0, u_1) \in W \times W \) satisfying \( \frac{\partial u_0}{\partial n} + I(u_1) = 0 \) on \( \Gamma_1 \), the system (3.1) admits a unique strong solution \( u : (0, \infty) \to \mathbb{R} \) such that
\[ u \in L^\infty(0, \infty; V), \quad u' \in L^\infty(0, \infty; V) \quad \text{and} \quad u'' \in L^\infty(0, \infty; L^2(\Omega)). \]
Moreover, if \( (u_0, u_1) \in V \times L^2(\Omega) \), then (3.1) possesses at least a weak solution in the space \( C([0, \infty); V) \cap C^1([0, \infty); L^2(\Omega)) \).

**Theorem 3.2.** Let \( u \) be a (strong or weak) solution to the system (3.1). Suppose in addition that \( f \) satisfies
\[ 2rF(s) \leq sf(s) \quad \forall s \in \mathbb{R} \quad \text{for some constant } r > 1, \]  
(3.11)
and (3.2). Also suppose that \( h \) satisfies (3.5) and Assumptions 3.2–3.4 hold. If \( \beta \) in (3.5) is sufficiently small, then the energy of the system (3.1) defined by
\[ E(t) = \int_\Omega \left[ |u'(t)|^2 + |\nabla_g u(t)|^2 + 2F(u(t)) \right] dx \]  
(3.12)
decays exponentially to zero in the sense that
\[ E(t) \leq Ce(0)e^{-\omega t} \quad \forall t \geq 0, \]  
(3.13)
for some positive constants \( c \) and \( \omega \) independent of initial values.

In order to prove the main results, we need several lemmas.

**Lemma 3.1** ([26, pp. 128, 138]). Let \( u, v \in C^1(\overline{\Omega}) \) and \( H \) be a vector field on \((\mathbb{R}^n, g)\). Then it has
(a) divergence theorem:
\[ \text{div}_g(uH) = u \text{div}_g(H) + H(u), \quad \int_\Omega \text{div}_g(H) dx = \int_{\partial \Omega} \langle H, \mu \rangle_g d\Gamma; \]
(b) the Green formula:
\[ \int_\Omega v \Delta_g u dx = \int_{\partial \Omega} v \frac{\partial u}{\partial n} d\Gamma - \int_\Omega \langle \nabla_g u, \nabla_g v \rangle_g dx, \]
where \( dx \) and \( d\Gamma \) stand for the volume elements of \( \Omega \) and \( \partial \Omega \), respectively.

**Lemma 3.2.** Let \( H \) be a vector field on \((\mathbb{R}^n, g)\). Let \( \{e_i\}_{i=1}^n \) be an orthonormal frame field on \( \overline{\Omega} \) and \( \{w^i\}_{i=1}^n \) be its dual frame field. Then we have
\[ \text{div}_g H = \sum_{i=1}^n (D_g H, e_i)_g = \sum_{i=1}^n DH(e_i, e_i). \]  
(3.14)
Proof. Let $D$ be the Levi-Civita connection in $(\mathbb{R}^n, g)$ and the operator $C^1$ be the contraction of tensor field. Denote by $DH$ the $(1, 1)$-type tensor field. By the definition of operator $\text{div}_g$, we have

$$\text{div}_g H = C^1(DH) = C^1(DH(w^i, e_i) e_i \otimes u^i) = \sum_{i=1}^n DH(w^i, e_i) = \sum_{i=1}^n D_{e_i} H(w^i) = \sum_{i=1}^n w^i (D_{e_i} H) = \sum_{i=1}^n \langle w^i (D_{e_i} H), e_i \rangle_g = \sum_{i=1}^n \langle D_{e_i} H, e_i \rangle_g.$$

(3.14) is then verified by virtue of the definition of $DH(e_i, e_i)$ in (2.6).

Remark 3.3. Combining (3.8) in Assumption 3.4 and Lemma 3.2, we have

$$nb \leq \text{div}_g H \leq nB \quad \text{in} \quad \Omega.$$

This observation is important for the proof of the exponential stability claimed by Theorem 3.2.

The following lemma is similar to (4) of Lemma 2.1 in [19].

Lemma 3.3. Let $H$ be a vector field on $(\mathbb{R}^n, g)$. For $u \in C^1(\Omega)$ and $x \in \mathbb{R}^n$ the following formula holds

$$\langle \nabla_g u, \nabla_g (H(u)) \rangle_g(x) = DH(\nabla_g u, \nabla_g (H(u)))(x) + \frac{1}{2} \text{div}_g (|\nabla_g u|_g^2 H)(x) - \frac{1}{2} |\nabla_g u|_g^2(x) \text{div}_g (H)(x).$$

(3.16)

Proof. Let $x \in \mathbb{R}^n$ be fixed. Let $\{E_i\}_{i=1}^n$ be a frame field normal at $x$ on the Riemannian manifold $(\mathbb{R}^n, g)$. This means that in some neighborhood of $x$, $\{E_i\}_{i=1}^n$ is a local basis satisfying

$$\langle E_i, E_j \rangle_g = \delta_{ij} \quad \text{and} \quad D_{E_i} E_j(x) = 0 \quad \text{for} \quad 1 \leq i, j \leq n.$$

Since $H$ is a vector field, there exist functions $h_1, h_2, \ldots, h_n$ such that $H = \sum_{i=1}^n h_i E_i$. Hence

$$H(u) = \sum_{i=1}^n h_i E_i(u)$$

(3.17)

and

$$\nabla_g u = \sum_{i=1}^n E_i(u) E_i \quad \text{and} \quad |\nabla_g u|_g^2 = \sum_{i=1}^n (E_i(u))^2,$$

(3.18)

where $E_i(u)(1 \leq i \leq n)$ is the covariant differential of $f$ with regard to $E_i$ in the Riemannian metric $g$.

Notice that $\langle E_i, E_j \rangle_g = \delta_{ij}$ and $D_{E_i} E_j(x) = 0$ for $1 \leq i, j \leq n$. We obtain

$$DH(E_i, E_j)(x) = \langle D_{E_i} H, E_j \rangle_g(x) = \langle D_{h_i} E_i, E_j \rangle_g(x) = \langle E_i(h_i) E_i + h_i D_{E_i} E_i, E_j \rangle_g(x) = E_i(h_j)(x).$$

\begin{equation}
DH(E_i, E_j)(x) = \langle D_{E_i} H, E_j \rangle_g(x) = \langle D_{h_i} E_i, E_j \rangle_g(x) = \langle E_i(h_i) E_i + h_i D_{E_i} E_i, E_j \rangle_g(x) = E_i(h_j)(x). \tag{3.19}
\end{equation}

Since $\text{div}_g(uH) = u \text{div}_g(H) + H(u)$, it follows from (3.17), (3.18) and (3.19) that

$$\langle \nabla_g u, \nabla_g (H(u)) \rangle_g(x) = E_i(f) E_i(H(u))(x) = E_i(u) \{E_i(h_j)E_i(u) + h_j E_j(u)\}(x) = E_i(h_j)E_i(u)E_i(u) + h_j E_j(u)E_i(u)(x) = \text{div}_g (\nabla_g u)(x) + \frac{1}{2} H(\nabla_g u)^2 - \frac{1}{2} |\nabla_g u|_g^2 \text{div}_g (H)(x),$$

(3.20)

where $E_i E_j$ is the second covariant derivatives with regard to $g$. Notice that in the last step of (3.20), we used (a) of Lemma 3.1. The formula (3.16) then follows from the arbitrariness of $x$. □

Proposition 3.1 ([27, pp. 390]). Suppose that there is a metric $\hat{g}$ such that $(\mathbb{R}^n, g)$ has zero curvature and there is a function $u$ on $\mathbb{R}^n$ to meet the relation

$$g = e^{2u} \hat{g} \quad \forall \ x \in \mathbb{R}^n.$$
Given \( x_0 \in \mathbb{R}^n \). Denote by \( \hat{\rho}(x) \) the distance function from \( x_0 \) to \( x \) in the metric \( \hat{g} \). Set

\[
H = \hat{\rho} \hat{D} \hat{\rho},
\]

where \( \hat{D} \) is the Levi-Civita connection on \((\mathbb{R}^n, \hat{g})\). Then

\[
DH(X, X) = [1 + H(u)]|X|_g^2 \quad \forall \ X \in \mathbb{R}^n, \ x \in \mathbb{R}^n. \tag{3.22}
\]

To end this section, we present an example which comes from [27] to meet Assumption 3.4.

**Example 3.1** ([27, pp. 391]). Let \((\mathbb{R}^2, g)\) be a Riemannian manifold with the metric

\[
g = \frac{1}{1 + ax^2 + by^2}(dx^2 + dy^2),
\]

where \( a \) and \( b \) are positive constants. For any \((x, y) \in \mathbb{R}^2\), the Gaussian curvature is

\[
k(x, y) = \frac{a + b + a(b - a)x^2 + b(a - b)y^2}{(1 + ax^2 + by^2)^3}.
\]

By (3.21) we have \( g = e^{2u} \hat{g} \) with \( u = -\frac{1}{2} \log(1 + x^2 + y^2) \) and \( \hat{g} = dx^2 + dy^2 \). \((\mathbb{R}^2, \hat{g})\) is of zero curvature. Take \( \hat{\rho} = \sqrt{x^2 + y^2} \). Then \( H = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \) and

\[
1 + H(u) = \frac{1}{1 + ax^2 + by^2} > 0 \quad \forall \ (x, y) \in \mathbb{R}^2.
\]

It follows from Proposition 3.1 that for any given annulus bounded by two concentric circles centered at the origin \( \Omega \subset \mathbb{R}^2 \), if the constants \( a \) and \( b \) are small enough, then Assumption 3.4 holds.

### 4. Proof of the main results

**Proof of Theorem 3.1.** We use the Galerkin approximation to prove the existence of solution to (3.1). A variational formulation for problem (3.1) is given by

\[
\int_{\Omega} u''w \, dx + \int_{\Omega} (\nabla g u, \nabla g w) \, dx + \int_{\Omega} h(\nabla u)w \, dx + \int_{\Omega} f(u)w \, dx + \int_{\Gamma_1} l(w')w \, d\Gamma = 0 \quad \forall \ w \in V.
\]

The change of variable

\[
v(x, t) = u(x, t) - \phi(x, t) \tag{4.1}
\]

where

\[
\phi(x, t) = u_0(x) + tu_1(x), \quad (x, t) \in Q \triangleq \Omega \times (0, T) \tag{4.2}
\]

gives the equivalent problem to (3.1):

\[
\begin{aligned}
  v'' - \Delta g v + h(\nabla v + \nabla \phi) + f(v + \phi) &= \mathcal{F} \quad \text{in } Q, \\
  v &= 0 \quad \text{on } \Gamma_0 \times (0, T), \\
  \frac{\partial v}{\partial \mu} + l(v' + \phi') &= \mathcal{B} \quad \text{on } \Gamma_1 \times (0, T), \\
  v(0) = v'(0) &= 0,
\end{aligned} \tag{4.3}
\]

where

\[
\mathcal{F} = \Delta g \phi \quad \text{and} \quad \mathcal{B} = -\frac{\partial \phi}{\partial \mu} \tag{4.4}
\]

Let \( \{w_i\}_{i \in \mathbb{N}} \) be a basis for \( W \) that is orthonormal in \( L^2(\Omega) \), and let \( V_m \) be the space spanned by \( w_1, \ldots, w_m \). Let

\[
v_m(t) = \sum_{j=1}^{m} \gamma_j(t)w_j
\]

If \( u_0 \in C^{1,2}(\Omega) \) and \( \partial u_0 / \partial \mu \in C^{1,2}(\Gamma) \), then we can treat \( \phi \) as the initial function. Otherwise, we must solve Problem 3.1 and take the solution \( \phi \) as the initial function. This can be done by the mean value theorem for double integrals.
be the solution to the Cauchy problem:

\[
\int_{\Omega} v''_m(t) \, dx + \int_{\Omega} (\nabla v_m(t), \nabla \phi) \, dx + \int_{\Omega} h(\nabla v_m(t) + \nabla \phi(t)) \, dx \\
+ \int_{\Omega} f(v_m(t) + \phi(t)) \, dx + \int_{\Gamma_1} l(v'_m(t) + \phi'(t)) \, d\Gamma \\
= \int_{\Omega} F(t) \, dx + \int_{\Gamma_1} B(t) \, d\Gamma \\
\forall \ w \in V_m, \text{ and } v_m(0) = v'_m(0) = 0. \tag{4.5}
\]

The existence of solution on some interval \([0, T_m]\) for the system (4.5) can be proven by standard methods of ordinary differential equations. The existence of the solution in the whole interval \([0, T]\) is a consequence of the first estimate in Step 1 below. The proof will be accomplished by splitting into several steps.

**Step 1: The first-order estimate of \(v_m\).**

Replace \(w\) by \(v'_m(t)\) in (4.5) to get

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left[ |v'_m(x, t)|^2 + |\nabla v_m(x, t)|^2 + 2F(v_m(x, t) + \phi(x, t)) \right] \, dx \\
+ \int_{\Omega} h(\nabla v_m + \nabla \phi)v'_m \, dx - \int_{\Omega} f(v_m + \phi) \phi' \, dx + \int_{\Gamma_1} l(v'_m + \phi')(v'_m + \phi') \, d\Gamma \\
= \int_{\Omega} F(t)v'_m(t) \, dx + \frac{d}{dt} \int_{\Gamma_1} B(t)v_m(t) \, d\Gamma - \int_{\Gamma_1} B'(t)v_m(t) \, d\Gamma + \int_{\Gamma_1} l(v'_m + \phi') \phi' \, d\Gamma. \tag{4.6}
\]

Firstly, by **Assumption 3.1**, the Sobolev imbedding theorem, and the regularities of the initial data, we have

\[
\int_{\Omega} f(v_m + \phi) \phi' \, dx \leq \mathcal{C} \int_{\Omega} |v_m + \phi|^\rho |u_1| \, dx + \mathcal{C} \int_{\Omega} |u_1| \, dx \\
\leq \mathcal{C} \left( \int_{\Omega} |v_m|^\rho |u_1| \, dx + \int_{\Omega} |\phi|^\rho |u_1| \, dx \right) + \mathcal{C} \\
\leq \mathcal{C} \left( \int_{\Omega} |v_m|^{2\rho} \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |u_1|^2 \, dx \right)^{\frac{1}{2}} + \mathcal{C} \int_{\Omega} (|u_0|^\rho |u_1| + t^\rho |u_1|^{\rho+1}) \, dx + \mathcal{C} \\
\leq \mathcal{C} \left( \int_{\Omega} |\nabla v_m(t)|^2 \, dx \right)^{\frac{1}{2}} + C t^\rho + \mathcal{C}. \tag{4.7}
\]

Here and in what follows, we use constant \( \mathcal{C} > 0 \) to denote some constants independent of functions involved although it may have different values in different contexts.

Secondly, by **Assumption 3.2**, it has

\[
\int_{\Omega} h(\nabla v_m(t) + \nabla \phi(t))v'_m \, dx \leq \frac{\beta^2}{2} \int_{\Omega} |\nabla v_m(t) + \nabla \phi(t)|^2 \, dx + \frac{1}{2} \int_{\Omega} |v'_m(t)|^2 \, dx. \tag{4.8}
\]

Thirdly, by **Assumption 3.3**, one has

\[
\int_{\Gamma_1} l(v'_m + \phi')(v'_m + \phi') \, d\Gamma \geq \mathcal{C} \int_{\Gamma_1} |v'_m(t) + u_1|^2 \, d\Gamma \tag{4.9}
\]

and

\[
\int_{\Gamma_1} l(v'_m + \phi') \phi' \, d\Gamma \leq \mathcal{C} \int_{\Gamma_1} |v'_m(t) + u_1| |u_1| \, d\Gamma \\
\leq \eta \int_{\Gamma_1} |v'_m(t) + u_1|^2 \, d\Gamma + \mathcal{C}(\eta) \int_{\Gamma_1} |u_1|^2 \, d\Gamma \\
\leq \eta \int_{\Gamma_1} |v'_m(t) + u_1|^2 \, d\Gamma + \mathcal{C}(\eta), \tag{4.10}
\]

where \( \eta > 0 \) is a constant that will be determined later.
Combining (4.6)–(4.10), and the trace theorem that \( \int_{\Gamma_1} |v_m|^2 d\Gamma \leq C_0 \int_{\Omega} |\nabla g v_m|^2 dx \) for some constant \( C_0 > 0 \), it follows that

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( |v_m'(t)|^2 + |\nabla g v_m(t)|^2 + 2F(v_m(t) + \phi(t)) \right) dx + (C - \eta) \int_{\Gamma_1} |v_m(t) + u_1|^2 d\Gamma
\]

\[
\leq C \left( \int_{\Omega} |\nabla g v_m(t)|^2 dx \right)^\frac{1}{2} + C \eta^\rho + \frac{\beta^2}{2} \int_{\Omega} |\nabla g v_m(t) + \nabla g \phi(t)|^2 dx + \frac{1}{2} \int_{\Omega} |v_m'(t)|^2 dx
\]

\[
+ \frac{d}{dt} \int_{\Gamma_1} \mathcal{B}(t)v_m(t) d\Gamma + \frac{1}{2} \int_{\Omega} |\mathcal{F}(t)|^2 dx + \frac{1}{2} \int_{\Omega} |v_m(t)|^2 dx
\]

\[
\frac{C^2}{2} \int_{\Gamma_1} |\mathcal{B}'(t)|^2 d\Gamma + \frac{1}{2} \int_{\Omega} |\nabla g v_m(t)|^2 d\Gamma + C(\eta) + C. \tag{4.11}
\]

Integrate (4.11) over the interval \((0, t)\) and notice \( v_m(0) = v_m'(0) = 0 \) and \( \frac{d}{dt} \leq 1 \), to obtain

\[
\int_{\Omega} \left( |v_m'(t)|^2 + |\nabla g v_m(t)|^2 + 2F(v_m(t) + \phi(t)) \right) dx + 2(C - \eta) \int_{0}^{t} \int_{\Gamma_1} |v_m(t) + u_1|^2 d\Gamma ds
\]

\[
\leq C_0 \int_{0}^{t} \left( \int_{\Omega} |\nabla g v_m(s)|^2 dx \right)^\frac{1}{2} ds + C \eta^\rho + (1 + 2\beta^2) \int_{0}^{t} \int_{\Omega} |\nabla g v_m(s)|^2 dx ds + 2 \int_{0}^{t} \int_{\Omega} |v_m'(s)|^2 dx ds
\]

\[
+ 2\beta^2 \int_{0}^{t} \int_{\Omega} |\nabla g \phi(s)|^2 dx ds + \int_{0}^{t} \int_{\Omega} |\mathcal{F}(s)|^2 dx ds + \int_{\Gamma_1} \mathcal{B}(t)v_m(t) d\Gamma + C^2 t \int_{\Gamma_1} \left| \frac{\partial u_1}{\partial \nu} \right|^2 d\Gamma + Ct + C
\]

\[
\leq Ct + (C + 1 + 2\beta^2) \int_{0}^{t} \int_{\Omega} |\nabla g v_m(s)|^2 dx ds + 2 \int_{0}^{t} \int_{\Omega} |v_m'(s)|^2 dx ds + \eta \int_{\Omega} |\nabla g v_m(t)|^2 dx + C(t^\rho + 1 + t^3) + C. \tag{4.12}
\]

Finally, choosing \( \eta \) sufficiently small, by Gronwall’s lemma and the fact \( \mathcal{F}(s) \geq 0 \) for any \( s \in \mathbb{R} \), we obtain the first-order estimate of \( v_m \)

\[
\int_{\Omega} \left( |v_m'(t)|^2 + |\nabla g v_m(t)|^2 + 2F(v_m(t) + \phi(t)) \right) dx + \int_{0}^{t} \int_{\Gamma_1} |v_m'(t) + u_1|^2 d\Gamma ds \leq C_1, \tag{4.13}
\]

where \( C_1 > 0 \) is a constant independent of \( m \in \mathbb{N} \) and \( t \in [0, T] \).

**Step 2:** The second-order estimate of \( v_m \).

We first estimate the term \( \|u_m''(0)\|_{L^2(\Omega)} \). Take \( t = 0 \) in (4.5) and notice the fact \( v_m(0) = v_m'(0) = 0 \) in (4.4), to obtain

\[
\int_{\Omega} u_m''(0) w dx + \int_{\Omega} h(\nabla u_0) w dx + \int_{\Omega} f(u_0) w dx + \int_{\Gamma_1} l(u_1) w d\Gamma
\]

\[
= \int_{\Omega} \Delta g u_0 w dx + \int_{\Gamma_1} \left( -\frac{\partial u_0}{\partial \mu} \right) w d\Gamma \quad \forall w \in V_m. \tag{4.14}
\]

Since \( \frac{\partial u_0}{\partial \mu} + l(u_1) = 0 \) on \( \Gamma_1 \), it has

\[
\int_{\Omega} u_m''(0) w dx + \int_{\Omega} h(\nabla u_0) w dx + \int_{\Omega} f(u_0) w dx = \int_{\Omega} \Delta g u_0 w dx \quad \forall w \in V_m. \tag{4.15}
\]

This together with Assumptions 3.1 and 3.2 and the regularity of the initial value gives

\[
\|u_m''(0)\|_{L^2(\Omega)} \leq C_2, \tag{4.16}
\]

where \( C_2 \) is a positive constant independent of \( m \in \mathbb{N} \).

Next, differentiate (4.5) with respect to \( t \) and replace \( w \) by \( v_m'' \), to obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( |v_m'(t)|^2 + |\nabla g v_m'(t)|^2 \right) dx + \int_{\Omega} \left( \nabla h(\nabla v_m + \nabla \phi) \cdot (\nabla v_m' + \nabla \phi') \right) v_m'' dx
\]

\[
+ \int_{\Omega} f'(v_m + \phi)(v_m' + \phi') v_m'' dx + \int_{\Gamma_1} f'(v_m' + \phi')(v_m'')^2 d\Gamma = \int_{\Omega} \mathcal{F}'(t)v_m'(t) dx + \frac{d}{dt} \int_{\Gamma_1} \mathcal{B}'(t)v_m'(t) d\Gamma. \tag{4.17}
\]
By Assumption 3.2, we have

$$\int_{\Omega} (\nabla h(\nabla v_m + \nabla \phi) \cdot (\nabla v_m' + \nabla \phi')) v_m'' \, dx \leq C \left( 1 + \int_{\Omega} |\nabla g v_m'(t)|^2 \, dx + \int_{\Omega} |v_m''(t)|^2 \, dx \right). \tag{4.18}$$

Applying Hölder's inequality, (4.13), and the Sobolev imbedding theorem, we deduce, by noticing the assumption on \(f'\), that

$$\int_{\Omega} f'(v_m + \phi)(v_m' + \phi') v_m'' \, dx \leq \int_{\Omega} C |v_m|^{\rho-1} + |\phi|^{\rho-1} + C |v_m'| + |\phi'| |v_m''| \, dx$$

$$\leq C \int_{\Omega} |v_m|^{2(\rho-1)} |v_m'|^2 \, dx + C \int_{\Omega} |\phi|^{2(\rho-1)} |v_m'|^2 \, dx + C \int_{\Omega} |v_m''|^2 \, dx + C$$

$$\leq C \left( \int_{\Omega} |v_m|^{\frac{2(\rho-1)}{\rho-2}} \, dx \right)^{\frac{\rho-2}{\rho-1}} \left( \int_{\Omega} |v_m'|^{\frac{2}{\rho-2}} \, dx \right)^{\frac{\rho-1}{\rho-2}} + C \int_{\Omega} |\phi|^{2(\rho-1)} |v_m'|^2 \, dx + C \int_{\Omega} |v_m''|^2 \, dx + C$$

$$\leq C \left( \int_{\Omega} |\nabla g v_m'(t)|^2 \, dx + \int_{\Omega} |v_m''(t)|^2 \, dx \right) + C, \tag{4.19}$$

and

$$\int_{\Gamma_1} \mathcal{F}'(t)v_m''(t)\,dx \leq C \int_{\Omega} |v_m'(t)|^2 \, dx + C, \tag{4.20}$$

$$\int_{\Gamma_1} \mathcal{G}'(t)v_m'(t)\,d\Gamma \leq C \frac{C_1}{\eta} + \eta \int_{\Omega} |v_m'(t)|^2 \, dx. \tag{4.21}$$

Finally, combining (4.18)–(4.21), integrating (4.17) over \((0, t)\) and choosing \(\eta\) sufficiently small, by Gronwall’s lemma and the fact \(l'(s) \geq 0\) for any \(s \in \mathbb{R}\), we obtain the second-order estimate of \(v_m\)

$$\int_{\Omega} |v_m''(t)|^2 \, dx + \int_{\Omega} |\nabla g v_m'(t)|^2 \, dx + \int_0^t \int_{\Gamma_1} l'(v_m'(s) + \phi'(s))(v_m''(s))^2 \, d\Gamma \, ds \leq C_3, \tag{4.22}$$

where \(C_3\) is a positive constant independent of \(m \in \mathbb{N}\) and \(t \in [0, T]\).

**Step 3: Analysis of the nonlinear term.**

By (4.13) and Assumption 3.2, we know that

$$\{h(\nabla v_m + \nabla \phi)\} \text{ is bounded in } L^\infty(0, T; L^2(\Omega)). \tag{4.23}$$

By the growth condition of \(f\) in Assumption 3.1 and (4.13), we have

$$\{f(v_m + \phi)\} \text{ is bounded in } L^\infty(0, T; L^2(\Omega)). \tag{4.24}$$

Denoting still by \(\{v_m\}\) its convergent subsequence obviously from the context without confusion, it follows from (4.23) and (4.24) that there exist \(\chi, \zeta \in L^2(\Omega)\) such that

$$h(\nabla v_m + \nabla \phi) \to \chi \text{ weakly in } L^2(0, T; L^2(\Omega)), \tag{4.25}$$

and

$$f(v_m + \phi) \to \zeta \text{ weakly in } L^2(0, T; L^2(\Omega)). \tag{4.26}$$

Next, by (4.22) and Assumption 3.3, we have that

$$\{l(v_m' + \phi')\} \text{ is bounded in } L^\infty(0, T; L^2(\Gamma_1)). \tag{4.27}$$

Hence there exists \(\psi \in L^2(0, T; L^2(\Gamma_1))\) such that

$$l(v_m' + \phi') \to \psi \text{ weakly in } L^2(0, T; L^2(\Gamma_1)). \tag{4.28}$$

On the other hand, since \(H^{\frac{1}{2}}(\Gamma_1) \hookrightarrow L^2(\Gamma_1)\) is continuous and compact and that

$$\|v_m(t)\|_{H^{\frac{1}{2}}(\Gamma_1)}^2 \leq C \int_{\Omega} |\nabla g v_m(t)|^2 \, dx \quad \text{and} \quad \int_{\Gamma_1} |v_m'(t)|^2 \, dx \leq C \int_{\Omega} |\nabla g v_m'(t)|^2 \, dx,$$

it follows from the Aubin–Lions theorem (see Theorem 5.1 on the pages 57–58 of [28]), (4.13), and (4.22) that

$$v_m \to v \text{ strongly in } L^2(0, T; L^2(\Gamma_1)). \tag{4.30}$$
Furthermore, \((4.13)\) and \((4.22)\) also imply that
\[
\begin{align*}
  v''_m &\to v'' \quad \text{weakly in } L^2(0, T; L^2(\Omega)), \\
  \nabla g v_m &\to \nabla g v \quad \text{and} \quad \nabla v_m \to \nabla v \quad \text{weakly in } L^2(0, T; (L^2(\Omega))^n),
\end{align*}
\]  
(4.31)  
(4.32)

Passing to the limit as \(m \to \infty\) in \((4.5)\), we obtain
\[
v'' - \Delta g v + \chi + \zeta = F \quad \text{in } D'(Q).
\]
Since \(u', \chi, \zeta \in L^2(Q)\), the above equality shows that \(\Delta g v \in L^2(Q)\) and
\[
v'' - \Delta g v + \chi + \zeta = F \quad \text{in } L^2(0, T; L^2(\Omega)).
\]

Thirdly, by the generalized Green formula (see the Appendix) and \((4.33)\), we have
\[
\frac{\partial v}{\partial \mu} + \psi = B \quad \text{in } D'(0, T; H^{-\frac{1}{2}}(\Gamma_1)).
\]

This together with the fact \(\psi, B \in L^2(0, T; L^2(\Gamma_1))\) shows that
\[
\frac{\partial v}{\partial \mu} + \psi = B \quad \text{in } L^2(0, T; L^2(\Gamma_1)).
\]

Now, we are in a position to show that
\(a\) \(f(v + \phi) = \zeta\);
\(b\) \(h(\nabla v + \nabla \phi) = \chi\);
\(c\) \(l(v' + \phi') = \psi\).

To this end, replace \(w\) by \(v_m\) in \((4.5)\) and integrate over \([0, T]\) in \(t\) to yield
\[
\begin{align*}
\int_Q v''_m(t)v_m(t)\,dx\,dt &+ \int_Q (\nabla g v_m(t), \nabla g v_m(t))_g\,dx\,dt + \int_Q h(\nabla v_m(t) + \nabla \phi(t))v_m(t)\,dx\,dt \\
&+ \int_Q f(v_m(t) + \phi(t))v_m(t)\,dx\,dt + \int_0^T \int_{\Gamma_1} l(v'_m(t) + \phi'(t))v_m(t)\,d\Gamma\,dt \\
&= \int_Q F(t)v_m(t)\,dx\,dt + \int_0^T \int_{\Gamma_1} B(t)v_m(t)\,d\Gamma\,dt, \quad v_m(0) = v'_m(0) = 0.
\end{align*}
\]

This together with \((4.13), (4.22)\), and the Aubin–Lions theorem (see Theorem 5.1 on the pages 57–58 of [28]) implies that
\[
\begin{align*}
  v_m &\to v \quad \text{strongly in } L^2(0, T; L^2(\Omega)), \\
  v'_m &\to v' \quad \text{strongly in } L^2(0, T; L^2(\Omega)),
\end{align*}
\]
(4.37)  
(4.38)

Furthermore, by \((4.37)\), \(v_m \to v\) a.e. in \(\Omega \times [0, T]\), and hence
\[
f(v_m + \phi) \to f(v + \phi) \quad \text{a.e. in } \Omega \times [0, T].
\]

Combine \((4.24), (4.39)\) and apply the Lions lemma (Lemma 1.3 of [28]), to obtain
\[
f(v_m + \phi) \to \xi = f(v + \phi) \quad \text{weakly in } L^2(0, T; L^2(\Omega)).
\]

The first claim \(a\) is thus proved. Similarly, by virtue of \((4.25), (4.28), (4.30), (4.37)\), and passing to the limit as \(m \to \infty\) in \((4.36)\) gives
\[
\lim_{m \to \infty} \int_Q |\nabla g v_m(t)|_g^2\,dx\,dt = -\int_Q v''(t)v(t)\,dx\,dt - \int_Q \chi(t)v(t)\,dx\,dt - \int_Q f(v(t) + \phi(t))v(t)\,dx\,dt \\
- \int_0^T \int_{\Gamma_1} \psi(t)v(t)\,d\Gamma\,dt + \int_Q F(t)v(t)\,dx\,dt + \int_0^T \int_{\Gamma_1} B(t)v(t)\,d\Gamma\,dt.
\]

Applying the generalized Green formula (see \((A.4)\) of the Appendix) again, we have, from \((4.33), (4.35)\) and \((4.41)\), that
\[
\lim_{m \to \infty} \int_Q |\nabla g v_m(t)|_g^2\,dx\,dt = \int_Q |\nabla g v(t)|_g^2\,dx\,dt.
\]

By \((4.32)\) and \((4.42)\), it has
\[
\nabla g v_m \to \nabla g v \quad \text{and} \quad \nabla v_m \to \nabla v \quad \text{strongly in } L^2(0, T; (L^2(\Omega))^n),
\]
(4.42)  
(4.43)
which leads to the convergence of the following
\[ \nabla_g v_m + \nabla_g \phi \to \nabla_g v + \nabla_g \phi \quad \text{a.e. in } \Omega, \]
\[ \nabla v_m + \nabla \phi \to \nabla v + \nabla \phi \quad \text{a.e. in } \Omega. \]

Consequently
\[ h(\nabla v_m + \nabla \phi) \to h(\nabla v + \nabla \phi) \quad \text{a.e. in } \Omega. \] (4.44)

With the application of the Lions lemma (Lemma 1.3 of [28]), (4.44) and (4.23) it guarantees that
\[ h(\nabla v_m + \nabla \phi) \to \chi = h(\nabla v + \nabla \phi) \quad \text{weakly in } L^2(0, T; L^2(\Omega)). \] (4.45)

This is claim (b).

In order to prove claim (c), replace \( w \) by \( v'_m \) in (4.5) and integrate over \([0, T]\), to obtain
\[
\int_0^T v'_m(t)v'(t)dt + \int_Q (\nabla_g v_m(t), \nabla_g v'_m(t))_g dt = \int_Q h(\nabla v_m(t) + \nabla \phi(t))v'_m(t)dt \]
\[ + \int_0^T f(v_m(t) + \phi(t))v'_m(t)dt + \int_0^T \int_{\Gamma_1} I(v_m(t) + \phi(t))v'_m(t)d\Gamma dt \]
\[ = \int_0^T F(t)v'_m(t)dt + \int_0^T \int_{\Gamma_1} B(t)v'_m(t)d\Gamma dt, \quad v_m(0) = v'_m(0) = 0. \] (4.46)

Since
\[ v'_m \to v' \quad \text{weakly in } L^2(0, T; L^2(\Omega)), \] (4.47)
\[ \nabla_g v'_m \to \nabla_g v' \quad \text{weakly in } L^2(0, T; (L^2(\Omega))'), \] (4.48)

combine (4.46) with (4.30), (4.31), (4.38), (4.43), and (4.45) to deduce
\[
\lim_{m \to \infty} \int_0^T \int_{\Gamma_1} l(v'_m(t) + \phi'(t))v'_m(t)d\Gamma dt = -\int_0^T \int_0^T \int_{\Gamma_1} l(v(t) + \phi(t))v'(t)dt dt \]
\[ - \int_0^T \int_0^T \int_0^T \int_{\Gamma_1} F(t)v'(t)dt dt + \int_0^T \int_{\Gamma_1} B(t)v'(t)d\Gamma dt. \] (4.49)

Applying the generalized Green formula (see (A.4) of the Appendix) and noticing (4.33) and (4.35), it deduces from the above equality that
\[
\lim_{m \to \infty} \int_0^T \int_{\Gamma_1} l(v'_m(t) + \phi'(t) - l(\varphi)) (v'_m(t) + \phi'(t) - \varphi) d\Gamma dt \geq 0, \quad \forall \varphi \in L^2(\Gamma_1). \] (4.50)

On the other hand, by the monotonicity of \( l \), we have
\[
\int_0^T \int_{\Gamma_1} l(v'_m(t) + \phi'(t)) \varphi d\Gamma dt \leq \int_0^T \int_{\Gamma_1} l(\varphi)(v'_m(t) + \phi'(t) - \varphi) d\Gamma dt \]
or equivalently,
\[
\int_0^T \int_{\Gamma_1} l(v'_m(t) + \phi'(t)) \varphi d\Gamma dt \leq \int_0^T \int_{\Gamma_1} l(\varphi)(v'_m(t) + \phi'(t) - \varphi) d\Gamma dt \]
\[ \leq \int_0^T \int_{\Gamma_1} l(v'_m(t) + \phi'(t))(v'_m(t) + \phi'(t)) d\Gamma dt. \] (4.51)

Hence
\[
\liminf_{m \to \infty} \int_0^T \int_{\Gamma_1} l(v'_m(t) + \phi'(t)) \varphi d\Gamma dt + \liminf_{m \to \infty} \int_0^T \int_{\Gamma_1} l(\varphi)(v'_m(t) + \phi'(t) - \varphi) d\Gamma dt \]
\[ \leq \liminf_{m \to \infty} \int_0^T \int_{\Gamma_1} l(v'_m(t) + \phi'(t))(v'_m(t) + \phi'(t)) d\Gamma dt. \] (4.52)
This together with (4.28), (4.47), and (4.50) gives
\[
\int_0^T \int_{\Gamma_1^t} (\psi(t) - l(\theta)) \left( u(t) + \phi'(t) - \phi \right) d\Gamma^t dt \geq 0 \quad \forall \phi \in L^2(\Gamma_1).
\] (4.53)

Replace \( \phi \) by \( u' + \phi' + \varepsilon \xi \) in (4.53) for arbitrary \( \xi \in L^2(\Gamma_1) \) and \( \varepsilon > 0 \), to get
\[
\int_0^T \int_{\Gamma_1^t} (\psi(t) - l((u'(t) + \phi'(t)) + \varepsilon \xi))(-\varepsilon \xi) d\Gamma^t dt \geq 0,
\]
and hence,
\[
\int_0^T \int_{\Gamma_1^t} (\psi(t) - l((u'(t) + \phi'(t)) + \varepsilon \xi))\xi d\Gamma^t dt \leq 0 \quad \forall \xi \in L^2(\Gamma_1).
\] (4.54)

Furthermore, since the operator \( L : L^2(\Gamma_1^t) \to (L^2(\Gamma_1))^t = L^2(\Gamma_1); \ v \mapsto l(v) \) is hemicontinuous, (4.54) implies that
\[
\int_0^T \int_{\Gamma_1^t} (\psi(t) - l(u'(t) + \phi'(t)))\xi d\Gamma^t dt \leq 0 \quad \forall \xi \in L^2(\Gamma_1).
\]

Therefore,
\[
\int_0^T \int_{\Gamma_1^t} (\psi(t) - l(u'(t) + \phi'(t)))\xi d\Gamma^t dt = 0 \quad \forall \xi \in L^2(\Gamma_1),
\]
which implies that \( \psi = l(u' + \phi') \). The claim (c) follows.

**Step 4: Uniqueness.**

Suppose \( z_1 \) and \( z_2 \) are two smooth solutions to the system (3.1). Then \( z = z_1 - z_2 \) satisfies
\[
\int_\Omega z''(t) w dt + \int_\Omega (\nabla_s z(t), \nabla_s w) dt + \int_{\Gamma_1} (l(z'_1) - l(z'_2)) w d\Gamma^t \]
\[
= \int_\Omega (h(z_2) - h(z_1)) w dt + \int_\Omega (f(z_2) - f(z_1)) w dt, \quad \forall w \in V.
\] (4.55)

Choose \( w = z'(t) \) in (4.55) and notice the monotonicity of \( l \), to get
\[
\frac{d}{dt} \left[ \frac{1}{2} \int_\Omega |z'(t)|^2 + |\nabla_s z(t)|^2 dt \right] \leq \int_\Omega (h(z_2) - h(z_1))z'(t) dt + \int_\Omega (f(z_2) - f(z_1))z'(t) dt.
\] (4.56)

First, by Assumption 3.3, the first term of the right-hand side of (4.56) satisfies
\[
\int_\Omega (h(z_2) - h(z_1))z'(t) dt \leq C \left( \int_\Omega |\nabla_s z(t)|^2 dt + \int_\Omega |z(t)|^2 dt \right).
\] (4.57)

Next, notice Assumption 3.1 and the fact \( 0 \leq \rho - 1 \leq \frac{2}{n-2} \) when \( n \geq 4 \), and \( 0 \leq \rho - 1 \leq 1 \) when \( n \leq 3 \). By a simple application of Hölder’s inequality and the Sobolev imbedding theorem, we can get the estimate for the second term on the right-hand side of (4.56) that
\[
\int_\Omega (f(z_2) - f(z_1))z'(t) dt \leq C \int_\Omega \left( 1 + |z_1|^{\rho - 1} + |z_2|^{\rho - 1} \right) |z(t)|^2 dt + C \int_\Omega |z'(t)|^2 dt
\]
\[
\leq C \left( \int_\Omega \left( 1 + |z_1|^{\rho - 1} + |z_2|^{\rho - 1} \right)^{\frac{3}{2}} |z(t)|^2 dt \right)^{\frac{2}{3}} + C \int_\Omega |z'(t)|^2 dt
\]
\[
\leq C \left( \int_\Omega |\nabla_s z(t)|^2 dt + \int_\Omega |z'(t)|^2 dt \right),
\] (4.58)

where \( q = \frac{n}{2}, p = \frac{n}{n - 2} \).

Finally, substituting (4.57) and (4.58) into (4.56), and then integrating over \( (0, t) \), after applying Gronwall’s lemma, we can get the uniqueness that \( \int_\Omega |z'(t)|^2 dt = \int_\Omega |\nabla_s z(t)|^2 dt = 0 \).

**Step 5: Existence of weak solution.**

Now suppose that the initial data for the system (3.1) satisfy
\[
\{u_0, u_1\} \in V \times L^2(\Omega).
\] (4.59)
Since $D(-\Delta_g) = \{u \in V \cap H^2(\Omega) | \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_1\}$ is dense in $V$ and $H^1_0(\Omega) \cap H^2(\Omega)$ is dense in $L^2(\Omega)$, there exist two sequences $\{u_{0m}\} \subset D(-\Delta_g)$ and $\{u_{1m}\} \subset H^1_0(\Omega) \cap H^2(\Omega)$ such that

$$u_{0m} \to u_0 \text{ strongly in } V,$$

$$u_{1m} \to u_1 \text{ strongly in } L^2(\Omega).$$

Under Assumption 3.3, it has

$$\frac{\partial u_{0m}}{\partial \mu} + l(u_{1m}) = 0 \quad \forall m \in \mathbb{N}.$$  

By the first part of Theorem 3.1 that has just been proven, for each $m \in \mathbb{N}$, there exists a smooth solution $u_m : \Omega \times (0, \infty) \to \mathbb{R}$ to the system:

$$\begin{cases}
u_{m}'' - \Delta_g u_m + h(\nabla u_m) + f(u_m) = 0 & \text{in } L^2(0, \infty; L^2(\Omega)), \\
u_m = 0 & \text{on } \Gamma_0 \times (0, \infty), \\
\frac{\partial u_m}{\partial \mu} + l(u_{1m}) = 0 & \text{in } L^2(0, \infty; L^2(\Gamma_1)), \\
\frac{\partial u_m}{\partial t}(x, 0) = u_{0m}(x), \quad u_m'(x, 0) = u_{1m}(x) & \text{in } \Omega.
\end{cases}

By Assumptions 3.2 and 3.3, using the same arguments as in obtaining (4.13), we can get

$$\int_\Omega |u_m'(t)|^2 \, dx + \int_0^t \int_\Omega |\nabla u_m(t)|^2 \, dx \, dt + \int_0^t \int_{\Gamma_1} |u_m'(s)|^2 \, d\Gamma \, ds \leq C,$$

which implies

$$\int_0^t \int_\Omega |h(\nabla u_m(s))|^2 \, dx \, ds \leq C,$$

$$\int_0^t \int_\Omega |f(u_m(s))|^2 \, dx \, ds \leq C,$$

$$\int_0^t \int_{\Gamma_1} |l(u_{1m}(s))|^2 \, d\Gamma \, ds \leq C.$$  

Consider $z_{mi} = u_m - u_i$ for all $m, i \in \mathbb{N}$ where $u_m$ and $u_i$ are smooth solutions to (4.63). Repeating the process of the proof for the uniqueness of the strong solution to (3.1) and making use of (4.60) and (4.61), we conclude that there exists $u : \Omega \times (0, \infty) \to \mathbb{R}$ such that

$$u_m \to u \text{ strongly in } C^0([0, T]; V),$$

$$u_m' \to u' \text{ strongly in } C^0([0, T]; L^2(\Omega)).$$

On the other hand, by (4.64)-(4.67) we have

$$u_m' \to u' \text{ weakly in } L^2(0, T; L^2(\Omega)),$$

$$h(\nabla u_m) \to \chi \text{ weakly in } L^2(0, T; L^2(\Omega)),$$

$$f(u_m) \to \zeta \text{ weakly in } L^2(0, T; L^2(\Omega)),$$

$$l(u_m') \to \psi \text{ weakly in } L^2(0, T; L^2(\Gamma_1)).$$

Now, from (4.68) we can easily get that $\chi = h(\nabla u)$ and $\zeta = f(u)$. Next, we show that $\psi = l(u')$. To this purpose, first, multiply both sides of the first equation in (4.63) by $u_m'$ and integrate over $\Omega$ in $x$, to obtain

$$\frac{1}{2} \frac{d}{dt} \int_\Omega (u_m'(t))^2 + |\nabla u_m(t)|^2 \, dx + \int_\Omega h(\nabla u_m(t))u_m'(t) \, dx$$

$$+ \int_\Omega f(u_m(t))u_m'(t) \, dx + \int_{\Gamma_1} l(u_m'(t))u_m'(t) \, d\Gamma = 0.$$  

Integrate (4.74) over $(0, t)$ in $s$ to get

$$\frac{1}{2} \int_\Omega (u_m'(t))^2 + |\nabla u_m(t)|^2 \, dx + \int_0^t \int_\Omega h(\nabla u_m(s))u_m'(s) \, dx \, ds$$

$$+ \int_0^t \int_\Omega f(u_m(s))u_m'(s) \, dx \, ds + \int_0^t \int_{\Gamma_1} l(u_m'(s))u_m'(s) \, d\Gamma \, ds = \frac{1}{2} \int_\Omega (u_{1m})^2 + |\nabla u_{0m}|^2 \, dx.$$
This together with (4.60), (4.61), and (4.70)–(4.73) deduces that
\[
\lim_{m \to \infty} \int_0^t \int_{\Gamma_1} l(u_m(s))u_m(s) d\Gamma ds = -\frac{1}{2} \int_{\Omega} (|u'(t)|^2 + |\nabla_y u(t)|_g^2) dx - \int_0^t \int_{\Omega} h(\nabla u(s))u'(s) dx ds \\
- \int_0^t \int_{\Omega} f(u(s))u'(s) dx ds + \frac{1}{2} \int_{\Omega} (|u'_1|^2 + |\nabla_y u_0|^2_g) dx. \tag{4.76}
\]

Next, suppose that \( y \) is a weak solution to the system
\[
\begin{cases}
y'' - \Delta_y y + h(\nabla y) + f(y) = 0 & \text{in } \Omega \times (0, \infty), \\
y = 0 & \text{on } \Gamma_0 \times (0, \infty), \\
\frac{\partial y}{\partial \mu} + \psi = 0 & \text{on } \Gamma_1 \times (0, \infty), \\
y(x, 0) = u_0(x), & y'(x, 0) = u_1(x) \quad \text{in } \Omega.
\end{cases} \tag{4.77}
\]

Then applying the same techniques as in the proofs of (4.76)–(4.77) gives
\[
\int_0^t \int_{\Gamma_1} \psi(s)y'(s) d\Gamma ds = -\frac{1}{2} \int_{\Omega} (|y'(t)|^2 + |\nabla_y y(t)|_g^2) dx - \int_0^t \int_{\Omega} h(\nabla y(s))y'(s) dx ds \\
- \int_0^t \int_{\Omega} f(y(s))y'(s) dx ds + \frac{1}{2} \int_{\Omega} (|u'_1|^2 + |\nabla_y u_0|^2_g) dx. \tag{4.78}
\]

Since \( u \) is a weak solution to (4.77), it follows from (4.76) and (4.78) that
\[
\lim_{m \to \infty} \int_0^t \int_{\Gamma_1} l(u'_m(s))u'_m(s) d\Gamma ds = \int_0^t \int_{\Gamma_1} \psi(s)u'(s) d\Gamma ds. \tag{4.79}
\]

Now the same arguments after (4.50) as in Step 3 can be applied to conclude that \( \psi = l(u') \). The proof is complete. \( \blacksquare \)

**Proof of Theorem 3.2.** We only prove the exponential stability for the strong solution to (3.1) since by a density argument, the same result holds true for the weak solution.

By the first equation and the boundary conditions in (3.1), it has
\[
\dot{E}(t) = \int_{\Omega} (2u'' u + 2(u_h u, \nabla_y u) + 2f(u)u') dx \\
= \int_{\Omega} (2u'' u - 2u'h(\nabla u) + 2(\nabla_y u, \nabla_y u')) dx \\
= \int_{\Gamma_1} 2u' \frac{\partial u}{\partial \mu} d\Gamma - \int_{\Omega} 2u'h(\nabla u) dx = -2 \int_{\Gamma_1} u'u' l'(u') d\Gamma - 2 \int_{\Omega} u'h(\nabla u) dx. \tag{4.80}
\]

This together with Assumption 3.3 and (3.5) gives
\[
\dot{E}(t) \leq \beta \int_{\Omega} (|u'|^2 + |\nabla_y u' u'|_g^2) dx - 2c_2 \int_{\Gamma_1} |u'|^2 d\Gamma - \beta E(t) - 2c_2 \int_{\Gamma_1} |u'|^2 d\Gamma, \tag{4.81}
\]

where \( E(t) \) defined by (3.12) is the energy function of the system (3.1). From the above inequality, it seems that the system (3.1) is not dissipative, which was claimed by [22] for constant coefficients case. However, this is a wrong impression. Actually, by introducing an equivalent energy function, we will find that the system (3.1) is essentially dissipative.

To this end, for any \( \varepsilon > 0 \), define a new energy function for the system (3.1) by
\[
E_\varepsilon(t) = E(t) + \varepsilon p(t), \tag{4.82}
\]

where
\[
p(t) = 2 \int_{\Omega} u'(x, t)H(u)(x, t) dx + (nb - \varepsilon_0) \int_{\Omega} u'(x, t)u(x, t) dx \quad \text{for some } \varepsilon_0 \in (0, b). \tag{4.83}
\]

Observe that
\[
|E_\varepsilon(t) - E(t)| = \varepsilon|p(t)| \leq \varepsilon C_1 E(t) \quad \forall \ t \geq 0 \tag{4.84}
\]

for some constant \( C_1 > 0 \). So, \( E_\varepsilon(t) \) is an equivalent energy function of (3.1) for small \( \varepsilon \). By this equivalence, the proof of exponential stability will be accomplished if we can show that there exists some constant \( C_2 > 0 \) such that
\[
\dot{E}_\varepsilon(t) \leq -C_2 E(t) \quad \forall \ t \geq 0. \tag{4.85}
\]
To this end, we estimate \( \hat{p}(t) \). Differentiate (4.83) with respect to \( t \), to get
\[
\hat{p}(t) = 2 \int_\Omega \Delta_g u H(u) dx - 2 \int_\Omega h(\nabla u) H(u) dx - 2 \int_\Omega f(u) H(u) dx + 2 \int_\Omega u' H(u') dx \\
+ (nb - \varepsilon_0) \int_\Omega \left( \Delta_g u - h(\nabla u) - f(u) \right) u dx + (nb - \varepsilon_0) \int_\Omega |u'|^2 dx \]
\[
= I_1(t) + I_2(t) + I_3(t) + I_4(t),
\]
where
\[
I_1(t) = \int_\Omega u' \left( 2H(u') + (nb - \varepsilon_0)u' \right) dx, \\
I_2(t) = \int_\Omega \Delta_g u (2H(u) + (nb - \varepsilon_0)u) dx, \\
I_3(t) = -\int_\Omega h(\nabla u)(2H(u) + (nb - \varepsilon_0)u) dx, \\
I_4(t) = -\int_\Omega f(u)(2H(u) + (nb - \varepsilon_0)u) dx.
\]

Now we estimate \( I_i(t), i = 1, 2, 3, 4 \), respectively.
\[
I_1(t) = \int_\Omega H(|u'|^2) dx + \int_\Omega (nb - \varepsilon_0) |u'|^2 dx \\
= \int_{\Gamma_1} |u'|^2 (H, \mu_g) \, d\Gamma - \int_\Omega \left( \text{div}_g H - nb \right) |u'|^2 dx - \varepsilon_0 \int_\Omega |u'|^2 dx,
\]
where in the second step, we used the following identity
\[
H(|u'|^2) = \text{div}_g (|u'|^2 H) - |u'|^2 \text{div}_g H
\]
and the divergence theorem in Lemma 3.1. Denoting by \( M = \max_{\Omega} |H| \_g \) and noticing the fact \( \text{div}_g H \geq nb \) from (3.15), we obtain
\[
I_1(t) \leq M \int_{\Gamma_1} |u'|^2 \, d\Gamma - \varepsilon_0 \int_\Omega |u'|^2 dx.
\]

Next, we estimate \( I_2(t) \). By the divergence theorem of Lemmas 3.1 and 3.3, we have
\[
I_2(t) = 2 \int_\Omega \text{div}_g \nabla_g u H(u) dx - 2 \int_\Omega \left( \nabla \langle \nabla_g (H(u)), \nabla_g u \rangle_g \right) dx + (nb - \varepsilon_0) \int_\Omega \text{div}_g (u \nabla_g u) dx - (nb - \varepsilon_0) \int_\Omega \delta_g \nabla_g u^2 dx \\
= 2 \int_{\partial \Omega} \frac{\partial u}{\partial \mu} H(u) d\Gamma - (nb - \varepsilon_0) \left( \int_\Omega \delta_g \nabla_g u^2 dx - \int_{\Gamma_1} \frac{\partial u}{\partial \mu} \, d\Gamma \right) \\
- 2 \int_\Omega \left( DH(\nabla_g u, \nabla_g u) + \frac{1}{2} \text{div}_g (|\nabla_g u|^2 H) - \frac{1}{2} |\nabla_g u|^2 \text{div}_g H \right) dx \\
= -2 \int_\Omega DH(\nabla_g u, \nabla_g u) dx + \int_{\Gamma_0} (\text{div}_g H - nb + \varepsilon_0) |\nabla_g u|^2 dx \\
+ \int_{\Gamma_1} \left( 2 \frac{\partial u}{\partial \mu} H(u) - |\nabla_g u|^2 (H, \mu_g) + (nb - \varepsilon_0) u \frac{\partial u}{\partial \mu} \right) \, d\Gamma + \int_{\Gamma_0} |\nabla_g u|^2 (H, \mu_g) \, d\Gamma,
\]
where the validity of the last step comes from the fact \( u = 0 \) on \( \Gamma_0 \) and hence
\[
\nabla_g u = \nabla \langle \nabla_g u, \mu_g \rangle, \quad \frac{\partial u}{\partial \mu} H(u) = \frac{\partial u}{\partial \mu} \nabla \langle \nabla_g u, H \rangle_g = \nabla \langle \nabla_g u, \mu_g \rangle (H, \mu_g) = |\nabla_g u|^2 \langle H, \mu_g \rangle \quad \text{on} \ \Gamma_0.
\]

Since
\[
\int_{\Gamma_1} 2 \frac{\partial u}{\partial \mu} H(u) \, d\Gamma \leq \int_{\Gamma_1} 2 \left| \frac{\partial u}{\partial \mu} \right| |H|_g |\nabla_g u|_g d\Gamma \\
\leq \int_{\Gamma_1} \left( \frac{\delta}{A} |\nabla_g u|_g^2 + \frac{\Lambda}{\delta} M^2 \left| \frac{\partial u}{\partial \mu} \right|^2 \right) d\Gamma,
\]
\[
(4.90)
\]
by (3.15) and the fact \( \langle H, \mu \rangle_g = \frac{1}{|v_c|^2} H \cdot v > \frac{\delta}{A} \) on \( \Gamma_1 \), we have, from (4.89), that

\[
I_2(t) \leq \int_{\Omega} (\text{div}_g H - (n + 2)b + \varepsilon_0)|\nabla g u|^2_g dx + \int_{\Gamma_1} \left( -\frac{\delta}{A} |\nabla g u|^2_g + (nb - \varepsilon_0)u \frac{\partial u}{\partial \mu} + \left( \frac{\delta}{A} |\nabla g u|^2_g + \frac{\Lambda}{M^2} \right) \frac{\partial u}{\partial \mu} \right)^2 d\Gamma
\leq (nB - (n + 2)b + \varepsilon_0) \int_{\Omega} |\nabla g u|^2_g dx + \int_{\Gamma_1} \left( (nb - \varepsilon_0)u \frac{\partial u}{\partial \mu} + \frac{\Lambda}{M^2} \right)^2 d\Gamma.
\]

Using the inequality \( \int_{\Gamma_1} |v|^2 d\Gamma \leq \frac{\tilde{c}}{\eta} \int_{\Omega} |\nabla g v|^2_g dx \) for any \( v \in V \), the boundary condition, Assumption 3.3, and the inequality \( ab \leq \eta a^2 + \frac{1}{\eta} b^2 \), \( \eta > 0 \), we estimate the last term on the right-hand side of (4.91) as

\[
\int_{\Gamma_1} \left( (nb - \varepsilon_0)u \frac{\partial u}{\partial \mu} + \frac{\Lambda}{M^2} \right)^2 d\Gamma = -\int_{\Gamma_1} \left( (nb - \varepsilon_0)ul(u') - \frac{\Lambda}{\delta} M^2 |u'(u')|^2 \right) d\Gamma
\leq c_2 (nb - \varepsilon_0) \int_{\Gamma_1} \left( \eta |u|^2 + \frac{1}{4\eta} |u'|^2 \right) dx + \frac{c_2 A M^2}{\delta} \int_{\Gamma_1} |u'|^2 d\Gamma
= \tilde{c} c_2 (nb - \varepsilon_0) \int_{\Omega} |\nabla g u|^2_g dx + \frac{c_2 (nb - \varepsilon_0)}{4\eta} + \frac{c_2 A M^2}{\delta} \int_{\Gamma_1} |u'|^2 d\Gamma
= \varepsilon_0 \int_{\Omega} |\nabla g u|^2_g dx + M_1 \int_{\Gamma_1} |u'|^2 d\Gamma.
\]

(4.92)

where \( \eta := \frac{\varepsilon_0}{c_2 (nb - \varepsilon_0)} + \frac{c_2 A M^2}{\delta} \) were used in the last step. Substitute (4.92) into (4.91), to obtain

\[
I_2(t) \leq (nB - (n + 2)b + 2\varepsilon_0) \int_{\Omega} |\nabla g u|^2_g dx + M_1 \int_{\Gamma_1} |u'|^2 d\Gamma.
\]

(4.93)

The estimation of \( I_3(t) \) comes from (3.5) and the Cauchy inequality

\[
I_3(t) \leq 2\beta M \int_{\Omega} |\nabla g u|^2_g dx + \beta (nb - \varepsilon_0) \int_{\Omega} |\nabla g u||u| dx
\leq \beta (2M + \tilde{c} nb) \int_{\Omega} |\nabla g u|^2_g dx,
\]

(4.94)

where \( \tilde{c} \) is the positive constant verifying

\[
\int_{\Omega} |v|^2 dx \leq \tilde{c} \int_{\Omega} |\nabla g v|^2_g dx \quad \forall \ v \in V.
\]

Finally, we estimate \( I_4(t) \). By (3.11), the nonnegativity of \( F, F(0) = 0 \), and the divergence formula, we have

\[
I_4(t) \leq -(nb - \varepsilon_0)r \int_{\Omega} 2F(u) dx - \int_{\Omega} 2H(F(u)) dx
= -\int_{\Gamma_1} ((nb - \varepsilon_0)r - \text{div}_g H)2F(u) dx - \int_{\Gamma_1} 2F(u) \langle H, \mu \rangle_g d\Gamma
\leq (nb - (n - \varepsilon_0)r) \int_{\Omega} 2F(u) dx.
\]

(4.95)

Let \( 0 < \beta < \frac{\varepsilon_0}{2M + \tilde{c} nb} \). Combine (4.87), (4.93), (4.94), (4.95) and (4.86) to obtain

\[
\dot{p}(t) \leq -\gamma E(t) + (M + M_1) \int_{\Gamma_1} |u'|^2 d\Gamma,
\]

(4.96)

where \( \gamma := \min((n + 2)b - nB - 3\varepsilon_0, (nb - \varepsilon_0)r - nB, \varepsilon_0) \) is positive by (3.9) in Assumption 3.4.

By (4.81), (4.82) and (4.96), we finally obtain

\[
\dot{\delta}(t) = \dot{E}(t) + \epsilon \dot{p}(t)
\leq -(\epsilon \gamma - \beta)E(t) - (2c_2 - \epsilon (M + M_1)) \int_{\Gamma_1} |u'|^2 d\Gamma.
\]
Taking $\varepsilon > 0$ sufficiently small such that $2c_2 - \varepsilon(M + M_1) > 0$ and choosing $\beta > 0$ small enough such that $\varepsilon\gamma - \beta > 0$, we obtain the desired inequality (4.85). The proof is complete. ■

To end this paper, we point out some facts for (1.1). Since (1.1) is a special case of system (3.1), the energy of system (1.1) is

$$
\tilde{E}(t) = \int_{\Omega} \left[ |u'(t)|^2 + |\nabla u(t)|^2 + 2F(u(t)) \right] dx,
$$

(4.97)

and from the proof of Theorem 3.2 we know that system (1.1) is actually dissipative under an equivalent energy function

$$
\tilde{E}(t) + 2\varepsilon \int_{\Omega} u'(x, t)(x - x_0) \cdot \nabla u(x, t) dx + \varepsilon(n - \varepsilon_0) \int_{\Omega} u'(x, t)u(x, t) dx
$$

(4.98)

for some constants $\varepsilon_0 \in (0, 1)$ and sufficiently small $\varepsilon > 0$. We specially state the following Corollary 4.1 as a consequence of Theorem 3.2, which is the main result of [22] obtained by regarding (1.1) as a nondissipative system.

**Corollary 4.1.** Let $u \in C([0, \infty); V) \cap C^1([0, \infty); L^2(\Omega))$ be a solution of (1.1). Under the conditions (3.2), (3.6), (3.7), (3.11), and (3.5) with $\lambda = 1$ and sufficiently small $\beta$, the energy $\tilde{E}(t)$ of system (1.1), defined by (4.97), decays exponentially in the sense

$$
\tilde{E}(t) \leq \tilde{c}\tilde{E}(0)e^{-\tilde{\omega}t} \quad \forall \ t \geq 0,
$$

(4.99)

for some constants $\tilde{c}, \tilde{\omega}$ independent of $t$.

**Appendix. Explanation of (4.34) and the generalized Green formula**

Similar to (4.5), for any $w \in V$, the following variation formula holds true

$$
\int_0^T \int_{\Omega} \frac{d}{dt}(\nabla_v v(t) w(t)) \psi(t) dx dt + \int_0^T \int_{\Omega} \langle \nabla_v v(t), \nabla_g w \rangle_g \psi(t) dx dt
$$

$$
= \int_0^T \int_{\Gamma_1} B(t)w(t) \psi(t) d\Gamma dt \quad \forall \ \psi \in D(0, T) \text{ and } v_m(0) = v'_m(0) = 0.
$$

(A.1)

Making use of (4.25), (4.26), (4.28), (4.31) and (4.32), and passing to the limit as $m \to \infty$ in (A.1) yield

$$
\int_0^T \int_{\Omega} \frac{d}{dt}(\nabla_v v(t) w(t)) \psi(t) dx dt + \int_0^T \int_{\Omega} \langle \nabla_v v(t), \nabla_g w \rangle_g \psi(t) dx dt
$$

$$
+ \int_0^T \int_{\Omega} \xi w(t) \psi(t) dx dt + \int_0^T \int_{\Gamma_1} \phi(t) w(t) \psi(t) d\Gamma dt
$$

$$
= \int_0^T \int_{\Omega} F(t)w(t) \psi(t) dx dt + \int_0^T \int_{\Gamma_1} B(t)w(t) \psi(t) d\Gamma dt.
$$

(A.2)

On the other hand, multiply both sides of (4.33) by $w(t)$ and integrate over $\Omega \times [0, T]$ to give

$$
\int_0^T \int_{\Omega} \frac{d}{dt}(v'(t) w(t)) \psi(t) dx dt - \int_0^T \int_{\Omega} \Delta_v v w(t) \psi(t) dx dt + \int_0^T \int_{\Omega} \chi w(t) \psi(t) dx dt
$$

$$
+ \int_0^T \int_{\Omega} \xi w(t) \psi(t) dx dt = \int_0^T \int_{\Omega} F(t)w(t) \psi(t) dx dt.
$$

(A.3)

Since the term $\int_0^T \int_{\Omega} \langle \nabla_v v(t), \nabla_g w \rangle_g \psi(t) dx dt$ makes sense, one can define

$$
\int_0^T \int_{\Omega} \Delta_v v w(t) \psi(t) dx dt + \int_0^T \int_{\Omega} \langle \nabla_v v(t), \nabla_g w \rangle_g \psi(t) dx dt \equiv \int_0^T \int_{\Gamma_1} \frac{\partial w}{\partial \mu} \psi(t) d\Gamma dt
$$

$$
\equiv \left\{ \frac{\partial w}{\partial \mu}(\psi, w) \right\}_{H^{-\frac{1}{2}}(\Gamma_1) \times H^{\frac{1}{2}}(\Gamma_1)},
$$

(A.4)

which means that $\frac{\partial w}{\partial \mu} \in \mathcal{D}'(0, T; H^{-\frac{1}{2}}(\Gamma_1))$ because $w \in H^{\frac{1}{2}}(\Gamma_1)$ and $\psi \in \mathcal{D}(0, T)$. When $v$ is smooth enough, the above equality is just the Green formula. We call (A.4) the generalized Green formula. (4.34) is a consequence of (A.2), (A.3), and (A.4).
References