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Controllability and stability of a second-order hyperbolic system with collocated sensor/actuator

Bao-Zhu Guo[a,*], Yue-Hu Luo[b]

[a] *Institute of Systems Science, Academy of Mathematics and System Sciences, Academia Sinica, Beijing 100080, People's Republic of China*

[b] *Department of Applied Mathematics, Nanjing University of Science and Technology, Nanjing 210094, People's Republic of China*

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Abstract

A second-order hyperbolic system with collocated sensor/actuator is considered. The semigroup generation is shown for the closed-loop system under the feedback of a generic unbounded observation operator. The equivalence between the exponential stability of the closed-loop system and exact controllability of the open-loop system is established in the general framework of well-posed linear systems. Finally, the conditions are weakened for the diagonal semigroups with finite dimensional inputs. Example of beam equation is presented to display the application. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

In the past two decades, much effort has been made for the stability analysis of flexible systems under boundary feedback controls (see [3,4]). Because of the simplicity and effectiveness of the direct output feedback, the methods of collocated sensor/actuator control strategy have been applied in the control of vibration of flexible robot arms ([15,16,30]) and smart structures [5]. Such kind of systems can be modeled by the following infinite dimensional system in a Hilbert space H:

$$\dot{x}(t) = Ax(t) + Bu(t),$$

$$y(t) = B^*x(t), \tag{1}$$

where the system operator A is skew-adjoint in H and hence generates a C_0-group on H, $u(t)$ is the control input and $y(t)$ is the output of the system. The closed-loop system is produced by the output feedback control

$$u(t) = -Ky(t) = -KB^*x(t) \tag{2}$$

* Corresponding author. Fax: +86-10-62587343.
E-mail address: bzguo@iss03.iss.ac.cn (B.-Z. Guo).

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for some bounded feedback operator K from the output space to the input space, and hence the closed-loop system becomes

$$\dot{x}(t) = (A - BKB^*)x(t). \tag{3}$$

System (1) is called output feedback stabilizable if there is a K such that system (3) is stable in some sense. When B is a bounded operator from the control space to the state space, it is shown in [2] that the weak controllability of system (1) implies the weak stability of system (3) and also the strong stability provided that A has compact resolvent. Slemrod [24] showed that if system (1) is exactly controllable on some $[0,T]$, $T > 0$, then it is exponentially stabilizable. The well-known Russell's "controllability via stability" principle [20] shows, on the other hand, that the exponential stability of system (3) implies the exact controllability of system (1) on some $[0,T]$, $T > 0$.

It should be indicated that for linear systems with bounded control operator B in Hilbert spaces, many profound results are already known in earlier literatures. The limitation of the characterization of exact controllability by bounded control is obtained in [17]. A recent nice summary can be found in [8]. For instance, Theorem 4.15 of [8] says that if A generates a C_0-semigroup in a Hilbert space H and B is a bounded operator and the control space is finite dimensional, then the linear system of the following:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

is not exactly controllable in $[0,t]$ for any finite $t > 0$. On the other hand, when B is unbounded but admissible [27], the exact controllability is possible even for scalar control. We refer to [11,12] for recent consideration on this respect.

However, for some given K, few results are known in the literature on the relationship between the exponential stability of system (3) and the exact controllability of system (1) when the control operator B is admissible. Although some efforts have been made from the point of view of optimal feedback control [18], the whole matter could not be understood until the generic theory of linear well-posed system theory was developed (see [6] and the references therein). In this paper, we discuss such a relation for a second-order hyperbolic system with generic admissible input B of the following:

$$\begin{aligned} &y_{tt} + Ay + Bu(t) = 0, \\ &O(t) = B^* y_t, \end{aligned} \tag{4}$$

where (i) $A : D(A)(\subset X) \to X$ is an unbounded positive self adjoint operator in the Hilbert space X. $D(A)$ as well as $D(A^{1/2})$ is dense in X. We identify X with its dual X', so the following relations hold:

$$D(A^{1/2}) \subset X \subset D(A^{1/2})'.$$

(ii) $B \in \mathscr{L}(U, D(A^{1/2})')$, where U is the control Hilbert space.

(iii) $B^* \in \mathscr{L}(D(A^{1/2}), U)$ is defined as

$$\langle B^*x, u\rangle_{U\times U} = \langle x, Bu\rangle_{D(A^{1/2})\times D(A^{1/2})'} \quad \forall x \in D(A^{1/2}). \tag{5}$$

(iv) An extension $\tilde{A} \in \mathscr{L}(D(A^{1/2}), D(A^{1/2})')$ of A is defined by

$$\langle \tilde{A}x, z\rangle_{D(A^{1/2})'\times D(A^{1/2})} = \langle A^{1/2}x, A^{1/2}z\rangle_{X\times X}, \quad \forall x,\ z \in D(A^{1/2}). \tag{6}$$

$\tilde{A}$ is an isometry from $D(A^{1/2})$ to $D(A^{1/2})'$ by virtue of the Lax–Milgram theorem.

By the closed-loop form of system (4), we mean system (4) under the direct output feedback control $u(t) = O(t)$. Our first result in Section 2 shows that the closed-loop system is well-posed without admissibility assumption. In Section 3, it is shown that the exponential stability of the closed-loop system implies the exact controllability of the open-loop system. In Section 4, by assuming the boundedness of the transfer function of system (4), it is shown that the exact controllability of the open-loop system also implies the exponential stability of the closed-loop system. Finally, in Section 5, we discuss the application to diagonal semigroups where the admissibility is shown to guarantee the boundedness of the transfer function under some spectral condition of A. This avoids the verification of boundedness of transfer function which is difficult in applications. An example of beam equation is presented to display the application of our results in the vibration control of flexible systems.

2. Semigroup generation

Let A, B, B^* be defined as in (4). It is well-known that for any $\lambda \in \rho(A)$, $\lambda - \tilde{A}$ is an isometric from $D(A^{1/2})$ to $D(A^{1/2})'$ and

$$\tilde{A}x = Ax \quad \text{for any } x \in D(A).$$

Moreover, it is easy to verify that $\tilde{A}^{-1}B \in \mathscr{L}(U, D(A^{1/2}))$ and more generally

$$B^*(\lambda - \tilde{A})^{-1}B \in \mathscr{L}(U) \quad \text{for any } \lambda \in \rho(A).$$

By these definitions, we may formulate (4) to be

$$y_{tt} + \tilde{A}y + Bu(t) = 0 \quad \text{in } D(A^{1/2})'. \tag{7}$$

Design the feedback control

$$u(t) = B^* y_t. \tag{8}$$

Then the closed-loop system becomes

$$y_{tt} + \tilde{A}y + BB^* y_t = 0 \quad \text{in } D(A^{1/2})', \tag{9}$$

which can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y \\ y_t \end{pmatrix} = \begin{pmatrix} 0 & I \\ -\tilde{A} & -BB^* \end{pmatrix}\begin{pmatrix} y \\ y_t \end{pmatrix} \quad \text{in } D(A^{1/2}) \times D(A^{1/2})'. \tag{10}$$

However, we want to consider system (10) in the energy state space $H = D(A^{1/2}) \times \mathrm{X}$. To this purpose, define

$$\mathscr{A}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 & I \\ -\tilde{A} & -BB^* \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} g \\ -\tilde{A}f - BB^*g \end{pmatrix} \tag{11}$$

with

$$D(\mathscr{A}) = \{(f, g) \mid f, g \in D(A^{1/2}), -\tilde{A}f - BB^*g \in X\}. \tag{12}$$

Theorem 1. *$\mathscr{A}$ generates a C_0-semigroup of contractions on H.*

Proof. First, we show that $\mathscr{A}$ is dissipative. Let $(f, g) \in D(\mathscr{A})$. Then

$$\begin{aligned} Re\langle \mathscr{A}(f,g),(f,g)\rangle &= Re\langle A^{1/2}g, A^{1/2}f\rangle_{X\times X} - Re\langle \tilde{A}f + BB^*g, g\rangle_{X\times X} \\ &= Re\langle A^{1/2}g, A^{1/2}f\rangle_{X\times X} - Re\langle \tilde{A}f + BB^*g, g\rangle_{D(A^{1/2})'\times D(A^{1/2})} \\ &= Re\langle \tilde{A}f, g\rangle_{D(A^{1/2})\times D(A^{1/2})'} - Re\langle \tilde{A}f + BB^*g, g\rangle_{D(A^{1/2})'\times D(A^{1/2})} \end{aligned}$$

$$= -Re\langle BB^* g, g\rangle_{D(A^{1/2})'\times D(A^{1/2})}$$

$$= -Re\langle B^* g, B^* g\rangle_{U\times U} = -\|B^* g\|_U^2 \leqslant 0.$$

So $\mathscr{A}$ is dissipative. Next, we show that $\mathscr{A}^{-1}$ exists. Solving

$$\mathscr{A}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} g \\ -\tilde{A}f - BB^* g \end{pmatrix} = \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in H$$

we have $g = \phi \in D(A^{1/2})$ and $-\tilde{A}f - BB^* g = \psi$. The later is equivalent to

$$\tilde{A}f = -BB^*\phi - \psi \in D(A^{1/2})'.$$

However, since $\tilde{A}$ is an isometry from $D(A^{1/2})$ to $D(A^{1/2})'$, the above equation is solvable in $D(A^{1/2})$ with

$$f = \tilde{A}^{-1}(-BB^*\phi - \psi).$$

So

$$\mathscr{A}^{-1}\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \tilde{A}^{-1}(-BB^*\phi - \psi) \\ \phi \end{pmatrix}. \tag{13}$$

Therefore, $\mathscr{A}$ generates a C_0-semigroup on H by the Lumer–Phillips theorem [19]. $\square$

We claim that for any $\psi \in X$, $\tilde{A}^{-1}\psi = A^{-1}\psi$. Indeed, let $\tilde{A}^{-1}\psi = w$. Then $w \in D(A^{1/2})$ and $\tilde{A}w = \psi$. For any $x \in D(A)$, we have

$$\langle w, Ax\rangle = \langle A^{1/2}w, A^{1/2}x\rangle = \langle \tilde{A}w, x\rangle = \langle \psi, x\rangle.$$

Hence $w \in D(A^*) = D(A)$ and so $Aw = \psi$, that is $w = A^{-1}\psi$. So (13) can be written as

$$\mathscr{A}^{-1}\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} -\tilde{A}^{-1}BB^*\phi - A^{-1}\psi \\ \phi \end{pmatrix}. \tag{14}$$

Corollary 1. *B^* satisfies*

$$\int_0^T \|B^* y_t\|_U^2\,\mathrm{d}t \leqslant \frac{1}{2}\|(y_0, y_1)\|_H^2 \quad \forall (y_0, y_1) \in D(\mathscr{A}),\ \ T > 0.$$

Proof. By assumption, $(y, y_t) \in D(\mathscr{A})$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y \\ y_t \end{pmatrix} = \mathscr{A}\begin{pmatrix} y \\ y_t \end{pmatrix} \in H.$$

Taking inner product with (y, y_t) on both sides of the above in H, one has

$$\langle A^{1/2}y_t, A^{1/2}y\rangle + \langle y_{tt}, y_t\rangle = \langle A^{1/2}y_t, A^{1/2}y\rangle - \langle \tilde{A}y + BB^* y_t, y_t\rangle$$

$$= \langle A^{1/2}y_t, A^{1/2}y\rangle - \langle A^{1/2}y, A^{1/2}y_t\rangle - \langle BB^* y_t, y_t\rangle.$$

Hence,

$$\langle y_{tt}, y_t\rangle + \langle A^{1/2}y, A^{1/2}y_t\rangle = -\langle BB^* y_t, y_t\rangle.$$

That is

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\|B^* y_t\|_U^2,$$

where

$$E(t) = \tfrac{1}{2}\,[\|A^{1/2}y\|^2 + \|y_t\|^2]. \tag{15}$$

Hence,

$$\int_0^T \|B^* y_t\|_U^2 \,\mathrm{d}t = E(0) - E(T) \leqslant E(0). \qquad \Box \tag{16}$$

Corollary 1 shows that although $(y, y_t) \in H$ provided that $(y_0, y_1) \in H, B^* y_t$, however, is well-defined in $L^2(0, T; U)$ for any $T > 0$. Namely

$$\Pi(y_0, y_1) = B^* y_t \tag{17}$$

is a well-defined linear-bounded operator from H to $L^2(0, T; U)$ for any $T > 0$. In other words, $(0, B)^{\mathrm{T}}$ is always admissible with respect to the semigroup generated by $\mathscr{A}$ [27]. By this reason, we always understand $B^* y_t$ in the sense of (17) for any $(y_0, y_1) \in H$.

3. Controllability via stability

From generic well-posed linear system theory [6], the basic assumption for the following observation system

$$\phi_{tt} + A\phi = 0,$$

$$O(t) = B^* \phi_t \tag{18}$$

making sense is the admissibility assumption

$$\int_0^T \|B^* \phi_t\|_U^2 \,\mathrm{d}t \leqslant C_T \|(\phi_0, \phi_1)\|_H^2 \quad \forall (\phi_0, \phi_1) \in D(\mathbf{A}) \tag{19}$$

for some $T > 0$ and $C_T > 0$, where (ϕ_0, ϕ_1) is the initial condition of (18) and the operator $\mathbf{A}$ is defined as

$$\mathbf{A} = \begin{pmatrix} 0 & I \\ -\tilde{A} & 0 \end{pmatrix}, \quad D(\mathbf{A}) = \{(f, g) \in H,\ \mathbf{A}(f, g) \in H\}. \tag{20}$$

Suppose that $(f, g) \in H$, $\tilde{A}(f, g) \in$H. Then $f \in D(A)$ and $-Af = g$. That is

$$\mathbf{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}. \tag{21}$$

When condition (19) is satisfied, we say that B is admissible. In this case, for any initial condition (ϕ_0, ϕ_1), the output $O(t)$ makes sense in the sense of (19), or by [26]

$$O(t) = \mathbf{B}_L^* \mathrm{e}^{\mathbf{A}t} (\phi_0, \phi_1)^{\mathrm{T}}, \quad t \geqslant 0 \text{ a.e.}, \tag{22}$$

where

$$\mathbf{B} = \begin{pmatrix} 0 \\ B \end{pmatrix} \tag{23}$$

$\mathbf{B}^* = (0, B^*)$ and $\mathbf{B}_L^*$ is the Lebesgue extension of $\mathbf{B}^*$.

Admissibility is a fundamental assumption for the well-posedness of the controlled system (7). Actually write (7) to be

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y \\ y_t \end{pmatrix} = \mathbf{A}\begin{pmatrix} y \\ y_t \end{pmatrix} + \mathbf{B}u(t) \quad \text{in } D(A^{1/2}) \times D(A^{1/2})'. \tag{24}$$

It is seen that $\mathbf{B} \in \mathscr{L}(U, D(\mathbf{A}^*)' = D(\mathbf{A})')$ with

$$\left\langle \mathbf{B}u, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle_{D(\mathbf{A})' \times D(\mathbf{A})} = \langle Bu, g \rangle_{D(A^{1/2})' \times D(A^{1/2})}.$$

Therefore, the admissibility assumption (19) is nothing but the admissibility of $\mathbf{B}$ with respect to $\mathrm{e}^{\mathbf{A}t}$ (note that $\mathbf{A}$ is skew-adjoint). Under assumption (19), for every $u \in L^2(0,T;U)$, $T > 0$ and the initial condition $(y_0, y_1) \in H$, there exists a unique solution to (24)

$$\begin{pmatrix} y(t) \\ y_t(t) \end{pmatrix} = \mathrm{e}^{\mathbf{A}t}\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} + \Phi(t)u, \tag{25}$$

where $\Phi(t) : L^2(0,T;U) \to H$ is the strongly continuous family of bounded operators given by

$$\langle \Phi(t)u, Z \rangle = \int_0^t \langle \mathbf{B}u(s), \mathrm{e}^{-\mathbf{A}(t-s)}Z \rangle_{D(\mathbf{A})' \times D(\mathbf{A})}\, \mathrm{d}s \quad \forall Z \in D(\mathbf{A}).$$

The generic well-posed linear abstract system theory tells us that $\Phi(t)u$ is continuous simultaneously with respect to (t,u) (see [9,27]).

The next result is the infinite dimensional version of Russell's "controllability via stability" principle.

Theorem 2. *Suppose that the control operator B is admissible. If $\mathscr{A}$ generates an exponential stable C_0-semigroup, then system (18) is exact observable on some $[0,T]$, $T > 0$, in H, namely, there exists $D_T > 0$ such that*

$$\int_0^T \|O(t)\|_U^2\, \mathrm{d}t \geqslant D_T \|(\phi_0, \phi_1)\|_H^2 \quad \forall (\phi_0, \phi_1) \in D(\mathbf{A}). \tag{26}$$

Proof. By duality principle, we only need to show that system (24) is exactly controllable on some $[0,T]$, $T > 0$. That is, for any given $(y_0^*, y_1^*) \in H$, there exists an $T > 0$ such that the solution to (24) satisfies

$$y(T) = y_0^*, \quad y_t(T) = y_1^*.$$

Since $\mathscr{A}$ is exponentially stable, there is an $T_0 > 0$ such that for all $T > T_0$

$$\|\mathrm{e}^{\mathscr{A}T}\| < 1. \tag{27}$$

Let

$$\begin{pmatrix} w(t) \\ w_t(t) \end{pmatrix} = \mathrm{e}^{\mathscr{A}t}\begin{pmatrix} w_0 \\ w_1 \end{pmatrix}, \quad 0 \leqslant t \leqslant T, \tag{28}$$

which defines a solution to Eq. (10) with initial condition $(w_0, w_1) \in H$ that will be determined later. From Corollary 1 and (25), $u_1 = B^* y_t \in L^2(0,T;U)$ and

$$\begin{pmatrix} w(t) \\ w_t(t) \end{pmatrix} = \mathrm{e}^{\mathbf{A}t}\begin{pmatrix} w_0 \\ w_1 \end{pmatrix} + \Phi(t)u_1. \tag{29}$$

Next, consider

$$z_{tt} + \tilde{A}z - BB^* z_t = 0,$$

$$z(T) = y(T), \quad z_t(T) = y_t(T). \tag{30}$$

Let $\eta(t) = z(T - t)$. Then η satisfies

$$\eta_{tt} + \tilde{A}\eta + BB^*\eta_t = 0,$$

$$\eta(0) = w(T), \quad \eta_t(0) = -w_t(T). \tag{31}$$

So, $u_2(t) = -B^*\eta_t(T - t) = -B^* z_t(t) \in L^2(0, T; U)$. Hence, the solution of (30) can be written as

$$\begin{pmatrix} z \\ z_t \end{pmatrix} = \mathrm{e}^{\mathbf{A}t} \begin{pmatrix} z(0) \\ z_t(0) \end{pmatrix} - \Phi(t)u_2. \tag{32}$$

Set

$$y(t) = w(t) - z(t), \quad u = u_1 + u_2 \in L^2(0, T; U).$$

Then it follows from (29) and (32) that

$$\begin{pmatrix} y \\ y_t \end{pmatrix} = \mathrm{e}^{\mathbf{A}t} \left[\begin{pmatrix} w_0 \\ w_1 \end{pmatrix} - \begin{pmatrix} z(0) \\ z_t(0) \end{pmatrix} \right] + \Phi(t)u \tag{33}$$

and $y(T) = y_t(T) = 0$. The proof is complete if we can show that for any given (y_0, y_1), there is a (w_0, w_1) such that

$$\begin{pmatrix} w_0 \\ w_1 \end{pmatrix} - \begin{pmatrix} z(0) \\ z_t(0) \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}. \tag{34}$$

This is true since from (31)

$$\begin{pmatrix} \eta(t) \\ \eta_t(t) \end{pmatrix} = \mathrm{e}^{\mathscr{A}t} \begin{pmatrix} w(T) \\ -w_t(T) \end{pmatrix}.$$

Hence,

$$\left\| \begin{pmatrix} z(0) \\ z_t(0) \end{pmatrix} \right\| \leqslant \|\mathrm{e}^{\mathscr{A}T}\| \left\| \begin{pmatrix} w(T) \\ -w_t(T) \end{pmatrix} \right\| \leqslant \|\mathrm{e}^{\mathscr{A}T}\|^2 \left\| \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} \right\|.$$

So the map

$$\begin{pmatrix} w_0 \\ w_1 \end{pmatrix} \to \begin{pmatrix} z(0) \\ z_t(0) \end{pmatrix} = \mathscr{P} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}$$

is a contraction map. Therefore,

$$\begin{pmatrix} w_0 \\ w_1 \end{pmatrix} - \begin{pmatrix} z(0) \\ z_t(0) \end{pmatrix} = (I - \mathscr{P}) \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$$

has a unique solution

$$\begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = (I - \mathscr{P})^{-1} \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}. \qquad \square \tag{35}$$

4. Stability via controllability

In this section, we consider the reversion of Theorem 2, namely, whether the exact observability of system (18) implies the exponential stability of the semigroup $e^{\mathscr{A}t}$?

Let us study the following control-observation system

$$y_{tt} + \tilde{A}y + Bu(t) = 0,$$

$$O_u(t) = B^* y_t \tag{36}$$

or its first-order form

$$\frac{d}{dt}\begin{pmatrix} y \\ y_t \end{pmatrix} = \mathbf{A}\begin{pmatrix} y \\ y_t \end{pmatrix} + \mathbf{B}u(t), \quad \text{in } D(A^{1/2}) \times D(A^{1/2})',$$

$$O_u(t) = \mathbf{B}^* \begin{pmatrix} y \\ y_t \end{pmatrix}. \tag{37}$$

Suppose B is admissible. Then the generic well-posed abstract linear system theory (see e.g. [7]) tells us that (37) defines a well-posed linear system if and only if its transfer function $\mathbf{H}(s) \in \mathscr{L}(U)$ determined (up to a constant bounded linear operator of $\mathscr{L}(U)$) by

$$\frac{\mathbf{H}(s) - \mathbf{H}(\beta)}{s - \beta} = -\mathbf{B}^*(s - \mathbf{A})^{-1}(\beta - \tilde{\mathbf{A}})^{-1}\mathbf{B} \quad \textit{for any } s,\ \beta \in \rho(\mathbf{A}) \text{ with } s \neq \beta \tag{38}$$

is uniformly bounded on some vertical line parallel to the imaginary axis:

$$\sup_{\mathrm{Re}s = \alpha} \|\mathbf{H}(s)\| < \infty \quad \text{for some } \alpha > 0, \tag{39}$$

where

$$\tilde{\mathbf{A}} = \begin{pmatrix} 0 & I \\ -\tilde{A} & 0 \end{pmatrix}. \tag{40}$$

Remark 1. In the original paper [7], (39) is replaced by

$$\sup_{\mathrm{Re}s \geqslant \alpha} \|\mathbf{H}(s)\| < \infty \quad \text{for some } \alpha > 0. \tag{41}$$

However, the arguments there can be changed a little bit so that assumption (39) is sufficient for the validity of the results. Moreover, when B is admissible, (39) implies (41). This can also be shown by the Lindelöf theorem in complex analysis (see e.g. [25]). Indeed, by (38)

$$\|\mathbf{H}(s)\| \leqslant \|\mathbf{H}(\alpha)\| + |\alpha - s|\, \|\mathbf{B}^*(\alpha - \mathbf{A})^{-1}\|\, \|(s - \tilde{\mathbf{A}})^{-1}\mathbf{B}\|.$$

However, from [28]

$$\|(s - \tilde{\mathbf{A}})^{-1}\mathbf{B}\| \leqslant \frac{K}{\sqrt{\mathrm{Re}s}} \quad \text{for some } K > 0 \text{ and } \mathrm{Re}s > \alpha.$$

Hence the Lindelöf theorem can be applied to get the implication of (41) and (39).

Now, it is easy to find

$$(s-\tilde{\mathbf{A}})^{-1}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} (s^2+\tilde{A})^{-1}(g+sf) \\ s(s^2+\tilde{A})^{-1}(g+sf)-f \end{pmatrix}, \quad \forall (f,g)\in D(A^{1/2})\times D(A^{1/2})'. \tag{42}$$

Define

$$h(s)=\mathbf{B}^*(s-\tilde{\mathbf{A}})^{-1}\mathbf{B}=sB^*(s^2+\tilde{A})^{-1}B. \tag{43}$$

It is obvious that for any s with $\mathrm{Re}s>0$, $H(s)\in\mathscr{L}(U)$ and

$$\frac{h(s)-h(\beta)}{s-\beta}=-\mathbf{B}^*(s-\mathbf{A})^{-1}(\beta-\tilde{\mathbf{A}})^{-1}\mathbf{B} \quad \text{for all } s,\ \beta\in\rho(\mathbf{A}),\ s\neq\beta.$$

So condition (39) is satisfied if and only if assumption (H) holds:

Assumption (H).

$$\sup_{\mathrm{Re}s=\alpha}\|h(s)\|<\infty \quad \text{for some } \alpha>0. \tag{44}$$

Therefore, under Assumption (H) and the admissibility of B (A,B,B^*) is well posed in the sense of [7]. Furthermore, since for any $s\in\rho(\tilde{\mathbf{A}})$

$$(s-\tilde{\mathbf{A}})^{-1}\mathbf{B}u=\begin{pmatrix} (s^2+\tilde{A})^{-1}Bu \\ s(s^2+\tilde{A})^{-1}Bu \end{pmatrix}\in D(\mathbf{B}^*) \quad \forall u\in U \tag{45}$$

it follows from the appendix that the transfer function of (37) is

$$\mathbf{H}(s)=h(s)=sB^*(s^2+\tilde{A})^{-1}B. \tag{46}$$

For any initial condition $(y_0,y_1)\in H$, the solution of (37) is found to be

$$\begin{pmatrix} y \\ y_t \end{pmatrix}=T(t)\begin{pmatrix} y_0 \\ y_1 \end{pmatrix}+\Phi(t)u\in H,$$

$$O_u(t)=\mathbf{B}_L^*\begin{pmatrix} y \\ y_t \end{pmatrix}=L_\infty\begin{pmatrix} y_0 \\ y_1 \end{pmatrix}+F_\infty u, \tag{47}$$

where $F_\infty\in\mathscr{L}(L^2_{\mathrm{loc}}(0,\infty;U))$, $L_\infty\in\mathscr{L}(H,L^2_{\mathrm{loc}}(0,\infty;U))$ are bounded linear operators. In particular, when $(y_0,y_1)=0$ and the Laplace transform $\hat{u}(s)$ of u exists, it holds that

$$\hat{O}(s)=\mathbf{H}(s)\hat{u}(s) \quad \text{for all } \mathrm{Re}s>0. \tag{48}$$

Theorem 3. *Suppose*

(i) *B is admissible.*
(ii) *Assumption* (H) *holds.*

Then the reversion Theorem 2 *holds true. That is, if system* (18) *is exactly observable on some* $[0,T]$, $T>0$, *in* H, *then* $\mathscr{A}$ *generates an exponential stable* C_0*-semigroup.*

Proof. Suppose without loss of generality that $(y_0, y_1) \in D(\mathscr{A})$. Let $E(t)$ be defined by (15). Then $e^{\mathscr{A}t}$ is exponentially stable if and only if

$$E(0) - E(T) \geqslant cE(0) \quad \text{for some } c > 0 \text{ and } T > 0. \tag{49}$$

However, since

$$\int_0^T \|B^* y_t\|_U^2 \, dt = E(0) - E(T)$$

(49) holds if and only if

$$\int_0^T \|B^* y_t\|_U^2 \, dt \geqslant cE(0). \tag{50}$$

Now, decompose y into $y = \phi + \psi$, where ϕ satisfies

$$\begin{aligned} &\phi_{tt} + A\phi = 0, \quad t \in (0, T], \\ &\phi(0) = y_0, \quad \phi_t(0) = y_1 \end{aligned} \tag{51}$$

and ψ satisfies

$$\begin{aligned} &\psi_{tt} + \tilde{A}\psi = -Bu(t), \quad u(t) = B^* y_t, \quad t \in (0, T], \\ &\psi(0) = 0, \quad \psi_t(0) = 0. \end{aligned} \tag{52}$$

Note that $B^* y_t = B^* \phi_t + B^* \psi_t$. Since (18) is exactly observable by output $B^* \phi_t$, in order to show (50), we need only show that

$$\int_0^T \|B^* y_t\|_U^2 \, dt \geqslant \tilde{c} \int_0^T \|B^* \phi_t\|_U^2 \, dt \quad \text{for some } \tilde{c} > 0. \tag{53}$$

Note that $B^* \phi_t$ and $B^* \psi_t$ make sense under the admissibility of B and

$$\int_0^T \|B^* \phi_t\|_U^2 \, dt \leqslant 2 \int_0^T \|B^* y_t\|_U^2 \, dt + 2 \int_0^T \|B^* \psi_t\|_U^2 \, dt.$$

The proof is complete if we can show that

$$\int_0^T \|B^* \psi_t\|_U^2 \, dt \leqslant \bar{c} \int_0^T \|B^* y_t\|_U^2 \, dt \quad \text{for some } \bar{c} > 0. \tag{54}$$

Set $\tilde{u}(t) = 0$, as $t > T$ and $\tilde{u}(t) = B^* y_t(t)$ as $t \in [0, T)$. It follows from (48) that the solution of (52) satisfies

$$\hat{O}_\psi(s) = \mathbf{H}(s)\hat{u}(s) \quad \text{for all } s \text{ with } \operatorname{Re} s > 0,$$

where $\hat{O}_\psi(t) = B^* \tilde{\psi}(t)$, $\tilde{\psi}(t)$ is the solution of (52) with $u(t) = \tilde{u}(t)$. Hence,

$$\|\hat{O}_\psi(s)\| = \|H(s)\| \, \|\hat{u}(s)\| \leqslant C_\alpha \|\hat{u}(s)\| \quad \text{for some } C_\alpha > 0 \text{ and all } s \text{ with } \operatorname{Re} s = \alpha.$$

By the Plancherel theorem, we get

$$\int_0^\infty e^{-2\alpha t} \|\hat{O}_\psi(t)\|^2 \, dt \leqslant C_\alpha \int_0^\infty e^{-2\alpha t} \|u(t)\|^2 \, dt \leqslant C_\alpha \int_0^T \|u(t)\|^2 \, dt.$$

Therefore,

$$\int_0^T \|\hat{O}_\psi(t)\|^2\,\mathrm{d}t \leqslant \tilde{c}\int_0^T \|u(t)\|^2\,\mathrm{d}t.$$

Noting that

$$\Phi(t)v = \int_0^t T(t-\tau)\begin{bmatrix} 0 \\ BB^* \end{bmatrix} v(\tau)\,\mathrm{d}\tau \quad \text{for any } v \in L^2(U; D(A^{1/2})')$$

in (47), we see that $\psi(t) = \tilde{\psi}(t)$ for $t \in [0,T)$. So, $O_\psi(t) = B^*\psi(t)$ for $t \in [0,T)$ and (53) then follows from the above inequality. □

Remark 2. By (50), $e^{\mathscr{A}t}$ is exponentially stable if and only if $(\mathscr{A}, \mathbf{B})$ is exactly controllable. This fact was stated as Proposition 8 of [30]. From the property of transfer function, we know that the negative identity operator in control space $-I \in \mathscr{L}(U)$ is an admissible feedback operator [29] for system (37). Since system (37) is a well-posed system and is assumed being exactly controllable, for any given $(y_0, y_1), (y_0^*, y_1^*)$, there exists an $t^* > 0$ and control $u_0 \in L^2(0, t^*; U)$ such that the solution of (37) with initial condition (y_0, y_1) satisfies $(y(\cdot, t^*), y_t(\cdot, t^*)) = (y_0^*, y_1^*)$, the feedback control $u(t) = -O_u(t) + u_0(t)$ for system (37) will steel (y_0, y_1) to (y_0^*, y_1^*), which shows that $(\mathscr{A}^*, \mathbf{B})$ is exactly controllable. However, at this stage, we are not sure mathematically if system (37) under the feedback control $u(t) = -\mathbf{O}_u(t) + u_0(t)$ is just $(\mathscr{A}, \mathbf{B})$. If this is true, the proof can be significantly simplified.

Remark 3. In the reviewing process, we found in Proposition 10 of the survey paper [30] (the proof was not presented there) that Assumption (H) implies the admissibility of B. Hence condition (i) of Theorem 3 can be removed.

To end this section, we present a result on the compactness of $\mathscr{A}$.

Proposition 1. *Suppose that A^{-1} is compact on X. Then $\mathscr{A}^{-1}$ is compact on H.*

Proof. By assumption, $A^{-1/2}$ is compact on X. Suppose $(\phi_n, \psi_n) \in H$, $\|A^{1/2}\phi_n\| \leqslant C$, $\|\psi_n\| \leqslant C$ are bounded sequence. Since

$$\phi_n = A^{-1/2}A^{1/2}\phi_n,$$

we see that ϕ_n has a subsequence which (still denoted by ϕ_n) is convergent on $X: \phi_n \to \phi$. Let

$$x_n = -\tilde{A}^{-1}BB^*\phi_n - A^{-1}\psi_n.$$

We want to show that x_n has a subsequence converging in $D(A^{1/2})$. This is true for the second term $A^{-1}\psi_n$ since $A^{1/2}A^{-1}\psi_n = A^{-1/2}\psi_n$ has a convergent subsequence. So we consider only the sequence

$$w_n = \tilde{A}^{-1}BB^*\phi_n.$$

Note that $\tilde{A}^{-1}B \in \mathscr{L}(U, D(A^{1/2}))$, $B^* \in \mathscr{L}(D(A^{1/2}), U)$. So $\tilde{A}^{-1}BB^* \in \mathscr{L}(D(A^{1/2}))$. Hence,

$$\tilde{A}^{-1}BB^*\phi_n \to \tilde{A}^{-1}BB^*\phi \quad \text{in } D(A^{1/2}).$$

The proof is complete. □

Since system (36) or (37) is a regular system under conditions (i) and (ii) of Theorem 3, $B^* y_t$ makes sense in $L^2(0,T;U)$ for any $T>0$. We can then discuss the following optimal control problem for the cost functional:

$$J(u) = E(T) + \frac{1}{2}\int_0^T \|u(t)\|^2\,\mathrm{d}t + \frac{1}{2}\int_0^T \|B^* y_t\|^2(t)\,\mathrm{d}t.$$

It can be easily shown that the feedback $u = B^* y_t$ is the optimal control of system (36) for the above cost function. We omit the details here. A special example can be found in [14].

5. Application to diagonal semigroup

In this section, we have limited ourselves to a kind of diagonal semigroups. Let A be a positive self-adjoint operator in X

$$Ae_n = \omega_n^2 e_n, \quad \omega_n > 0, \tag{55}$$

where $\{e_n\}$ is an orthonormal basis of X. Suppose that $U = \mathbb{C}^m$ is a finite dimensional space, and u_k is an orthonormal basis of U. Rewrite (36) here as

$$\begin{aligned} &y_{tt}(t) + Ay(t) + Bu = 0,\\ &O_u(t) = B^* y_t. \end{aligned} \tag{56}$$

Then for any $B \in \mathscr{L}(U, D(A^{1/2})')$, there are $B_k \in D(A^{1/2})'$, $1 \leqslant k \leqslant m$ such that

$$B = \sum_{k=1}^m u_k B_k. \tag{57}$$

Here we understand $u_k \in U'$ such that $\langle u_k, u_j\rangle = \delta_{kj}$, the notation of Dirac delta. So $B \in \mathscr{L}(U, D(A^{1/2})')$ if and only if

$$B_k = \sum_{n=1}^\infty b_{kn} e_n \quad \text{with} \quad \sum_{n=1}^\infty \frac{|b_{kn}|^2}{\omega_n^2} < \infty,\ 1 \leqslant k \leqslant m. \tag{58}$$

In this case

$$B^*(e_n) = (B_1(e_n), \ldots, B_m(e_n)) = (b_{1n}, \ldots, b_{mn}). \tag{59}$$

Now,

$$\mathbf{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}$$

has eigenelements $\{\pm \mathrm{i}\omega_n, \Phi_{\pm n}\}_{n=1}^\infty$:

$$\mathbf{A}\Phi_{\pm n} = \pm \mathrm{i}\omega_n \Phi_{\pm n}.$$

$\{\Phi_n\}$ forms an orthonormal basis for $D(A^{1/2}) \times X$:

$$\Phi_n = \begin{pmatrix} -\mathrm{i}\omega_n^{-1} e_n \\ e_n \end{pmatrix}, \quad \Phi_{-n} = \begin{pmatrix} \mathrm{i}\omega_n^{-1} e_n \\ e_n \end{pmatrix}. \tag{60}$$

Let

$$\begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} = \sum_{n=1}^\infty a_n \Phi_n + \sum_{n=1}^\infty c_n \Phi_{-n}.$$

Then the solution of (18) can be written as

$$\begin{pmatrix} \phi \\ \phi_t \end{pmatrix} = \sum_{n=1}^{\infty} a_n e^{i\omega_n t} \Phi_n + \sum_{n=1}^{\infty} c_n e^{-i\omega_n t} \Phi_{-n}.$$

Hence,

$$B^* \phi_t = \sum_{n=1}^{\infty} (a_n e^{i\omega_n t} + c_n e^{-i\omega_n t})(b_{1n}, \ldots, b_{mn}). \tag{61}$$

Proposition 2. *Let A and B be defined by* (55) *and* (57), *respectively. Suppose that*

$$|\omega_n - \omega_{n-1}| \geqslant \alpha > 0 \quad for\ all\ n \geqslant 1. \tag{62}$$

Then

(i) *B is admissible if and only if $|b_{kn}| \leqslant M$ for some $M > 0$ and all $n \geqslant 1$ and $1 \leqslant k \leqslant$*m.
(ii) *System* (56) *is exactly observable if and only if $|b_{kn}| > M_0$ for some $M_0 > 0$ and all $n \geqslant 1$ and $1 \leqslant k \leqslant m$.*

Proof. The necessities come from Propositions 4.1 and 4.2 of [22]. The sufficiencies follow from Ingham's class result [10]. □

Now, we are in a position to study $h(\lambda)$ which can be found to be

$$h(\lambda) = \sum_{n=1}^{\infty} \frac{\lambda}{\lambda^2 + \omega_n^2} \left(b_{1n} \sum_{k=1}^{m} u_k b_{kn}, \ldots, b_{mn} \sum_{k=1}^{m} u_k b_{kn} \right). \tag{63}$$

Theorem 4. *Suppose that U is finite dimensional. Let A and B be defined by* (55) *and* (57), *respectively. If there are constants $\beta > 0$ and $\delta > 0$ such that*

$$\omega_{n+1} - \omega_n \geqslant \delta \omega_{n+1}^{\beta}, \quad \forall n \geqslant 1, \tag{64}$$

then $h(\lambda)$ defined by (63) *satisfies assumption* (H) *provided that B is admissible. Therefore, the corresponding operator $\mathscr{A}$ of the closed-loop system of* (56) *generates an exponential stable C_0-semigroup if and only if*

(i) *B is admissible.*
(ii) *System* (18) *is exactly observable on some* $[0, T]$, $T > 0$.

According to Proposition 2, (i) and (ii) can be replaced by

$$M_0 < |b_{kn}| < M \quad \text{for all } n \geqslant 1 \quad \text{and} \quad 1 \leqslant k \leqslant m$$

where $M_0, M > 0$ are two constants independent of k and n. Moreover,

$$\lim_{\lambda \to \infty} h(\lambda) = 0. \tag{65}$$

Therefore, under assumption (64), the feedthrough operator $D = 0$ and hence system (56) is regular [29].

Proof. By (64), we see that ω_n is increasing with respect to n. Moreover, it is easy to show by induction that

$$\omega_{n+1} \geqslant \omega_1 + n\delta\omega_1^{\beta}, \quad \forall n \geqslant 1. \tag{66}$$

Hence,

$$\sum_{n=1}^{\infty} \omega_n^{-2} < \infty. \tag{67}$$

When B is admissible, we have, by Proposition 2, that $|b_{kn}| \leqslant M$ for some $M > 0$ and all $n \geqslant 1$ and $1 \leqslant k \leqslant m$. Hence,

$$\|h(\lambda)\| \leqslant m^2 M^2 \sum_{n=1}^{\infty} \frac{|\lambda|}{|\lambda^2 + \omega_n^2|}. \tag{68}$$

Because of (67), in order to show the boundedness of the right-hand side of (68) on some vertical line $Re\,\lambda = \alpha > 0$ of the complex plane, we need only consider those λ with $|\lambda|$ large enough. Suppose that $\lambda = \alpha + \mathrm{i}y$. Suppose without loss of generality that $y > 0$. First, from (68)

$$\|h(\lambda)\| \leqslant 2m^2 M^2 \sum_{n=1}^{\infty} \frac{y}{|y^2 - \alpha^2 - \omega_n^2| + 2\alpha y} \quad \text{for all } y > \alpha. \tag{69}$$

Suppose that $y^2 > \alpha^2 + \omega_1^2$, $y/2\sqrt{y^2 - \alpha^2 \pm 2\alpha y} < C$ for some $C > 0$, and n_0 is the largest integer such that $\omega_{n_0}^2 < y^2 - \alpha^2 \leqslant \omega_{n_0+1}^2$. Then we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{y}{|y^2 - \alpha^2 - \omega_n^2| + 2\alpha y} \\
&\quad = \sum_{y^2-\alpha^2<\omega_n^2}^{\infty} \frac{y}{\omega_n^2 + \alpha^2 - y^2 + 2\alpha y} + \sum_{y^2-\alpha^2 \geqslant \omega_n^2}^{\infty} \frac{y}{y^2 - \alpha^2 - \omega_n^2 + 2\alpha y} \\
&\quad = \sum_{n=n_0+2}^{\infty} \frac{y}{\omega_n^2 + \alpha^2 - y^2 + 2\alpha y} + \frac{y}{\omega_{n_0+1}^2 + \alpha^2 - y^2 + 2\alpha y} \\
&\qquad + \sum_{n=1}^{n_0-1} \frac{y}{y^2 - \alpha^2 - \omega_n^2 + 2\alpha y} + \frac{y}{y^2 - \alpha^2 - \omega_{n_0}^2 + 2\alpha y} \\
&\quad \leqslant \frac{1}{\alpha} + \sum_{n=n_0+2}^{\infty} \frac{y}{\omega_n^2 + \alpha^2 - y^2 + 2\alpha y} + \sum_{n=1}^{n_0-1} \frac{y}{y^2 - \alpha^2 - \omega_n^2 + 2\alpha y} \\
&\quad \leqslant \frac{1}{\alpha} + \sum_{n=n_0+1}^{\infty} \frac{1}{\omega_{n+1} - \omega_n} \int_{\omega_n}^{\omega_{n+1}} \frac{y}{x^2 + \alpha^2 - y^2 + 2\alpha y} + \sum_{n=1}^{n_0-1} \frac{1}{\omega_{n+1} - \omega_n} \int_{\omega_n}^{\omega_{n+1}} \frac{y}{y^2 - \alpha^2 - x^2 + 2\alpha y} \\
&\quad \leqslant \frac{1}{\alpha} + \frac{1}{\delta\omega_{n_0+1}^{\beta}} \int_{\omega_{n_0+1}}^{\infty} \frac{y}{x^2 + \alpha^2 - y^2 + 2\alpha y} + \frac{1}{\delta\omega_1^{\beta}} \int_{\omega_1}^{\omega_{n_0}/2} \frac{y}{y^2 - \alpha^2 - x^2 + 2\alpha y} \\
&\qquad + \frac{1}{\omega_{n_0} - \omega_{n_0-1}} \int_{\omega_{n_0}/2}^{\omega_{n_0}} \frac{y}{y^2 - \alpha^2 - x^2 + 2\alpha y} = \frac{1}{\alpha} + S_1 + S_2 + S_3,
\end{aligned}$$

where

$$
\begin{aligned}
S_1 &= \frac{1}{\delta\omega_{n_0+1}^{\beta}} \frac{y}{2\sqrt{y^2-\alpha^2-2\alpha y}} \log \frac{\omega_{n_0+1}+\sqrt{y^2-\alpha^2-2\alpha y}}{\omega_{n_0+1}-\sqrt{y^2-\alpha^2-2\alpha y}} \\
&\leqslant \frac{C}{\delta\omega_{n_0+1}^{\beta}} \log \frac{4\omega_{n_0+1}^2}{\omega_{n_0+1}^2-y^2+\alpha^2+2\alpha y} \leqslant \frac{C}{\delta\omega_{n_0+1}^{\beta}} \log \frac{2\omega_{n_0+1}^2}{\alpha y} \\
&\leqslant \frac{C}{\delta\omega_{n_0+1}^{\beta}} \log \frac{2\omega_{n_0+1}^2}{\alpha\omega_{n_0}} \leqslant \frac{C}{\delta\omega_{n_0+1}^{\beta}} \log \frac{2\omega_{n_0+1}^2}{\alpha\omega_1} \leqslant M_1 \leqslant \infty \quad \text{for some } M_1>0, \\
S_2 &= \frac{1}{\delta\omega_1^{\beta}} \int_{\omega_1}^{\omega_{n_0}/2} \frac{y}{y^2-\alpha^2-x^2+2\alpha y}\,\mathrm{d}x \leqslant \frac{1}{\delta\omega_1^{\beta}} \frac{y}{2\sqrt{y^2-\alpha^2-2\alpha y}} \log \frac{\sqrt{y^2-\alpha^2+2\alpha y}+\omega_{n_0}/2}{\sqrt{y^2-\alpha^2+2\alpha y}-\omega_{n_0}/2} \\
&\leqslant \frac{C}{\delta\omega_1^{\beta}} \log\left(1+\frac{\omega_{n_0}}{\sqrt{y^2-\alpha^2+2\alpha y}-\omega_{n_0}/2}\right) \leqslant \frac{C}{\delta\omega_1^{\beta}} \log\left(1+\frac{\omega_{n_0}}{\sqrt{y^2-\alpha^2}-\omega_{n_0}/2}\right) \\
&\leqslant \frac{C}{\delta\omega_1^{\beta}} \log(3) \leqslant M_2 \leqslant \infty \quad \text{for some } M_2>0, \\
S_3 &= \frac{1}{\omega_{n_0}-\omega_{n_0-1}} \int_{\omega_{n_0}/2}^{\omega_{n_0}} \frac{y}{y^2-\alpha^2-x^2+2\alpha y}\,\mathrm{d}x \leqslant \frac{C}{\delta\omega_{n_0}^{\beta}} \log \frac{\sqrt{y^2-\alpha^2+2\alpha y}+\omega_{n_0}}{\sqrt{y^2-\alpha^2+2\alpha y}-\omega_{n_0}} \\
&\leqslant \frac{C}{\delta\omega_{n_0}^{\beta}} \log\left(1+\frac{2\omega_{n_0}}{\sqrt{y^2-\alpha^2+2\alpha y}-\omega_{n_0}}\right) \leqslant \frac{C}{\delta\omega_{n_0}^{\beta}} \log\left(1+\frac{2\sqrt{2}\omega_{n_0}}{\sqrt{y^2-\alpha^2}-\omega_{n_0}+\sqrt{2\alpha y}}\right) \\
&\leqslant \frac{C}{\delta\omega_{n_0}^{\beta}} \log\left(1+\frac{2\omega_{n_0}}{\sqrt{\alpha\omega_{n_0}}}\right) \leqslant M_3 \leqslant \infty \quad \text{for some } M_3>0.
\end{aligned}
$$

Note that in the above derivations, we used frequently the following inequality:

$$\tfrac{1}{2}(a+b)^2 \leqslant a^2+b^2 \leqslant (a+b)^2 \quad \text{for all } a,b>0.$$

Therefore,

$$\|h(\lambda)\| \leqslant 2m^2M^2\sum_{n=1}^{3}\left(\frac{1}{\alpha}+M_i\right) < \infty \quad \text{for all } \lambda=\alpha+\mathrm{i}y,\ \alpha>0 \text{ is fixed.} \tag{70}$$

Now we show (65). Let $\lambda>0$. Then for any $N>1$

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{\lambda}{\lambda^2+\omega_n^2} &\leqslant \sum_{n=1}^{N-1}\frac{\lambda}{\lambda^2+\omega_n^2}+\sum_{n=N}^{\infty}\frac{1}{\omega_n-\omega_{n-1}}\int_{\omega_{n-1}}^{\omega_n}\frac{\lambda}{\lambda^2+x^2}\,\mathrm{d}x \\
&\leqslant \sum_{n=1}^{N-1}\frac{\lambda}{\lambda^2+\omega_n^2}+\sum_{n=N}^{\infty}\frac{1}{\delta\omega_N^{\beta}}\int_{\omega_{n-1}}^{\omega_n}\frac{\lambda}{\lambda^2+x^2}\,\mathrm{d}x = \sum_{n=1}^{N-1}\frac{\lambda}{\lambda^2+\omega_n^2}+\frac{1}{\delta\omega_N^{\beta}}\int_{\omega_{N-1}}^{\infty}\frac{\lambda}{\lambda^2+x^2}\,\mathrm{d}x \\
&\leqslant \sum_{n=1}^{N-1}\frac{\lambda}{\lambda^2+\omega_n^2}+\frac{1}{\delta\omega_N^{\beta}}\int_{0}^{\infty}\frac{1}{1+x^2}\,\mathrm{d}x.
\end{aligned}
$$

Letting first $\lambda\to\infty$ and then $N\to\infty$ in the above inequality, we see that (65) holds true. The proof is complete. □

Remark 4. Condition (64) is satisfied when

$$\omega_n = \gamma n^{1+\beta}[1 + \mathcal{O}(n^{-\vartheta})], \quad \gamma, \beta, \vartheta > 0 \text{ as } n \to \infty. \tag{71}$$

The boundedness of $h(\lambda)$ in the case of $\beta = 1$ was proved in [21].

For a quite while we conjecture that the transfer function in Theorem 4 is always bounded on some right half complex plane without the spectral assumption (64), which is based on the Corollary 9 of [23] that if conditions (i) and (ii) of Theorem 4 are satisfied, then

$$\sup N_k < \infty,$$

where N_k denotes the number of elements in the set $\{n \in \mathbb{N} \mid k \leqslant \omega_n \leqslant k + 1\}$. However, the following counterexample gives a negative answer.

Example 1. Take $m = 1$, $b_n = 1$;

$$\omega_n = \begin{cases} n & \text{if } n \in [4^k, \frac{5}{2}4^k], \\ n^2 & \text{otherwise}, \end{cases} \qquad n \geqslant 1, \ k \geqslant 1.$$

Then $\omega_n \geqslant n$, $n \geqslant 1$, so b is always admissible and the associated system is exactly controllable by Proposition 2. However, ω_n here do not satisfy (64) because ω_n in this example is not even monotonically increasing which is implied automatically by condition (64). We show that the transfer function in this case is not bounded on any right complex plane. Indeed, (63) now becomes

$$h(\lambda) = \sum_{k=1}^{\infty} \frac{\lambda b_k^2}{\lambda^2 + \omega_k^2} = \sum_{k=1}^{\infty} \sum_{n=4^k}^{(5/2)4^k} \frac{\lambda}{\lambda^2 + n^2} + h_2(\lambda) = h_1(\lambda) + h_2(\lambda), \tag{72}$$

where $h_2(\lambda)$ is bounded on any vertical line $Re\,\lambda = \alpha > 0$ as we proved in Theorem 4. We claim that

$$|h_1(\alpha + 4^n i)| \to \infty \quad \text{as } n \to \infty \tag{73}$$

for any $\alpha > 0$. Therefore, h does not satisfy Assumption (H).

Indeed, set $\lambda_n = a + 4^n i$. Then $|\lambda_n| > 4^n$ and

$$\begin{aligned} |h_1(\alpha + 4^n i)| &\geqslant \left| \sum_{k=1}^{\infty} \sum_{m=4^k}^{(5/2)4^k} \frac{4^n}{\lambda_n^2 + m^2} \right| \\ &\geqslant \left| \sum_{m=4^n}^{(5/2)4^n} \frac{4^n}{\lambda_n^2 + m^2} \right| - \left| \sum_{k=n+1}^{\infty} \sum_{m=4^k}^{(5/2)4^k} \frac{4^n}{\lambda_n^2 + m^2} \right| - \left| \sum_{k=1}^{n-1} \sum_{m=4^k}^{(5/2)4^k} \frac{4^n}{\lambda_n^2 + m^2} \right| \\ &\geqslant \left| \sum_{m=4^n}^{(5/2)4^n} \frac{4^n}{\lambda_n^2 + m^2} \right| - \sqrt{2} \sum_{k=n+1}^{\infty} \sum_{m=4^k}^{(5/2)4^k} \frac{4^n}{m^2 - 4^{2n} + \alpha^2 + 2\alpha 4^n} \\ &\quad - \sqrt{2} \sum_{k=1}^{n-1} \sum_{m=4^k}^{(5/2)4^k} \frac{4^n}{4^{2n} - \alpha^2 - m^2 + 2\alpha 4^n} = S_{11} - S_{12} - S_{13}. \end{aligned} \tag{74}$$

First, we show that

$$S_{12}+S_{13}\leqslant M_4<\infty \quad \text{for some } M_4>0. \tag{75}$$

In fact, as $4^n\geqslant \alpha^2$, it has

$$\begin{aligned}
S_{12}+S_{13} &= \sum_{k=1}^{n-1}\sum_{m=4^k}^{(5/2)4^k}\frac{4^n}{4^{2n}-m^2-\alpha^2+2\alpha 4^n}+\sum_{k=n+1}^{\infty}\sum_{m=4^k}^{(5/2)4^k}\frac{4^n}{m^2-4^{2n}+\alpha^2+2\alpha 4^n}\\
&\leqslant \sum_{k=1}^{n-1}\sum_{m=4^k}^{(5/2)4^k}\int_m^{m+1}\frac{4^n\,\mathrm{d}x}{4^{2n}-x^2-\alpha^2+2\alpha 4^n}+\sum_{k=n+1}^{\infty}\sum_{m=4^k}^{(5/2)4^k}\int_{m-1}^{m}\frac{4^n\,\mathrm{d}x}{x^2-4^{2n}+\alpha^2+2\alpha 4^n}\\
&\leqslant \int_4^{(5/2)4^{n-1}+1}\frac{4^n\,\mathrm{d}x}{4^{2n}-x^2-\alpha^2+2\alpha 4^n}+\int_{4^{n+1}-1}^{\infty}\frac{4^n\,\mathrm{d}x}{x^2-4^{2n}+\alpha^2+2\alpha 4^n}\\
&\leqslant \frac{4^n}{2\sqrt{4^{2n}-\alpha^2+2\alpha 4^n}}\log\frac{\sqrt{4^{2n}-\alpha^2+2\alpha 4^n}+x}{\sqrt{4^{2n}-\alpha^2+2\alpha 4^n}-x}\Bigg|_{x=(5/2)4^{n-1}+1}\\
&\quad +\frac{4^n}{2\sqrt{4^{2n}-\alpha^2-2\alpha 4^n}}\log\frac{x+\sqrt{4^{2n}-\alpha^2-2\alpha 4^n}}{x-\sqrt{4^{2n}-\alpha^2-2\alpha 4^n}}\Bigg|_{x=4^{n+1}-1}<M_4<\infty \quad \forall n\geqslant 1.
\end{aligned}$$

Now, we estimate S_{11}. Let n_0 be such an integer so that $n_0^2<4^{2n}-\alpha^2+2^{2n+1}\alpha\leqslant(n_0+1)^2$. It is seen that $[n_0,n_0+1]\subset[4^n,\frac{5}{2}4^n]$. Note that as $m\in[4^n,\frac{5}{2}4^n]$

$$Re\,\frac{4^n}{\lambda_n^2+m^2}=\frac{4^n(m^2-4^{2n}+\alpha^2)}{(m^2-4^{2n}+\alpha^2)^2+4^{2n+1}\alpha^2}\geqslant 0,\quad m\in\left[4^n,\frac{5}{2}4^n\right]$$

and the function $f(y)=y/(y^2+b^2)$ is a convex function as $y>0$ and $f(y)$ attains its unique maximum $f(b)=1/(2b)$ at $y=b$ and hence

$$\begin{aligned}
&\frac{4^n(m^2-4^{2n}+\alpha^2)}{(m^2-4^{2n}+\alpha^2)^2+4^{2n+1}\alpha^2}\\
&\quad\geqslant\begin{cases}\dfrac{4^n(x^2-4^{2n}+\alpha^2)}{(x^2-4^{2n}+\alpha^2)^2+4^{2n+1}\alpha^2} & \text{when } m^2-4^{2n}+\alpha^2\leqslant 2^{2n+1}\alpha,\ x\in[m-1,m],\\[2ex] \dfrac{4^n(x^2-4^{2n}+\alpha^2)}{(x^2-4^{2n}+\alpha^2)^2+4^{2n+1}\alpha^2} & \text{when } m^2-4^{2n}+\alpha^2\geqslant 2^{2n+1}\alpha,\ x\in[m,m+1].\end{cases}
\end{aligned}$$

Therefore,

$$\begin{aligned}
S_{11} &\geqslant \sum_{m=4^n}^{(5/2)4^n}\frac{4^n(m^2-4^{2n}+\alpha^2)}{(m^2-4^{2n}+\alpha^2)^2+4^{2n+1}\alpha^2}\\
&\geqslant \sum_{m=4^n+1}^{n_0}\int_{m-1}^{m}\frac{4^n(x^2-4^{2n}+\alpha^2)\,\mathrm{d}x}{(x^2-4^{2n}+\alpha^2)^2+4^{2n+1}\alpha^2}+\sum_{m=n_0+1}^{(5/2)4^n}\int_m^{m+1}\frac{4^n(x^2-4^{2n}+\alpha^2)\,\mathrm{d}x}{(x^2-4^{2n}+\alpha^2)^2+4^{2n+1}\alpha^2}
\end{aligned}$$

$$= \int_{4^n}^{(5/2)4^n} \frac{4^n(x^2-4^{2n}+\alpha^2)\,dx}{(x^2-4^{2n}+\alpha^2)^2+4^{2n+1}\alpha^2} - \int_{n_0}^{n_0+1} \frac{4^n(x^2-4^{2n}+\alpha^2)\,dx}{(x^2-4^{2n}+\alpha^2)^2+4^{2n+1}\alpha^2}$$

$$\geqslant \int_{4^n}^{(5/2)4^n} \frac{4^n(x^2-4^{2n}+\alpha^2)\,dx}{(x^2-4^{2n}+\alpha^2)^2+4^{2n+1}\alpha^2} - \frac{1}{4\alpha}.$$

Denote $\alpha_n = \alpha/4^n$. Then

$$\int_{4^n}^{(5/2)4^n} \frac{4^n(x^2-4^{2n}+\alpha^2)\,dx}{(x^2-4^{2n}+\alpha^2)^2+4^{2n+1}\alpha^2} = \int_1^{5/2} \frac{(x^2-1+\alpha_n^2)\,dx}{(x^2-1+\alpha_n^2)^2+4\alpha_n^2}$$

$$\geqslant \frac{1}{5}\int_1^2 \frac{(x-1+\alpha_n^2)\,dx}{(x-1+\alpha_n^2)^2+4\alpha_n^2} = \frac{1}{5}\int_{\alpha_n^2}^{1+\alpha_n^2} \frac{u}{u^2+4\alpha_n^2}\,du$$

$$\geqslant \frac{1}{5}\int_{\alpha_n^2}^{1} \frac{u}{u^2+\alpha_n^2}\,du = \frac{1}{10}\log\frac{1+4\alpha_n^2}{\alpha_n^4+4\alpha_n^2} \to \infty \quad (\text{as } n\to\infty),$$

which implies that

$$S_{11} \to \infty \quad \text{as } n \to \infty. \tag{76}$$

This concludes (73) by combining (74)–(76).

Finally, as an application of the result, we give a different proof for the exponential stability of the following example of beam equation which was first discussed in [4].

Example 2. Consider the beam equation with boundary control

$$\begin{aligned} &y_{tt}(x,t) + y_{xxxx}(x,t) = 0,\\ &y(0,t) = y_x(0,t) = y_{xx}(1,t) = 0,\\ &y_{xxx}(1,t) = u(t). \end{aligned} \tag{77}$$

Using the method in [9] or [21], we can write (77) to be

$$\begin{aligned} &y_{tt}(x,t) + y_{xxxx}(x,t) + \delta(x-1)u(t) = 0,\\ &y(0,t) = y_x(0,t) = y_{xx}(1,t) = y_{xxx}(1,t) = 0. \end{aligned} \tag{78}$$

Define $X = L^2(0,1)$, $U = \mathbb{C}$,

$$\begin{aligned} &A\phi = \phi^{(4)}(x), \quad D(A) = \{\phi \in H^4(0,1) \,|\, \phi(0) = \phi'(0) = \phi''(1) = \phi'''(1) = 0\},\\ &B\phi = k\delta(x-1). \end{aligned} \tag{79}$$

Then

$$B^*\phi = k\phi(1) \tag{80}$$

for any $\phi \in D(A^{1/2}) = \{\phi \in H^2(0,1) \,|\, \phi(0) = \phi'(0) = 0\}$. The closed-loop equation is the well-known beam equation with shear force feedback control

$$\begin{aligned} &y_{tt}(x,t) + y_{xxxx}(x,t) = 0,\\ &y(0,t) = y_x(0,t) = y_{xx}(1,t) = 0,\\ &y_{xxx}(1,t) = k^2 y_t(1,t) \end{aligned} \tag{81}$$

and operator $\mathscr{A}$ is defined as

$$\mathscr{A}(f,g)=(g,-f^{(4)}),$$

$$D(\mathscr{A})=\{(f,g)\in(D(A^{1/2})\cap H^4(0,1))\times D(A^{1/2})|,$$

$$f'''(1)=k^2g(1), f''(1)=0\}. \tag{82}$$

It is well known that A^{-1} is compact on X and hence $\mathscr{A}^{-1}$ is compact on $H=D(A^{1/2})\times X$ in terms of Proposition 1. The eigenpairs of $\{(\omega_n^2,e_n)\}_1^\infty$ of A can be easily found to be

$$\omega_n=[n-1/2)\pi]^2+\mathscr{O}(n^{-1}),$$

$$e_n=\mathrm{e}^{-(n-1/2)\pi x}+(-1)^n\mathrm{e}^{-(n-1/2)\pi(1-x)}+\sin(n-1/2)\pi x-\cos(n-1/2)\pi x+\mathscr{O}(n^{-1}).$$

Now, it is easily shown that $b_n\neq 0$ and

$$b_n=2k(-1)^{n-1}+\mathscr{O}(n^{-1}),\quad n\geqslant 1.$$

By virtue of Theorem 4, system (81) is exponentially stable. Moreover, system (77) is a regular system with the output $O(t)=y_t(1,t)$.

Dedication

The first author would like to express his deep sadness for the sudden death of the co-author of this paper Dr. Yue-Hu Luo during the revising of this paper and would like to dedicate this work to his memory.

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In the reviewing process of this paper, we learnt from Prof.G. Weiss by suggestion of an anonymous referee that some better results of the same problem indicated in Remarks 2, 3 have already been announced in a recent survey paper [30]. In the second review of the paper, one reviewer indicated that Theorem 1 is already available in Proposition 7.6.7.1 in volume 2 of [13]; Corollary 1 can be found at p. 665 of [13] and Proposition 2.1 of [1]; Theorem 3 was announced in Proposition 3.3 of [1]. The authors are very grateful to referees for their careful reading and helpful suggestions and comments for the revision of the paper. The idea of Remark 2 comes from one referee's comments. The support of the National Natural Science Foundation of China is gratefully acknowledged.

Appendix A

The following result is due to [31].

Theorem A. *Let H,U,Y be the Hilbert spaces. Suppose that the following system*

$$\dot{x}(t)=Ax(t)+Bu(t),$$

$$y(t)=Cx(t) \tag{A.1}$$

is a well-posed system in H, where A is the generator of a C_0-semigroup on H, and $B\in\mathscr{L}(U,[D(A^*)]')$, $C\in\mathscr{L}(D(A),Y)$ be the admissible control and observe operators. If $(\lambda-A)^{-1}B\subset D(C)$ for all $\lambda\in\rho(A)$, then the transfer function $H(s)$ of system (A.1) *is*

$$H(s)=C(s-A)^{-1}B.$$

Proof. Suppose that $C_\alpha = \{s|\mathrm{Re}s \geqslant \alpha\} \subset \rho(A)$ for some $\alpha > 0$. For any $\beta \in C_\alpha$ and $u_0 \in U$, setting $u(t) = \mathrm{e}^{\beta t}u_0$ and solving Eq. (83), we find that

$$x(t) = \mathrm{e}^{\beta t}(\beta - A)^{-1}Bu_0, \quad y(t) = \mathrm{e}^{\beta t}C(\beta - A)^{-1}Bu_0$$

satisfies Eq. (A.1) with the initial condition $x(0) = (\beta - A)^{-1}Bu_0$. In view of the general well-posed system theory, the Laplace transforms of x, y, u satisfy

$$\hat{y}(s) = C(s - A)^{-1}(\beta - A)^{-1}Bu_0 + H(s)\hat{u}(s). \tag{A.2}$$

Now, it is found directly that

$$\hat{y}(s) = \frac{1}{s-\beta}C(\beta - A)^{-1}Bu_0, \;\; \hat{x}(s) = \frac{1}{s-\beta}(\beta - A)^{-1}Bu_0, \;\; \hat{u}(s) = \frac{1}{s-\beta}u_0 \;\; \forall \mathrm{Re}s > \beta.$$

Substituting the above into (A.2) and dividing by $s - \beta$ on both sides and letting $s \to \beta$ gives

$$H(\beta)u_0 = C(\beta - A)^{-1}Bu_0.$$

The result is proved due to the arbitrariness of β.

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