

The active disturbance rejection control approach to stabilisation of coupled heat and ODE system subject to boundary control matched disturbance

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We consider stabilisation for a linear ordinary differential equation system with input dynamics governed by a heat equation, subject to boundary control matched disturbance. The active disturbance rejection control approach is applied to estimate, in real time, the disturbance with both constant high gain and time-varying high gain. The disturbance is cancelled in the feedback loop. The closed-loop systems with constant high gain and time-varying high gain are shown, respectively, to be practically stable and asymptotically stable.

Keywords: coupled PDE-ODE system; boundary control; disturbance rejection; active disturbance rejection control; stability

1. Introduction

In recent years, stabilisation for systems described by partial differential equations (PDEs) subject to external disturbance has received much attention. Many different approaches have been applied to deal with disturbance such as the internal model principle for output regulation (Immonen & Pohjolainen, 2006); robust control for systems with uncertainties from both internal un-modelled dynamics and external disturbance; sliding mode control in various situations (Cheng, Radisavljevic, & Su, 2011; Drakunov, Barbieri, & Silver, 1996; Guo & Jin, 2013a, 2013b; Guo & Liu, 2014; Orlov & Utkin, 1982, 1987; Pisano, Orlov, & Usai, 2011; Utkin, 2008); adaptive control for systems with unknown parameters (Guo & Guo, 2013a, 2013b, 2013c; Guo, Guo, & Shao, 2011; Krstic, 2009; Krstic & Smyshlyaev, 2008); and the Lyapunov approach for distributed disturbance (He, Ge, How, Choo, & Hong, 2011; Orlov, 1983), to name just a few.

On the other hand, there are many coupled ODE-PDE systems that appeared in engineering from different aspects such as electromagnetic coupling, mechanical coupling, and coupled chemical reactions. An ordinary differential equation (ODE)-wave system and an ODE-heat system, without considering disturbance, have been considered in Krstic (2009) and Tang and Xie (2011), respectively. For the motivation, let us consider a controlled ODE:

$$\dot{Y}(t) = AY(t) + Bu(t - \tau), \tau > 0, \quad (1)$$

where A is an $n \times n$ matrix, B is the appropriate sized control matrix, u is the control input, and τ is the time delay in control. Set

$$z(x, t) = u(t + \tau(x - 1)), \quad 0 < x < 1.$$

Then, z satisfies

$$\tau z_t(x, t) = z_x(x, t), \quad 0 < x < 1, t > 0. \quad (2)$$

So, the control problem (1) can be formulated as the following coupled ODE-PDE control system:

$$\begin{cases} \dot{Y}(t) = AY(t) + Bz(0, t), \\ \tau z_t(x, t) = z_x(x, t), \\ z(1, t) = u(t), \end{cases} \quad (3)$$

where the PDE part is considered as the controller and the original control plant ODE is connected with PDE through the boundary output of the PDE. It is seen that the time delay disappears in the state of the new formulated system (3). This point of view clearly shows the infinite-dimensional nature of the delay systems.

In this paper, we are concerned with stabilisation for the following PDE-ODE cascade system through Dirichlet

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interconnection (see Figures 1):

$$\begin{cases} \dot{X}(t) = AX(t) + Bu(0, t), & t > 0, \\ u_t(x, t) = u_{xx}(x, t), & x \in (0, 1), t > 0, \\ u_x(0, t) = 0, & t \geq 0, \\ u_x(1, t) = U(t) + d(t), & t \geq 0, \\ X(0) = X_0, & u(x, 0) = u_0(x), \end{cases} \quad (4)$$

where $X \in \mathbb{R}^{n \times 1}$ and $u \in L^2(0, 1)$ are the states of ODE and PDE, respectively, $U \in L^2_{\text{loc}}(0, \infty)$ is the control input of the entire system, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times 1}$ are matrices, d is the external disturbance at the control end, X_0 and $u_0(x)$ are the initial value of ODE and PDE, respectively. It is supposed that the pair (A, B) is stabilisable, and both d and its derivative are uniformly bounded, i.e., $|d(t)| \leq M_1$ and $|\dot{d}(t)| \leq M_2$ for some $M_1, M_2 > 0$ and all $t \geq 0$.

The objective of this paper is to design a state feedback control to achieve stabilisation for system (4) by attenuating the disturbance. Our approach is the active disturbance rejection control (ADRC) approach that was proposed by Han (2009) to deal with large uncertainty for general nonlinear lumped parameter systems, and the backstepping approach which was originally developed for PDEs in Smyshlyayev and Krstic (2004, 2005). The backstepping method enables us to transform the system (4) into a target system where the ODE part is stable and the control is only used to cope with the disturbance and stabilise the PDE part. The ADRC is used to build an estimator to estimate the disturbance and then cancel the disturbance in the feedback loop. The ADRC has been successfully applied to one-dimensional PDEs in previous studies (Guo & Jin, 2013a, 2013b; Guo & Liu, 2014).

The rest of the paper is organised as follows. In Section 2, we design a disturbance estimator with constant high gain by ADRC approach. The practical stability is developed. The constant high-gain disturbance estimator shears the simple tuning in practice and noise filtering function but causes peaking value problem in the initial stage. To overcome the peaking value problem, we design a time-varying disturbance estimator in Section 3 and obtain the asymptotic stability. Section 4 presents some numerical simulations

for illustration. Some concluding remarks are presented in Section 5.

2. Constant high-gain estimator based feedback

In this section, we design a disturbance estimator-based state feedback control with constant high gain. This is motivated from the extended state observer (ESO) by ADRC approach.

We first introduce a feedback stabilising mechanism to ODE part by the transformation $u \mapsto w$ in the form (K. Krstic, 2009):

$$w(x, t) = u(x, t) - \int_0^x q(x, y)u(y, t)dy - \gamma(x)X(t), \quad (5)$$

where

$$q(x, y) = \int_0^{x-y} \gamma(\sigma)Bd\sigma, \quad \gamma(x) = [K0]e^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix}x} \begin{bmatrix} I \\ 0 \end{bmatrix},$$

with I being the $n \times n$ identity matrix and K being chosen so that $A + BK$ is Hurwitz. This transforms system (4) into the following system:

$$\begin{cases} \dot{X}(t) = (A + BK)X(t) + Bw(0, t), \\ w_t(x, t) = w_{xx}(x, t), \\ w_x(0, t) = 0, \\ w_x(1, t) = U(t) + d(t) - \int_0^1 q_x(1, y)u(y, t)dy - \gamma'(1)X(t), \\ X(0) = X_0, \quad w(x, 0) = w_0(x). \end{cases} \quad (6)$$

It is seen from (6) that if the PDE part is stable, then so is for the ODE part. This is the crucial role played by backstepping transformation (5) after which we need only to consider the stabilisation of the PDE part. To go back to system (4) from (6), the transformation (5) must be invertible. This is true by solving u from (5) that

$$u(x, t) = w(x, t) + \int_0^x l(x, y)w(y, t)dy + \psi(x)X(t),$$

where

$$l(x, y) = \int_0^{x-y} \psi(\xi)Bd\xi, \quad \psi(x) = [K0]e^{\begin{bmatrix} 0 & A+BK \\ I & 0 \end{bmatrix}x} \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

We can stabilise system (6) by designing boundary state feedback control. However, to increase the decay rate, we

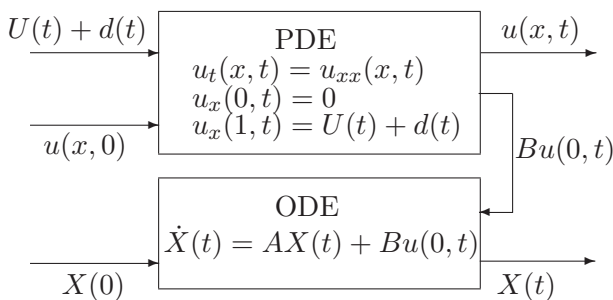


Figure 1. Block diagram of coupled ODE-PDE system (4).

introduce further another transformation $w \mapsto z$:

$$z(x, t) = w(x, t) - \int_0^x k(x, y)w(y, t)dy, \quad (7)$$

where

$$k(x, y) = -cx \frac{I_1(\sqrt{c(x^2 - y^2)})}{\sqrt{c(x^2 - y^2)}}, \quad 0 \leq y \leq x \leq 1, \quad (8)$$

with I_1 being the modified Bessel function. By transformation (7), system (6) is transformed into the following target system:

$$\begin{cases} \dot{X}(t) = (A + BK)X(t) + Bz(0, t), \\ z_t(x, t) = z_{xx}(x, t) - cz(x, t), \\ z_x(0, t) = 0, \\ z_x(1, t) = U(t) + d(t) - \int_0^1 q_x(1, y)u(y, t)dy \\ \quad - \gamma'(1)X(t) - k(1, 1)w(1, t) \\ \quad - \int_0^1 k_x(1, y)w(y, t)dy, \\ X(0) = X_0, \quad z(x, 0) = z_0(x). \end{cases} \quad (9)$$

The transformation (7) is also invertible since

$$w(x, t) = z(x, t) + \int_0^x p(x, y)z(y, t)dy, \quad (10)$$

where

$$p(x, y) = -cx \frac{J_1(\sqrt{c(x^2 - y^2)})}{\sqrt{c(x^2 - y^2)}}, \quad 0 \leq y \leq x \leq 1,$$

with J_1 being the Bessel function. Therefore, under two transformations (5) and (7), systems (4) and (9) are equivalent. So, we need only consider system (9) in what follows.

The state space for system (9) is chosen as $\mathcal{H} = \mathbb{R}^n \times L^2(0, 1)$ with the inner product given by

$$\begin{aligned} \langle [X, f], [Y, g] \rangle &= X^\top Y + \int_0^1 f(x)g(x)dx, \\ \forall [X, f], [Y, g] &\in \mathcal{H}. \end{aligned} \quad (11)$$

First, introduce a new control variable U_0 by designing

$$\begin{aligned} U(t) &= \int_0^1 q_x(1, y)u(y, t)dy + \gamma'(1) + k(1, 1)w(1, t) \\ &\quad + \int_0^1 k_x(1, y)w(y, t)dy + U_0(t), \end{aligned} \quad (12)$$

under which, system (9) becomes

$$\begin{cases} \dot{X}(t) = (A + BK)X(t) + Bz(0, t), \\ z_t(x, t) = z_{xx}(x, t) - cz(x, t), \\ z_x(0, t) = 0, \\ z_x(1, t) = U_0(t) + d(t), \\ X(0) = X_0, \quad z(x, 0) = z_0(x). \end{cases} \quad (13)$$

We write system (13) as

$$\frac{d}{dt}Z(\cdot, t) = \mathcal{A}_D Z(\cdot, t) + \mathcal{B}_D[U_0(t) + d(t)], \quad (14)$$

where $Z(\cdot, t) = [X(t), z(\cdot, t)]$, $\mathcal{B}_D = [0, \delta(x - 1)]$, and \mathcal{A}_D is a linear operator defined in $\mathbb{R}^n \times L^2(0, 1)$ as

$$\begin{cases} \mathcal{A}_D[X, f] = [(A + BK)X + Bf(0), f'' - cf], \\ \quad \quad \quad \forall [X, f] \in D(\mathcal{A}_D), \\ D(\mathcal{A}_D) = \{[X, f] \in \mathbb{R}^n \times H^2(0, 1) \mid f'(0) = f'(1) = 0\}. \end{cases} \quad (15)$$

We compute \mathcal{A}_D^* , the adjoint operator of \mathcal{A}_D , to obtain

$$\begin{cases} \mathcal{A}_D^*[Y, g] = [(A + BK)^\top Y, g'' - cg], \quad \forall [Y, g] \in D(\mathcal{A}_D^*), \\ D(\mathcal{A}_D^*) = \{[Y, g] \in \mathbb{R}^n \times H^2(0, 1) \mid g'(0) \\ \quad \quad \quad + B^\top Y = 0, g'(1) = 0\}. \end{cases} \quad (16)$$

Proposition 2.1: The operator \mathcal{A}_D defined by (15) generates an exponential stable C_0 -semigroup on \mathcal{H} , and the control operator \mathcal{B}_D is admissible to the semigroup $e^{\mathcal{A}_D t}$. Hence, for any $Z(x, 0) \in \mathcal{H}$, there exists a unique (weak) solution to (14), which can be written as

$$\begin{aligned} Z(\cdot, t) &= e^{\mathcal{A}_D t} Z(\cdot, 0) + \int_0^t e^{\mathcal{A}_D(t-s)} \mathcal{B}_D[U_0(s) + d(s)]ds, \\ \forall U_0 &\in L_{loc}^2(0, \infty), \end{aligned} \quad (17)$$

which is equivalent to saying

$$\begin{aligned} \frac{d}{dt} \langle Z(\cdot, t), f \rangle &= \langle Z(\cdot, t), \mathcal{A}_D^* f \rangle + [U_0(t) + d(t)] \mathcal{B}_D^* f, \\ \forall f &\in D(\mathcal{A}_D^*). \end{aligned} \quad (18)$$

Proof: Consider the observation problem for the dual system of (13), which is produced by the operator \mathcal{A}_D^* as follows:

$$\begin{cases} \frac{d}{dt}[Y(t), w(\cdot, t)] = \mathcal{A}_D^*[Y(t), w(\cdot, t)], \\ y(t) = \mathcal{B}_D^*[Y(t), w(\cdot, t)], \end{cases} \quad (19)$$

that is,

$$\begin{cases} \dot{Y}(t) = (A + BK)^\top Y(t), \\ w_t(x, t) = w_{xx}(x, t) - cw(x, t), \\ w_x(0, t) = -B^\top Y(t), \quad w_x(1, t) = 0, \\ Y(0) = Y_0, \quad w(x, 0) = w_0(x), \\ y(t) = w(1, t). \end{cases} \quad (20)$$

It is seen that the 'ODE part' of (20) is always well posed, that is,

$$Y(t) = e^{(A+BK)^\top t} Y_0$$

and $Y(t)$ is exponentially stable: $\|Y(t)\| \leq M e^{-\omega t} \|Y_0\|$ for some $M, \omega > 0$ and all $t \geq 0$. For the 'PDE part' of (20), set $\theta(x, t) = w(x, t) - \frac{(x-1)^2}{2} B^\top Y(t)$. Then, $\theta(x, t)$ satisfies

$$\begin{cases} \theta_t(x, t) = \theta_{xx}(x, t) - c\theta(x, t) + \left[B^\top - \frac{c(x-1)^2}{2} B^\top \right. \\ \quad \left. - \frac{(x-1)^2}{2} B^\top (A + BK)^\top \right] Y(t), \\ \theta_x(0, t) = 0, \quad \theta_x(1, t) = 0, \\ \theta(x, 0) = \theta_0(x) = w_0(x) - \frac{(x-1)^2}{2} B^\top Y_0, \\ y(t) = \theta(1, t). \end{cases} \quad (21)$$

The homogeneous part of (21) associates with a C_0 -semigroup $S(t)$ on $L^2(0, 1)$:

$$\begin{aligned} S(t)\theta_0(x) &= \sum_{m=0}^{\infty} a_m e^{(-c-(m\pi)^2)t} \cos m\pi x, \\ \forall \theta_0(x) &= \sum_{m=0}^{\infty} a_m \cos m\pi x \in L^2(0, 1) \end{aligned} \quad (22)$$

because $\{\cos m\pi x\}_{m=0}^{\infty}$ forms an orthonormal basis for $L^2(0, 1)$. Hence,

$$\begin{aligned} w(x, t) &= \theta(x, t) + \frac{(x-1)^2}{2} B^\top Y(t) \\ &= S(t) \left[w_0(x) - \frac{(x-1)^2}{2} B^\top Y_0 \right] \\ &\quad + \int_0^t S(t-s) \left[B^\top - \frac{c(x-1)^2}{2} B^\top \right. \\ &\quad \left. - \frac{(x-1)^2}{2} B^\top (A + BK)^\top \right] Y(s) ds \\ &\quad + \frac{(x-1)^2}{2} B^\top Y(t). \end{aligned} \quad (23)$$

By Theorem 1.4 of Pazy (1983, p.102), \mathcal{A}_D^* generates a C_0 -semigroup $e^{\mathcal{A}_D^* t}$ on \mathcal{H} , and so does \mathcal{A}_D (Corollary 10.6 of (Pazy, 1983, p.41)).

Furthermore, since $S(t)$ is an exponential stable C_0 -semigroup on $L^2(0, 1)$ and $\|Y(t)\| \leq M e^{-\omega t} \|Y_0\|$, we can easily obtain from (23) that there are constants $M_0, \omega_0 > 0$ such that $\|w(\cdot, t)\|_{L^2(0,1)} \leq M_0 e^{-\omega_0 t} [\|Y_0\| + \|w_0\|_{L^2(0,1)}]$. Hence, $e^{\mathcal{A}_D^* t}$ is exponentially stable and so is $e^{\mathcal{A}_D t}$. This proves the first part of the results.

Now, by (23),

$$y(t) = \theta(1, t) = \sum_{m=0}^{\infty} a_m (-1)^m e^{(-c-(m\pi)^2)t}.$$

So, for any $T > 0$,

$$\begin{aligned} \int_0^T y^2(t) dt &\leq \sum_{m=0}^{\infty} a_m^2 \int_0^T e^{(-2c-2(m\pi)^2)t} dt \leq \frac{1}{2c} \sum_{m=0}^{\infty} a_m^2 \\ &= \frac{1}{2c} \|\theta_0\|_{L^2(0,1)}^2 \leq C [\|w_0\|_{L^2(0,1)}^2 + \|Y_0\|^2] \end{aligned}$$

for some constant $C > 0$. By Definition 4.3.1 of Tucsnak and Weiss (2009, p. 122), the observation operator \mathcal{B}_D^* of system (19) is admissible, and so is \mathcal{B}_D for $e^{\mathcal{A}_D t}$ by the duality principle Proposition 4.4.1 of Tucsnak and Weiss (2009, p. 126). The other results are consequences of the C_0 -semigroup generation and the admissibility of \mathcal{B}_D . \square

By (18), system (14) is equivalent to a system of infinitely many ODEs (18), where $f(x)$ is called a test function. Taking specially $f(x) = (0, 2x^3 - 3x^2) \in D(\mathcal{A}_D^*)$ in (18), we obtain

$$\dot{y}_1(t) = -U_0(t) - d(t) + y_2(t), \quad (24)$$

where

$$\begin{aligned} y_1(t) &= \int_0^1 (2x^3 - 3x^2) z(x, t) dx, \\ y_2(t) &= \int_0^1 (-2cx^3 + 3cx^2 + 12x - 6) z(x, t) dx. \end{aligned} \quad (25)$$

From the resulting ODE (24), we see that the trick of choosing the special test function $f(x) = (0, 2x^3 - 3x^2) \in D(\mathcal{A}_D^*)$ is to make the disturbance $d(t)$ appear in the resulting ODE so that we can estimate the disturbance by the ADRC approach to ODEs. Directly estimating the disturbance in the PDE is extremely difficult.

Now, we design a linear ESO for ODE system (24) by ADRC approach as follows (Guo & Zhao, 2011):

$$\begin{cases} \dot{\hat{y}}_\varepsilon(t) = -U_0(t) - \hat{d}_\varepsilon(t) + y_2(t) - \frac{1}{\varepsilon}(\hat{y}_\varepsilon(t) - y_1(t)), \\ \dot{\hat{d}}_\varepsilon(t) = \frac{1}{\varepsilon^2}(\hat{y}_\varepsilon(t) - y_1(t)), \end{cases} \quad (26)$$

where $\varepsilon > 0$ is the design small parameter and \hat{d}_ε is regarded as an approximation of d by the following Lemma 2.2, which is never well explained in the existing literature.

Lemma 2.2: Let $(\hat{y}_\varepsilon, \hat{d}_\varepsilon)$ be the solution of (26) and let y_1 be defined in (25). Then,

(i) For any $a > 0$,

$$|\hat{y}_\varepsilon(t) - y_1(t)| + |\hat{d}_\varepsilon(t) - d(t)| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \text{ uniformly in } t \in [a, \infty). \quad (27)$$

(ii) For any $a > 0$,

$$\int_0^a [|\hat{y}_\varepsilon(t) - y_1(t)| + |\hat{d}_\varepsilon(t) - d(t)|] dt \text{ is uniformly bounded as } \varepsilon \rightarrow 0. \quad (28)$$

(iii) For any $a > 0$,

$$\int_0^a [|\dot{\hat{y}}_\varepsilon(t)|^2 + |\dot{\hat{d}}_\varepsilon(t)|^2] dt = \mathcal{O}(\varepsilon^{-1}) \text{ as } \varepsilon \rightarrow 0. \quad (29)$$

Proof: Let

$$\tilde{y}_\varepsilon(t) = \hat{y}_\varepsilon(t) - y_1(t), \quad \tilde{d}_\varepsilon(t) = \hat{d}_\varepsilon(t) - d(t) \quad (30)$$

be the errors. Then, $(\tilde{y}_\varepsilon, \tilde{d}_\varepsilon)$ satisfies

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} \tilde{y}_\varepsilon(t) \\ \tilde{d}_\varepsilon(t) \end{pmatrix} &= \begin{pmatrix} -\frac{1}{\varepsilon} & -1 \\ \frac{1}{\varepsilon^2} & 0 \end{pmatrix} \begin{pmatrix} \tilde{y}_\varepsilon(t) \\ \tilde{d}_\varepsilon(t) \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \dot{d}(t) \\ &= A \begin{pmatrix} \tilde{y}_\varepsilon(t) \\ \tilde{d}_\varepsilon(t) \end{pmatrix} + B \dot{d}(t). \end{aligned} \quad (31)$$

The eigenvalues of A are found to be

$$\lambda_1 = -\frac{1}{2\varepsilon} + \frac{\sqrt{3}}{2\varepsilon}j, \quad \lambda_2 = -\frac{1}{2\varepsilon} - \frac{\sqrt{3}}{2\varepsilon}j. \quad (32)$$

A straightforward computation shows that

$$\begin{aligned} e^{At} &= \begin{pmatrix} \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{\lambda_1 t} - \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{\lambda_2 t} & \frac{\lambda_1 \lambda_2}{C_\varepsilon (\lambda_2 - \lambda_1)} (e^{\lambda_2 t} - e^{\lambda_1 t}) \\ \frac{C_\varepsilon}{\lambda_2 - \lambda_1} (e^{\lambda_1 t} - e^{\lambda_2 t}) & -\frac{\lambda_2}{\lambda_2 - \lambda_1} e^{\lambda_1 t} + \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{\lambda_2 t} \end{pmatrix}, \\ e^{At} B &= -\begin{pmatrix} \frac{\lambda_1 \lambda_2}{C_\varepsilon (\lambda_2 - \lambda_1)} (e^{\lambda_2 t} - e^{\lambda_1 t}) \\ -\frac{\lambda_2}{\lambda_2 - \lambda_1} e^{\lambda_1 t} + \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{\lambda_2 t} \end{pmatrix}, \quad C_\varepsilon = \frac{1}{\varepsilon^2}. \end{aligned} \quad (33)$$

By (33), we see that there exists a constant $\hat{L} > 0$ such that

$$\|e^{At}\| \leq \frac{\hat{L}}{\varepsilon} e^{-\frac{1}{2\varepsilon}t}, \quad \|e^{At} B\| \leq \hat{L} e^{-\frac{1}{2\varepsilon}t}. \quad (34)$$

From estimations (34), we obtain immediately the convergence (27)–(29). \square

By Lemma 2.2, we see that the design of ESO (26) is to make $\|e^{At}\|$ have arbitrary decay rate and then make us of special structure of B . So, ADRC is hard to apply directly to PDEs because it is difficult to make a PDE system have arbitrary decay rate. This also explains why \hat{d} must be uniformly bounded.

The point (29) brings trouble to PDEs (see (44) later) because usually we only have admissibility with L_{loc}^2 control. The admissibility with L_{loc}^1 leads to the bounded control operator, see, for instance, Theorem 4.8 of Weiss (1989). To overcome this difficulty in PDEs, we design the following feedback control law to system (13), which is a slight change of associated controls in papers Guo and Jin (2013a, 2013b); Guo and Liu (2014):

$$U_0(t) = -\text{sat}(\hat{d}_\varepsilon(t)), \quad (35)$$

where

$$\text{sat}(x) = \begin{cases} M_1, & x \geq M_1 + 1, \\ -M_1, & x \leq -M_1 - 1, \\ x, & x \in (-M_1 - 1, M_1 + 1). \end{cases} \quad (36)$$

Since $|d(t)| \leq M_1$, for any given $a > 0$, by (27), when ε is sufficiently small, we have $U_0(t) = -\hat{d}_\varepsilon(t)$ for all $t \in [a, \infty)$. So, the feedback control law (35) is just used to cancel the disturbance d since \mathcal{A}_D generates an exponential stable C_0 -semigroup. This estimation/cancellation strategy is just the nature of ADRC.

Under feedback (35), the closed-loop of system (13) is

$$\begin{cases} \dot{X}(t) = (A + BK)X(t) + Bz(0, t), \\ z_t(x, t) = z_{xx}(x, t) - cz(x, t), \\ z_x(0, t) = 0, \\ z_x(1, t) = -\text{sat}(\hat{d}_\varepsilon(t)) + d(t), \\ \dot{\hat{y}}_\varepsilon(t) = y_2(t) - \frac{1}{\varepsilon}(\hat{y}_\varepsilon(t) - y_1(t)), \\ \dot{\hat{d}}_\varepsilon(t) = \frac{1}{\varepsilon^2}(\hat{y}_\varepsilon(t) - y_1(t)). \end{cases} \quad (37)$$

Lemma 2.3: Assume that $|d(t)| \leq M_1$ and $\dot{d}(t)$ is measurable, $|\dot{d}(t)| \leq M_2$ for all $t \geq 0$. Then, for any initial value $(X(0), z(\cdot, 0), \hat{y}_\varepsilon(0), \hat{d}_\varepsilon(0)) \in \mathcal{H} \times \mathbb{R}^2$, the closed-loop system (37) admits a unique solution $(X, z, \hat{y}_\varepsilon, \hat{d}_\varepsilon)^\top \in C(0, \infty; \mathcal{H} \times \mathbb{R}^2)$, and

$$\lim_{\substack{t \rightarrow \infty \\ \varepsilon \rightarrow 0}} \|(X(t), z(\cdot, t), \hat{y}_\varepsilon(t), \hat{d}_\varepsilon(t) - d(t))\|_{\mathcal{H} \times \mathbb{R}^2} = 0.$$

Proof: Using the error variables $(\tilde{y}_\varepsilon, \tilde{d}_\varepsilon)$ defined in (30), we can write the equivalent system of (37) as follows:

$$\begin{cases} \dot{X}(t) = (A + BK)X(t) + Bz(0, t), \\ z_t(x, t) = z_{xx}(x, t) - cz(x, t), \\ z_x(0, t) = 0, \\ z_x(1, t) = -\text{sat}(\tilde{d}_\varepsilon(t) + d(t)) + d(t), \\ \dot{\tilde{y}}_\varepsilon(t) = -\frac{1}{\varepsilon}\tilde{y}_\varepsilon(t) - \tilde{d}_\varepsilon(t), \\ \dot{\tilde{d}}_\varepsilon(t) = \frac{1}{\varepsilon^2}\tilde{y}_\varepsilon(t) - \dot{d}(t). \end{cases} \quad (38)$$

It is seen from (38) that $(\tilde{y}_\varepsilon, \tilde{d}_\varepsilon)$ is an external model for the ' (X, z) ' part of the system (Medvedev & Hillerström, 1995), which is just (31). By Lemma 2.2,

$$(\tilde{y}_\varepsilon(t), \tilde{d}_\varepsilon(t)) \rightarrow 0 \text{ as } t \rightarrow \infty, \varepsilon \rightarrow 0. \quad (39)$$

Now, we consider the ' (X, z) ' part of system (38), which is re-written as

$$\begin{cases} \dot{X}(t) = (A + BK)X(t) + Bz(0, t), \\ z_t(x, t) = z_{xx}(x, t) - cz(x, t), \\ z_x(0, t) = 0, \\ z_x(1, t) = -\text{sat}(\tilde{d}_\varepsilon(t) + d(t)) + d(t), \end{cases} \quad (40)$$

System (40) can be rewritten as an evolution equation in \mathcal{H} as

$$\frac{d}{dt}Z(\cdot, t) = \mathcal{A}_D Z(\cdot, t) + \mathcal{B}_D[-\text{sat}(\tilde{d}_\varepsilon(t) + d(t)) + d(t)], \quad (41)$$

where $Z(\cdot, t) = [X(t), z(\cdot, t)]$, and $\mathcal{A}_D, \mathcal{B}_D$ are the same as that in (14).

By Proposition 2.1, for any initial value $[X(0), z(\cdot, 0)] \in \mathcal{H}$, there exists a unique (weak) solution $[X, z] \in C(0, \infty; \mathcal{H})$ which can be written as

$$Z(\cdot, t) = e^{\mathcal{A}_D t} Z(\cdot, 0) + \int_0^t e^{\mathcal{A}_D(t-s)} \mathcal{B}_D[-\text{sat}(\tilde{d}_\varepsilon(s) + d(s)) + d(s)] ds. \quad (42)$$

By (39), for any given $\varepsilon_0 > 0$, there exist $t_0 > 0$ and $\varepsilon_1 > 0$ such that $|\text{sat}(\tilde{d}_\varepsilon(t) + d(t)) + d(t)| < \varepsilon_0$ for all $t > t_0$ and $0 < \varepsilon < \varepsilon_1$. We rewrite solution of (42) as

$$\begin{aligned} Z(\cdot, t) &= e^{\mathcal{A}_D t} Z(\cdot, 0) + e^{\mathcal{A}_D(t-t_0)} \int_0^{t_0} e^{\mathcal{A}_D(t_0-s)} \mathcal{B}_D[-\text{sat}(\tilde{d}_\varepsilon(s) \\ &\quad + d(s)) + d(s)] ds \\ &\quad + \int_{t_0}^t e^{\mathcal{A}_D(t-s)} \mathcal{B}_D[-\text{sat}(\tilde{d}_\varepsilon(s) + d(s)) + d(s)] ds. \end{aligned} \quad (43)$$

The admissibility of \mathcal{B}_D implies that

$$\begin{aligned} &\left\| \int_0^{t_0} e^{\mathcal{A}_D(t-s)} \mathcal{B}_D[-\text{sat}(\tilde{d}_\varepsilon(s) + d(s)) + d(s)] ds \right\|_{\mathcal{H}}^2 \\ &\leq C_{t_0} \|\text{sat}(\tilde{d}_\varepsilon + d) + d\|_{L^2(0, t_0)}^2 \\ &\leq t_0 C_{t_0} (2M_1 + 1)^2, \end{aligned} \quad (44)$$

for some constant C_{t_0} that is independent of \tilde{d}_ε and d . Since $e^{\mathcal{A}_D t}$ is exponentially stable, and \mathcal{B} is admissible to $e^{\mathcal{A}_D t}$ with L_{loc}^2 control and hence is admissible to $e^{\mathcal{A}_D t}$ with L_{loc}^∞ control, it follows from Proposition 2.5 of Weiss (1989) that

$$\begin{aligned} &\left\| \int_{t_0}^t e^{\mathcal{A}_D(t-s)} \mathcal{B}_D[\text{sat}(\tilde{d}_\varepsilon(s) + d(s)) + d(s)] ds \right\| \\ &= \left\| \int_0^t e^{\mathcal{A}_D(t-s)} \mathcal{B}_D(0 \diamond [\text{sat}(\tilde{d}_\varepsilon(s) + d(s)) + d(s)]) ds \right\| \\ &\leq L \|\text{sat}(\tilde{d}_\varepsilon + d) + d\|_{L^\infty(0, \infty)} \leq L\varepsilon_0, \end{aligned} \quad (45)$$

where L is a constant that is independent of \tilde{d}_ε and d , and (Weiss, 1989)

$$(d_1 \diamond_{\tau} d_2)(t) = \begin{cases} d_1(t), & 0 \leq t \leq \tau, \\ d_2(t - \tau), & t > \tau. \end{cases} \quad (46)$$

Suppose that $\|e^{\mathcal{A}_D t}\| \leq L_0 e^{-\omega t}$ for some $L_0, \omega > 0$. By (43)–(45),

$$\begin{aligned} \|Z(\cdot, t)\| &\leq L_0 e^{-\omega t} \|Z(\cdot, 0)\| \\ &\quad + L_0 t_0 (2M_1 + 1)^2 C_{t_0} e^{-\omega(t-t_0)} + L\varepsilon_0. \end{aligned} \quad (47)$$

The first two terms on the right-hand side of (47) tend to zero as $t \rightarrow \infty$. This shows that $\|z(\cdot, t)\|_{L^2(0,1)} \rightarrow 0$ as $t \rightarrow \infty$. Hence, by (25), $y_1(t) = \int_0^1 (2x^3 - 3x^2)z(x, t)dx \rightarrow 0$ as $t \rightarrow \infty$. The result then follows with (39) and (30). \square

Returning back to system (4) by the inverse transformations (5) and (7), feedback control (12) and (35), we have proved, the following Theorem 2.4, the main result of this section.

Theorem 2.4: Assume that $|d(t)| \leq M_1$ and $\dot{d}(t)$ is measurable, $|\dot{d}(t)| \leq M_2$ for all $t \geq 0$. Then, for any initial value $(X(0), u(\cdot, 0), \hat{y}_\varepsilon(0), \hat{d}_\varepsilon(0)) \in \mathcal{H} \times \mathbb{R}^2$, the closed-loop of system (4) following:

$$\begin{cases} \dot{X}(t) = AX(t) + Bu(0, t), \quad t > 0, \\ u_t(x, t) = u_{xx}(x, t), \quad x \in (0, 1), \quad t > 0, \\ u_x(0, t) = 0, \\ u_x(1, t) = U(t) + d(t), \\ \dot{\hat{y}}_\varepsilon(t) = y_2(t) - \frac{1}{\varepsilon}(\hat{y}_\varepsilon(t) - y_1(t)), \\ \dot{\hat{d}}_\varepsilon(t) = \frac{1}{\varepsilon^2}(\hat{y}_\varepsilon(t) - y_1(t)), \end{cases} \quad (48)$$

admits a unique solution $(X, u, \hat{y}_\varepsilon, \hat{d}_\varepsilon)^\top \in C(0, \infty; \mathcal{H} \times \mathbb{R}^2)$, and

$$\lim_{\substack{t \rightarrow \infty \\ \varepsilon \rightarrow 0}} \|(X(t), u(\cdot, t), \hat{y}_\varepsilon(t), \hat{d}_\varepsilon(t) - d(t))\|_{\mathcal{H} \times \mathbb{R}^2} = 0,$$

where the feedback control is

$$\begin{aligned} U(t) = & \int_0^1 q_x(1, x)u(x, t)dx + \gamma'(1) + k(1, 1)w(1, t) \\ & + \int_0^1 k_x(1, x)w(x, t)dx - \text{sat}(\hat{d}_\varepsilon(t)), \quad t \geq 0, \end{aligned} \quad (49)$$

and

$$\begin{cases} y_1(t) = \int_0^1 (2x^3 - 3x^2)z(x, t)dx, \\ y_2(t) = \int_0^1 (-2cx^3 + 3cx^2 + 12x - 6)z(x, t)dx, \\ \begin{cases} z(x, t) = w(x, t) - \int_0^x k(x, y)w(y, t)dy, \\ k(x, y) = -cx \frac{I_1(\sqrt{c(x^2 - y^2)})}{\sqrt{c(x^2 - y^2)}}, \\ w(x, t) = u(x, t) - \int_0^x q(x, y)u(y, t)dy - \gamma(x)X(t), \\ q(x, y) = \int_0^{x-y} \gamma(\sigma)Bd\sigma. \end{cases} \end{cases} \quad (50)$$

3. Time-varying high-gain estimator-based feedback

In previous section, we estimate the disturbance $d(t)$ by constant high gain. This brings the notorious peaking value problem in estimator at $t = 0$ as indicated by (29). In this section, we propose a novel disturbance estimator by time varying high gain. This improves the performance through four aspects: (1) the practical stability claimed by Theorem 2.4 becomes the asymptotic stability; (2) the boundedness of derivative of disturbance is relaxed in much extend; (3) the peaking value is reduced significantly; and (4) the possible non-smooth control (35) becomes smooth. The possible trouble brought by this approach is the high-frequency noise sensitivity.

Now, we design the following ESO with time varying high gain for system (24) as follows:

$$\begin{cases} \dot{\hat{y}}(t) = -U_0(t) - \hat{d}(t) - g(t)[\hat{y}(t) - y_1(t)], \\ \dot{\hat{d}}(t) = -g^2(t)[\hat{y}(t) - y_1(t)], \end{cases} \quad (52)$$

where $g \in C^1[0, \infty)$ is a time-varying gain function satisfying:

$$\begin{cases} g(t) > 0, \quad \dot{g}(t) > 0, \quad \forall t \geq 0, \\ g(t) \rightarrow \infty \quad \text{as } t \rightarrow \infty, \quad \sup_{t \in [0, \infty)} \left| \frac{\dot{g}(t)}{g(t)} \right| < \infty. \end{cases} \quad (53)$$

In addition, we assume that the disturbance $d(t) \in H_{\text{loc}}^1(0, \infty)$ satisfies

$$\lim_{t \rightarrow \infty} \frac{|\dot{d}(t)| + |d(t)|}{g(t)} = 0. \quad (54)$$

By condition (54), both $d(t)$ and $\dot{d}(t)$ are allowed (at least mathematically) to grow exponentially at any rate by choosing properly the gain function $g(t)$. This relaxes the condition in the previous section where $d(t)$ and $\dot{d}(t)$ are assumed to be uniformly bounded. Once again, $\hat{d}(t)$ is used to estimate $d(t)$, which is confirmed by the following lemma.

Lemma 3.1: Let (\hat{y}, \hat{d}) be the solution of (52). Then,

$$\lim_{t \rightarrow \infty} |\hat{y}(t) - y(t)| = 0, \quad \lim_{t \rightarrow \infty} |\hat{d}(t) - d(t)| = 0. \quad (55)$$

Proof: Set

$$\tilde{y}(t) = g(t)[\hat{y}(t) - y_1(t)], \quad \tilde{d}(t) = \hat{d}(t) - d(t). \quad (56)$$

Then, the error (\tilde{y}, \tilde{d}) is governed by

$$\begin{cases} \dot{\tilde{y}}(t) = -g(t)[\tilde{y}(t) - \tilde{d}(t)] + \frac{\dot{g}(t)}{g(t)}\tilde{y}(t) - g(t)y_2(t), \\ \dot{\tilde{d}}(t) = -g(t)\tilde{y}(t) - \dot{d}(t). \end{cases} \quad (57)$$

The existence of the local classical solution to (57) is guaranteed by the local Lipschitz condition of the right-hand side of (57). The global solution is ensured by the following Lyapunov function argument. Define

$$V(t) = \tilde{y}^2(t) + \frac{3}{2}\tilde{d}^2(t) - \tilde{y}(t)\tilde{d}(t). \quad (58)$$

Then,

$$\frac{1}{2}V(t) \leq \tilde{y}^2(t) + \tilde{d}^2(t) \leq 2V(t). \quad (59)$$

Differentiate V along the solution of (57) to obtain

$$\begin{aligned}
 \dot{V}(t) &= 2\tilde{y}(t)\dot{\tilde{y}}(t) + 3\tilde{d}(t)\dot{\tilde{d}}(t) - \dot{\tilde{y}}(t)\tilde{d}(t) - \dot{\tilde{d}}(t)\tilde{y}(t) \\
 &= 2\tilde{y}(t) \left\{ -g(t)[\tilde{y}(t) - \tilde{d}(t)] + \frac{\dot{g}(t)}{g(t)}\tilde{y}(t) - g(t)y_2(t) \right\} \\
 &\quad + 3\tilde{d}(t) \left\{ -g(t)\tilde{y}(t) - \dot{d}(t) \right\} \\
 &\quad - \tilde{d}(t) \left\{ -g(t)[\tilde{y}(t) - \tilde{d}(t)] + \frac{\dot{g}(t)}{g(t)}\tilde{y}(t) - g(t)y_2(t) \right\} \\
 &\quad - \tilde{y}(t) \left\{ -g(t)\tilde{y}(t) - \dot{d}(t) \right\} \\
 &= \left[-g(t) + \frac{2\dot{g}(t)}{g(t)} \right] \tilde{y}^2(t) - g(t)\tilde{d}^2(t) - \frac{\dot{g}(t)}{g(t)}\tilde{y}(t)\tilde{d}(t) \\
 &\quad + \dot{d}(t)[\tilde{y}(t) - 3\tilde{d}(t)] + g(t)y_2(t)[\tilde{d}(t) - 2\tilde{y}(t)] \\
 &\leq -\frac{1}{2}\kappa(t)V(t) + [4|\dot{d}(t)| + 3g(t)|y_2(t)|] \|(\tilde{y}(t), \tilde{d}(t))\| \\
 &\leq -\frac{1}{2}\kappa(t)V(t) + \sqrt{2} [4|\dot{d}(t)| + 3g(t)|y_2(t)|] \sqrt{V(t)},
 \end{aligned} \tag{60}$$

where, by assumptions (53) and (54),

$$\kappa(t) = g(t) - \sup_{t \in [0, \infty)} \left| \frac{3\dot{g}(t)}{g(t)} \right| \rightarrow \infty \text{ as } t \rightarrow \infty, \tag{61}$$

and hence, there exists $t_0 > 0$ such that

$$\kappa(t) > 0, \quad \forall t \geq t_0.$$

This, together with (60), shows that

$$\begin{aligned}
 \frac{\sqrt{V(t)}}{dt} &\leq -\frac{1}{4}\kappa(t)\sqrt{V(t)} + \frac{\sqrt{2}}{2}[4|\dot{d}(t)| \\
 &\quad + 3g(t)|y_2(t)|], \quad \forall t \geq 0.
 \end{aligned} \tag{62}$$

It then follows that

$$\begin{aligned}
 \sqrt{V(t)} &\leq \sqrt{V(0)}e^{-\frac{1}{4}\int_0^t \kappa(s)ds} \\
 &\quad + \frac{\int_0^t [\sqrt{2}(4|\dot{d}(s)| + 3g(s)|y_2(s)|)]e^{\frac{1}{4}\int_0^s \kappa(\tau)d\tau}ds}{2e^{\frac{1}{4}\int_0^t \kappa(s)ds}}.
 \end{aligned} \tag{63}$$

The first term on the right-hand side of (63) is obviously convergent to zero as $t \rightarrow \infty$ owing to (61). Apply the L'Hospital rule to the second term on the right-hand side of (63) to obtain $\lim_{t \rightarrow \infty} \sqrt{V(t)} = 0$, which amounts to

$$\lim_{t \rightarrow \infty} [\tilde{y}^2(t) + \tilde{d}^2(t)] = 0. \tag{64}$$

This leads to

$$\lim_{t \rightarrow \infty} [|\tilde{y}(t)| + |\tilde{d}(t)|] = 0.$$

The proof is complete. \square

By Lemma 3.1, we design naturally the feedback control

$$U_0(t) = -\hat{d}(t), \tag{65}$$

under which, the closed-loop of system (13) is

$$\begin{cases} \dot{X}(t) = (A + BK)X(t) + Bz(0, t), \\ z_t(x, t) = z_{xx}(x, t) - cz(x, t), \\ z_x(0, t) = 0, \\ z_x(1, t) = -\hat{d}(t) + d(t), \\ \dot{\hat{y}}(t) = -g(t)[\hat{y}(t) - y_1(t)], \\ \dot{\hat{d}}(t) = -g^2(t)[\hat{y}(t) - y_1(t)]. \end{cases} \tag{66}$$

Proposition 3.2: Assume that the time-varying gain $g(t) \in C^1[0, \infty)$ satisfies (53) and the disturbance $d(t) \in H_{loc}^1(0, \infty)$ satisfies (54). Then, for any initial value $(X(0), z(0), \hat{y}(0), \hat{d}(0)) \in \mathcal{H} \times \mathbb{R}^2$, there exists a unique solution $(X, z, \hat{y}, \hat{d}) \in C(0, \infty; \mathcal{H} \times \mathbb{R}^2)$ to system (66) and system (66) is asymptotically stable in the sense that

$$\lim_{t \rightarrow \infty} \|(X(t), z(\cdot, t), \hat{y}(t), \hat{d}(t) - d(t))\|_{\mathcal{H} \times \mathbb{R}^2} = 0.$$

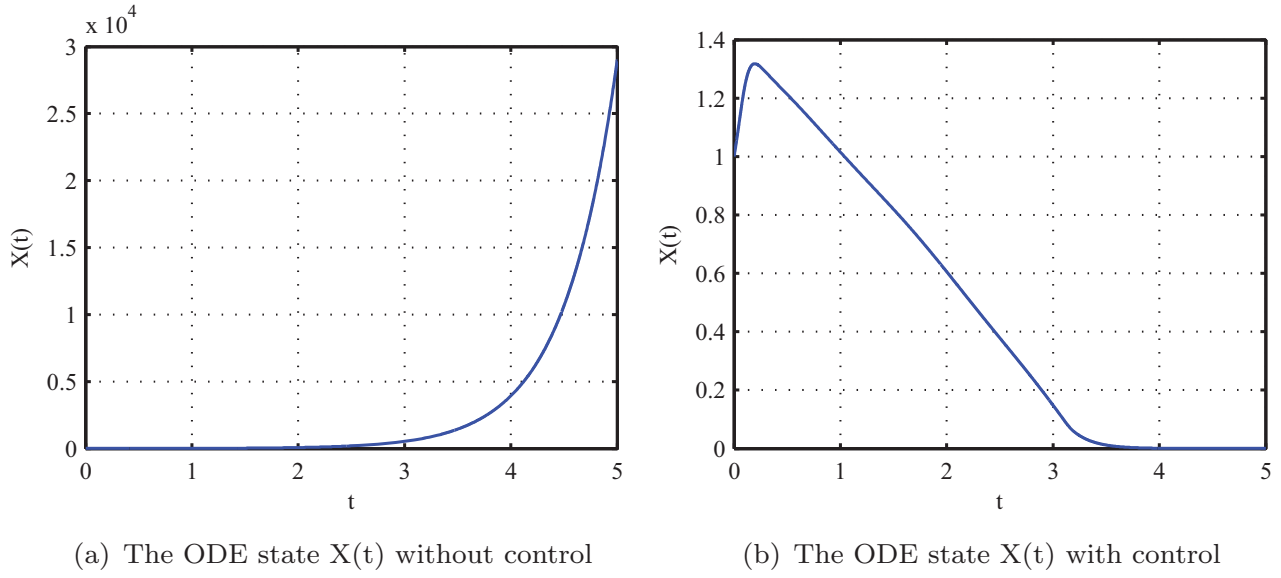
Proof: Using the error variables (\tilde{y}, \tilde{d}) defined in (56), we can write the equivalent system of (66) as follows:

$$\begin{cases} \dot{X}(t) = (A + BK)X(t) + Bz(0, t), \\ z_t(x, t) = z_{xx}(x, t) - cz(x, t), \\ z_x(0, t) = 0, \\ z_x(1, t) = -\hat{d}(t) + d(t), \\ \dot{\tilde{y}}(t) = -g(t)[\tilde{y}(t) - \tilde{d}(t)] + \frac{\dot{g}(t)}{g(t)}\tilde{y}(t) - g(t)y_2(t), \\ \dot{\tilde{d}}(t) = -g(t)\tilde{y}(t) - \dot{d}(t). \end{cases} \tag{67}$$

The 'ODE part' of (67) is just the system (57), which is shown to be convergent by Lemma 3.1. The '(X, z) part' of (67) is similar to (40) and the proof hence becomes similar to the proof of Theorem 2.4. The details are omitted. \square

Returning back to system (4) by the inverse transformations (5) and (7), feedback control (12) and (35), we have proved, from Proposition 3.2, the following Theorem 3.3.

Theorem 3.3: Assume that the time-varying gain $g(t) \in C^1[0, \infty)$ satisfies (53) and the disturbance $d(t) \in H_{loc}^1(0, \infty)$ satisfies (54). Then, for any initial value $(X(0), u(\cdot, 0), \hat{y}(0), \hat{d}(0)) \in \mathcal{H} \times \mathbb{R}^2$, the closed-loop of system (4)

Figure 2. The ODE state $X(t)$ for the open-loop and the closed-loop counterparts.

following:

$$\begin{cases} \dot{X}(t) = AX(t) + Bu(0, t), & t > 0, \\ u_t(x, t) = u_{xx}(x, t), & x \in (0, 1), t > 0, \\ u_x(0, t) = 0, \\ u_x(1, t) = -\hat{d}(t) + \int_0^1 q_x(1, y)u(y, t)dy + \gamma'(1) \\ \quad + k(1, 1)w(1, t) + \int_0^1 k_x(1, y)w(y, t)dy + d(t), \\ \dot{\hat{y}}(t) = -g(t)[\hat{y}(t) - y_1(t)], \\ \dot{\hat{d}}(t) = -g^2(t)[\hat{y}(t) - y_1(t)], \end{cases} \quad (68)$$

admits a unique solution $(X, u, \hat{y}, \hat{d})^\top \in C(0, \infty; \mathcal{H} \times \mathbb{R}^2)$, and system (68) is asymptotically stable:

$$\lim_{t \rightarrow \infty} \|(X(t), u(\cdot, t), \hat{y}(t), \hat{d}(t) - d(t))\|_{\mathcal{H} \times \mathbb{R}^2} = 0,$$

where

$$\begin{cases} y_1(t) = \int_0^1 (2x^3 - 3x^2)z(x, t)dx, \\ y_2(t) = \int_0^1 (-2cx^3 + 3cx^2 + 12x - 6)z(x, t)dx, \end{cases} \quad (69)$$

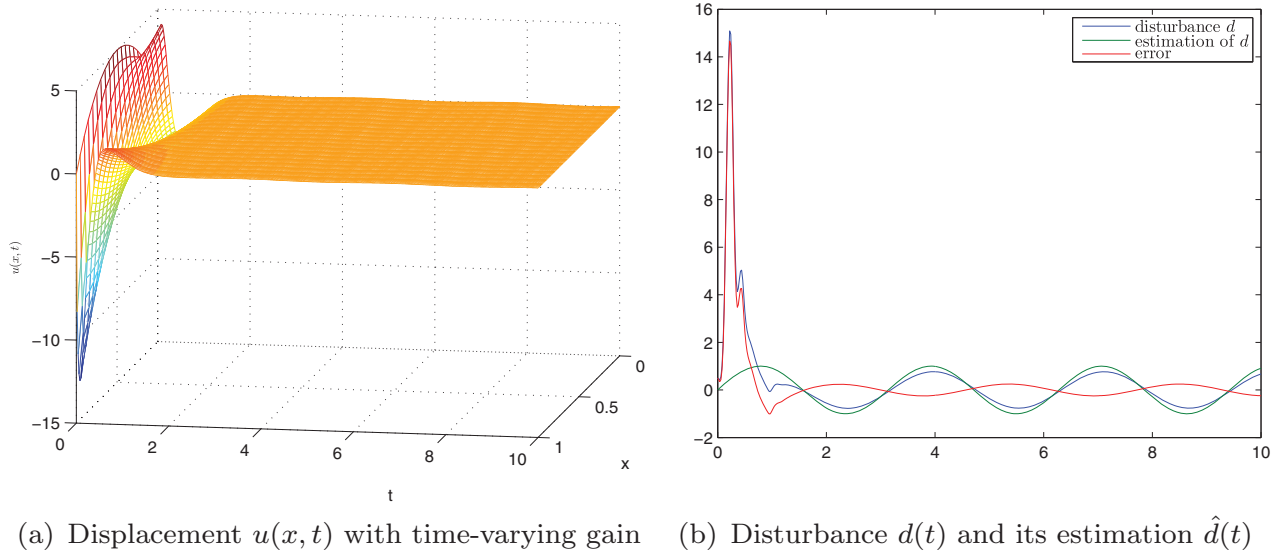


Figure 3. PDE displacement and disturbance tracking by time-varying gain.

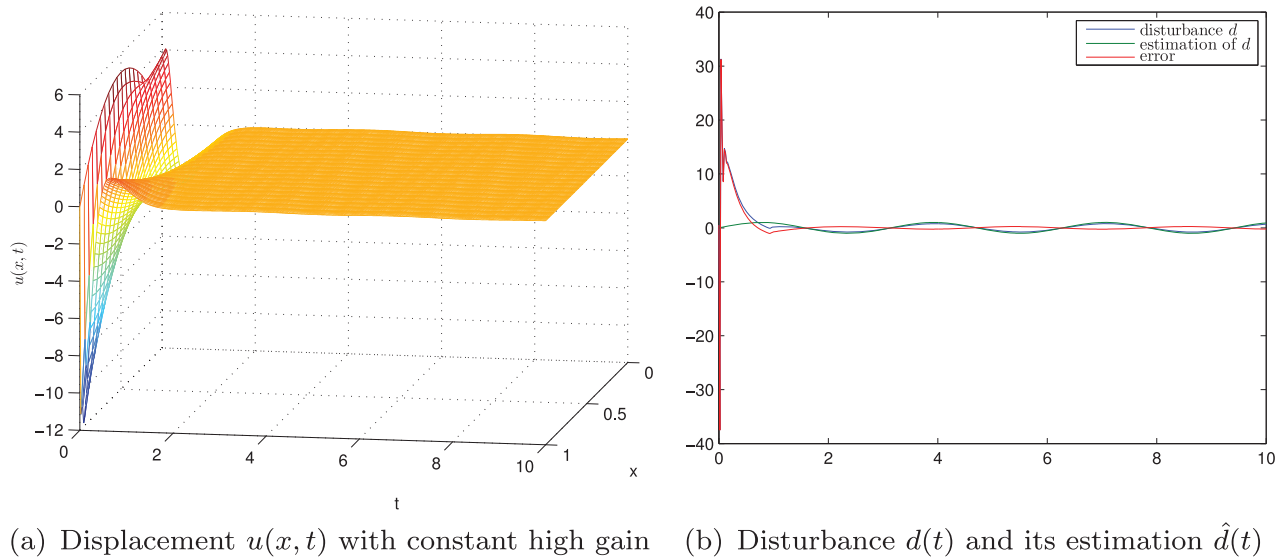


Figure 4. PDE displacement and disturbance tracking by constant high gain.

$$\begin{cases} z(x, t) = w(x, t) - \int_0^x k(x, y)w(y, t)dy, & k(x, y) \\ & = -cx \frac{I_1(\sqrt{c(x^2 - y^2)})}{\sqrt{c(x^2 - y^2)}}, \\ w(x, t) = u(x, t) - \int_0^x q(x, y)u(y, t)dy - \gamma(x)X(t), \\ q(x, y) = \int_0^{x-y} \gamma(\sigma)Bd\sigma. \end{cases} \quad (70)$$

4. Numerical simulation

In this section, we present some numerical simulations to show visually the effectiveness of the proposed controller for the ODE–PDE cascade system (68).

We choose $A = 2$, $B = 1$, $d(t) = \cos 5t$, and the initial values as $X(0) = 1$, $u(x, 0) = x$ ($0 < x \leq 1$). The time-varying gain is taken as

$$g(t) = \begin{cases} t + 1, & t + 1 \leq 10, \\ 10, & t + 1 \geq 10. \end{cases} \quad (71)$$

The numerical results are plotted in Figures 2 and 3, respectively. In Figure 2 (a), the ODE state $X(t)$ without control is shown and in Figure 2(b), the ODE state $X(t)$ with control in system (68) is demonstrated. It is seen that the control effect is very satisfactorily. Figure 3(a) shows the displacement $u(x, t)$ of system (68), and Figure 3(b) demonstrates the disturbance estimation $\hat{d}(t)$ compared with the disturbance $d(t)$. The convergences are all shown to be fast.

Figure 4 demonstrates the displacement $u(x, t)$ of system (68) (Figure 4(a)), and the disturbance estimation $\hat{d}(t)$ compared with $d(t)$ (Figure 4(b)), with the constant high

gain $g(t) \equiv 10$. The peaking value around 40 for disturbance tracking is clearly observed in Figure 4(b).

Compared with Figure 3(b), it is seen from Figure 4(b) that the peaking value is dramatically reduced.

5. Concluding remarks

In this paper, the active disturbance rejection control is applied to stabilisation for a cascade ODE–PDE system. Both disturbance estimators with constant high gain and time-varying gain are designed, respectively. The practical stability for the closed-loop system with constant high gain and asymptotic stability with time-varying gain are proved. The constant high gain is easily tuning in practice but produces peaking value problem. The time-varying gain reduces peaking value significantly but it brings sensitivity for high-frequency noise. The last point comes from the fact of noise tracking with disturbance together instead of noise filtering, for which we refer to the discussion in numerical simulation of Guo and Zhou (2014). A recommended scheme is to apply the time-varying gain in the initial stage so that the peaking value can reach a reasonable area, and then apply the constant high gain.

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