The Lyapunov approach to boundary stabilization of an anti-stable one-dimensional wave equation with boundary disturbance

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SUMMARY

In this paper, we are concerned with the boundary stabilization of a one-dimensional anti-stable wave equation with the boundary external disturbance. The backstepping method is first applied to transform the anti-stability from the free end to the control end. A variable structure feedback stabilizing controller is designed by the Lyapunov function approach. It is shown that the resulting closed-loop system is associated with a nonlinear semigroup and is asymptotically stable. In addition, we show that this controller is robust to the external disturbance in the sense that the vibrating energy of the closed-loop system is also convergent to zero as time goes to infinity in the presence of bounded deterministic disturbance at the control end. The existence and uniqueness of the solution are also developed by the Galerkin approximation scheme.

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1. INTRODUCTION

In the last three decades, the boundary control, due to its easily physical implementations in engineering, is widely applied as the major control strategy for the systems governed by partial differential equations (PDEs) (see for instance [1–8] and the references therein). In many situations, the control is used not only to guarantee the system to be normally operated in an ideal operation environment but also to be normally operated in the environment with uncertainties coming from either the internal or the external disturbance. This requires the controller to be robust against the uncertainty in some extent. There are many works that contributed to this aspect. In [2, 9], the stabilization of a one-dimensional wave equation with harmonic uncertainty that suffered from input or output is considered. Based on semigroup theory, the sliding mode control method is applied to deal with a class of abstract infinite-dimensional systems in [10] where the control operator and disturbance operator are all assumed to be bounded, which represents mainly the distributed control. The boundary stabilization for a one-dimensional heat equation with boundary disturbance is studied in [11] by sliding mode control also, where an integral transformation is used to transform the heat equation that is the second order in spacial variable into first-order PDEs. Very recently, the sliding mode boundary stabilizer is designed for a one-dimensional unstable heat equation in [1], and the stabilization of a wave equation with distributed control and uncertainty by the variable structure control is considered in [12] on the basis of the Lyapunov function method.

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Motivated mainly by [12], we are concerned with, in this paper, the stabilization of a one-dimensional wave equation that suffered from anti-stability in the free end and external disturbance from the input end by the variable structure control based on the Lyapunov function approach. The system is governed by the following PDEs:

\[
\begin{align*}
&u_{tt}(x, t) = u_{xx}(x, t), \ x \in (0, 1), t > 0, \\
&u_x(0, t) = -qu_t(0, t), \ t \geq 0, \\
&u_x(1, t) = U(t) + d(t), \ t \geq 0,
\end{align*}
\]

where \( u \) is the state, \( q > 0, q \neq 1 \) is a constant, \( U \) is the control input. The unknown disturbance \( d \) and its derivative \( d \) are supposed to be bounded, that is, \( |d(t)| \leq M|d(t)| \leq M_1 \) for some constants \( M, M_1 < \infty \) and all \( t \geq 0 \). We refer to [3] for the physical interpretation of the anti-stable wave equation. Throughout the paper, we use \( u_t \) or \( u \) to stand for the partial derivative of \( u \) with respect to \( t \), and \( u_x \) the the partial derivative with respect to \( x \).

We proceed as follows. In the next section, we introduce a backstepping transformation that transforms system (1.1) to a target system where the anti-stability term from the free end is transformed into the control end. Then using the Lyapunov function method, we design a boundary stabilizing state feedback controller for the target system, and the boundary controller to the original system is thus designed. In Section 3, we show that the closed-loop system without disturbance is associated with a nonlinear semigroup solution and is asymptotically stable. Subsection 4.1 is devoted to the existence and uniqueness of the classical solution for the closed-loop system in the presence of the disturbance. The convergence for the vibrating energy of the closed-loop system with disturbance is presented in Subsection 4.2. An example with numerical simulation is illustrated in Section 5. Some concluding remarks are presented in Section 6.

2. STATE FEEDBACK CONTROLLER DESIGN

In view of [7], we introduce a transformation as follows:

\[
w(x, t) = u(x, t) - \frac{q + c}{q^2 - 1} \int_0^x u_t(y, t)dy - \frac{q(q + c)}{q^2 - 1} \int_0^x u_x(y, t)dy,
\]

which transforms system (1.1) into the following target system

\[
\begin{align*}
&w_{tt}(x, t) = w_{xx}(x, t), \ x \in (0, 1), \ t > 0, \\
&w_x(0, t) = cw_t(0, t), \ t \geq 0, \\
&w_x(1, t) = -\frac{c^2 - 1}{1 + qc}(U + d(t)) + \frac{q + c}{1 + qc}w_t(1, t), \ t \geq 0,
\end{align*}
\]

where \( c > 1 \) is the design parameter and the recommended value is \( c \approx 1 \). It is seen that under the transformation (2.1), the anti-stable term \(-qu_t(0, t)\) at the free end of (1.1) is transformed into the anti-stable term \( \frac{q + c}{1 + qc}w_t(1, t) \) at the control end of (2.2). This is the role played by the backstepping transformation (2.1).

The transformation (2.1) is invertible that

\[
u(x, t) = w(x, t) + \frac{q + c}{c^2 - 1} \int_0^x w_t(y, t)dy - \frac{c(q + c)}{c^2 - 1} \int_0^x w_x(y, t)dy.
\]

So the systems (1.1) and (2.2) are equivalent to each other under the transformation (2.1) and its inverse (2.3). In what follows, we need only focus on the target system (2.2), which has a more simpler form.

We design the following boundary state feedback controller for system (2.2):

\[
U(t) = \frac{q + c + (1 + qc)c_2}{c^2 - 1}w_t(1, t) + c_1 \frac{qc + 1}{c^2 - 1}w(1, t) + (c^2 - 1)K\text{sign}(w_t(1, t)),
\]

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where $c_1$ and $c_2 > 0$ are the design parameter and $K > 0$ is a constant to be chosen so that $(c^2 - 1)K > M$, and

$$\text{sign}(f(x)) = \begin{cases} 1, & f(x) > 0, \\ 0, & f(x) = 0, \\ -1, & f(x) < 0. \end{cases} \quad (2.5)$$

Under the feedback control (2.4), the closed-loop system of (2.2) becomes

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), x \in (0,1), t > 0, \\ w_x(0,t) = c w_t(0,t), t \geq 0, \\ w_x(1,t) = -c_1 w(1,t) - c_2 w_t(1,t) - \frac{c^2 - 1}{1 + q c} [(c^2 - 1)K \text{sign}(w_t(1,t)) + d(t)], t \geq 0. \end{cases} \quad (2.6)$$

The motivation for the control design is that under control (2.4), the closed-loop system of (2.2) has the following Lyapunov function:

$$V(t) = \frac{1}{2} \int_0^1 [w_t^2(x,t) + w_x^2(x,t)] dx + \frac{c_1}{2} w^2(1,t). \quad (2.7)$$

Actually, differentiating (2.7) formally along the solution of (2.2) gives

$$\dot{V}(t) = \int_0^1 [w_t(x,t)w_{xx}(x,t) + w_x(x,t)w_{xt}(x,t)] dx + c_1 w(1,t)w_t(1,t)$$

$$= -w_t(1,t) \frac{c^2 - 1}{1 + q c} [(c^2 - 1)K \text{sign}(w_t(1,t)) + d(t)] - cw_t^2(0,t) - c_2 w_t^2(1,t) \leq 0. \quad (2.8)$$

3. THE WELL-POSEDNESS AND STABILITY OF SYSTEM (2.6)

WITHOUT DISTURBANCE

In order to see the efficiency of the controller (2.4), we consider in this section the system (2.6) without disturbance, which is $d(t) \equiv 0$. In this case, (2.6) reads

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), x \in (0,1), t > 0, \\ w_x(0,t) = c w_t(0,t), t \geq 0, \\ w_x(1,t) = -c_1 w(1,t) - c_2 w_t(1,t) - \frac{(c^2 - 1)^2}{1 + q c} K \text{sign}(w_t(1,t)), t \geq 0. \end{cases} \quad (3.1)$$

We investigate system (3.1) in the energy state space $\mathbb{H} = H^1(0,1) \times L^2(0,1)$ defined in the real number field $\mathbb{R}$. The norm of $\mathbb{H}$ is endowed by the induced inner product:

$$\langle (f_1, g_1), (f_2, g_2) \rangle = \frac{1}{2} \int_0^1 [f_1'(x) f_2'(x) + g_1(x) g_2(x)] dx + \frac{c_1}{2} f_1(1) f_2(1), \quad (3.2)$$

$$\forall \ (f_i, g_i) \in \mathbb{H}, i = 1, 2.$$ 

The energy of system (3.1) is $V$ as defined by (2.8). Formally, the time derivative of $V$ along the solution of (3.1) is given by (2.7) with $d \equiv 0$, that is,

$$\dot{V}(t) = -cw_t^2(0,t) - c_2 w_t^2(1,t) - \frac{(c^2 - 1)^2}{1 + q c} K |w_t(1,t)| \leq 0. \quad (3.3)$$

So the nonlinear system (3.1) is dissipative. In order to associate system (3.1) with a nonlinear semigroup solution, we define the (nonlinear) system operator $A$ for (3.1) as

$$\begin{cases} A(f, g) = (g, f''), \forall \ (f, g) \in D(A), \\ D(A) = \left\{ (f, g) \in H^2(0,1) \times H^1(0,1) \mid f'(0) = cg(0) \right\}, \quad (3.4) \\ f'(1) = -c_1 f(1) - c_2 g(1) - \frac{(c^2 - 1)^2}{1 + q c} K \text{sign}(g(1)) \right\}. \end{cases}$$
With the operator $A$ at hand, system (3.1) can be written as a nonlinear evolution equation in $\mathbb{H}$:

$$
\frac{d}{dt}(w(\cdot, t), w_t(\cdot, t)) = A(w(\cdot, t), w_t(\cdot, t)), \quad (w(\cdot, 0), w_t(\cdot, 0)) = (w_0, w_1) \in \mathbb{H}. 
$$

(3.5)

**Lemma 3.1**

Suppose that $c_1, c_2 > 0$, $(c^2 - 1)K > M$. Then the operator $A$ defined by (3.4) generates a nonlinear semigroup of contractions on $\mathbb{H}$. Therefore, for any $(w_0, w_1) \in D(A)$, there exists a unique classical solution to system (3.5). Moreover, $\overline{D(A)} = \mathbb{H}$, and hence for any initial value $(w_0, w_1) \in \mathbb{H}$, there exists a unique weak solution to system (3.5).

**Proof**

We first show that $A$ is dissipative in $\mathbb{H}$. Actually, for any given $(f_i, g_i) \in D(A), i = 1, 2$, it has

$$
\langle A(f_1, g_1) - A(f_2, g_2), (f_1, g_1) - (f_2, g_2) \rangle = \frac{1}{2} \int_0^1 \left[ \left( g_1'(x) - g_2'(x) \right) \left( f_1'(x) - f_2'(x) \right) + \left( f_1''(x) - f_2''(x) \right) (g_1(x) - g_2(x)) \right] dx \\
+ \frac{c_1}{2} (g_1(1) - g_2(1))(f_1(1) - f_2(1)) \\
= \frac{1}{2} \left( f_1'(x) - f_2'(x) \right) (g_1(x) - g_2(x)) \Bigg|_0^1 + \frac{c_1}{2} (g_1(1) - g_2(1))(f_1(1) - f_2(1)) \\
= -\frac{1}{2} \frac{(c^2 - 1)^2}{1 + qc} K \left[ \text{sign}(g_1(1)) - \text{sign}(g_2(1)) \right] \left[ g_1(1) - g_2(1) \right] \\
- \frac{c}{2} |g_1(0) - g_2(0)|^2 - \frac{c_2}{2} |g_1(1) - g_2(1)|^2 \\
\leq -\frac{c}{2} |g_1(0) - g_2(0)|^2 - \frac{c_2}{2} |g_1(1) - g_2(1)|^2 \leq 0.
$$

(3.6)

Next, we show that

$$
\mathcal{R}(\lambda I - A) = \mathbb{H} \quad \text{for any } \lambda > 0.
$$

(3.7)

To this end, it suffices to show that for any $\lambda > 0$ and $(\tilde{u}, \tilde{v}) \in \mathbb{H}$, there exists a $(u, v) \in D(A)$, such that $(\lambda I - A)(u, v) = (\tilde{u}, \tilde{v})$, which is equivalent to

$$
\begin{align*}
\lambda u - v &= \tilde{u}, \\
\lambda v - u'' &= \tilde{v}, \\
u'(0) &= cv(0), \\
u'(1) &= -c_1 u(1) - c_2 v(1) - \frac{(c^2 - 1)^2}{1 + qc} K \text{sign}(v(1)),
\end{align*}
$$

(3.8)

so $v = \lambda u - \tilde{u}$ and $u$ satisfy

$$
\begin{align*}
u'' - \lambda^2 u &= -\lambda \tilde{u} - \tilde{v}, \\
u'(0) &= c \lambda u(0) - \tilde{v}(0), \\
u'(1) &= -c_1 u(1) - c_2 (\lambda u(1) - \tilde{u}(1)) - \frac{(c^2 - 1)^2}{1 + qc} K \text{sign}(\lambda u(1) - \tilde{u}(1)).
\end{align*}
$$

(3.9)

Define a functional $J(\cdot)$ on $H^1(0, 1)$ by

$$
J(\psi) = \frac{1}{2} \int_0^1 \left[ \psi''^2(x) + \lambda^2 \psi^2(x) \right] dx - \int_0^1 \left[ \lambda \tilde{u}(x) + \tilde{v}(x) \right] \psi(x) dx + \frac{c}{2\lambda} [\lambda \psi(0) - \tilde{u}(0)]^2 \\
+ \frac{c_1}{2} \psi^2(1) + \frac{c_2}{2\lambda} (c^2 - 1) K \text{sign}(\lambda u(1) - \tilde{u}(1)) \psi(1) + \frac{c_2}{2\lambda} (\lambda \psi(1) - \tilde{u}(1))^2.
$$

(3.10)
It is easy to check that \( J(\cdot) \) is convex and coercive on \( H^1(0, 1) \). In addition, \( J(\cdot) \) is continuous in \( H^1(0, 1) \). By virtue of the minimization theorem [13, Theorems 6.1.1, 6.1.3, p.301], there exists a function \( u \in H^1(0, 1) \) such that

\[
J(u) = \inf_{\psi \in H^1(0, 1)} J(\psi).
\]

This implies that the function \( \mu \to J(\mu) = J(u + \mu \psi) \) admits a minimum at \( \mu = 0 \) and thus

\[
\frac{d(J(u + \mu \psi))}{d\mu} \bigg|_{\mu=0} = 0, \forall \psi \in H^1(0, 1).
\]

This means that for any \( \psi \in H^1(0, 1) \), it has

\[
\begin{align*}
\int_0^1 \lambda^2 u(x) \psi(x) dx + \int_0^1 \psi'(x)u'(x) dx - \int_0^1 (\lambda \dot{u}(x) + \ddot{u}(x)) \psi(x) dx + c(\lambda u(0) - \dot{u}(0)) \psi(0) \\
+ \left[ c_1 u(1) + c \frac{c^2 - 1}{1 + q c} (c^2 - 1) K \text{sign}(\lambda u(1) - \dot{u}(1)) + c_2 (\lambda u(1) - \dot{u}(1)) \right] \psi(1) = 0.
\end{align*}
\]

(3.11)

In particular, for any \( \psi \in C^\infty_0(0, 1) \),

\[
\int_0^1 \lambda^2 u(x) \psi(x) dx + \int_0^1 \psi'(x)u'(x) dx - \int_0^1 (\lambda \dot{u} + \ddot{u}) \psi dx = 0,
\]

which implies that

\[
u \in H^2(0, 1), \lambda^2 u - u'' = \lambda \dot{u} + \ddot{u}.
\]

Integrate (3.11) by parts and make use of the aforementioned fact to give

\[
\begin{align*}
u'(0) &= c(\lambda u(0) - \dot{u}(0)), \\
u'(1) &= -c_1 u(1) - c_2 (\lambda u(1) - \dot{u}(1)) - \frac{(c^2 - 1)^2}{1 + q c} K \text{sign}(\lambda u(1) - \dot{u}(1)).
\end{align*}
\]

Combining the previous results, we can obtain a solution \((u, v) \in D(A)\) that satisfies

\[
(\lambda I - A)(u, v) = (\ddot{u}, \dot{u}).
\]

This is (3.7). Finally, we show that \( D(A) = \mathbb{H} \). In fact, if there exists a \( Z \in \mathbb{H} \) so that \( \langle Z, U \rangle_{\mathbb{H}} = 0 \) for all \( U \in D(A) \), and because \( R(\lambda I - A) = \mathbb{H} \) for \( \lambda > 0 \), there exists an \( X \in D(A) \), such that \((\lambda I - A)X = Z\). This implies that \( \langle (\lambda I - A)X, U \rangle_{\mathbb{H}} = 0 \). Take \( U = X \) to obtain \( \lambda \|X\|_{\mathbb{H}}^2 = \langle AX, X \rangle \). Because \( A \) is dissipative, it follows that \( X = 0 \). Therefore, \( D(A) = \mathbb{H} \). The proof is complete.

Lemma 3.2 is used to show that any trajectory of system (3.5) is pre-compact in the state space \( \mathbb{H} \).

**Lemma 3.2**

\( 0 \in R(A) \) and \((I - \lambda A)^{-1}\) is compact for some \( \lambda > 0 \).

**Proof**

\( 0 \in R(A) \) is trivial. We need only prove the second assertion. Let \( V_n \subset \mathbb{H}, \|V_n\| \leq K \) be a bounded sequence and \( U_n = (u_n, v_n) \in D(A) \) satisfy \((I - \lambda A)U_n = V_n \). Because \( \langle V_n, U_n \rangle = \langle (I - \lambda A)U_n, U_n \rangle = \langle U_n, U_n \rangle - \lambda \langle AU_n, U_n \rangle \), by dissipativity of \( A \), it follows that \( \langle U_n, U_n \rangle \leq \text{Re} \langle V_n, U_n \rangle \), and hence \( \|U_n\| \leq \|V_n\| \leq K \). These imply that

\[
\|U_n\|_{H^2} \leq C_1, \|v_n\|_{H^1} \leq C_2
\]
for some constant $C_1, C_2 < \infty$ independent of $n$. By the Sobolev imbedding theorem, there is a subsequence of $U_n$, still denoted by $U_n$ without confusion, and $U_0 \in \mathbb{H}$, such that

$$U_n \to U_0.$$  

This is the required result.

\section*{Proposition 3.1}

Suppose that $c_1, c_2 > 0, (c^2 - 1)K > M$. Then system (3.1) is asymptotically stable.

\section*{Proof}

By Lemma 3.1, we need only consider the classical solution. Owing to Lemma 3.2, it follows from [6, Theorem 3.65, p.162] that the trajectory of (3.5)

$$\gamma(w_0, w_1) = \{(w(\cdot, t), \dot{w}(\cdot, t))|t \geq 0\}$$

is pre-compact in $\mathbb{H}$. In light of Lasalle’s invariance principle, any solution of system (3.5) tends to the maximal invariant set of the following:

$$S = \{(w, w_t)|\dot{V}(t) = 0\}.$$

Now by $\dot{V}(t) = 0$, it follows from (3.3) that $w_t(1, t) = w_t(0, t) = 0$. So, $\dot{V}(t) = 0$ reduces to

$$\begin{cases} 
  w_{tt}(x, t) = w_{xx}(x, t), \ x \in (0, 1), t > 0, \\
  w_x(0, t) = 0, \ t \geq 0, \\
  w_t(0, t) = 0, \ t \geq 0, \\
  w_x(1, t) = -c_1 w(1, t), \ t \geq 0, \\
  w_t(1, t) = 0, \ t \geq 0,
\end{cases} \quad (3.12)$$

The proof will be accomplished if we can show that (3.12) admits zero solution only. Let the Lyapunov function $V$ for system (3.12) be also defined by (2.8), and a multiplier $g$ be defined as

$$g(t) = \int_0^1 (x - 2)w_x(x, t)w_t(x, t)dx.$$  

Then

$$|g(t)| \leq 2V(t)$$

and

$$\dot{g}(t) = \int_0^1 (x - 2)[w_x(x, t)w_{tt}(x, t) + w_t(x, t)w_{xt}(x, t)]dx$$

$$= -\frac{1}{2}w_t^2(1, t) - \frac{1}{2}w_x^2(1, t) + w_t^2(0, t) + w_x^2(0, t) - \frac{1}{2} \int_0^1 [w_t^2(x, t) + w_x^2(x, t)]dx$$

$$\leq -K_0V(t),$$  

where $K_0 = \min\{c_1, 1\}$ is a constant. Let $E = V + \varepsilon g$ and take $0 < \varepsilon < \frac{1}{2}$. Then we have

$$(1 - 2\varepsilon)V(t) \leq E(t) \leq (1 + 2\varepsilon)V(t)$$

$$\dot{E}(t) = \dot{V}(t) + \varepsilon \dot{g}(t) = \varepsilon \dot{g}(t) \leq -\frac{K_0\varepsilon}{1 + 2\varepsilon}E(t)$$

This shows that $\lim_{t \to \infty} V(t) = 0$ and hence $V(t) \equiv 0$. So (3.12) admits zero solution only. The system (3.5) is hence asymptotically stable. \ \square
Returning to system (1.1) by the transformations (2.1) and (2.3), we have the following theorem.

**Theorem 3.1**

Suppose that \( d \equiv 0 \) and \( c_1, c_2 > 0, (c^2 - 1)K > M \). Then the closed-loop system of (1.1)

\[
\begin{align*}
  u_{tt}(x,t) &= u_{xx}(x,t), \quad x \in (0,1), \ t > 0, \\
  u_x(0,t) &= -q u_t(0,t), \quad t \geq 0, \\
  u_x(1,t) &= \frac{q + c + (1 + qc)c_2}{c^2 - 1}g(t) \\
  &\quad + c_1 \frac{qc + 1}{c^2 - 1} \left( u(1,t) - \frac{q + c}{q^2 - 1} \int_0^1 u_r(y,t) \, dy - \frac{q(q + c)}{q^2 - 1} \int_0^1 u_x(y,t) \, dy \right) \\
  &\quad + (c^2 - 1)K \text{sign}(g(t)), \quad t \geq 0,
\end{align*}
\]

where

\[ g(t) = -\frac{1 + cq}{q^2 - 1} u_t(1,t) + \frac{q(q + c)}{q^2 - 1} u_r(0,t) - \frac{q + c}{q^2 - 1} (u_x(1,t) - u_x(0,t)) \]

associates with a nonlinear semigroup solution in \( \mathbb{H} \) and is asymptotically stable.

**4. ENERGY CONVERGENCE OF THE CLOSED-LOOP SYSTEM WITH DISTURBANCE**

In the last section, we have shown that when there is no external disturbance \( d \), the closed-loop system (3.1) is asymptotically stable. In this section, we show that the controller is actually robust to the disturbance in some extent. Precisely, we show that the energy of closed-loop system (2.6) is convergent to zero as time goes to infinity in the presence of disturbance \( d \).

**4.1. Existence and uniqueness of the classical solution**

Because the control is discontinuous, the first problem we are facing is the well posedness of the solution. In this subsection, we give the existence and uniqueness of the solution to (2.6).

**Theorem 4.1**

Given any \( T > 0 \). Assume that the initial value \((w(\cdot,0), w_t(\cdot,0)) = (w_0, w_1) \in H^2(0,1) \times H^1(0,1)\) and satisfies the compatible conditions:

\[
\begin{align*}
  w_0'(0) = c w_1(0), \\
  w_1'(0) = -c_1 w_0(1) - c_2 w_1(1) - \frac{c^2 - 1}{1 + qc}[(c^2 - 1)K \text{sign}(w_1(1)) + d(0)].
\end{align*}
\]

Then (2.6) admits a unique classical solution in \([0,T]\).

**Proof**

We start by showing the uniqueness of the classical solution. Suppose otherwise that there are two solutions \((w, w_t)\) and \((\tilde{w}, \tilde{w}_t)\) to (2.6). Set \( p(x,t) = w(x,t) - \tilde{w}(x,t) \). Then \( p(x,t) \) satisfies

\[
\begin{align*}
  p_{tt}(x,t) &= p_{xx}(x,t), \quad x \in (0,1), \ t > 0, \\
  p_x(0,t) &= c p_t(0,t), \quad t \geq 0, \\
  p_x(1,t) &= -c_1 p(1,t) - c_2 p_t(1,t) \tag{4.2} \\
  &\quad - \frac{c^2 - 1}{1 + qc}[(c^2 - 1)K \text{sign}(w_t(1,t)) - (c^2 - 1)K \text{sign}(\tilde{w}_t(1,t))].
\end{align*}
\]

Define the Lyapunov-like functional as follows:

\[
E(t) = \frac{1}{2} \int_0^1 \left[ p_t^2(x,t) + p_x^2(x,t) \right] \, dx + \frac{c_1}{2} p^2(1,t). \tag{4.3}
\]
A direct computation shows that the time derivative of $E(t)$ along the solution of (4.2) satisfies

\[
\dot{E}(t) = p_t(1, t) p_x(1, t) - p_t(0, t) p_x(0, t) + c_1 p(1, t) p_t(1, t)
\]
\[
= -c p_t^2(0, t) - c_2 p_t^2(1, t)
\]
\[
- \frac{(c^2 - 1)^2}{1 + qc} K[\text{sign}(w_t(1, t)) - \text{sign}(\dot{w}_t(1, t))] [w_t(1, t) - \dot{w}_t(1, t)]
\]
\[
\leq -c p_t^2(0, t) - c_2 p_t^2(1, t) \leq 0,
\]

hence if $(w(\cdot, 0), w_t(\cdot, 0)) \equiv (\dot{w}(\cdot, 0), \dot{w}_t(\cdot, 0))$, then $E(t) \leq E(0) = 0$ or $(w, w_t) \equiv (\dot{w}, \dot{w}_t)$.

Now we turn to the existence. Multiply the first equation of (2.6) by $\phi \in H^1(0, 1)$ and integrate over $[0, 1]$ with respect to $t$ to obtain

\[
\langle w_{\text{tt}}(\cdot, t), \phi \rangle + \langle w_x(\cdot, t), \phi_x \rangle = \phi(1) \left\{ -c_1 w(1, t) - \frac{c^2 - 1}{1 + qc} [(c^2 - 1) K \text{sign}(w_t(1, t)) + d(t)] \right\}
\]
\[
- c \phi(0) w_t(0, t) - c_2 \phi(1) w_t(1, t).
\]

Suppose that $\{\phi_n\}_{n=1}^\infty$ is an orthogonal basis for $H^1(0, 1)$. Because $(w_0, w_1) \in H^2(0, 1) \times H^1(0, 1)$, we may assume without loss of generality that $(w_0, w_1) \in \text{span}\{\phi_1, \phi_2\}$. For each $N \in \mathbb{Z}^+$, let $V_N = \text{span}\{\phi_1, \phi_2, \ldots, \phi_N\}$. Find the Galerkin approximation solution $w^N$ to (2.6):

\[
w^N(x, t) = \sum_{n=1}^N g_{nN}(t) \phi_n(x)
\]

which satisfies

\[
\begin{aligned}
\langle w_{\text{tt}}^N(\cdot, t), \phi \rangle + \langle w_x^N(\cdot, t), \phi_x \rangle &= \phi(1) \left\{ -c_1 w^N(1, t) - c_2 w_t^N(1, t) - \frac{c^2 - 1}{1 + qc} [(c^2 - 1) K \text{sign}(w_t^N(1, t)) + d(t)] \right\} \\
&- c \phi(0) w_t^N(0, t), \forall \phi \in V_N,
\end{aligned}
\]
\[
w^N(\cdot, 0) = w_0(\cdot), \dot{w}^N(\cdot, 0) = w_1(\cdot).
\]

The existence and uniqueness of the solution to (4.5) in some interval $[0, t_N)$, $t_N > 0$ are ensured by the local Lipschitz condition. Lemma 4.1 ensures that $t_N = \infty$. The proof for the existence of the solution will be split into several lemmas.

\[\square\]

Lemma 4.1

\[
\begin{aligned}
\dot{w}^N(1, t) &\in L^2(0, \infty), \\
\max_{t \geq 0} \sup_N \left[ \|\dot{w}^N(\cdot, t)\| + \|w_x^N(\cdot, t)\| + |w^N(1, t)| \right] &< \infty.
\end{aligned}
\]

\[\text{Proof}\]

Take $\phi = \dot{w}^N(\cdot, t)$ in (4.5) to obtain

\[
\frac{d}{dt} \|w_x^N(\cdot, t)\|^2 + \frac{d}{dt} \|w_t^N(\cdot, t)\|^2
\]
\[
= 2 \left\{ -c_1 w^N(1, t) - c_2 \dot{w}^N(1, t) - \frac{c^2 - 1}{1 + qc} [(c^2 - 1) K \text{sign}(\dot{w}^N(1, t)) + d(t)] \right\} \dot{w}^N(1, t) \\
- 2c [\dot{w}_0^N(0, t)]^2.
\]
Define the Lyapunov function $Y_N(t) = \|\dot{w}^N(\cdot,t)\|^2 + \|w_N^N(\cdot,t)\|^2 + c_1[w^N(1,t)]^2$. It is found along the solution of (4.5) that

$$\dot{Y}_N(t) \leq -2c[\dot{w}^N(0,t)]^2 - 2c_2[\dot{w}^N(1,t)]^2 - 2\frac{c^2 - 1}{1 + qc}[(c^2 - 1)K\|\dot{w}^N(1,t)\| \leq 0.$$  

This concludes (4.6). In addition, it shows that (4.5) admits a unique global classical solution.

**Lemma 4.2**

$$\sup_N \|\dot{w}^N(\cdot,0)\| < \infty.$$  

**Proof**

Let $t = 0$ in the first equation of (4.5). Then for any $\phi \in V_N$, we have

$$\langle w'^N_{tt}(\cdot,0), \phi \rangle + \langle w'^N_{tt}(\cdot,0) \rangle$$

$$= \phi(1) \left\{ -c_1 w_0(1) - c_2 w_1(1) - \frac{c^2 - 1}{1 + qc}[(c^2 - 1)K\text{sign}(w_1(1)) + d(0)] \right\}$$

$$- \phi(0)cw_1(0).$$

Take $\phi = w'^N_{tt}(\cdot,0)$ in (4.8) to give

$$\|w'^N_{tt}(\cdot,0)\|^2 + \langle w'^N_{tt}(\cdot,0) \rangle$$

$$= w'^N_{tt}(1,0) \left\{ -c_1 w_0(1) - c_2 w_1(1) - \frac{c^2 - 1}{1 + qc}[(c^2 - 1)K\text{sign}(w_1(1)) + d(0)] \right\} - w'^N_{tt}(0,0)cw_1(0).$$

Because by compatible condition (4.1),

$$\langle w'^N_{tt}(\cdot,0) \rangle = w'^N_{tt}(1,0) - w'^N_{tt}(0,0) - \langle w'^N_{tt}(\cdot,0) \rangle$$

$$= w'^N_{tt}(1,0) \left\{ -c_1 w_0(1) - c_2 w_1(1) - \frac{c^2 - 1}{1 + qc}[(c^2 - 1)K\text{sign}(w_1(1)) + d(0)] \right\}$$

$$- w'^N_{tt}(0,0)cw_1(0) - \langle w'^N_{tt}(\cdot,0) \rangle,$$

we obtain

$$\|\dot{w}^N(\cdot,0)\|^2 = \langle w'^N_{tt}(\cdot,0) \rangle, \|\dot{w}^N(\cdot,0)\| \leq \|w'^N_{tt}(\cdot,0)\|.$$  

This proves Lemma 4.2.

**Lemma 4.3**

$$\sup_N \left[ \|\dot{w}^N(\cdot,t)\| + \|\dot{w}^N_X(\cdot,t)\| + |\dot{w}^N(1,t)| \right] < \infty, \forall t \in [0,T] \text{ a.e.}.$$  

**Proof**

We adopt the method used in [14] for this estimation. Fix $t, \xi > 0$ so that $\xi < T - t$. Replace $t$ by $t + \xi$ in (4.5) and subtract the first equation of (4.5) to yield

$$\langle \dot{w}^N(\cdot,t + \xi) - \dot{w}^N(\cdot,t), \phi \rangle + \langle w'^N_X(\cdot,t + \xi) - w'^N_X(\cdot,t), \phi \rangle$$

$$= \phi(1) \left\{ -c_1 w^N(1,t) - c_2 \dot{w}^N(1,t + \xi) + \xi - \frac{c^2 - 1}{1 + qc}[(c^2 - 1)K\text{sign}(\dot{w}^N(1,t + \xi)) + d(t + \xi)] \right\}$$

$$- \phi(0) \left\{ -c_1 w^N(1,t) - c_2 \dot{w}^N(1,t) - \frac{c^2 - 1}{1 + qc}[(c^2 - 1)K\text{sign}(\dot{w}^N(1,t)) + d(t)] \right\}$$

$$- c\phi(0)\dot{w}^N(0,t + \xi) + c\phi(0)\dot{w}^N(0,t).$$  

(4.10)
Take $\phi = \dot{w}^N(\cdot, t + \xi) - \dot{w}^N(\cdot, t)$ in (4.10) to obtain

$$\frac{1}{2} \frac{d}{dt} \left\{ \| \dot{w}^N(\cdot, t + \xi) - \dot{w}^N(\cdot, t) \|^2 + \| w_x^N(\cdot, t + \xi) - w_x^N(\cdot, t) \|^2 + c_1 [w^N(1, t + \xi) - w^N(1, t)]^2 \right\} = I_1 + I_2,$$

where

$$I_1 = -c [\dot{w}^N(0, t + \xi) - \dot{w}^N(0, t)]^2 - c_2 [\dot{w}^N(1, t + \xi) - \dot{w}^N(1, t)]^2 \leq 0,$$

$$I_2 = -\frac{c^2 - 1}{1 + qc} \left[ (c^2 - 1) K \text{sign} \dot{w}^N(1, t + \xi) + d(t + \xi) \right] [\dot{w}^N(1, t + \xi) - \dot{w}^N(1, t)].$$

It is seen that

$$I_2 \leq -\frac{c^2 - 1}{1 + qc} [d(t + \xi) - d(t)] [\dot{w}^N(1, t + \xi) - \dot{w}^N(1, t)]^2.$$

Let

$$\psi^N(t, \xi) = \frac{1}{2} \left\{ \| \dot{w}^N(\cdot, t + \xi) - \dot{w}^N(\cdot, t) \|^2 + \| w_x^N(\cdot, t + \xi) - w_x^N(\cdot, t) \|^2 + c_1 [w^N(1, t + \xi) - w^N(1, t)]^2 \right\}.$$

Then

$$\frac{d}{dt} \psi^N(t, \xi) \leq -\frac{c^2 - 1}{1 + qc} [d(t + \xi) - d(t)] [\dot{w}^N(1, t + \xi) - \dot{w}^N(1, t)] - c_2 [\dot{w}^N(1, t + \xi) - \dot{w}^N(1, t)]^2.$$

Integrate the aforementioned inequality over $[0, t]$ to obtain

$$\psi^N(t, \xi) - \psi^N(0, 0) \leq -\int_0^t \frac{c^2 - 1}{1 + qc} [d(t + \xi) - d(t)] [\dot{w}^N(1, t + \xi) - \dot{w}^N(1, t)] dt - c_2 \int_0^t [\dot{w}^N(1, t + \xi) - \dot{w}^N(1, t)]^2 dt \leq \frac{(c^2 - 1)^2}{4c_2(1 + qc)^2} \int_0^t [d(t + \xi) - d(t)]^2 dt.$$

Divide the aforementioned inequality by $\xi^2$ and pass to the limit as $\xi \to 0$ to obtain

$$\| \dot{w}^N(\cdot, t) \|^2 + \| w_x^N(\cdot, t) \|^2 + c_1 [\dot{w}^N(1, t)]^2 \leq \| \dot{w}^N(\cdot, 0) \|^2 + \| w_x^N(\cdot, 0) \|^2 + c_1 [\dot{w}^N(1, 0)]^2$$

$$+ \frac{(c^2 - 1)^2}{4c_2(1 + qc)^2} \int_0^t |d(\tau)|^2 d\tau$$

$$= \| \dot{w}^N(\cdot, 0) \|^2 + \| w_x^N(\cdot, 0) \|^2 + c_1 w_x^2(0) + \frac{(c^2 - 1)^2}{4c_2(1 + qc)^2} \int_0^t |d(\tau)|^2 d\tau.$$
Because by Lemma 4.2 $\|\tilde{w}^N(\cdot,0)\|^2 < \infty$, it follows from the aforementioned inequality that
$$\|\tilde{w}^N(\cdot,t)\|^2 + \|\tilde{w}_x^N(\cdot,t)\|^2 + c_1 [\tilde{w}^N(1,t)]^2 < \infty, \forall t \in [0,T] \text{ a.e.}$$
This completes the proof of Lemma 4.3.

Continuation of proof of Theorem 4.1
Thanks to Lemmas 4.1 and 4.3 and lemma 4 of [14], we may extract a subsequence $N_k$, which is still denoted by $N$ without diffusion, such that
$$\begin{align*}
\tilde{w}^N \rightharpoonup \tilde{w} & \quad \text{in } L^\infty(0, T; H^1(0, 1)) \text{ weak*}, \\
\tilde{w}_x^N \rightharpoonup \tilde{w}_x & \quad \text{in } L^\infty(0, T; H^1(0, 1)) \text{ weak*}, \\
\tilde{w}^N \rightharpoonup \tilde{w} & \quad \text{in } L^\infty(0, T; L^2(0, 1)) \text{ weak*}.
\end{align*}$$
Because $\tilde{w}^N \rightharpoonup \tilde{w}$ in $L^\infty(0, T; H^1(0, 1))$ weak star topology and for any fixed $t \in [0,T]$, $\{\tilde{w}^N(1,t)\}$ is a compact sequence of $L^2(0,T)$, we may extract a subsequence $\{\tilde{w}^{N_k}(1,t)\}$ of $\{\tilde{w}^N(1,t)\}$ such that $\tilde{w}^{N_k}(1,t) \to \tilde{w}(1,t)$ in $L^2(0,T)$ as $k \to \infty$. For brevity in notation, we still denote the sequence $\tilde{w}^{N_k}(1,t)$ as $\tilde{w}^N(1,t)$.

Now for any $\psi \in D(0, T)$ and $\phi \in H^1(0, 1)$, it has
\[
\int_0^T \langle w_{tt}^N(\cdot,t), \phi(\cdot) \rangle \psi(t)dt + \int_0^T \langle w_x^N(\cdot,t), \phi_x \rangle \psi(t)dt \\
= \phi(1) \int_0^T \left\{-c_1 w^N(1,t) - c_2 w_x^N(1,t) - \frac{c^2 - 1}{1 + q_c}[(c^2 - 1)K \text{sign}(w_t^N(1,t)) + d(t)]\right\} \psi(t)dt \\
- \phi(0) \int_0^T c w_t^N(0,t) \psi(t)dt.
\]
Setting $N \to \infty$ in the aforementioned equality, we obtain for all $\phi \in H^1(0, 1)$ and $t \in [0, T]$ that
\[
\langle w_{tt}, \phi \rangle + \langle w_x, \phi' \rangle = \phi(1) \left\{-c_1 w(1,t) - c_2 w_t(1,t) - \frac{c^2 - 1}{1 + q_c}[(c^2 - 1)K \text{sign}(w_t(1,t)) + d(t)]\right\} \\
- \phi(0)c w_t(0,t), \forall t \in [0, T] \text{ a.e.}.
\]
In particular, taking $\phi \in D(0, 1)$ in (4.12) gives
\[
\langle w_{tt}, \phi \rangle + \langle w_x, \phi' \rangle = 0, \forall \phi \in D(0, 1).
\]
This shows that the generalized derivative $w_{xx}$ exists and $w_{xx}(\cdot,t) = w_{tt}(\cdot,t)$ for $t \in [0,T]$ almost everywhere. In particular, $w(\cdot,t) \in H^2(0,1)$. Integration by parts for (4.13) over $[0,1]$ with respect to $x$ yields
\[
\langle w_{tt}, \phi \rangle + w_x(1)\phi(1) - w_x(0)\phi(0) - \langle w_{xx}, \phi \rangle \\
= \phi(1) \left\{-c_1 w(1,t) - c_2 w_t(1,t) - \frac{c^2 - 1}{1 + q_c}[(c^2 - 1)K \text{sign}(w_t(1,t)) + d(t)]\right\} - \phi(0)c w_t(0,t).
\]
Therefore
\[
w_x(1,t) = -c_1 w(1,t) - c_2 w_t(1,t) - \frac{c^2 - 1}{1 + q_c}[(c^2 - 1)K \text{sign}(w_t(1,t)) + d(t)], \quad w_x(0,t) = c w_t(0,t).
\]
The existence of the solution is thus proved.

4.2. Convergence of the vibrating energy
In this subsection, we first define the weak solution of (2.6). Obviously, all the states that satisfy the compatible conditions in (4.1) are dense in $\mathbb{H}$. 

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Definition 4.1

For any \((w(\cdot), w_t(\cdot, 0)) = (w_0, w_1) \in \mathbb{H}\), let \((w_{n0}, w_{n1}) \to (w_0, w_1)\) in \(\mathbb{H}\) where \((w_{n0}, w_{n1})\) satisfies the compatible conditions in (4.1). Let \((w_n, w_{n1})\) be the classical solution to (2.6) corresponding to the \((w_{n0}, w_{n1})\). Then it follows from (4.2) to (4.4) that \(\{(w_n, w_{n1})\}\) is a Cauchy sequence in \(\mathbb{H}\). Its limit \((w, w_t)\) is defined as the weak solution to (2.6) corresponding to the initial value \((w_0, w_1)\). Obviously, \((w, w_t)\) is independent of the choice of \((w_{n0}, w_{n1})\). So the weak solution to (2.6) is well defined.

Now we show that the vibrating energy of the system (2.6) is convergent to zero. The main idea comes from [12].

Proposition 4.1

Suppose that \(|d(t)| \leq M, |d(t)| \leq M_1\) for some constants \(M, M_1 < \infty\) and all \(t \geq 0\). Let \(c_1 > 0, c_2 > 0, (c^2 - 1)K > M\). Then for any initial value \((w(\cdot, 0), w_t(\cdot, 0)) = (w_0, w_1) \in H^1(0, 1) \times L^2(0, 1)\), the (weak) solution of (2.6) satisfies

\[
\lim_{t \to \infty} \int_0^1 \left[w_t^2(x, t) + w_x^2(x, t)\right] dx = 0. \tag{4.14}
\]

Proof

By Definition 4.1 (see also (4.2)–(4.4)), we only need to show (4.14) for the classical solution with the compatible condition (4.1). In what follows, we denote \(\| \cdot \|_2\) as the \(L^2\)-norm. Consider the Lyapunov function \(V\) defined by (2.8). Then the time derivative of \(V\) along the solution of (2.6) is given by (2.7)

\[
\dot{V}(t) = -c w_t^2(0, t) - c_2 w_x^2(1, t) - \frac{c^2 - 1}{1 + q \epsilon} [(c^2 - 1)K \text{sign}(w_t(1, t)) + d(t)]w_t(1, t) \leq 0. \tag{4.15}
\]

Therefore

\[
V(t_2) \leq V(t_1), \quad \forall t_2 \geq t_1 \geq 0. \tag{4.16}
\]

In particular

\[
V(t) \leq V(0), \quad \forall t \geq 0. \tag{4.17}
\]

Consequently, \(|w(1, t)|\) is bounded. Moreover, it follows from (4.15) that \(w_t(1, t) \in L^1(0, \infty)\).

Now consider the ‘augmented’ function

\[
V_R(t) = \frac{1}{2} \int_0^1 \left[w_t^2(x, t) + w_x^2(x, t)\right] dx + K_R \int_0^1 (x - 2)w_tw_{xx} dx,
\]

where \(K_R\) is a positive constant. It is obvious that \(V_R\) is positive definite:

\[
V_R(t) \geq \left(\frac{1}{2} - K_R\right) \left(\| w_t \|_2^2 + \| w_x \|_2^2\right) \tag{4.18}
\]

provided that \(K_R < \frac{1}{2}\). On the other hand, it is seen that

\[
V_R(t) \leq \left(\frac{1}{2} + K_R\right) \left(\| w_t \|_2^2 + \| w_x \|_2^2\right). \tag{4.19}
\]
Now, finding the time derivative of $V_R$ along the solution of (2.6) gives

$$V_R(t) = \int_0^1 \left[ w_t(x,t)w_{tt}(x,t) + w_x(x,t)w_{xt}(x,t) \right] dx$$

$$+ K_R \int_0^1 (x-2)[w_{tt}(x,t)w_t(x,t) + w_t(x,t)w_{xt}(x,t)] dx$$

$$= \int_0^1 \left[ w_t(x,t)w_{xx}(x,t) + w_x(x,t)w_{xt}(x,t) \right] dx$$

$$+ K_R \int_0^1 (x-2)[w_{xx}(x,t)w_x(x,t) + w_t(x,t)w_{xt}(x,t)] dx$$

$$= w_t(x,t)w_x(x,t) \bigg|_0^1 + \frac{K_R}{2} ((x-2)w_x^2(x,t)_0^1 + \frac{K_R}{2} ((x-2)w_t^2(x,t)_0^1$$

$$- \frac{K_R}{2} \int_0^1 \left[ w_t^2(x,t) + w_x^2(x,t) \right] dx \right) \tag{4.20}$$

$$= -w_t(1,t) \frac{c^2 - 1}{1 + qc} \left[ \frac{c}{1 + qc} + 1 \right] w(1,t) + (c^2 - 1)K \text{sign}(w_t(1,t)) + d(t) - c_2 w_t^2(1,t)$$

$$- c w_t^2(0,t) + K_R c^2 w_t^2(0,t) + K_R w_t^2(0,t) - \frac{K_R}{2} \int_0^1 \left[ w_t^2(x,t) + w_x^2(x,t) \right] dx$$

$$- \frac{K_R}{2} w_x^2(1,t) - \frac{K_R}{2} w_t^2(1,t) \leq - \frac{K_R}{2} \int_0^1 \left[ w_t^2(x,t) + w_x^2(x,t) \right] dx$$

$$- (c - K_R c^2 - K) w_t^2(1,t) - \frac{c^2 - 1}{1 + qc} [(c^2 - 1)K-M] |w_t(1,t)|$$

$$- \frac{K_R}{2} w_x^2(1,t) - \left( \frac{K_R}{2} + c_2 \right) w_t^2(1,t) - c_1 w(1,t)w_t(1,t).$$

Suppose that $K_R$ is chosen so that

$$0 < K_R < \min \left\{ \frac{1}{2}, \frac{c}{c^2 + 1} \right\}.$$

Then

$$V_R(t) \leq - \frac{K_R}{2} \left( \left\| w_t \right\|^2 + \left\| w_x \right\|^2 \right) - c_1 w(1,t)w_t(1,t)$$

$$\leq - \frac{K_R}{1 + 2K_R} V_R(t) - c_1 w(1,t)w_t(1,t). \tag{4.21}$$

For notation simplicity, we denote the constant $\frac{K_R}{1+2K_R} = \mu$. By making use of Gronwall’s inequality, we obtain

$$V_R(t) \leq e^{-\mu t} V_R(0) - c_1 \int_0^t e^{-\mu (t-\tau)} w(1,\tau)w_t(1,\tau) d\tau$$

$$\leq e^{-\mu t} V_R(0) + c_1 \int_0^t e^{-\mu (t-\tau)} |w(1,\tau)w_t(1,\tau)| d\tau$$

$$\leq e^{-\mu t} V_R(0) + c_1 \int_0^t e^{-\mu (t-\tau)} |w(1,\tau)w_t(1,\tau)| d\tau$$

$$+ c_1 \int_0^t e^{-\mu (t-\tau)} |w(1,\tau)w_t(1,\tau)| d\tau$$

$$\leq e^{-\mu t} V_R(0) + c_1 \int_0^t e^{-\mu t} |w(1,\tau)w_t(1,\tau)| d\tau$$
This is the result required. The proof is complete.

Suppose that $j \text{ Theorem 4.2}$ \text{Theorem 4.1, Proposition 4.1, and transformations (2.1) and (2.3).}

where $c = 0$. u there exists a unique classical solution system of (1.1): $w, w_t, w_{tt}$. Let $c_1 > 0$, $c_2 > 0$, $(c^2 - 1)K > M$. For the initial value $(u(\cdot, 0), u_t(\cdot, 0)) \in H^2(0, 1) \times H^1(0, 1)$ that satisfies the compatible conditions:

\[
\begin{align*}
    u_x(0, 0) &= -qu_t(0, 0), \\
    u_x(1, 0) &= \frac{g + c + (1 + qc)c_2}{c^2 - 1} g(0) \\
    &+ c_1 \frac{qc + 1}{c^2 - 1} \left( u(1, 0) - \frac{g + c}{q^2 - 1} \int_0^1 u_t(x, 0)dx - \frac{q(q + c)}{q^2 - 1} \int_0^1 u_x(x, 0)dx \right) \\
    &+ (c^2 - 1)K \text{sign}(g(0)) + d(0),
\end{align*}
\]

where $g(0) = -\frac{1 + qc}{q^2 - 1} u_t(1, 0) + \frac{q(q + c)}{q^2 - 1} u_t(0, 0) - \frac{g + c}{q^2 - 1} (u_x(1, 0) - u_x(0, 0)),$

there exists a unique classical solution $(u, u_t) \in H^2(0, 1) \times H^1(0, 1)$ to the following closed-loop system of (1.1):

\[
\begin{align*}
    u_{tt}(x, t) &= u_{xx}(x, t), \ x \in (0, 1), t > 0, \\
    u_x(0, t) &= -qu_t(0, t), \ t \geq 0, \\
    u_x(1, t) &= \frac{g + c + (1 + qc)c_2}{c^2 - 1} g(t) \\
    &+ c_1 \frac{qc + 1}{c^2 - 1} \left( u(1, t) - \frac{g + c}{q^2 - 1} \int_0^1 u_t(x, t)dx - \frac{q(q + c)}{q^2 - 1} \int_0^1 u_x(x, t)dx \right) \\
    &+ (c^2 - 1)K \text{sign}(g(t)) + d(t), \ t \geq 0,
\end{align*}
\]

where $g(t) = -\frac{1 + qc}{q^2 - 1} u_t(1, t) + \frac{q(q + c)}{q^2 - 1} u_t(0, t) - \frac{g + c}{q^2 - 1} (u_x(1, t) - u_x(0, t)).$

In addition, the vibrating energy of of system (1.1) is convergent for all (weak) solutions:

\[
\lim_{t \to \infty} \frac{1}{2} \int_0^1 \left[ u_t^2(x, t) + u_x^2(x, t) \right] dx = 0, \ \forall (w(\cdot, 0), w_t(\cdot, 0)) \in \mathbb{H}.
\]
5. NUMERICAL SIMULATION

In this section, we take an example of system (4.24) with the parameters and initial values as follows:

\[ q = 2, c = 1.1, c_1 = 4, c_2 = 2, K = 1/2, M = 1, M_1 = 1, u(x, 0) = x, u_t(x, 0) = -x, \forall x \in [0, 1]. \]

We apply the finite difference method to compute the displacements of system (4.24) with the disturbance \( d(t) = \sin t \) and \( d(t) = 0 \), respectively. The latter corresponds to the system (3.14). Here the steps of space and time are taken as 0.01 and 0.001. The results are plotted in Figure 1(a) and (b), respectively. It is seen that both cases are convergent but the system without the disturbance converges faster than that with the disturbance.

6. CONCLUDING REMARKS

In this paper, the stabilization of an anti-stable one-dimensional wave equation with boundary control and disturbance is considered by the Lyapunov functional approach. The anti-stable term at the free end is first transformed into the control end of a target system where a general disturbance with uniformly bounded derivative is suffered. The stabilizing variable structure controller is designed on the basis of the Lyapunov functional method. The existence and uniqueness of the solution are proved via the Galerkin approximation scheme. In particular, if the system has no disturbance, then it is associated with a nonlinear dissipative semigroup in the state space and is asymptotically stable. In the presence of the disturbance, the vibrating energy of the system is shown to be convergent to zero as time goes to infinity. The latter is the best possible result in the sense that when the disturbance \( d(t) = d \) is a constant, the closed-loop system (2.6) has a solution \( (w, w_t) = \left( -\frac{c_1^2 - 1}{c_1^2 (t_0 + q_0)} d, 0 \right) \).

This shows that due to the presence of disturbance, the system may have a rigid motion. The convergence for the non-constant disturbance remains an open question. Finally, an example is given and the numerical simulation illustrates the claimed convergence.

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