The First Real Eigenvalue of a One-Dimensional Linear Thermoelastic System

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(Received and accepted October 1998)

Abstract—In this note, we show, for a one-dimensional linear thermoelastic equation with Dirichlet-Dirichlet boundary conditions, that there is at least one real eigenvalue which is greater than the dominant eigenvalue of the "pure" heat equation with the same boundary conditions. The result concludes the spectrum-determined growth condition for the system by virtue of a result of Renardy [1]. Moreover, this property is shown to be preserved for the same system with boundary vibration control. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords—Thermoelasticity, Spectrum-determined growth condition, Eigenvalues.

1. INTRODUCTION

Many literatures are concerned with the following one-dimensional linear thermoelastic equation:

\[ u_{tt}(x, t) - u_{xx}(x, t) + \gamma u_{xt}(x, t) = 0, \quad 0 < x < 1, \quad t > 0, \]
\[ \theta_t(x, t) + \gamma u_{xt}(x, t) - k \theta_{xx}(x, t) = 0, \quad 0 < x < 1, \quad t > 0, \]
\[ u(i, t) = \theta(i, t) = 0, \quad i = 0, 1, \quad t \geq 0, \]  

where \( 0 < \gamma \leq 1, \) \( k > 0 \) are constants and \( \gamma \) is usually much smaller than \( 1, \) (see [1-5] and the references therein). The following results were collected from [1,2].

Theorem 1.

(i) System (1) associates with a solution of \( C_0 \)-semigroup of contractions \( T(t) = e^{At} \) on the state Hilbert space \( \mathcal{H} = H_0^1(0, 1) \times (L^2(0, 1))^2, \) where \( A : D(A) \to \mathcal{H}: \)

\[
A = \begin{pmatrix}
0 & 1 & 0 \\
D^2 & 0 & -\gamma D \\
0 & -\gamma D & kD^2
\end{pmatrix}, \quad \text{with} \quad D = \frac{\partial}{\partial x}, \quad D(A) = (H^2(0, 1) \times H^1_0(0, 1) \times H^1_0(0, 1)) \cap \mathcal{H}.
\]

(ii) The growth order \( \omega(A) \) and the spectral bound \( S(A) \) have the following relation

\[
\omega(A) \leq \max \{ S(A), -k\pi^2 \},
\]

where \( S(A) = \sup \{ \Re \lambda \mid \lambda \in \sigma(A) \}, \) \( \omega(A) = \lim_{t \to \infty}(1/t) \log \| T(t) \|. \)
(iii) \( \lambda \in \sigma(A) \), the spectrum of \( A \), if and only if \( \lambda \neq 0 \) satisfies the characteristic equation

\[
8\gamma^2 \sqrt{k\lambda} + [\exp(a_1 - a_2) + \exp(a_2 - a_1)] \left( k\lambda + \gamma^2 + 1 + 2\sqrt{k\lambda} \right) \left( 1 - \sqrt{k\lambda} \right)^2
- [\exp(a_1 + a_2) + \exp(-a_1 - a_2)] \left( k\lambda + \gamma^2 + 1 - 2\sqrt{k\lambda} \right) \left( 1 + \sqrt{k\lambda} \right)^2 = 0,
\]

where

\[
a_1 = \frac{\lambda}{2k} \left[ k\lambda + \gamma^2 + 1 + \sqrt{(k\lambda + \gamma^2 + 1)^2 - 4k\lambda} \right], \quad a_2 = \frac{\lambda \sqrt{\lambda/k}}{a_1}.
\]

(iv) The eigenvalues of \( A \) consist of a real sequence \( \{\sigma_n\} \) and a sequence of conjugate pairs \( \{\lambda_n, \overline{\lambda_n}\} \) with the following asymptotic properties:

\[
\sigma_n = -k(n\pi)^2 + \frac{\gamma^2}{k} + O\left( n^{-2} \right),
\]

\[
\lambda_n = -\frac{\gamma^2}{2k} + in\pi + O\left( n^{-1} \right).
\]

It was shown in [3] that the number \(-\gamma^2/2k\) in equation (4) is just the essential spectral bound of \( A : \omega_{\text{ess}}(A) = -\gamma^2/2k \). Since it is well known that \( \omega(A) = \max\{\mathcal{S}(A), \omega_{\text{ess}}(A)\} \), it follows from (4) that the spectrum-determined growth condition

\[
\omega(A) = \mathcal{S}(A)
\]

is always true for the system. Physically, the real eigenvalue sequence is aroused by the heat equation, while conjugate pair eigenvalues reflect the vibration. As indicated in [1], \(-k\pi^2\) is the first eigenvalue of the "pure heat equation" (i.e., the second equation of (1) in which \( \gamma = 0 \)) with Dirichlet-Dirichlet boundary conditions. Because of the mechanical vibration, we cannot expect the decay rate of system (1) is less than that of "pure" heat equation. In other words, \( S(A) < -k\pi^2 \). Naturally, it is conjectured that there is at least one real eigenvalue of system (1) which is greater than \(-k\pi^2\). The purpose of this note is to give a rigorous proof of this conjecture. The validity of this fact trivially leads to the spectrum-determined growth condition of system (1) again. Moreover, this property is shown to be preserved for the same system with exertion of vibration control.

2. THE FIRST REAL EIGENVALUE OF THE FREE SYSTEM

Because the real eigenvalues of \( A \) consist of the negative real roots of equation (2), we substitute \( k\lambda = -x, \ x > 0 \) into (2) to obtain

\[
8\gamma^2 \sqrt{x} + [\exp(if + g) + \exp(-if - g)] (-x + \gamma^2 + 1 + 2i\sqrt{x}) \left( 1 - i\sqrt{x} \right)^2
- [\exp(if - g) + \exp(g - if)] (-x + \gamma^2 + 1 - 2i\sqrt{x}) \left( 1 + i\sqrt{x} \right)^2 = 0,
\]

where

\[
f(x) = \frac{\sqrt{x}}{2k} h(x), \quad g(x) = \frac{\sqrt{2}}{k} \frac{x}{h(x)},
\]

\[
h(x) = \sqrt{-x + \gamma^2 + 1 + \sqrt{(-x + \gamma^2 + 1)^2 + 4x}} > 0, \quad \text{for all } x \geq 0.
\]

Comparing the real and imaginary parts, we find that (5) is equivalent to \( F(x) = 0 \), where

\[
F(x) = 4\gamma^2 \sqrt{x} - 2\gamma^2 \sqrt{x} \left[ \exp(g) + \exp(-g) \right] \cos f
+ \left[ (1 + x)^2 + (1 - x)\gamma^2 \right] \left[ \exp(g) - \exp(-g) \right] \sin f.
\]
Set $a = k\pi$. Note that
\[(x + 1 - \gamma^2)^2 < (-x + \gamma^2 + 1)^2 + 4x = x^2 + 2(1 - \gamma^2)x + (\gamma^2 + 1)^2 \leq (x + \gamma^2 + 1)^2,
\]
and hence, $\sqrt{2} < h(x) \leq \sqrt{2}\sqrt{1 + \gamma^2} \leq 2$ while $0 \leq \gamma \leq 1$. In particular,
\[
\pi < f(a^2) \leq \sqrt{2}\pi.
\]
Expressing $\gamma^2$ and $g$ in terms of $f$ by virtue of (6) with $x = a^2$, we have
\[
\gamma^2 = -\frac{\pi^2 a^2}{f^2} + \frac{f^2}{\pi^2} + a^2 - 1, \quad g = \frac{\pi^2 a}{f}.
\]
Substituting (9) into $F(a^2)$, we have
\[
f^2\pi^2 e^{\pi a/f} F(a^2) = 4a \left[f^4 - \pi^4 a^2 + \pi^2 (a^2 - 1) f^2\right] e^{\pi a/f}
- 2a \left[f^4 - \pi^4 a^2 + \pi^2 (a^2 - 1) f^2\right] \left[e^{\pi a/f} + 1\right] \cos f
+ [4\pi^2 a^2 f^2 + (1 - a^2) (f^4 - \pi^4 a^2)] \left[e^{\pi a/f} - 1\right] \sin f.
\]
We will show in this section that the right-hand-side of (10) is always negative for $\gamma \in (0, 1]$. 

**Lemma 1.** Let $p = 2uy(y^2 - 1)(u^2 + 1)$, $q = y^2(u^2 + 1)^2 - u^2(y^2 - 1)^2$. Then
\[
p \cos \pi(y - 1) - q \sin \pi(y - 1) < 0
\]
for all $u \geq 0$, $y \in (1, \sqrt{2}]$.

**Proof.** It is obvious that $p \geq 0$ and $q > 0$ for all $u, y$ concerned. What we are going to prove is just $\tan \pi(y - 1) > p/q$. But
\[
\frac{p}{q} = \frac{2uy(y^2 - 1)(u^2 + 1)}{y^2(u^2 + 1)^2 - u^2(y^2 - 1)^2} = 2y(y^2 - 1) \frac{u(u^2 + 1)}{u^4y^2 - u^2(y^4 - 4y^2 + 1) + y^2}.
\]
Setting its derivative with respect to $u$ to be 0, we have
\[
(1 + 3u^2) [u^4y^2 - u^2(y^4 - 4y^2 + 1) + y^2] = u(u^2 + 1) \left[4u^2y^2 - 2u(y^4 - 4y^2 + 1)\right],
\]
which can be simplified to be
\[
(u^2 - 1) [y^2(u^4 + u^2 + 1) + u^2(y^4 - y^2 + 1)] = 0.
\]
Hence, $u = 1$. Since $p/q = 0$ for $u = 0$ and $u = \infty$, we see $p/q$, as a function of $u$, attains its maximum at $u = 1$, that is,
\[
\max_{u \geq 0} \frac{p}{q} = \frac{4y(y^2 - 1)}{y^4 - 6y^2 + 1}.
\]
Therefore, it is sufficient to show
\[
\tan \pi(y - 1) > \frac{4y(y^2 - 1)}{y^4 - 6y^2 + 1}.
\]
Set $f(y) = \tan \pi(y - 1) + (4y(y^2 - 1))/(y^4 - 6y^2 + 1)$. We find that $f(1) = 0$, the proof will be finished if $f'(y) > 0$ for $y \in (1, \sqrt{2}]$. Now
\[
f'(y) = \frac{\pi}{\cos^2 \pi(y - 1)} - \frac{4(y^2 + 1)^3}{(y^4 - 6y^2 + 1)^2} > \pi - \frac{4(y^2 + 1)^3}{(y^4 - 6y^2 + 1)^2} = \frac{3 - 4(y^2 + 1)^3}{(y^4 - 6y^2 + 1)^2}.
\]
Define $g(z) = 3(z^2 - 6z + 1)^2 - 4(z + 1)^3$. Then $f'(y) > 0$ for $y \in (1, \sqrt{2}]$ if and only if $g(z) > 0$ for $z \in (1, 2]$. Find
\[
g'(z) = \frac{12}{z^2 - 6z + 1}(z - 3) - (z + 1)^2,
\]
\[
g''(z) = 3(z - 3)^2 - 2(z - 3) - 16 = 3 \left(z - 3 - \frac{8}{3}\right)(z - 1) < 0,
\]
for $z \in (1, 2]$. Hence, $g'(z)$ is decreasing in $z \in (1, 2]$. Since $g'(1) = 48 > 0$, $g'(2) = -24 < 0$, we know that $g(z)$ increases from $g(1) = 16$ at $z = 1$ to its maximum and then decreases to $g(2) = 39 > 0$. So $g(z) > 0$ for all $z \in (1, 2]$. The proof is complete.
LEMMA 2. For all \( u > 0 \),
\[
e^{\pi u} > \frac{\pi(u^2 + 1) + 4u}{\pi(u^2 + 1) - 4u}.
\]

**Proof.** It is seen that \( \pi(u^2 + 1) - 4u = (\pi - 2)(u^2 + 1) + 2(u - 1)^2 > 0 \). Let \( f(u) = e^{\pi u}[\pi(u^2 + 1) - 4u] - [\pi(u^2 + 1) + 4u] \). Then \( f(0) = 0 \), and
\[
f'(u) = e^{\pi u}[\pi^2(u^2 + 1) - 2\pi u - 4] - 2\pi u - 4
\geq (1 + \pi u)[\pi^2(u^2 + 1) - 2\pi u - 4] - 2\pi u - 4
= \pi^2(u^2 + 1) - 4\pi u - 8 + (\pi u)^3 + \pi^3 u - 2(\pi u)^2 - 4\pi u
= \pi^2 - 8\pi u - 8 + (\pi u)^3 + \pi^3 u - (\pi u)^2
= (\pi^2 - 8)(1 + \pi u) + (\pi u)^2(\pi u - 1).
\]

If \( \pi u > 1 \), then \( f'(u) > 0 \), while \( \pi u < 1 \), \( f'(u) > 1 + \pi u + \pi u - 1 = 2\pi u > 0 \) for \( u > 0 \). This shows that \( f(u) > 0 \) as \( u > 0 \).

LEMMA 3. For all \( u > 0 \), \( y \in (1, \sqrt{2}] \),
\[
\frac{c+d}{c-d} > \frac{(a+b)\sin \pi(y-1) + p}{(a-b)\sin \pi(y-1) - p\cos \pi(y-1)},
\]
where
\[
a = y^2(u^2 + 1)^2, \quad b = u^2(y^2 - 1)^2, \quad c = \pi(u^2 + 1), \quad d = 4u,
\]
\[
p = 2uy(y^2 - 1)(u^2 + 1).
\]

**Proof.** As in Lemma 1, \( c - d > 0 \). What we want to show is just
\[
(c + d)[(a - b)\sin \pi(y-1) - p\cos \pi(y-1)] > (c - d)[(a + b)\sin \pi(y-1) + p],
\]
that is
\[
2(ad - bc)\sin \pi(y-1) - p[(c + d)\cos \pi(y-1) + (c - d)] > 0.
\]
This is equivalent to
\[
4y^2(u^2 + 1) - \pi u(y^2 - 1)^2 \sin \pi(y-1)
- y(y^2 - 1)\left[\pi(u^2 + 1) + 4u\right] \cos \pi(y-1) + \pi(u^2 + 1) - 4u > 0.
\]
(12)

Set
\[
A = 4y^2 \sin \pi(y-1) - \pi y(y^2 - 1)[\cos \pi(y-1) + 1],
B = -\pi(y^2 - 1)^2 \sin \pi(y-1) - 4y(y^2 - 1)[\cos \pi(y-1) - 1].
\]
Then (12) is just \( Au^2 + Bu + A > 0 \) which holds if \( A > 0 \), \( B > 0 \). But both \( A > 0 \) and \( B > 0 \) are equivalent to \( \tan(\pi(y-1))/2 > \pi(y^2 - 1)/4y = (\pi(y-1))/2((y + 1)/2y) \) which is obviously valid since \( (y + 1)/2y < 1 \), \( \tan(\pi(y-1))/2 > (\pi(y-1))/2 \) for \( y \in (1, \sqrt{2}] \).

LEMMA 4. For all \( a > 0 \) and \( f \in (\pi, \sqrt{2}\pi] \),
\[
4a\left[f^4 - \pi^4a^2 + \pi^2(a^2 - 1)f^2\right]e^{\pi^2a/f} - 2a\left[f^4 - \pi^4a^2 + \pi^2(a^2 - 1)f^2\right]\left[e^{2\pi^2a/f} + 1\right] \cos f
+ \left[4\pi^2a^2f^2 + (1 - a^2)(f^4 - \pi^4a^2)\right]\left[e^{2\pi^2a/f} - 1\right] \sin f < 0.
\]
PROOF. Set \( f = \pi y \). Then we need to show that for all \( y \in (1, \sqrt{2}) \),
\[
4a \left[ y^4 - a^2 + (a^2 - 1) y^2 \right] e^{\pi a/y} \\
-2a \left[ y^4 - a^2 + (a^2 - 1) y^2 \right] \left( e^{2\pi a/y} + 1 \right) \cos \pi y \\
+ \left[ 4a^2 y^2 + (1 - a^2) (y^4 - a^2) \right] \left( e^{2\pi a/y} - 1 \right) \sin \pi y < 0.
\]

Let \( a = uy \). Then (13) is equivalent to saying that for all \( u > 0 \) and \( y \in (1, \sqrt{2}) \),
\[
2uy \left( y^2 - 1 \right) (u^2 + 1) \left[ 2e^{\pi u} - (e^{2\pi u} + 1) \cos \pi y \right] \\
+ \left[ y^2 (u^2 + 1)^2 - u^2 (y^2 - 1)^2 \right] \left( e^{2\pi u} - 1 \right) \sin \pi y < 0.
\]

Let \( p = 2uy[y^2 - u^2 + (u^2 y^2 - 1)], q = y^2(u^2 + 1)^2 - u^2(y^2 - 1)^2 \) defined as in Lemma 1. Observe that
\[
p \left[ 2e^{\pi u} - (e^{2\pi u} + 1) \cos \pi y \right] + q \left( e^{2\pi u} - 1 \right) \sin \pi y \\
= \left[ p \cos \pi (y - 1) - q \sin \pi (y - 1) \right] \left( e^{\pi u} - L_1 \right) \left( e^{\pi u} - L_2 \right),
\]
where
\[
L_1 = \frac{-p + \sqrt{p^2 + q^2} \sin \pi (y - 1)}{p \cos \pi (y - 1) - q \sin \pi (y - 1)} = \frac{y^2 (u^2 + 1)^2 + u^2 (y^2 - 1)^2}{p \cos \pi (y - 1) - q \sin \pi (y - 1)}, \\
L_2 = \frac{-p - \sqrt{p^2 + q^2} \sin \pi (y - 1)}{p \cos \pi (y - 1) - q \sin \pi (y - 1)} = \frac{-y^2 (u^2 + 1)^2 + u^2 (y^2 - 1)^2}{p \cos \pi (y - 1) - q \sin \pi (y - 1)}.
\]

By Lemma 1 and Lemma 3, it follows that \( L_2 < (\pi (u^2 + 1) + 4u)/(\pi (u^2 + 1) - 4u) \), and hence, \( L_2 < e^{\pi u} \) by Lemma 2, that is \( e^{\pi u} - L_2 > 0 \). Again by Lemma 1, \( p \cos \pi (y - 1) - q \sin \pi (y - 1) < 0 \), and hence, \( p^2 \cos^2 \pi (y - 1) < q^2 \sin^2 \pi (y - 1) \), we have
\[
L_1 = \frac{-p + \sqrt{p^2 + q^2} \sin \pi (y - 1)}{p \cos \pi (y - 1) - q \sin \pi (y - 1)} < -\frac{p - \sqrt{p^2 \sin^2 \pi (y - 1) + p^2 \cos^2 \pi (y - 1)}}{p \cos \pi (y - 1) - q \sin \pi (y - 1)} = 0.
\]

Therefore, \( e^{\pi u} - L_1 > 0 \). But this shows that
\[
p \left[ 2e^{\pi u} - (e^{2\pi u} + 1) \cos \pi y \right] + q \left( e^{2\pi u} - 1 \right) \sin \pi y \\
= \left[ p \cos \pi (y - 1) - q \sin \pi (y - 1) \right] \left( e^{\pi u} - L_1 \right) \left( e^{\pi u} - L_2 \right) < 0
\]
by Lemma 1 again. So (14) is valid. The proof is complete.

**Theorem 2.** For any \( 0 < \gamma \leq 1 \), there is at least one solution \( x \in (0, k^2 \pi^2) \) to \( F(x) = 0 \) which is defined by (7). That is, there is at least one eigenvalue of system (1) greater than \(-k^2 \pi^2\), the first eigenvalue of the "pure" heat equation.

**Proof.** It is found \( \lim_{x \to 0} F(x)/\sqrt{e x} = (2(\gamma^2 + 1)^2)/k^2 \) and so \( F(x) > 0 \) for \( x > 0 \) sufficiently small. By Lemma 4 and (10), it follows that \( F(k^2 \pi^2) \leq 0 \). But this completes the proof since \( F(x) \) is a continuous function on \([0, k^2 \pi^2]\) with respect to \( x \).

It should be pointed out that Theorem 2 is also true for equation (1) with natural Dirichlet-Neumann and Neumann-Dirichlet boundary conditions since the characteristic equations in such cases reduce to a series of cubic algebraic equations (see [4]). Moreover, Theorem 2, together with Theorem 1, trivially lead again to the spectrum-determined growth condition for system (1) as indicated in the beginning of this paper.
3. THE CASE WITH BOUNDARY VIBRATION CONTROL

In this section, we shall consider a case where a boundary control is exerted to suppress the vibration of the thermoelastic system with Dirichlet-Dirichlet boundary conditions. The analysis below shows that the property that the first eigenvalue of the coupled system is always greater than that of "pure" heat equation is still preserved. This might reveal a fact which is apparently from a physics point of view that only vibration control cannot decrease the decay rate of the thermoelastic system to be smaller than that of "pure" heat equation.

To make the analysis a little simpler, we take the diffusion coefficient \( k \) being 1 and consider the following thermoelastic system with boundary vibration control

\[
\begin{align*}
\frac{\partial u}{\partial t}(x, t) - \Delta u(x, t) + \gamma \frac{\partial u}{\partial t}(x, t) &= 0, \\
\frac{\partial \theta}{\partial t}(x, t) + \gamma \Delta \theta(x, t) - \theta(x, t) &= 0, \\
u(1, t) &= \theta(1, t) = 0, \\
u(0, t) &= \alpha u_t(0, t), \\
u(0, t) &= \alpha u_t(0, t), \\
0 < x < 1, & \quad t > 0,
\end{align*}
\]

(15)

System (15) is dissipative and the relationship as (ii) of Theorem 1 between the growth order and spectral bound as well as well-posedness can be established without difficulty by the approach, e.g., used in [1]. Actually, we have derived in [6] the following characteristic equation for system (15):

\[
G(x) = 4\gamma^2 \sqrt{x} - 2\gamma^2 \sqrt{x} (e^g - e^{-g}) \cos f + \left( \gamma^2 - 1 + x \right) (e^g + e^{-g}) \sin f
\]

(17)

where \( \alpha_1 \) and \( \alpha_2 \) are defined as in (3).

Now we consider the real solutions of (16). Because the system (15) is dissipative, all eigenvalues locate on the open left half complex plane. Hence, all real solutions of (16) are in negative real axis. Substituting \( \lambda = -x \), \( x > 0 \) into (16), we have, similar to (7), that \( \lambda = -x \) is a real eigenvalue if and only if

\[
G(x) = 4\gamma^2 \sqrt{x} - 2\gamma^2 \sqrt{x} (e^g - e^{-g}) \cos f + \left( \gamma^2 - 1 + x \right) (e^g + e^{-g}) \sin f
\]

(17)

where \( \alpha_1 \) and \( \alpha_2 \) are defined as in (3).

Theorem 3. Let \( G(x) \) be defined as in (17). Then \( G(x) \) has at least one zero point in \((0, \pi^2)\) for any \( \alpha > 0 \) and \( 0 < \gamma \leq 1 \). That is, system (15) has at least one real eigenvalue which is greater than \(-\pi^2\), the first eigenvalue of "pure" heat equation with the same Dirichlet-Dirichlet boundary conditions.

As in the previous section, we split our analysis into several lemmas.
Lemma 5. Suppose $\gamma \in (0,1]$. Let $y(x)$ be defined by (18) and

\[ f_1(x) = \left( y + \frac{x}{y} \right) \left( e^{2\pi/x} + 1 \right) \sin \left( \frac{\pi y}{x} \right) - \sqrt{x} \left( y - \frac{1}{y} \right) \left( e^{2\pi/y} - 1 \right) \cos \left( \frac{\pi y}{x} \right). \]

Then $f_1(x)$ has at least one zero point $x_0$ in $(0, \pi^2)$ satisfying $\pi < \sqrt{x_0}y(x_0) < 3/2\pi$.

Proof. Since $(x + 1 - \gamma^2)^2 < (\gamma^2 + 1 - x)^2 + 4x < (x + 1 + \gamma^2)^2$, we see $1 < y(x) < \sqrt{2}$ and so $0 < \sqrt{x_0}y < \sqrt{2\pi} < 3/2\pi$ for all $x \in (0, \pi^2)$. Note that $\tan \theta > 0$ for any $\theta \in (0, \pi/2)$. Note that for any $\pi/2 < \sqrt{x_0}y < \pi$, since $\sin \sqrt{x_0}y > 0$, $\cos \sqrt{x_0}y \leq 0$, and $\sin \sqrt{x_0}y$ and $\cos \sqrt{x_0}y$ cannot be zero simultaneously, it must have $f_1(x) > 0$. To complete the proof, it is sufficient to show that $f_1(\pi^2) < 0$.

Let $z = y(\pi^2)$. Then

\[ f_1(\pi^2) = \left( z + \frac{\pi^2}{z} \right) \left( e^{2\pi^2/z} + 1 \right) \sin (\pi z) - \pi \left( z - \frac{1}{z} \right) \left( e^{2\pi^2/z} - 1 \right) \cos (\pi z) \]

\[ = \frac{1}{z} \cos (z - 1)\pi \left[ (z^2 + \pi^2) \left( e^{2\pi^2/z} + 1 \right) \tan (z - 1)\pi - \pi (z^2 - 1) \left( e^{2\pi^2/z} - 1 \right) \right]. \]

But

\[ (z^2 + \pi^2) \left( e^{2\pi^2/z} + 1 \right) \tan (z - 1)\pi > (z^2 + \pi^2) \left( e^{2\pi^2/z} - 1 \right) (z - 1)\pi \]

\[ > (z + 1) \left( e^{2\pi^2/z} - 1 \right) (z - 1)\pi = \pi (z^2 - 1) \left( e^{2\pi^2/z} - 1 \right), \]

it follows immediately that $f_1(\pi^2) < 0$. The lemma follows then.

Lemma 6. Let $x_0$ be defined as in Lemma 5. Then $f_2(x_0) > 0$, where

\[ f_2(x) = 4\gamma^2\sqrt{x}e^g - 2\gamma^2\sqrt{x} \left( e^{2g} + 1 \right) \cos f + \left[ \gamma^2(1 - x) + (x + 1)^2 \right] \left( e^{2g} - 1 \right) \sin f, \]

with $f$ and $g$ being defined in (18).

Proof. Set $p = 2\gamma^2\sqrt{x} > 0$, $q = \gamma^2(1 - x) + (x + 1)^2 = x^2 + (2 - 3\gamma)x + 1 + \gamma^2 > 0$. Then a calculation shows that

\[ p = 2 \frac{(y^2 - 1)(y^2 + x)\sqrt{x}}{y^2}, \quad q = \frac{(x + y^2)^2 - x(y^2 - 1)^2}{y^2}. \]

Let $y_0 = y(x_0)$, $f_0 = \sqrt{x_0}y_0$, $p_0 = p(x_0)$, $q_0 = q(x_0)$, $g_0 = x_0/y_0$. Then

\[ f_2(x_0) = 2p_0e^{g_0} - p_0 \left( e^{2g_0} + 1 \right) \cos f_0 + q_0 \left( e^{2g_0} - 1 \right) \sin f_0 \]

\[ = \left[ -p_0 \cos f_0 + q_0 \sin f_0 \right] \left( e^{g_0} - c_1 \right) \left( e^{g_0} - c_2 \right), \]

where

\[ c_1 = \frac{-p_0 + \sqrt{p_0^2 + q_0^2} \sin f_0}{-p_0 \cos f_0 + q_0 \sin f_0}, \quad c_2 = \frac{-p_0 - \sqrt{p_0^2 + q_0^2} \sin f_0}{-p_0 \cos f_0 + q_0 \sin f_0}. \]

In order to show $f_2(x_0) > 0$, it suffices to show that

\[ q_0 \sin f_0 - p_0 \cos f_0 > 0. \]

Indeed, if the above holds, then $q_0^2 \sin f_0^2 < p_0^2 \cos f_0^2$ and so

\[ \left| \sqrt{p_0^2 + q_0^2} \sin f_0 \right| < p_0. \]
Hence, \( c_1 < 0, \) \( c_2 < 0 \) which conclude certainly \( f_2(x_0) > 0. \) Now, from Lemma 5, \( f_1(x_0) = 0, \) that is

\[
\left( y_0 + \frac{x_0}{y_0} \right) (e^{2g_0} + 1) \sin f_0 = \sqrt{x_0} \left( y_0 - \frac{1}{y_0} \right) (e^{2g_0} - 1) \cos f_0,
\]

hence,

\[
e^{2g_0} = \frac{-\sqrt{x_0} (y_0^2 - 1) \cos f_0 - (y_0^2 + x_0) \sin f_0}{-\sqrt{x_0} (y_0^2 - 1) \cos f_0 + (y_0^2 + x_0) \sin f_0} > 0.
\]

But since \( \sin f_0 < 0, \cos f_0 < 0, \) the above gives

\[
\sqrt{x_0} (y_0^2 - 1) \cos f_0 < (y_0^2 + x_0) \sin f_0.
\]

Therefore,

\[
q_0 \sin f_0 - p_0 \cos f_0 = y_0^{-2} \left[ (x_0 + y_0^2)^2 - x_0 (y_0^2 - 1)^2 \right] \sin f_0 - 2y_0^{-2} \left( y_0^2 - 1 \right) \left( y_0^2 + x_0 \right) \sqrt{x_0} \cos f_0 \\
= -y_0^{-2} \left[ (x_0 + y_0^2)^2 + x_0 (y_0^2 - 1)^2 \right] \sin f_0 > 0.
\]

Lemma 6 is proved.

**Proof of Theorem 3.** Note that

\[
e^g G(x) = f_2(x) - \frac{1}{\alpha} \sqrt{(\gamma^2 + 1 - x)^2 + 4x f_1(x)}.
\]

By the Taylor expansion, it is easily shown that

\[
\lim_{x \to 0} \frac{f_2(x)}{x^\sqrt{x}} = 2(\gamma^2 + 1)^2, \quad \lim_{x \to 0} \frac{f_1(x)}{\sqrt{x}} = 2(\gamma^2 + 1),
\]

and hence, \( \lim_{x \to 0} e^g G(x)/x = -2/\alpha(\gamma^2 + 1)^2 < 0. \) On the other hand, \( e^g G(x_0) = f_2(x_0) > 0 \) that follows from Lemma 5 and Lemma 6. Therefore, \( G(x) = 0 \) has at least one root in \((0, x_0)\) so does in \((0, 1)\). The proof is complete.

Theorem 3 will lead to the spectrum-determined growth condition for system (15) (for at least \( \alpha \neq 1, \) \( 0 \leq \gamma \leq 1, [1] \)).

**References**