[12] G. Strang, *Linear Algebra and Its Applications*, 2nd ed. New York: Academic, 1980.
[13] G. Tao, *Adaptive Control Design and Analysis*. New York: Wiley, 2003.
[14] S. R. Weller and G. C. Goodwin, "Hysteresis switching adaptive control of linear multivariable systems," *IEEE Trans. Autom. Control*, vol. 39, no. 7, pp. 1360–1375, Jul. 1994.
[15] E. Zergeroglu, D. Dawson, M. de Queiroz, and A. Behal, "Vision-based nonlinear tracking controllers in the presence of parametric uncertainty," *IEEE/ASME Trans. Mechatron.*, vol. 6, no. 3, pp. 322–337, Sep. 2001.

The Stabilization of a One-Dimensional Wave Equation by Boundary Feedback With Noncollocated Observation

Bao-Zhu Guo and Cheng-Zhong Xu

***Abstract*—This note addresses the stabilization of a one-dimensional wave equation with control at one end and noncollocated observation at another end. A simple exponentially convergent observer is constructed. The dynamical stabilizing boundary output feedback is designed via the observed state. While the closed-loop system is nondissipative, we show its exponential stability using Riesz basis approach.**

***Index Terms*—Exact controllability, exponential stability, feedback control, observer, wave equation.**

I. Introduction

It has been known by engineers for a long time that a partial differential equation describing a mechanical system, like a flexible structure in which the power flow into the system is the scalar product $\langle u, y\rangle$ (e.g., when u is force and y is velocity), leads to a positive-real system if actuators and sensors are designed in a "collocated" fashion. Collocated systems always relate with "passivity" which was introduced in connection with circuit theory in 1950s [6]. By "passivity," we mean that the increase of energy stored in the system does not exceed the energy that enters from the external world. For such a system, the transfer function is positive real and negative output feedback produces a dissipative system, which is stable in the sense of Lyapunov.

However, collocated control design is not always feasible in practice and its performance is not always good enough [2]. Actually, the noncollocated control has been widely used in engineering systems control, see [1], [12], [10], [15], [18], [20], [21], [23], [25], and [27]. It is well-known that the noncollocated systems are usually not minimum-phase. As a result, a small increment of feedback controller gains can easily make the closed-loop system unstable [18]. Hence, the control design for the stabilization of noncollocated systems is much harder than collocated ones. Compared with the huge works on the stabilization of collocated PDEs in literature, the study for noncollocated PDEs (systems described by partial differential equations) control is still fairly scarce. Very recently, the observed state feedbacks are designed through backstepping observers in [19] to stabilize a class of one-dimensional parabolic PDEs. The abstract observers design for a class of well-posed regular infinite-dimensional systems can be found in [5] but the stabilization is not addressed.

In this note, a simple observer for a one-dimensional wave equation with boundary control and noncollocated observation is constructed. This observer design is closely related with the theory of well-posed linear infinite-dimensional systems that has been studied extensively in the last two decades [8]. This is because that we require output of the system and state of the observer to depend continuously on input. This seems very natural in design of the observers. The state space for our observer is larger than the state space of the original system. We prove that this observer is exponentially convergent. The boundary output feedback control is designed by state of the observer. This results in nondissipativity of the closed-loop system, which gives rise to difficulty of proof for the stability. Fortunately, using Riesz basis approach, we show that the closed-loop system is indeed exponentially stable. Another difficulty that should be pointed out is that the uncontrolled wave equation has infinite many of eigenvalues on the imaginary axis. This is different to the parabolic equation that has at most a finite number of eigenvalues on the closed right-half complex plane, for which the LQ approach can also be used to solve the problem [13]. Furthermore, our problem is also not the regular in the sense of [5] where the feedthrough operator is required to be zero.

The problem we are concerned with is the following one-dimensional wave equation with boundary control and noncollocated observation:

$$\begin{cases} w_{tt}(x,t) - w_{xx}(x,t) = 0, \quad 0 < x < 1, t > 0 \\ w(0,t) = 0, \quad t \geq 0 \\ w_x(1,t) = u(t), \quad t \geq 0 \\ y(t) = w_x(0,t), \quad t \geq 0 \end{cases} \tag{1.1}$$

where u is the control (input) and y is the observation (output). The state space is $H = H_L^1(0,1) \times L^2(0,1), H_L^1(0,1) = \{f \mid f \in H^1(0,1), f(0) = 0\}$. H is equipped with the norm $\|(f,g)\|_H^2 = \int_0^1 [|f'(x)|^2 + |g(x)|^2]\,dx$ for any $(f,g) \in H$. The input (output) space is $U = \mathbb{C}$. A simple spectral analysis shows that for any direct PI output feedback $u = k_1 y + k_2 \int_0^t y(s)\,ds$ with reals k_1, k_2, the closed loop system is always unstable.

Define the operator $\mathcal{A} : D(\mathcal{A})(\subset H) \to H$ as following:

$$\begin{aligned} &\mathcal{A}(f,g) = (g, f'') \\ &\quad D(\mathcal{A}) = \{(f,g) \in H \mid \mathcal{A}(f,g) \in H, f'(1) = 0\}. \end{aligned} \tag{1.2}$$

Then, (1.1) can be written as

$$\sum(\mathcal{A},\mathcal{B},\mathcal{C}) : \begin{cases} \frac{d}{dt}\begin{pmatrix} w \\ w_t \end{pmatrix} = \mathcal{A}\begin{pmatrix} w \\ w_t \end{pmatrix} + \mathcal{B}u(t) \\ y(t) = \mathcal{C}\begin{pmatrix} w \\ w_t \end{pmatrix} = w_x(0,t) \end{cases} \tag{1.3}$$

where

$$\mathcal{B} = \begin{pmatrix} 0 \\ \delta(x-1) \end{pmatrix}, \quad \mathcal{C} = (-\langle \delta'(x), \cdot\rangle, 0)$$

and $\delta(\cdot)$ denotes the Dirac function. Obviously, both $\mathcal{B}$ and $\mathcal{C}$ are unbounded operators.

Manuscript received August 5, 2006; revised July 25, 2006 and October 25, 2006. Recommended by Associate Editor D. Dochain. This work was supported by the National Natural Science Foundation of China and the National Research Foundation of South Africa.

B. Z. Guo is with the Academy of Mathematics and System Sciences, Academia Sinica, Beijing 100080, P.R. China, and also with the School of Computational and Applied Mathematics, the University of the Witwatersrand, Wits 2050, Johannesburg, South Africa (e-mail: bzguo@iss.ac.cn).

C.-Z. Xu is with the LAGEP, Bât. 308G, Université Claude Bernard Lyon I et ESCPE 43, Bd du 11 Novembre 1918, Villeurbanne cedex France, and also with the School of Mathematics and Computer Science, Ludong University, Yantai, China (e-mail: xu@lagep.cpe.fr).

Digital Object Identifier 10.1109/TAC.2006.890385

Theorem 1: For each $u \in L^2_{\text{loc}}(0,\infty)$ and initial datum $(w(\cdot,0), w_t(\cdot,0)) \in H$, there exists a unique solution $(w, w_t) \in C(0,\infty; H)$ to (1.1), and for each $T > 0$, there exists a $C_T > 0$ independent of u and $(w(\cdot,0), w_t(\cdot,0))$ such that

$$\|(w(\cdot,T), w_t(\cdot,T))\|_H^2 + \int_0^T |y(\tau)|^2\, d\tau \le C_T \left[\|(w(\cdot,0), w_t(\cdot,0))\|_H^2 + \int_0^T |u(\tau)|^2 d\tau\right].$$

Proof: By the well-posed linear infinite-dimensional system theory [3], [8], it is equivalent to showing that $\mathcal{C}$ is admissible for $e^{\mathcal{A}t}$, $\mathcal{B}^*$ is admissible for $e^{\mathcal{A}^*t}$ and the transfer function is bounded on some right half complex plane (see [8, Defs. 2.1 and 2.5]).

We consider only the real function case since for the complex function, the proof is similar. Let $u = 0$. Define

$$E_0(t) = \frac{1}{2}\int_0^1 [w_x^2(x,t) + w_t^2(x,t)]\, dx$$

and

$$\rho(t) = \int_0^1 (x-1) w_x(x,t) w_t(x,t)\, dx.$$

Then, $E_0(t) = E_0(0)$ and $|\rho(t)| \le E_0(t)$ for every $t \ge 0$. Notice that

$$\dot\rho(t) = \frac{1}{2} w_x^2(0,t) - E_0(t).$$

We have

$$2(T-2)E_0(0) \le \int_0^T w_x^2(0,t)\, dt \le 2(T+2)E_0(0).$$

A direct computation shows

$$\mathcal{A}^{-1}(f,g) = \left(\int_1^x (x-\tau) g(\tau)\, d\tau - \int_0^1 \tau g(\tau)\, d\tau, f\right)$$

$$\mathcal{C}\mathcal{A}^{-1}(f,g) = \int_0^1 g(x)\, dx \qquad \forall\, (f,g) \in H.$$

Hence $\mathcal{C}\mathcal{A}^{-1}$ is bounded. This together with the previous right-hand inequality shows that $\mathcal{C}$ is admissible for $e^{\mathcal{A}t}$ (see also [16, eq. (2.3)]). The left-hand side inequality shows that $\sum(\mathcal{C}, \mathcal{A})$ is exactly observable in $[0,T]$ for any $T > 2$ (see [16, eq. (2.6)], where there is a typo). It should be $\int_0^\tau \|C\mathbb{T}_t x\|^2\, dt \ge \kappa_\tau \|x\|^2$; see [16, Prop. 2.8]). The fact that $\mathcal{B}^*$ is admissible for $e^{\mathcal{A}^*t}$ is a well-known fact. Finally, the transfer function for the system (1.1) is found to be

$$H(s) = \frac{2}{e^s + e^{-s}}$$

which is obviously bounded on some complex plane with feedthrough operator zero. □

In [5], the well-posed and regularity with feedthrough operator zero (although not necessarily after more complicated computations) for the system $\sum(\mathcal{A}, \mathcal{B}^*, \mathcal{B})$ is required. A simple computation shows that this corresponds to the following system in H:

$$\begin{cases} w_{tt}(x,t) - w_{xx}(x,t) = 0, & 0 < x < 1, \quad t > 0 \\ w_t(0,t) = u(t), & t \ge 0 \\ w_x(1,t) = 0, & t \ge 0 \\ y(t) = w_x(0,t), & t \ge 0. \end{cases}$$

However, the transfer function for this system is computed to be

$$H_b(s) = \frac{e^s + e^{-s}}{e^s - e^{-s}}$$

and so its feedthrough operator is not zero.

The remaining part of this note is organized as follows. In Section II, we construct an observer for the system (1.1) and show that this observer is exponentially convergent. Section III is devoted to the output feedback stabilization via the observed state.

II. Observer Design

We design the observer for the system (1.1) as follows:

$$\begin{cases} \hat w_{tt}(x,t) - \hat w_{xx}(x,t) = 0, & 0 < x < 1, \quad t > 0 \\ \hat w_x(0,t) = \alpha \hat w_t(0,t) + \beta \hat w(0,t) + w_x(0,t), & t \ge 0 \\ \hat w_x(1,t) = u(t), & t \ge 0 \end{cases} \tag{2.1}$$

where $\alpha, \beta > 0$ are constants. The system (2.1) is considered in the space $\mathbf{H} = H^1(0,1) \times L^2(0,1)$ which is larger than H. The norm of $\mathbf{H}$ is induced by the inner product

$$\|(p,q)\|^2 = \int_0^1 [|p'(x)|^2 + |q(x)|^2]\, dx + \beta |p(0)|^2 \qquad \forall\, (p,q) \in \mathbf{H}.$$

Define the operator $\mathbf{A} : D(\mathbf{A}) (\subset \mathbf{H}) \to \mathbf{H}$

$$\begin{cases} \mathbf{A}(f,g) = (g, f'') & \forall\, (f,g) \in D(\mathbf{A}) \\ D(\mathbf{A}) = \{(f,g) \in \mathbf{H} \,|\, \mathbf{A}(f,g) \in \mathbf{H} \\ \qquad f'(0) = \alpha g(0) + \beta f(0), f'(1) = 0\}. \end{cases} \tag{2.2}$$

It is readily found that

$$\begin{cases} \mathbf{A}^*(\phi,\psi) = (-\psi, -\phi'') & \forall\, (\phi,\psi) \in D(\mathbf{A}^*) \\ D(\mathbf{A}^*) = \{(\phi,\psi) \in \mathbf{H} \,|\, \mathbf{A}^*(\phi,\psi) \in \mathbf{H} \\ \qquad \phi'(0) = -\alpha\psi(0) + \beta\phi(0), \phi'(1) = 0\}. \end{cases} \tag{2.3}$$

Take the product of $(\phi,\psi) \in D(\mathbf{A}^*)$ with (2.1) to obtain

$$\frac{d}{dt}\left\langle \begin{pmatrix} \hat w \\ \hat w_t \end{pmatrix} \quad \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} \hat w \\ \hat w_t \end{pmatrix} \quad \mathbf{A}^* \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} 0 \\ \delta(x-1) \end{pmatrix} u(t) \quad \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} 0 \\ -\delta(x) \end{pmatrix} w_x(0,t) \quad \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\rangle.$$

Hence, (2.1) can be written as

$$\begin{cases} \hat w_{tt}(x,t) - \hat w_{xx}(x,t) = \delta(x-1)u(t) - \delta(x) w_x(0,t) \\ \qquad 0 < x < 1, \quad t > 0 \\ \hat w_x(0,t) = \alpha \hat w_t(0,t) + \beta \hat w(0,t), \quad t \ge 0 \\ \hat w_x(1,t) = 0, \quad t \ge 0 \end{cases} \tag{2.4}$$

or

$$\frac{d}{dt}\begin{pmatrix} \hat w \\ \hat w_t \end{pmatrix} = \mathbf{A}\begin{pmatrix} \hat w \\ \hat w_t \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \delta(x-1) & -\delta(x) \end{pmatrix} \cdot \begin{pmatrix} u \\ w_x(0,t) \end{pmatrix} = \mathbf{A}\begin{pmatrix} \hat w \\ \hat w_t \end{pmatrix} + \mathbf{B}\begin{pmatrix} u \\ w_x(0,t) \end{pmatrix}. \tag{2.5}$$

Theorem 2: For any $u(\cdot), w_x(0,\cdot) \in L^2_{\rm loc}(0,\infty)$ and initial datum $(\hat{w}(\cdot,0), \hat{w}_t(\cdot,0)) \in \mathbf{H}$, there exists a unique solution $(\hat{w}, \hat{w}_t) \in C(0,\infty; \mathbf{H})$ to (2.1), and for all $T > 0$, there exists a $D_T > 0$ depending on T only such that

$$\|(\hat{w}(\cdot,T), \hat{w}_t(\cdot,T))\|^2_{\mathbf{H}} \le D_T \left\{ \|(\hat{w}(\cdot,0), \hat{w}_t(\cdot,0))\|^2_{\mathbf{H}} + \int_0^T [|u(\tau)|^2 + |w_x(0,\tau)|^2]\, d\tau \right\}. \quad (2.6)$$

Proof: It suffices to show that $\mathbf{B}^*$ is admissible for $e^{\mathbf{A}^* t}$ (see [8, Def. 2.1] and [24, Th. 6.9]). A simple computation shows that this is equivalent to saying that $\mathbf{B}^* \mathbf{A}^{*-1}$ is bounded and for any $T > 0$, there exists a $M_T > 0$ depending on T only such that the system of the following:

$$\begin{cases} \hat{w}_{tt}(x,t) - \hat{w}_{xx}(x,t) = 0, & 0 < x < 1, \quad t > 0 \\ \hat{w}_x(0,t) = \alpha \hat{w}_t(0,t) + \beta \hat{w}(0,t), & t \ge 0 \\ \hat{w}_x(1,t) = 0, & t \ge 0 \\ y_w(t) = (\hat{w}_t(1,t), -\hat{w}_t(0,t)), & t \ge 0 \end{cases} \quad (2.7)$$

satisfies (see also [16, eq. (2.3)])

$$\int_0^T [|\hat{w}_t(1,t)|^2 + |\hat{w}_t(0,t)|^2]\, dt \le M_T E_m(0)$$

where

$$E_m(t) = \frac{1}{2} \int_0^1 [|\hat{w}_x(x,t)|^2 + |\hat{w}_t(x,t)|^2]\, dx + \frac{\beta}{2} |\hat{w}(0,t)|^2.$$

First, a simple computation shows that

$$\begin{aligned} &\mathbf{A}^{*-1}(\phi,\psi) \\ &\quad = \left(-\int_1^x (x-\tau)\psi(\tau)\, d\tau + \int_0^1 (\beta^{-1} + \tau)\psi(\tau)\, d\tau \right. \\ &\qquad \left. - \frac{\alpha}{\beta}\phi(0), -\phi(x) \right) \qquad \forall\, (\phi,\psi) \in \mathbf{H} \\ &\mathbf{B}^* \mathbf{A}^{*-1}(\phi,\psi) \\ &\quad = (\phi(1), -\phi(0)) \qquad \forall\, (\phi,\psi) \in \mathbf{H}. \end{aligned}$$

Hence $\mathbf{B}^* \mathbf{A}^{*-1}$ is bounded on $\mathbf{H}$. Second, we consider once again the real function case only. Now, differentiate $E_m(t)$ in t to give

$$\dot{E}_m(t) = -\alpha \hat{w}_t^2(0,t) \le 0$$

and, hence, $E_m(T) \le E_m(0)$ for any $T > 0$ and

$$\int_0^T \hat{w}_t^2(0,t)\, dt \le \frac{1}{\alpha} E_m(0).$$

Next, let

$$\rho_0(t) = \int_0^1 x \hat{w}_x(x,t) \hat{w}_t(x,t)\, dx$$

Then, $|\rho_0(t)| \le E_m(t)$. Since

$$\dot{\rho}_0(t) = \frac{1}{2} \hat{w}_t^2(1,t) - \frac{1}{2} \int_0^1 [\hat{w}_x^2(x,t) + \hat{w}_t^2(x,t)]\, dx$$

it follows that

$$\int_0^T \hat{w}_t^2(1,t)\, dt \le (2+T) E_m(0) \qquad \forall\, T > 0.$$

The result is proved. □

To end this section, we show that the observer is exponentially convergent in $\mathbf{H}$. Actually, let

$$z(x,t) = \hat{w}(1-x,t) - w(1-x,t).$$

Then z satisfies

$$\begin{cases} z_{tt}(x,t) - z_{xx}(x,t) = 0, & 0 < x < 1, \quad t > 0 \\ z_x(0,t) = 0, & t \ge 0 \\ z_x(1,t) = -\alpha z_t(1,t) - \beta z(1,t), & t \ge 0. \end{cases} \quad (2.8)$$

It is well-known that the system (2.8) associates with a C_0-semigroup solution in $\mathbf{H}$ and is exponentially stable in the sense of

$$E_\varepsilon(t) \le M e^{-\omega t} E_\varepsilon(0)$$

for some $M, \omega > 0$, where

$$E_\varepsilon(t) = \frac{1}{2} \int_0^1 [|z_x(x,t)|^2 + |z_t(x,t)|^2]\, dx + \frac{\beta}{2} |z(1,t)|^2.$$

III. Output Feedback Stabilization

In this section, we design the output feedback control law

$$u(t) = -\alpha \hat{w}_t(1,t)$$

for the system (1.1) and (2.1). The closed-loop system becomes

$$\begin{cases} \hat{w}_{tt}(x,t) - \hat{w}_{xx}(x,t) = 0, & 0 < x < 1, \quad t > 0 \\ \hat{w}_x(0,t) = \alpha \hat{w}_t(0,t) + \beta \hat{w}(0,t) + w_x(0,t), & t \ge 0 \\ \hat{w}_x(1,t) = -\alpha \hat{w}_t(1,t), & t \ge 0 \\ w_{tt}(x,t) - w_{xx}(x,t) = 0, & 0 < x < 1, \quad t > 0 \\ w(0,t) = 0, & t \ge 0 \\ w_x(1,t) = -\alpha \hat{w}_t(1,t), & t \ge 0. \end{cases} \quad (3.1)$$

Consider (3.1) in the space $X = \mathbf{H} \times H$ with norm induced by the inner product

$$\|(f,g,\phi,\psi)\|^2 = \int_0^1 [|f'(x)|^2 + |g(x)|^2 + |\phi'(x)|^2 + |\psi(x)|^2]\, dx + \beta |f(0)|^2 \qquad \forall\, (f,g,\phi,\psi) \in X.$$

The system operator $\mathbb{A}(\subset X) \to X$ for (3.1) is defined by

$$\begin{cases} \mathbb{A}(f,g,\phi,\psi) = (g, f'', \psi, \phi'') \qquad \forall\, (f,g,\phi,\psi) \in D(\mathbb{A}) \\ D(\mathbb{A}) = \{(f,g,\phi,\psi) \in X \mid \mathbb{A}(f,g,\phi,\psi) \in X \\ \qquad\qquad f'(0) = \alpha g(0) + \beta f(0) + \phi'(0) \\ \qquad\qquad \phi(0) = 0, f'(1) = \phi'(1) = -\alpha g(1)\}. \end{cases} \tag{3.2}$$

A simple computation shows that $\mathbb{A}$ is not dissipative. So we use Riesz basis approach in the sequel to treat the C_0-semigroup generation and stability property of $\mathbb{A}$.

Lemma 1: $\mathbb{A}^{-1}$ is compact on X and hence $\sigma(\mathbb{A})$, the spectrum of $\mathbb{A}$, consists of isolated eigenvalues only.

Proof: For any $(p_1, q_1, p_2, q_2) \in X$, solve $\mathbb{A}(f,g,\phi,\psi) = (p_1, q_1, p_2, q_2)$ to obtain

$$\begin{cases} g(x) = p_1(x), \quad \psi(x) = p_2(x) \\ f(x) = \int_1^x (x-\tau) q_1(\tau) d\tau - \int_0^1 \tau q_1(\tau) d\tau - \alpha p_1(1) x + f(0) \\ f(0) = \frac{1}{\beta} \int_0^1 [q_2(\tau) - q_1(\tau)] d\tau - \frac{\alpha}{\beta} p_1(0) \\ \phi(x) = \int_1^x (x-\tau) q_2(\tau) d\tau - \int_0^1 \tau q_2(\tau) d\tau - \alpha p_1(1) x. \end{cases}$$

So, $\mathbb{A}^{-1}$ is defined on whole space X and $\mathbb{A}^{-1}$ maps X into a subset of space $(H^2(0,1) \times H^1(0,1))^2$, which is compact in X. By the Sobolev embedding theorem [14], $\mathbb{A}^{-1}$ is compact on X, proving the required result. □

Lemma 2: There are two families of eigenvalues of $\mathbb{A}$ which can be expressed asymptotically as

$$\begin{cases} \lambda_{1n} = \frac{1}{2} \log \left| \frac{1-\alpha}{1+\alpha} \right| + n_\alpha \pi i, \quad \operatorname{Re} \lambda_{1n} < 0, \quad n \in \mathbb{Z} \\ \lambda_{2n} = \frac{1}{2} \log \left| \frac{1-\alpha}{1+\alpha} \right| + n^\alpha \pi i + o\left(\frac{1}{|n|}\right), \quad |n| \to \infty \\ \lambda_{2n} < 0 \end{cases} \tag{3.3}$$

where

$$\begin{cases} n_\alpha = \begin{cases} n - 1/2, & 0 < \alpha < 1 \\ n, & \alpha > 1 \end{cases} \\ n^\alpha = \begin{cases} n, & 0 < \alpha < 1 \\ n - 1/2, & \alpha > 1. \end{cases} \end{cases} \tag{3.4}$$

The corresponding eigenfunctions for $\alpha \in (0,1)$ are (there are parallel results for $\alpha > 1$)

$$\begin{cases} W_{1n}(x) = (F_n(x), F_n(x)), \quad n \in \mathbb{Z} \\ W_{2n}(x) = (\tilde{F}_n(x) + G_n(x), \tilde{F}_n(x)) \end{cases} \tag{3.5}$$

where

$$\begin{cases} G_n(x) = c_{2n} \left(\lambda_{2n}^{-1} p_{2n}(1-x), p_{2n}(1-x) \right) \\ c_{2n} = -\dfrac{1}{2\alpha} e^{\lambda_{2n}} \left[1 + \alpha + (1-\alpha) \dfrac{(1+\alpha)\lambda_{2n} + \beta}{(1-\alpha)\lambda_{2n} - \beta} \right] \\ \quad = -\frac{\sqrt{1-\alpha^2}}{\alpha} (-1)^n + o\left(\frac{1}{|n|}\right) \end{cases} \tag{3.6}$$

$$\begin{cases} p_{2n}(x) = e^{\lambda_{2n} x} + e^{-\lambda_{2n} x} = \left(\sqrt{\left| \dfrac{1-\alpha}{1+\alpha} \right|} \right)^x e^{i n^\alpha \pi x} \\ \quad + \left(\sqrt{\left| \dfrac{1+\alpha}{1-\alpha} \right|} \right)^x e^{-i n^\alpha \pi x} + o\left(\dfrac{1}{|n|}\right) \\ \lambda_{2n}^{-1} p_{2n}'(x) = \left(\sqrt{\left| \dfrac{1-\alpha}{1+\alpha} \right|} \right)^x e^{i n^\alpha \pi x} \\ \quad - \left(\sqrt{\left| \dfrac{1+\alpha}{1-\alpha} \right|} \right)^x e^{-i n^\alpha \pi x} + o\left(\dfrac{1}{|n|}\right), \quad |n| \to \infty \end{cases} \tag{3.7}$$

$$\begin{cases} F_n(x) = (\lambda_{1n}^{-1} f_{1n}(x), f_{1n}(x)), \quad n \in \mathbb{Z} \\ f_{1n}(x) = e^{\lambda_{1n} x} - e^{-\lambda_{1n} x} = \left(\sqrt{\left| \dfrac{1-\alpha}{1+\alpha} \right|} \right)^x e^{i n_\alpha \pi x} \\ \quad - \left(\sqrt{\left| \dfrac{1+\alpha}{1-\alpha} \right|} \right)^x e^{-i n_\alpha \pi x} \\ \lambda_{1n}^{-1} f_{1n}'(x) = \left(\sqrt{\left| \dfrac{1-\alpha}{1+\alpha} \right|} \right)^x e^{i n_\alpha \pi x} \\ \quad + \left(\sqrt{\left| \dfrac{1+\alpha}{1-\alpha} \right|} \right)^x e^{-i n_\alpha \pi x} \end{cases} \tag{3.8}$$

$$\begin{cases} \tilde{F}_n(x) = (\lambda_{2n}^{-1} \phi_{2n}(x), \phi_{2n}(x)) \\ \phi_{2n}(x) = e^{\lambda_{2n} x} - e^{-\lambda_{2n} x} = \left(\sqrt{\left| \dfrac{1-\alpha}{1+\alpha} \right|} \right)^x e^{i n^\alpha \pi x} \\ \quad - \left(\sqrt{\left| \dfrac{1+\alpha}{1-\alpha} \right|} \right)^x e^{-i n^\alpha \pi x} + o\left(\dfrac{1}{|n|}\right) \\ \lambda_{2n}^{-1} \phi_{2n}'(x) = \left(\sqrt{\left| \dfrac{1-\alpha}{1+\alpha} \right|} \right)^x e^{i n^\alpha \pi x} \\ \quad + \left(\sqrt{\left| \dfrac{1+\alpha}{1-\alpha} \right|} \right)^x e^{-i n^\alpha \pi x} + o\left(\dfrac{1}{|n|}\right), \quad |n| \to \infty. \end{cases} \tag{3.9}$$

Proof: Suppose $\mathbb{A}(f,g,\phi,\psi) = \lambda(f,g,\phi,\psi) \neq 0$. Then, we have $g = \lambda f, \psi = \lambda \phi$ and (f, ϕ) satisfies

$$\begin{cases} f''(x) = \lambda^2 f(x) \\ f'(0) = \alpha\lambda f(0) + \beta f(0) + \phi'(0), f'(1) = -\alpha\lambda f(1) \\ \phi'' = \lambda^2 \phi \\ \phi(0) = 0, \phi'(1) = -\alpha\lambda f(1). \end{cases} \tag{3.10}$$

Let $p(x) = f(1-x) - \phi(1-x)$. Then, p satisfies

$$\begin{cases} p''(x) = \lambda^2 p(x) \\ p'(0) = 0, p'(1) = -\alpha\lambda p(1) - \beta p(1). \end{cases} \tag{3.11}$$

Now, if $p \equiv 0$, that is, $f \equiv \phi$. Then, $f \neq 0$ satisfies

$$\begin{cases} f''(x) = \lambda^2 f(x) \\ f(0) = 0, f'(1) = -\alpha\lambda f(1). \end{cases} \tag{3.12}$$

We get the eigenvalues

$$\lambda_{1n} = \frac{1}{2} \log \left| \frac{1-\alpha}{1+\alpha} \right| + n_\alpha \pi i, \quad \operatorname{Re} \lambda_{1n} < 0, \quad n \in \mathbb{Z}.$$

This is the first expression of (3.3). The corresponding eigenvectors are

$$W_{1n}(x) = (\lambda_{1n}^{-1} f_{1n}(x), \quad f_{1n}(x), \lambda_{1n}^{-1} f_{1n}(x), f_{1n}(x)) \qquad n \in \mathbb{Z}$$

where f_{1n} is given by (3.8). This is the first expression of (3.5).

When $p \neq 0$, multiply by $\bar{p}$ on both sides of the first equation of (3.11) and integrate by parts to give

$$\operatorname{Re} \lambda < 0. \tag{3.13}$$

The characteristic equation of (3.11) is

$$e^{2\lambda} = \frac{(1-\alpha)\lambda - \beta}{(1+\alpha)\lambda + \beta} = \frac{1-\alpha}{1+\alpha} - \frac{2\beta}{(1+\alpha)^2 \lambda} + o(|\lambda|^{-2}), \quad |\lambda| \to \infty.$$

Solve the previous equation by virtue of Rouché's theorem to give the second expression of (3.3)

$$\lambda_{2n} = \frac{1}{2} \log \left| \frac{1-\alpha}{1+\alpha} \right| + n^\alpha \pi i + o\left(\frac{1}{|n|}\right), \quad |n| \to \infty.$$

The corresponding solutions p_{2n} to (3.11) are expressed by (3.7), which holds for $x \in [0,1]$ uniformly. It follows from (3.10) that the corresponding eigenfunctions are

$$W_{2n}(x) = (\lambda_{2n}^{-1} f_{2n}(x), \ f_{2n}(x), \lambda_{2n}^{-1} \phi_{2n}(x), \phi_{2n}(x))$$

where $\phi_{2n}(x)$ is given in (3.9). A simple computation shows that

$$f_{2n}(x) = \phi_{2n}(x) + c_{2n} p_{2n}(1-x)$$

where c_{2n} is given in (3.6). The second expression of (3.5) is derived. The proof is complete. □

Let us recall that $W \in D(\mathbb{A})$ is said to be a generalized eigenfunction of $\mathbb{A}$ associated with the eigenvalue λ if there is an integer $\ell \geq 1$ such that $(\lambda - \mathbb{A})^\ell W = 0$. The integer $m_{(a)}(\lambda) = \sup_{l \in \mathbb{N}} \dim [\ker(\lambda - \mathbb{A})^\ell]$ is called the algebraic multiplicity of λ. An eigenvalue λ is said algebraically simple if $m_{(a)}(\lambda) = 1$.

The sequence $\{W_n\}_{n \in \mathbb{Z}}$ is called a basis for X if to each element $W \in X$ corresponds a unique sequence of numbers $\{c_n\}$ such that the series

$$W = \sum_{n \in \mathbb{Z}} c_n W_n \tag{3.14}$$

is convergent with respect to the norm of X. $\{W_n\}_{n \in \mathbb{Z}}$ is called a Riesz basis for X if

a) $\overline{\operatorname{span}}\{W_n\} = X$;
b) there exist some positive constants m_1 and m_2 such that for any numbers $c_n, n \in \mathcal{I}$, where $\mathcal{I}$ is any finite subset of $\mathbb{Z}$, the following holds:

$$m_1 \sum_{n \in \mathcal{I}} |c_n|^2 \leq \| \sum_{n \in \mathcal{I}} c_n W_n \|^2 \leq m_2 \sum_{n \in \mathcal{I}} |c_n|^2.$$

We refer to [26] for more details on Riesz basis.

Remark 1: In designing the output feedback control law, we use the same gain α as that in (2.1). Actually, we can design $u(t) = -\gamma \hat{w}_t(1,t)$ for any $\gamma < 0$. In this case, by the same spectral analysis as in the proof of Lemma 2, the spectrum are still as that in (3.3) but λ_{1n} is replaced by $\lambda_{1n} = (1/2) \log |(1-\gamma)/(1+\gamma)| + n_\gamma \pi i$. So the decay rate is still dominated by λ_{2n}. It does not bring any advantage for the stabilization.

Theorem 3: Suppose $\alpha \neq 1$. Then, the following assertions hold true:

i) $\mathbb{A}$ has a family of generalized eigenfunctions, which forms a Riesz basis for X;
ii) all eigenvalues of $\mathbb{A}$ with sufficiently large module are algebraic simple;
iii) $\mathbb{A}$ generates a C_0-semigroup on X;
iv) the spectrum-determined growth condition holds for $e^{\mathbb{A}t}$: $\omega(\mathbb{A}) = S(\mathbb{A}) < 0$, where $\omega(\mathbb{A})$ is the growth order of $e^{\mathbb{A}t}$;
v) the semigroup $e^{\mathbb{A}t}$ is exponentially stable.

Proof: The relationship between iii) and iv) and i) and ii) is well-known, which can be found for the explanations of (5.1) of [22]). Hence ii)–v) are consequences of (i) and (3.3), (3.13) and the following arguments for the proof of assertion (i). We consider here only $0 < \alpha < 1$ since the case of $\alpha > 1$ can be treated similarly.

It is known that there exists a set of generalized eigenfunctions for the operator $\mathbf{A}$ defined by (2.2), which forms a Riesz basis for $\mathbf{H}$ (it can be shown simply by the asymptotic expression obtained in Lemma 2 and the abstract result in [7]). Its characteristic equation is just (3.11) and all generalized eigenfunctions are denoted by $\{G_n(x)\}_{n \in \mathbb{Z}_0}$, where $\mathbb{Z}_0$ is a set of index with the property that there exists a large integer N such that $\mathbb{Z}_0 - \{n \mid |n| > N\}$ is a finite set. The asymptotic eigenfunctions for $G_n(x)$ are given by (3.6). From the arguments in the sequel, we may assume without loss of generality that all $W_{2n}(x)$ have the same form of (3.5): $\{W_{2n}(x) = (\tilde{F}_n(x) + G_n(x), \tilde{F}_n(x))\}_{n \in \mathbb{Z}_0}$. All eigenvalues of $\mathbf{A}$ with large modulus are algebraically simple.

Let us define another operator $\tilde{\mathbf{A}}$ in H

$$\begin{cases} \tilde{\mathbf{A}}(f,g) = (g, f'') \\ D(\tilde{\mathbf{A}}) = \{(f,g) \in H \mid \tilde{\mathbf{A}}(f,g) \in H, f'(1) = -\alpha g(1)\}. \end{cases}$$

Its characteristic equation is (3.12). It is well known that the eigenfunctions $\{F_n(x)\}_{n \in \mathbb{Z}}$ of $\tilde{\mathbf{A}}$ given by (3.8) form a Riesz basis for H [17].

Now, define an isometric isomorphism $\mathbb{T}_1 : X \to (L^2(0,1))^2 \times \mathcal{C} \times (L^2(0,1))^2$ by

$$\mathbb{T}_1(f,g,\phi,\psi) = (f', g, \sqrt{\beta} f(0), \phi', \psi) \qquad \forall\, (f,g,\phi,\psi) \in X.$$

Then, the proof will be accomplished if we can show that

$$\left\{ \left\{ \mathbb{T}_1 \begin{pmatrix} F_n(x) \\ F_n(x) \end{pmatrix} \right\}_{n \in \mathbb{Z}}, \left\{ \mathbb{T}_1 \begin{pmatrix} \tilde{F}_n(x) + G_n(x) \\ \tilde{F}_n(x) \end{pmatrix} \right\}_{n \in \mathbb{Z}_0} \right\}$$

forms a Riesz basis for $(L^2(0,1))^2 \times \mathcal{C} \times (L^2(0,1))^2$.

Next, define a bounded invertible transformation $\mathbb{T}_0$ on $L^2(0,1) \times L^2(0,1)$:

$$\mathbb{T}_0(f,g)(x) = (-g(1-x), f(1-x)) \qquad \forall\, (f,g) \in L^2(0,1) \times L^2(0,1).$$

Then, it is easy to see that (we use the column and row notations alternatively without confusion)

$$\begin{aligned}
&\begin{pmatrix} I & 0 & -I \\ 0 & 1 & 0 \\ c_{2n}^{-1}\mathbb{T}_0 & 0 & I - c_{2n}^{-1}\mathbb{T}_0 \end{pmatrix} \mathbb{T}_1 \begin{pmatrix} F_n(x) \\ F_n(x) \end{pmatrix} \\
&\quad = \mathbb{T}_1 \begin{pmatrix} 0 \\ F_n(x) \end{pmatrix} \\
&\begin{pmatrix} I & 0 & -I \\ 0 & 1 & 0 \\ c_{2n}^{-1}\mathbb{T}_0 & 0 & I - c_{2n}^{-1}\mathbb{T}_0 \end{pmatrix} \cdot \\
&\mathbb{T}_1 \begin{pmatrix} \tilde{F}_n(x) + G_n(x) \\ \tilde{F}_n(x) \end{pmatrix} \\
&\quad = \mathbb{T}_1 \begin{pmatrix} G_n(x) \\ 0 \end{pmatrix} + o\left(\frac{1}{|n|}\right)
\end{aligned}$$

where c_{2n} is defined in (3.6) and I is the identity on $(L^2(0,1))^2$. It is obvious that $\{\{(0, F_n(x))\}_{n\in\mathbb{Z}}, \{(G_n(x), 0)\}_{n\in\mathbb{Z}_0}\}$ forms a Riesz basis for X and so

$$\left\{ \left\{ \mathbb{T}_1 \begin{pmatrix} 0 \\ F_n(x) \end{pmatrix} \right\}_{n\in\mathbb{Z}}, \left\{ \mathbb{T}_1 \begin{pmatrix} G_n(x) \\ 0 \end{pmatrix} \right\}_{n\in\mathbb{Z}_0} \right\}$$

forms a Riesz basis for $(L^2(0,1))^2 \times \mathcal{C} \times (L^2(0,1))^2$. By classical Bari's theorem [11] and (3.15), it follows that

$$\begin{aligned}
&\left\{ \left\{ \begin{pmatrix} I & 0 & -I \\ 0 & 1 & 0 \\ c_{2n}^{-1}\mathbb{T}_0 & 0 & I - c_{2n}^{-1}\mathbb{T}_0 \end{pmatrix} \mathbb{T}_1 \begin{pmatrix} F_n(x) \\ F_n(x) \end{pmatrix} \right\}_{n\in\mathbb{Z}}, \right. \\
&\quad \left\{ \begin{pmatrix} I & 0 & -I \\ 0 & 1 & 0 \\ c_{2n}^{-1}\mathbb{T}_0 & 0 & I - c_{2n}^{-1}\mathbb{T}_0 \end{pmatrix} \mathbb{T}_1 \right. \\
&\quad \left. \left. \times \begin{pmatrix} \tilde{F}_n(x) + G_n(x) \\ \tilde{F}_n(x) \end{pmatrix} \right\}_{n\in\mathbb{Z}_0} \right\}
\end{aligned}$$

forms a Riesz basis for $(L^2(0,1))^2 \times \mathcal{C} \times (L^2(0,1))^2$. Notice that for all $n \in \mathbb{Z}$ or $n \in \mathbb{Z}_0$, the transformations

$$\begin{pmatrix} I & 0 & -I \\ 0 & 1 & 0 \\ c_{2n}^{-1}\mathbb{T}_0 & 0 & I - c_{2n}^{-1}\mathbb{T}_0 \end{pmatrix}$$

and their inverses are uniformly bounded in $(L^2(0,1))^2 \times \mathcal{C} \times (L^2(0,1))^2$ with respect to n. Therefore

$$\left\{ \left\{ \mathbb{T}_1 \begin{pmatrix} F_n(x) \\ F_n(x) \end{pmatrix} \right\}_{n\in\mathbb{Z}}, \left\{ \mathbb{T}_1 \begin{pmatrix} \tilde{F}_n(x) + G_n(x) \\ \tilde{F}_n(x) \end{pmatrix} \right\}_{n\in\mathbb{Z}_0} \right\}$$

forms a Riesz basis for $(L^2(0,1))^2 \times \mathcal{C} \times (L^2(0,1))^2$. The proof is complete. $\square$

Finally, we indicate that Theorem 3 should be still true even for $\alpha = 1$. There is some other method to treat the exponential stability for nondissipative PDEs. We refer to [9] for the progress on this aspect. In order to understand the case of $\alpha = 1$ intuitionally, we use an Legendre spectral method to approximate (3.11) and obtain two figures. Fig. 1 demonstrates the functional relation of $S(\mathcal{A})$ with respect to β where we take $\alpha = 0.7$. It recommends the choice of small β. Fig. 2 demonstrates the functional relation of $S(\mathcal{A})$ with respect to α where we take $\beta = 0.15$, which recommends the choice of α around one.

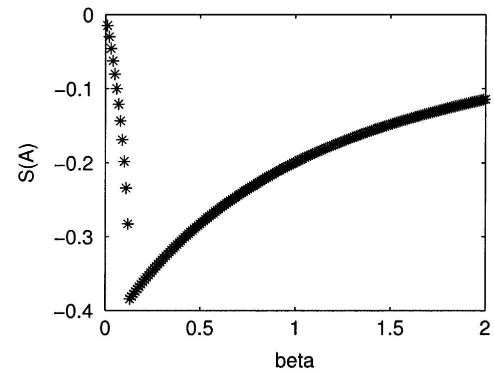

Fig. 1. Functional relation of $\boldsymbol{S}(\mathbb{A})$ with respect to $\boldsymbol{\beta}$ with $\boldsymbol{\alpha} = \mathbf{0.7}$.

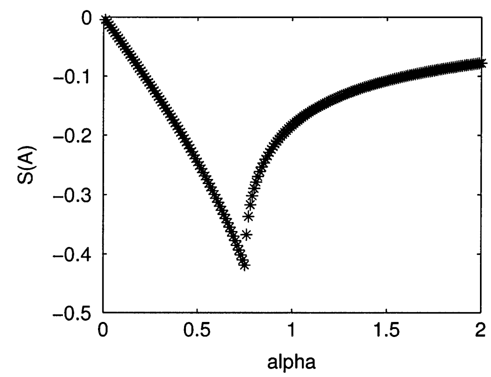

Fig. 2. Functional relation of $\boldsymbol{S}(\mathbb{A})$ with respect to $\boldsymbol{\alpha}$ with $\boldsymbol{\beta} = \mathbf{0.15}$.

ACKNOWLEDGMENT

This work was begun when B. Z. Guo was visiting LAGEP, Université Claude Bernard Lyon I, France, in 2006. The authors would like to thank the anonymous referees for useful suggestions.

REFERENCES

[1] R. H. Cannon and E. Schmitz, "Initial experiments on the end-point control of a flexible one-link robot," *Int. J. Robot. Res.*, vol. 3, pp. 62–75, 1984.
[2] P. A. Chodavarapu and M. W. Spong, "On Noncollocated control of a single flexible link," in *Proc. IEEE Int. Conf. Robotics Automation*, Minneapolis, MN, 1996, pp. 1101–1106.
[3] R. F. Curtain, "The Salamon-Weiss class of well-posed infinite dimensional linear systems: A survey," *IMA J. Math. Control Inform.*, vol. 14, pp. 207–223, 1997.
[4] R. F. Curtain and H. J. Zwart, *An Introduction to Infinite Dimensional Linear System Theory*. New York: Springer-Verlag, 1995.
[5] A. J. Deguenon, G. Sallet, and C. Z. Xu, "A Kalman observer for infinite-dimensional skew-symmetric systems with application to an elastic beam," in *Proc. 2nd Int. Symp. Communications, Control, Signal Processing*, Marrakech, Morocco, Mar. 2006.
[6] E. A. Guillemin, *Synthesis of Passive Networks*. New York: Wiley, 1957.
[7] B. Z. Guo and J. M. Wang, "Riesz basis generation of an abstract second-order partial differential equation system with general non-separated boundary conditions," *Numer. Funct. Anal. Optim.*, vol. 27, no. 3–4, pp. 291–328, 2006.
[8] B. Z. Guo and X. Zhang, "The regularity of the wave equation with partial Dirichlet control and colocated observation," *SIAM J. Control Optim.*, vol. 44, no. 5, pp. 1598–1613, 2005.
[9] A. Guesmia, "A new approach of stabilization of nondissipative distributed systems," *SIAM J. Control Optim.*, vol. 42, pp. 24–52, 2003.

[10] L. Y. Liu and K. Yuan, "Noncollocated passivity-based control of a single-link flexible manipulator," *Robotica*, vol. 21, pp. 117–135, 2003.

[11] I. C. Gohberg and M. G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*, ser. Trans. Math. Monographs. Providence, RI: AMS, 1969, vol. 18.

[12] W. Lacarbonara and H. Yabuno, "Closed-loop non-linear control of an initially imperfect beam with non-collocated input," *J. Sound Vibrat.*, vol. 273, pp. 695–711, 2004.

[13] I. Lasiecka and R. Triggiani, *Control Theory for Partial Differential Equations: Continuous and Approximation Theories*. Cambridge, U.K.: Cambridge Univ. Press, 2000, vol. 1.

[14] V. Mazja, *Sobolev Spaces*. New York: Springer-Verlag, 1985.

[15] M. S. de Queiroz and C. D. Rahn, "Boundary control and noise in distributed parameter systems: An overview," *Mech. Sys. Signal Process.*, vol. 16, pp. 19–38, 2002.

[16] D. L. Russell and G. Weiss, "A general necessary condition for exact observability," *SIAM J. Control Optim.*, vol. 32, pp. 1–23, 1994.

[17] P. Rideau, Contrôle d'un assemblage de poutres flexibles par des capteurs-actionneurs ponctuels: Étude du spectre du système, Thèse de l'École Nationale Supérieure des Mines de Paris. Sophia-Antipolis, France, Nov. 1985.

[18] J. H. Ryu, D. S. Kwon, and B. Hannaford, "Control of a flexible manipulator with nonconllocated feedback: Time-domain passivity approach," in *Control Problems in Robotics*, A. Bicchi, H. I. Christensen, and D. Prattichizzo, Eds. Berlin, Germany: Springer-Verlag, 2003, vol. 4, pp. 121–134, Springer Tracts in Advanced Robotics Series.

[19] A. Smyshlyaev and M. Krstic, "Backstepping observers for a class of parabolic PDEs," *Syst. Control Lett.*, vol. 54, pp. 613–625, 2005.

[20] V. A. Spector and H. Flashner, "Modelling and design implecations of non-collocated control in flexible systems," *ASME J. Dyna. Syst., Measure., Control*, vol. 112, pp. 186–193, 1990.

[21] F. E. Udwadia, "Noncollocated point control of nondispersive distributed-parameter systems using time delays," *Appl. Math. Comput.*, vol. 42, pp. 23–63, 1991.

[22] J. M. Wang, B. Z. Guo, and B. Chentouf, "Boundary feedback stabilization of a three-layer sandwich beam: Riesz basis approach," *ESAIM: Control, Optim. Calc. Variat.*, vol. 12, pp. 12–34, 2006.

[23] S. T. Wu, "Virtual passive control of flexible arms with collocated and noncollocated feedback," *J. Robot. Syst.*, vol. 18, pp. 645–655, 2001.

[24] G. Weiss, "Admissible observation operators for linear semigroups," *Israel J. Math.*, vol. 65, pp. 17–43, 1989.

[25] B. Yang and C. D. Mote, Jr., "On time delay in noncolocated control of flexible mechanical systems," *ASME J. Dyna. Syst., Measure., Control*, vol. 114, pp. 409–415, 1992.

[26] R. M. Young, *An Introduction to Nonharmonic Fourier Series*. London, U.K.: Academic, 1980.

[27] K. Yuan and L. Y. Liu, "On the stability of zero dynamics of a single-link flexible manipulator for a class of parametrized outputs," *J. Robot. Syst.*, vol. 20, pp. 581–586, 2003.