Brief paper

The strong stabilization of a one-dimensional wave equation by non-collocated dynamic boundary feedback control

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**A B S T R A C T**

The stabilization of a one-dimensional wave equation with non-collocated observation at its unstable free end and control at another end is considered. The controller comprises a state estimator which is designed in the case where the velocity is not available. The method of “backstepping” is adopted in our design of the feedback law. We use the theory of $C_0$-semigroups and Lyapunov functionals to prove the strong stability of the resulting closed-loop system.

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1. Introduction

In the last several decades, collocated boundary control for systems described by partial differential equations (PDEs) has been studied extensively, see Chen (1979), Lagnese (1983), Luo, Guo, and Morgul (1998) and Slemrod (1976), to name just a few. The collocated design requires that the actuators and sensors are placed in the same boundary parts and are associated in an adjoint way. The idea of collocated design comes from the "passivity" principle which was introduced in circuit theory in 1950s (Guillemin, 1957). The advantage of collocated design is obvious: the transfer function of the system is positive real and the negative output feedback makes the system dissipative, which is thereby stable in the sense of Lyapunov. On the other hand, it has been noticed for a long time in engineering practice that the performance of the collocated control design is not always good enough (Cannon & Schmitz, 1984). Actually, the non-collocated design has been widely used in engineering systems control (see Cannon and Schmitz (1984), Lacarbonara and Yabuno (2004), Liu and Yuan (1997), Spector and Flashner (1990), Udwadia (1991), Wu (2001), Yang and Mote (1992) and Yuan and Liu (2003)). However the non-collocated design gives rise to some difficulties in practice. For instance, the open-loop form of a non-collocated system is usually not minimum-phase. This leads to the closed-loop system being unstable with large feedback controller gain. Moreover, the closed-loop form of a non-collocated system is usually non-dissipative which makes the traditional Lyapunov method hard to apply to the analysis of its stability. Recently, estimated state feedbacks have been designed through backstepping observers in Smyshlyaev and Krstic (2005) to stabilize a class of one-dimensional parabolic PDEs. An observer-based compensator which exponentially stabilizes the string system with a non-collocated actuator/sensor configuration was proposed in Guo and Xu (2007). Recent progress was made in Krstic, Guo, Balogh, and Smyshlyaev (2008) where the controller and observer were designed using the backstepping method to stabilize an unstable one-dimensional wave equation in which the actuator and sensor are located at the opposite ends. It should be pointed out that in Krstic et al. (2008), the estimated feedback is actually a PD controller, that is, not only the displacement but also its time derivative (velocity) are used in observer design. When only the displacement is available, an abstract infinite-dimensional second-order collocated system was considered in Kobayashi (2004) where the feedback control was designed based on the displacement measurement only. The objective of this paper is to stabilize the non-collocated system considered in Krstic et al. (2008) by using only the displacement measurement. The
According to Krstić et al. (2008), the recommended choices of the control gains are $c_2$ around one, and $c_1$ relatively large.

Consider the invertible change of variable

$$\tilde{w}(x, t) = (I + P)\tilde{w}(x, t),$$

where $P$ is a Volterra transformation (Krstić et al., 2008). The inverse $(I + P)^{-1}$ is given by

$$\tilde{w}(x, t) = (I + P)^{-1}(w(x, t)).$$

It can be shown that (6) converts the system (5) into

$$\tilde{w}(x, t) = \tilde{w}_{x}(x, t) = (c_1 + q)\int_0^x e^{\eta(x-\xi)}\tilde{w}(\xi, t)d\xi,$$

where $\tilde{w}$ is given by (3) and

$$\tilde{w}(x, 0) = \tilde{w}_0(x), \quad \tilde{w}_1(x, 0) = \tilde{w}_1(x),$$

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Proof. By definition, a simple computation shows that
\[ \text{Re}(Af(f, g, \theta), (f, g, \theta))_{\mathcal{H}} = -k^2|f(0) + \theta|^2 \leq 0, \]
\[ \forall (f, g, \theta) \in D(A). \]

So \( A \) is dissipative in \( \mathcal{H} \). By the Lumer–Phillips theorem, the assertion (i) will be proved if we can show that \( A^{-1} \) exists and is bounded on \( \mathcal{H} \). Indeed, for any \((f, g, \xi) \in \mathcal{H}\), solve \( A(\phi, \psi, \theta) = (f, g, \xi) \), to obtain \( \psi = f \) with \( \phi \) satisfying
\[ \begin{cases} \phi''(x) = g(x), \\ \phi(1) = 0, \\ \phi'(0) = k(\phi(0) + \theta) = -\xi. \end{cases} \]  

(12)

Solve (12) to obtain
\[ \begin{align*}
\phi(x) &= -\xi(x - 1) + \int_0^x (x - s)g(s)ds \\
\theta &= -\frac{\xi}{k} \int_0^x (1 - x)g(x)dx.
\end{align*} \]  

(13)

It follows that \( A^{-1} \) exists and maps \( \mathcal{H} \) into \( H^2 \times H^1 \times C \). By expression of (13) and the Sobolev embedding theorem, \( A^{-1} \) is compact on \( \mathcal{H} \).

To prove the assertion (ii), it suffices to show that there is no eigenvalue of \( A \) located on the imaginary axis. Actually, suppose \( A(f, g, \theta) = \lambda(f, g, \theta) \) with \( \lambda = i\omega, \omega \in \mathbb{R} \). Then we have \( g = \lambda f \) with \( f \) satisfying
\[ \begin{align*}
f''(x) - \lambda^2 f(x) &= 0, \\
f(1) &= 0, \\
f'(0) &= k(f(0) + \theta) = -\lambda \theta.
\end{align*} \]  

(14)

If \( \lambda = 0 \), then \( f = g = 0 \) and so \( \theta = 0 \) by the last equality of (14).

Now suppose \( \lambda = i\omega \neq 0, \omega \in \mathbb{R} \). Solve the last equality of (14) to get
\[ \theta = -\frac{k}{\lambda + k} f(0). \]

Substitute the above into (14) to obtain
\[ \begin{align*}
f''(x) - \lambda^2 f(x) &= 0, \\
f(1) &= 0, \\
f'(0) &= \frac{k\lambda}{\lambda + k} f(0).
\end{align*} \]  

(15)

Take inner product with \( f \) on both sides of the first equation of (15) to get
\[ \begin{align*}
\frac{k\lambda}{\lambda + k} |f(0)|^2 + \int_0^1 |f'(x)|^2 dx - \lambda^2 \int_0^1 |f(x)|^2 dx &= 0.
\end{align*} \]

Consider the imaginary part of above equality to yield \( f(0) = 0 \). This together with (15) deduces \( f = 0 \) and so \( g = \theta = 0 \). Therefore, there is no eigenvalue of \( A \) which is located on the imaginary axis. The proof is complete. \( \blacksquare \)

We point out that (ii) of Theorem 3.1 is the best result. Precisely, \( e^{At} \) is not exponentially stable. In fact, for any \( \lambda \in \sigma(A) \), solving the eigenvalue problem
\[ A(f, g, \theta) = \lambda(f, g, \theta), \quad (f, g, \theta) \in D(A), \]

one has \( g = \lambda f \) with \( f \neq 0 \) satisfying (14). Solve (14) to give
\[ f(x) = -e^{\lambda x - 2} + e^{-\lambda x}, \]

where \( \lambda \) satisfies
\[ (\lambda + 2k)e^{\lambda} + \lambda e^{-\lambda} = 0. \]  

(16)

From (16), we can find a sequence of eigenvalues \( \lambda_n \) satisfying
\[ \lambda = \lambda_n = i \left( n - \frac{1}{2} \right) \pi + \theta \left( \frac{1}{|n|} \right) \]

as \( |n| \to \infty \), where \( n \)'s are integers, which implies that the system (3) is not exponentially stable.

Remark 3.1. It is noticed that in this paper we are only concerned with the closed-loop system of (1) under the feedback (4). For the closed-loop, we choose the natural state space \( H^1(0, 1) \times L^2(0, 1) \) because it is the energy space for the string vibration. In the energy space, it can be shown easily that the system is not exactly observable. However, for the open-loop system (1), a suitable state space is \( X = L^2(0, 1) \times (H^2(0, 1))^\prime \), where prime denotes duality with respect to the pivot space \( L^2(0, 1) \). In this space, the system can be shown probably to be exactly controllable and observable like Corollary 6.2.7 and Theorem 6.3.2 of Tucsnak and Weiss (submitted for publication). When the system is exactly controllable, observable, stabilizable (there is also one more technical assumption), the general dynamic stabilization was discussed for regular linear systems in Weiss and Curtain (1997).

4. Well-posedness and stability of transformed system

In this section, we consider the system (8) without dynamic equation for \( \theta(t) \) in the space \( \mathbb{H} = H^1 \times L^2 \), which has been determined by the error system (3):
\[ \begin{align*}
\tilde{w}_x(x, t) &= \tilde{w}_x(q, t) = (q + k)e^{\pi t} \\
\tilde{w}_x(0, t) &= \tilde{w}_x(q, t) + k\theta(t) \\
\tilde{w}_x(q, t) &= \tilde{w}_x(0, t) - k\theta(t) \\
\tilde{w}_x(q, t) &= -c_2 \tilde{w}_x(1, t) \\
\tilde{w}(x, 0) &= \tilde{w}_x(q, 0) = \tilde{w}_x(1, 0) = \tilde{w}_x(0, t) = \tilde{w}_x(q, t) = \tilde{w}_x(1, t) = 0.
\end{align*} \]  

(17)

The norm of \( \mathbb{H} \) is induced by the inner product
\[ \| (p, q) \|_{\mathbb{H}} = \int_0^1 \sqrt{[p'(x)]^2 + |q(x)|^2} dx + c_1 |p(0)|^2, \]
\[ \forall (p, q) \in \mathbb{H}. \]

Define the operator \( A : D(A) \subset H \to \mathbb{H} \) as follows:
\[ \begin{align*}
A(f, g) &= (f, g''), \quad \forall (f, g) \in D(A), \\
D(A) &= \{(f, g) \in \mathbb{H} | Af, g) \in H, \}
\end{align*} \]  

(18)

It is well-known that \( A \) generates a \( C_0 \)-semigroup of contractions \( e^{At} \) on \( H \) (Guo & Yu, 2001). It is readily found that
\[ \begin{align*}
A^\ast (\phi, \psi) &= (-\psi, -\phi), \quad \forall (\phi, \psi) \in D(A^\ast), \\
D(A^\ast) &= \{(\phi, \psi) \in \mathbb{H} | A^\ast(\phi, \psi) \in H, \}
\end{align*} \]  

(19)

Take the inner product of \( (\phi, \psi) \in D(A^\ast) \) with (17) to obtain
\[ \begin{align*}
\frac{d}{dt} \left( \begin{pmatrix} \tilde{w} \\ \tilde{w}_x \end{pmatrix}, (\phi, \psi) \right) &= \left( \begin{pmatrix} \tilde{w} \\ \tilde{w}_x \end{pmatrix}, A^\ast (\phi, \psi) \right) \\
&+ \left( \begin{pmatrix} 0 \\ \tilde{f}(x, t) \end{pmatrix}, (\phi, \psi) \right) \\
&+ \left( \begin{pmatrix} 0 \\ -\delta(x) \end{pmatrix}, \tilde{g}(t), (\phi, \psi) \right),
\end{align*} \]

where
\[ \tilde{f}(x, t) = (c_1 + q)e^{\pi t}(q + k)\epsilon(0, t) + k\theta(t), \]
\[ \tilde{g}(t) = -(q + k)\epsilon(0, t) = -k\theta(t), \]  

(20)
and \( \delta(\cdot) \) denotes the Dirac distribution. Hence (17) is equivalent to

\[
\begin{align*}
\frac{d}{dt} \tilde{w}_1(t) &= A \begin{pmatrix} \tilde{w}(t) \\ \tilde{u}_1(t) \end{pmatrix} + \begin{pmatrix} f(t) \\ \tilde{f}(t) \end{pmatrix} + B \tilde{g}(t), \\
\tilde{w}(0) &= \tilde{w}_0, \\
\tilde{u}_1(0) &= 0
\end{align*}
\]

(21)
or in operator form:

\[
\begin{align*}
\frac{d}{dt} \tilde{w}_1(t) &= A \tilde{w}_1(t) + \int_0^t \tilde{f}(\tau, t) \, d\tau + B \tilde{g}(t),
\end{align*}
\]

(22)

where \( B = (0, -\delta(x))^T \).

\textbf{Theorem 4.1.} For each initial datum \((\tilde{w}_0, \tilde{w}_1) \in H\), there exists a unique solution \((\tilde{w}, \tilde{u}_1) \in C(0, \infty; H)\) to (17), and for all \( T > 0 \), there exists a \( D_T > 0 \) depending on \( T \) only such that

\[
\| (\tilde{w}(\cdot, T), \tilde{u}_1(\cdot, T)) \|_H^2 \leq D_T \left\{ \| (\tilde{w}_0, \tilde{w}_1) \|_H^2 + \int_0^T \| \tilde{g}(\tau) \|^2_{L^2(0,1)} \, d\tau \right\}.
\]

Moreover, for each \((\tilde{w}_0, \tilde{w}_1) \in D(A)\) and \((\varepsilon_0, \varepsilon_1, \theta_0) \in D(A)\) with \( \theta_0 = \frac{1}{\varepsilon_0} \varepsilon_0(0) \), there exists a classical solution \((\tilde{w}, \tilde{u}_1) \in C^1(0, \infty; H)\) to (17).

\textbf{Proof.} We first prove the first part. By the well-posed linear infinite-dimensional system theory, it suffices to show that \( B^* \) is admissible for \( A^{*T} \) (see e.g., Definition 2.1 of Guo and Zhang (2005) or Theorem 6.9 of Weiss (1989)). This is equivalent to saying that \( B^*A^{*T} \) is bounded and for any \( T > 0 \), there exists a \( M_T > 0 \) depending on \( T \) only such that the system of the following:

\[
\begin{align*}
\frac{d}{dt} \tilde{u}_1(t) &= c_1 \tilde{w}_1(t), \\
\tilde{w}_1(0) &= 0, \\
\tilde{u}_1(t) &= c_2 \tilde{w}_1(t), \quad t \geq 0,
\end{align*}
\]

(23)
satisfies

\[
\int_0^T |\tilde{u}_1(0, t)|^2 \, dt \leq M_T E_{\tilde{u}}(0),
\]

where

\[
E_{\tilde{u}}(t) = \frac{1}{2} \int_0^t \left[ |\tilde{w}_1(x, t)|^2 + |\tilde{u}_1(x, t)|^2 \right] \, dx + \frac{c_1}{2} |\tilde{w}_1(0, t)|^2.
\]

First, a simple computation shows that

\[
A^{-1}(\phi, \psi) = \left( -c_2 \phi(1) \left( x - \frac{1}{c_1} \right) + \frac{1}{c_1} \int_0^1 \psi(\tau) \, d\tau \right),
\]

\[
B^*A^{-1}(\phi, \psi) = (0, \phi(0)), \quad (\phi, \psi) \in H.
\]

Hence \( B^*A^{-1} \) is bounded on \( H \). Secondly, differentiate \( E_{\tilde{u}}(t) \) with respect to \( t \) along the solution of (23) to give

\[
E_{\tilde{u}}(t) = -c_2 |\tilde{w}_1(1, t)|^2 \leq 0
\]

and hence, \( E_{\tilde{u}}(T) \leq E_{\tilde{u}}(0) \) for any \( T > 0 \). Next, let

\[
\rho(t) = \text{Re} \int_0^t (x - 1) \tilde{w}_1(x, t) \tilde{u}_1(x, t) \, dx.
\]

Then, \( |\rho(t)| \leq E_{\tilde{u}}(t) \). Differentiate \( \rho(t) \) to give

\[
\int_0^T |\tilde{u}_1(0, t)|^2 \, dt \leq (4 + 2T) E_{\tilde{u}}(0), \quad \forall \, T > 0.
\]

The first part is proved. Now we show the classical solution. Since \((\varepsilon_0, \varepsilon_1, \theta_0) \in D(A)\), the functions in (20) satisfy

\[
\tilde{f}(\cdot, t) \in C^1(0, \infty; L^2(0,1)), \quad \tilde{g}(t) \in C^1(0, \infty).
\]

Denote \( \tilde{W}(\cdot, t) = (\tilde{w}(\cdot, t), \tilde{w}_1(\cdot, t))^T \) and \( F(\cdot, s) = (0, \tilde{f}(\cdot, s))^T \). Then the solution of (22) can be written as

\[
\tilde{W}(\cdot, t) = e^{A^T} \tilde{W}(\cdot, 0) + \int_0^t e^{A^T} F(s, \cdot) \, ds + \int_0^t e^{A^T} B \tilde{g}(s) \, ds.
\]

(24)

For \( \tilde{W}(0, \cdot) = (\tilde{w}(0, \cdot), \tilde{w}_1(0, \cdot))^T \in D(A) \), it follows from (25) that

\[
\tilde{W}(\cdot, t) = e^{A^T} \tilde{W}(\cdot, 0) - \int_0^t \frac{d}{ds} \left[ e^{A^T} - I \right] A^{-1} F(s, \cdot) \, ds
\]

\[
+ \int_0^t e^{A^T} F(s, \cdot) - A^{-1} B \tilde{g}(s) \, ds
\]

\[
+ e^{A^T} A^{-1} B \tilde{g}(0) + A^{-1} \int_0^t e^{A^T} B \tilde{g}(s) \, ds.
\]

(25)

By the fact \( A^{-1} B \) is bounded and (24), the right-hand side of (26) makes sense. Notice that under the assumption \( \theta_0 = \frac{1}{\varepsilon_0} \varepsilon_0(0) \), \( \tilde{g}(0) = -(q + k) \varepsilon_0(0) - k \theta_0 = 0 \). Thus (26) becomes

\[
\tilde{W}(\cdot, t) = e^{A^T} \tilde{W}(\cdot, 0) - A^{-1} F(\cdot, t)
\]

\[
+ e^{A^T} A^{-1} F(\cdot, 0) - A^{-1} B \tilde{g}(t)
\]

\[
+ e^{A^T} A^{-1} B \tilde{g}(0) + A^{-1} \int_0^t e^{A^T} B \tilde{g}(s) \, ds.
\]

(27)

Differentiate (27) with respect to \( t \) to give

\[
\frac{d}{dt} \tilde{W}(\cdot, t) = A e^{A^T} \tilde{W}(\cdot, 0) + e^{A^T} F(\cdot, 0)
\]

\[
+ \int_0^t e^{A^T} F(s, \cdot) \, ds + \int_0^t e^{A^T} B \tilde{g}(s) \, ds.
\]

(28)

In view of (24) and the same arguments as the first part, we have \( \tilde{W}(\cdot, t) \in H \), which implies that (25) is a classical solution. The result is thus proved.

\textbf{Theorem 4.2.} The transformed system (17) is asymptotically stable, that is, for any \((\tilde{w}_0, \tilde{w}_1) \in H\), the (weak) solution of (17) justified by Theorem 4.1 satisfies

\[
\lim_{t \to \infty} E_{\tilde{u}}(t) = \lim_{t \to \infty} E_{\tilde{u}}(0) = 0.
\]

\textbf{Proof.} We first assume that

\[
(\varepsilon_0, \varepsilon_1, \theta_0) \in D(A), \quad (\tilde{w}_0, \tilde{w}_1) \in D(A),
\]

\[
\theta_0 = \frac{q}{k} \varepsilon_0(0).
\]

(29)
With (29), $\theta$ in (17) is uniquely determined by $\epsilon$. Now, Theorem 4.1 assures that the classical solution exists. For brevity in notation, we only consider the real solution since any complex solution can always be composed of two real parts with each real part satisfying the same equation. Let

$$
\rho_\omega(t) = \int_0^1 (1 + x) \bar{\omega}_\epsilon(x, t) \bar{\omega}_\epsilon(x, t) dx.
$$

Then there exists a $M_1 > 0$ such that

$$
|\rho_\omega(t)| \leq M_1 E_\omega(t). \tag{30}
$$

The time derivatives along the solution of $E_\omega$ and $\rho_\omega(t)$ of (17) are, respectively

$$
\dot{E}_\omega(t) = \int_0^1 \left[ \ddot{\omega}_\epsilon(x, t) \bar{\omega}_\epsilon(x, t) + \ddot{\omega}_\epsilon(x, t) \bar{\omega}_\epsilon(x, t) \right] dx
$$

and

$$
\dot{\rho}_\omega(t) = \int_0^1 (1 + x) \ddot{\omega}_\epsilon(x, t) \bar{\omega}_\epsilon(x, t) dx + \int_0^1 (1 + x) \ddot{\omega}_\epsilon(x, t) \bar{\omega}_\epsilon(x, t) dx
$$

Let

$$
F_\epsilon(t) = E_\omega(t) + \delta \rho_\omega(t).
$$

Then,

$$
\dot{F}_\epsilon(t) = \dot{E}_\omega(t) + \delta \dot{\rho}_\omega(t)
$$

Performing two completions of squares further gives

$$
\dot{F}_\epsilon(t) = \frac{\delta}{4} \int_0^1 \left[ \dddot{\omega}_\epsilon(x, t) x + \dddot{\omega}_\epsilon(x, t) x \right] dx
$$

where

$$
A = \frac{\delta}{2} + \frac{1}{\delta} + (c_1 + q)^2 \int_0^1 \left[ (1 + x)^2 + \frac{1}{\delta} \right] e^{2\delta x} dx.
$$

In view of (30), we have

$$
(1 - M_1 \delta) E_\omega(t) \leq F_\epsilon(t) \leq (1 + M_1 \delta) E_\omega(t). \tag{32}
$$

Take $\delta \leq \min \left\{ \frac{c_1}{1 + c_2^2}, \frac{1}{4k} \right\}$ to get

$$
\dot{F}_\epsilon(t) \leq -\frac{\delta}{2} E_\omega(t) + A [\theta(0, t) - \hat{\theta}(t)]^2
$$

where

$$
\hat{\theta}(t) = \frac{\delta}{4} \int_0^1 \ddot{\omega}_\epsilon(x, t) - (2c_1 \dot{\omega}_\epsilon(0, t) - \hat{\theta}(t))^2
$$

and

$$
\hat{\theta}(t) = -\frac{\delta}{4} \int_0^1 [\dddot{\omega}_\epsilon(x, t) - 2(\dot{c}_1 \dot{\omega}_\epsilon(0, t) - \hat{\theta}(t))]^2
$$

Now, by virtue of (32), we have

$$
\dot{F}_\epsilon(t) \leq -\frac{\delta}{2} E_\omega(t) + 2q^2 M_2 \hat{\dot{\theta}}^2(0, t) + 2 M_2 \hat{\dot{\theta}}^2(t). \tag{33}
$$

Let $\mu = \frac{\delta}{2 \pi + \delta}$ and apply the Gronwall inequality to (33) to conclude

$$
F_\epsilon(t) \leq e^{-\mu t} F_\epsilon(0) + 2M_2 \int_0^t e^{-\mu(t-r)} \hat{\dot{\theta}}^2(r) dr + 2q^2 M_2 \int_0^t e^{-\mu(t-r)} \hat{\dot{\theta}}^2(0, r) dr. \tag{35}
$$

We first show that

$$
\int_0^t e^{-\mu(t-r)} \hat{\dot{\theta}}^2(r) dr \to 0 \quad \text{as} \quad t \to \infty.
$$

In fact,

$$
\int_0^t e^{-\mu(t-r)} \hat{\dot{\theta}}^2(r) dr = \int_0^t \left[ e^{-\mu(t-r)} \hat{\dot{\theta}}^2(r) + \int_r^t e^{\mu(s-r)} \hat{\dot{\theta}}^2(s) ds \right] dr
$$

$$
\leq e^{-\mu t} \int_0^t \hat{\dot{\theta}}^2(r) dr + \max \int\frac{\mu}{\mu - \mu(t-r)} \hat{\dot{\theta}}^2(r) dr
$$

Define the Lyapunov function for the system (3):

$$
V_\epsilon(t) = \frac{1}{2} \int_0^1 [\epsilon_\epsilon^2(x, t) + \epsilon_\epsilon^2(x, t)] dx
$$

$$
+ \frac{k}{2} \epsilon(0, t) + \theta(t))^2. \tag{37}
$$
The time derivative of $V_e(t)$ along the solution of (3) is
\[ V_e(t) = \varepsilon_t (1, t) \varepsilon_s (1, t) - \varepsilon_t (0, t) \varepsilon_s (0, t) + k [\varepsilon_s (0, t) + \varepsilon_t (t)] [\varepsilon_t (0, t) + \dot{\theta} (t)] = k [\varepsilon_s (0, t) + \varepsilon_t (t)] [\dot{\varepsilon}_t (0, t)] = -\dot{\varepsilon}_s^2 (t) \leq 0. \] (38)

Hence
\[ V_e (t) \leq V_e (0), \quad \forall \, t \geq 0 \quad \text{and} \quad \int_0^\infty \dot{V}_e (\tau) d \tau \leq V_e (0) < \infty. \] (39)

This together with (36) gives
\[ \int_0^t e^{-\mu (t-\tau)} \dot{V}_e (\tau) d \tau \to 0 \quad \text{as} \quad t \to \infty. \] (40)

Next, we show that
\[ \int_0^\infty e^{-\mu (t-\tau)} e^2 (0, \tau) d \tau \to 0 \quad \text{as} \quad t \to \infty. \]

By the Poincaré inequality and Theorem 3.1,
\[ e^2 (0, t) \leq 4 \int_0^1 e^2 (x, t) dx \leq 8 V_e (t) \to 0 \quad \text{as} \quad t \to \infty. \]

Given $\eta > 0$, we choose $t_0$ such that for $t > t_0$, $e^2 (0, t) < \frac{4 \eta}{2}$. Then
\[ \begin{aligned}
\int_0^t e^{-\mu (t-\tau)} e^2 (0, \tau) d \tau &\leq \int_0^1 e^{-\mu (t-\tau)} e^2 (0, \tau) d \tau \\
&\leq \int_0^t e^{-\mu (t-\tau)} e^2 (0, \tau) d \tau + \int_0^\infty e^{-\mu (t-\tau)} e^2 (0, \tau) d \tau \\
&\leq V_e (0) \mu^{-1} e^{-\mu (t-t_0)} + \frac{\eta}{2}.
\end{aligned} \]

Choosing $t > t_0$ large enough, the first term on the righthand side above will be less than $\frac{\eta}{2}$ and thus for $t$ large enough
\[ \int_0^t e^{-\mu (t-\tau)} e^2 (0, \tau) d \tau \to 0 \quad \text{as} \quad t \to \infty. \] (41)

Combining (35), (41) and (40), we have limit $F_e (0) = 0$. This together with (32) gives
\[ \lim_{t \to \infty} E_s (t) = 0. \]

Finally since $D(A)$ is dense in $H$ and $D(A)$ is dense in $H$, for any $(\tilde{w}_0, \tilde{w}_1) \in H$ and $(\varepsilon_0, \varepsilon_1) \in H^2_k (0, 1) \times L^2 (0, 1)$. We take $(\tilde{w}_0, \tilde{w}_1) \in D(A)$ and $(\varepsilon_0, \varepsilon_1, \theta_0) \in D(A)$ such that
\[ (\tilde{w}_0, \tilde{w}_1) \to (\tilde{w}_0, \tilde{w}_1) \quad \text{in} \quad H, \]
\[ (\varepsilon_0, \varepsilon_1, \theta_0) \to (\varepsilon_0, \varepsilon_1, \theta_0) \quad \text{in} \quad H, \quad \theta_0 = \frac{-q - k}{k} \varepsilon_0 (0). \]

The result then follows from the density argument and the conclusion just justified for the classical solution. \( \blacksquare \)

5. Well-posedness and asymptotic stability of closed-loop system

We go back to the closed-loop system of (1) under the feedback (4):
\[ \begin{aligned}
\frac{w_{\hat{u}} (x, t) - w_{ax} (x, t)}{w (1, t) = \hat{w} (1, t),} \\
\frac{w (0, t) = -w (0, t)}{w_{\hat{u}} (x, x) = \hat{w}_{ax} (x, t),} \\
\frac{\hat{w}_{ax} (x, x) = 0}{\hat{w}_{ax} (0, t) = -w (0, t) - k [w (0, t) - \hat{w} (0, t) + \theta (t)],} \\
\frac{\hat{w}_{ax} (1, t) = -c_2 \hat{w}_{ax} (1, t) - (c_1 + q) \hat{w} (1, t)}{\hat{w} (x, x) = \hat{w} (x, 0),} \\
\frac{w_t (x, t) = w_t (x, 0),}{} \\
\frac{w (x, 0) = w_0 (x),}{} \\
\frac{w_t (x, 0) = w_t (x, 0),}{} \\
\frac{\hat{w} (0, 0) = \hat{w} (0, 0)}{} \\
\frac{\theta (x, t) = \theta (x, 0),}{}
\end{aligned} \] (42)

We consider the system (42) in the energy state space $X = \{ (\xi, \eta, \phi, \psi, \theta) \in (H^2 (0, 1) \times L^2 (0, 1))^2 \times \mathbb{R} | f (1) = \phi (1) \}$.

**Theorem 5.1.** For any initial value $(w_0, w_1, \hat{w}_0, \hat{w}_1, \theta_0) \in X$, there exists a unique (weak) solution to (42) such that $(w (x, t), \theta (x, t), \hat{w} (x, t), \hat{w} (t, t), \hat{w} (t, \theta (t))) \in C ([0, \infty) ; X)$. Moreover, the closed-loop solution $(w, \hat{w}, \theta)$ of (42) is asymptotically stable in the sense that
\[ \lim_{t \to \infty} \left[ \int_0^t \left[ |w_s (x, t)|^2 + |w_t (x, t)|^2 + |\hat{w}_{ax} (x, t)|^2 + |\theta (t)|^2 \right] dx + c_1 |\hat{w} (0, t)|^2 + |\theta (t)|^2 \right] = 0. \]

**Proof.** For any initial value $(w_0, w_1, \hat{w}_0, \hat{w}_1, \theta_0) \in X$, from (3) and (9), it is easy to verify that $(\varepsilon_0 (x), \varepsilon_1 (x), \theta) \in H$ and $(\tilde{w}_0 (x), \tilde{w}_1 (x)) \in H$, which imply that there exists a unique solution to (3) and (17), respectively. Let
\[ \begin{aligned}
w (x, t) = \hat{w} (x, t) - (c_1 + q) \int_0^t e^{-c_1 (t-s)} \hat{w} (\xi, \xi) d \xi + \theta (x, t), \\
\hat{w} (x, t) = \hat{w} (x, t) - (c_1 + q) \int_0^t e^{-c_1 (t-s)} \hat{w} (\xi, \xi) d \xi.
\end{aligned} \] (43)

Then a direct computation shows that such a defined $(w, \hat{w})$ satisfies (42) with initial value $(w_0, w_1, \hat{w}_0, \hat{w}_1)$. This solution is unique by the invertible transformation
\[ \left( \begin{array}{c}
\varepsilon \\
\varepsilon_1 \\
\hat{w} \\
\hat{w}_1 \\
\theta
\end{array} \right) = \left( \begin{array}{ccccc}
1 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{array} \right) \left( \begin{array}{c}
w \\
w_t \\
\hat{w} \\
\hat{w}_1 \\
\theta
\end{array} \right) \]

and the uniqueness of solution to (3) and (17). The asymptotic stability follows from (7), Theorems 3.1 and 4.2. \( \blacksquare \)

6. Numerical simulation

In this section, we used the same numerical method of Krstic et al. (2008) to simulate the error system (3) and closed-loop system (42). In order to compare with Figs. 5 and 6 of Krstic et al. (2008) where the velocity measurements were used, we take the same spatial grid size $N = 40$ and time step $dt = 10^{-4}$. The parameter values were set in the same values: $K = q = 1, c_1 = 600, c_2 = 1$. The initial conditions are also the same: $w_0 (x) = x - 1$ for $x \in [0, 1]$, $w_1 (x) = \begin{cases} 
1, & \text{if } 0.45 \leq x \leq 0.55, \\
0, & \text{otherwise},
\end{cases}$ $\hat{w}_0 (x) = \hat{w}_1 (x) = 0, \quad \theta_0 = \frac{-q - k}{k} [w_0 (0) - \hat{w}_0 (0)].$

Fig. 1 shows that the observed error converges well and Fig. 2 shows that the string displacement convergence is a little bit
Fig. 1. Observer error $e$ (denoted by $e$ in plot) in controlled case ($\theta \neq 0$, left) and uncontrolled case ($\theta = 0$, right).

Fig. 2. String response with observed displacement control only (left) and observed displacement and velocity control in Krstic et al. (2008) (right).

Fig. 3. Observer error (left) and string response with controlled end (right).

slower than and not smooth as that in Krstic et al. (2008). With the parameters as before, if we take the smooth initial values

$e_0(x) = e_1(x) = w_0(x) = w_1(x) = x - 1,$
$
\hat{w}_0(x) = \hat{w}_1(x) = 0.$

Fig. 3 shows that the convergence is very good and smooth.

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