Remarks on the application of the Keldysh theorem to the completeness of root subspace of non-self-adjoint operators and comments on “Spectral operators generated by Timoshenko beam model”

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Abstract

In this note, we show, by means of an example, that the assumption of compactness in the Keldysh theorem in proving the completeness of root subspaces of non-self-adjoint operators cannot be ignored. This indicates a gap of the above mentioned paper and many other related papers in the proof of completeness for the root subspaces of beam-related problems.

Keywords: Keldysh theorem; Completeness; Root subspace; Generalized eigenvectors

1. Introduction and the Keldysh theorem

Let $H$ be a Hilbert space and $A$ be a discrete linear operator in $H$ (i.e. there is a $\lambda \in \rho(A)$, the resolvent set of $A$, such that $(\lambda - A)^{-1}$ is compact on $H$). Let us recall that $x \in D(A)$ is called a generalized eigenvector of $A$ associated with eigenvalue $\lambda$ if there is an integer $n \geq 1$ such that $(\lambda - A)^nx = 0$. The root subspace $\text{Sp}(A)$ of $A$ is the closed subspace of $H$ that is spanned by all generalized eigenvectors of $A$.

It is well-known that the most important bases in Hilbert spaces are orthonormal bases, and the second in importance are Riesz bases that are bases equivalent to some orthonormal basis under a bounded invertible transformation. It has been turned out that Riesz bases play a very important role in the analysis of feedback control of flexible structures. A series of studies on this aspect have been carried out in [1,2,5–11,15] and the references therein. In order to get the Riesz basis property of a controlled flexible vibration system, one has to consider the completeness of the root subspace for the associated system operator. In most of applications, the system operators are non-self-adjoint in the state spaces, and hence the first task is to verify the completeness for the root subspace. However, this is not easy in most cases. Some of approaches that have been applied and developed successfully to the completeness are: (a) by means of the classical Bari’s theorem [5]; (b) using the resolvent estimate approach based on the Phragmén–Lindelöf theorem [12,13,15]; and (c) the Green’s function approach [6].

In all these studies, the completeness of the root subspace is a difficult problem leading to the Riesz basis generation of the generalized eigenvectors in the state space. However, we found in many other typical papers such as [8] (see also e.g. [2,7,10,11]) that the completeness of the root subspace becomes a simple consequence of the classical Keldysh theorem.

The aim of this paper is to indicate, using [8] as an example, that there is a gap, represented by paper [8], in the proof of completeness for the root subspaces of the beam-like systems with boundary feedbacks in the application of the Keldysh theorem.

The precise statement of the latest Keldysh theorem that is stated as Theorem 1 below can be found in [3] on p. 170. The original statement of the Keldysh theorem is presented in [4] on p. 257.
**Theorem 1 (Keldysh).** Let $K$ be a compact selfadjoint operator on a Hilbert space $H$ with $\text{Ker}(K) = \{0\}$ and assume that
\[
\sum_{j=1}^{\infty} |\lambda_j(K)|^r < \infty
\]
for some $r \geq 1$, where $\{\lambda_j(K)\}_{j=1}^{\infty}$ are all eigenvalues of $K$. Furthermore, let $S$ be a compact operator so that $I + S$ is invertible. Then the system of generalized eigenvectors of the operator $A = K(I + S)$ is complete in $H : \text{Sp}(A) = H$.

In Section 2, an example will be given to show that the assumption of compactness of $S$ in the Keldysh theorem cannot be ignored. In the last section, we show that there is a gap in paper [8] in proving the completeness of the root subspace of the system operator. Other baffling arguments appeared in [1,9] are also indicated.

2. **Compactness is necessary in the Keldysh theorem**

Consider the following one-dimensional wave equation with a boundary feedback control:
\[
\begin{align*}
\psi_{tt}(x, t) - \psi_{xx}(x, t) &= 0, \quad 0 < x < 1, \quad t > 0, \\
y(0, t) &= 0, \\
y_x(1, t) &= u(t), \\
y_{out}(t) &= y_{1}(t, 1), \\
u(t) &= -ky_{out}(t),
\end{align*}
\]
(2.1)
where $u(t)$ is the boundary control (input), $y(t)$ the boundary observation (output), and $k > 0$ is the feedback gain.

The state space of system (2.1) is taken naturally as the energy space of the following:
\[
\mathcal{H} = H^1_E(0, 1) \times L^2(0, 1),
\]

where $H^1_E(0, 1) = \{f \in H^1(0, 1) | f(0) = 0\}$. The system operator $\mathcal{A}_k$ associated with (2.1) is given by
\[
\mathcal{A}_k(f, g) = (g, f''),
\]
\[
D(\mathcal{A}_k) = \{(f, g) \in \mathcal{H} | f \in H^2(0, 1), g \in H^1_E(0, 1), f'(1) = -kg(1)\}.
\]

Let $\mathcal{A}_0$ be the operator $\mathcal{A}_k$ with $k = 0$. Then it is easy to find that $\mathcal{A}_0$ is a skew-adjoint operator in $\mathcal{H}$ and $\mathcal{A}_k^{-1}$ and $\mathcal{A}_0^{-1}$ can be represented explicitly:
\[
[\mathcal{A}_k^{-1}(f, g)](x) = \left( -\int_0^1 \tau g(\tau) \, d\tau + sk(1)x \right) + \int_1^x (x-\tau)g(\tau) \, d\tau, \quad f(x) \in \mathcal{H}
\]
and
\[
[\mathcal{A}_0^{-1}(f, g)](x) = \left( -\int_0^1 \tau g(\tau) \, d\tau + \int_1^x (x-\tau)g(\tau) \, d\tau, \quad f(x) \right) \forall (f, g) \in \mathcal{H}
\]

Hence, both $\mathcal{A}_k^{-1}$ and $\mathcal{A}_0^{-1}$ are compact on $\mathcal{H}$ and we have
\[
\mathcal{A}_k^{-1} = \mathcal{A}_0^{-1} + \mathcal{P}_k,
\]
where $\mathcal{P}_k$ is a rank-one compact operator on $\mathcal{H}$:
\[
[\mathcal{P}_k(f, g)](x) = (-k f(1)x), \quad x \in (0, 1).
\]

Thus, the above expressions will lead to
\[
\mathcal{A}_k^{-1} = \mathcal{A}_0^{-1}(I + \mathcal{A}_0 \mathcal{P}_k).
\]
It is easily found that $\mathcal{A}_0 \mathcal{P}_k = 0$ on its domain
\[
D(\mathcal{A}_0 \mathcal{P}_k) = \{(f, g) \in \mathcal{H} | f(1) = 0\} \neq \mathcal{H}.
\]

Therefore, $\mathcal{A}_0 \mathcal{P}_k$ is unbounded and hence is not compact in $\mathcal{H}$.

However, it is well-known that when $k = 1$,
\[
\sigma(\mathcal{A}_1) = \sigma_p(\mathcal{A}_1^{-1}) = \emptyset
\]
which says that $\text{Sp}(\mathcal{A}_1) = \text{Sp}(\mathcal{A}_1^{-1})$ is never complete in $\mathcal{H}$.

The failure of above example shows that although all other assumptions of the Keldysh theorem are satisfied with $\sigma(\mathcal{A}_0^{-1}) = \{\pm 1/(n-1)\} \cup H \mathcal{A}_0^{-1}$ and $A = \mathcal{A}_1$, but $S = \mathcal{A}_0 \mathcal{P}_k$ is not compact (even not bounded) on $\mathcal{H}$. It is zero on its domain but not on all $\mathcal{H}$. It should be indicated that in the Keldysh theorem, the compactness of $S$ implies automatically that $S$ is bounded, which implies, in particular that its domain is the whole space: $D(S) = H$ (see p. 1 of [4] or p. 277 of [16]).

Our example shows that even if $D(S)$ is dense in $H$ and $S$ is bounded on $D(S)$, the Keldysh theorem does not hold. An anonymous reviewer of this note indicated that in the book [14], a different definition for bounded operator is introduced.

To sum up, this example tells us that in order to apply the Keldysh theorem, the assumption that $S$ is compact cannot be ignored.

3. **Comments on “Spectral operators generated by Timoshenko beam model”**

In the paper [8], the author considered the Riesz basis generation of a non-self-adjoint operator associated with a Timoshenko beam under boundary feedback controls.

For brevity in notation, we assume that all system constants are identical to 1; that is $\rho = K = R = EI = L = 1$ in the model considered in [8]. Under this assumption, the closed-loop system given by (2.1) and (2.2) of [8] now becomes
\[
\begin{align*}
W_{tt}(x, t) + \Phi(x, t) - W_{xx}(x, t) &= 0, \quad x \in (-1, 0), \\
\Phi_{tt}(x, t) - \Phi_{xx}(x, t) + \Phi(x, t) &= 0, \quad x \in (-1, 0), \\
W(-1, t) &= \Phi(-1, t) = 0, \quad t \geq 0, \\
\Phi(0, t) - W_{x}(0, t) &= xW_{x}(0, t), \\
\Phi_{x}(0, t) &= -\beta \Phi_{x}(0, t),
\end{align*}
\]
(3.1)
where $x$ and $\beta$ are arbitrary complex numbers.
In the state Hilbert space 
\[ H' = (H_E^1(-1, 0) \times L^2(-1, 0))^2, \]
\[ H_E^1(-1, 0) = \{ f \in H^1(-1, 0) \mid f(-1) = 0 \}, \]
the system operator \( \Omega_{z\beta} \) associated with (3.1) is given by (see (2.11) of [8]):
\[
\Omega_{z\beta} = -i \begin{pmatrix}
 0 & 1 & 0 & 0 \\
 \frac{d^2}{dx^2} - 1 & 0 & \frac{d}{dx} & 0 \\
 0 & 0 & 0 & 1 \\
 -\frac{d}{dx} & 0 & \frac{d^2}{dx^2} & 0
\end{pmatrix},
\]
with domain
\[
D(\Omega_{z\beta}) = \begin{cases}
  \Phi = (\varphi_0, \varphi_1, \varphi_2, \varphi_3)^T \in H' \\
  \varphi_0, \varphi_2 \in H^2((-1, 0), \varphi_1, \varphi_3 \in H_E^1(-1, 0), \\
  \varphi_0(0) - \varphi_2(0) = \varphi_3(0), \varphi_0'(0) = -\beta \varphi_1(0)
\end{cases}
\]
It should be noted that our definition \( D(\Omega_{z\beta}) \) is actually the same as (2.12) and its explanation below (2.12) in [8].

The following arguments and Theorems 6.1, 6.2 are quoted from p. 256 of [8].

**Theorem 6.1.** If both \( z \) and \( \beta \) are nonzero, then there exists a rank-two operator \( T_{z\beta} \) on \( H' \) such that
\[
\Omega_{z\beta}^{-1} = \Omega_{00}^{-1} + T_{z\beta}. \tag{6.1}
\]
If \( z = 0 \) or \( \beta = 0 \), then there exists a rank-one operator \( T_{z\beta} \) in \( H' \) such that (6.1) is valid.

Theorem 6.1 means that for any \( z \) and \( \beta \), the operator \( \Omega_{z\beta} \)
is a finite-dimensional perturbation of the self-adjoint operator \( \Omega_{00} \). Combining Theorem 6.1 with the well-known Keldysh theorem (see [4, p. 257]), we can prove the main result of this section.

**Theorem 6.2.** The set of the generalized eigenvectors of the operator \( \Omega_{z\beta} \) is complete in the energy space \( H' \).

**Claim.** When \(|z|^2 + |\beta|^2 \neq 0\), there is a gap between Theorems 6.1 and 6.2 in the application of the Keldysh theorem. Indeed, from Theorem 6.1, it follows that
\[
\Omega_{z\beta}^{-1} = \Omega_{00}^{-1} + T_{z\beta} = \Omega_{00}^{-1}(I + \Omega_{00}T_{z\beta}),
\]
but the domain of the operator \( \Omega_{00}T_{z\beta} \) is not the whole space and hence is unbounded in \( H' \). Therefore, the Keldysh theorem cannot be applied directly to get the result of Theorem 6.2.

**Explanation of the Claim:** First, we compute directly \( \Omega_{z\beta}^{-1} \). For any \( G := (g_0, g_1, g_2, g_3)^T \in H' \), find \( F := (f_0, f_1, f_2, f_3)^T \in D(\Omega_{z\beta}) \) such that
\[
\Omega_{z\beta} F = G;
\]
that is, \( f_1(x) = ig_0(x), f_3(x) = ig_2(x) \) with \( f_0, f_2 \) satisfying
\[
\begin{align*}
  f_0''(x) - f_0(x) + f_2''(x) &= ig_1(x), \\
  f_0(-1) &= 0, \quad f_0'(0) = -i\beta g_0(0)
\end{align*}
\tag{3.2}
\]
and
\[
\begin{align*}
  -f_0''(x) + f_2''(x) &= ig_3(x), \\
  f_2(-1) &= 0, \quad f_2(0) - f_2'(0) = izg_2(0).
\end{align*}
\tag{3.3}
\]
Integrate by parts to give the solutions to (3.2) and (3.3), respectively
\[
f_0(x) = \tilde{f}_0(x) + \frac{i}{2} zg_2(0)(x^2 - 1) - i\beta g_0(0)(x + 1) \tag{3.4}
\]
and
\[
f_2(x) = \tilde{f}_2(x) + \frac{i}{6} zg_2(0)(x^3 - 3x - 2) - \frac{i}{2} \beta g_0(0)(x + 1)^2 - izg_2(0)(x + 1) \tag{3.5}
\]
where
\[
\tilde{f}_0(x) := i \int_0^x (x + 1)g_1(\tau) d\tau - i \int_0^x (\tau + 1)g_1(\tau) d\tau
\]
\[
- i \int_{-1}^x \left( \frac{1}{2} \tau^2 + \tau + \frac{1}{2} \right) g_3(\tau) d\tau
\]
\[
- i \int_0^x \left[ \frac{1}{2} \tau^2 - \frac{1}{2} - \tau(x + 1) \right] g_3(\tau) d\tau \tag{3.6}
\]
and
\[
\tilde{f}_2(x) := \int_{-1}^x \tilde{f}_0(\tau) d\tau - i \int_{-1}^x (\tau + 1)g_3(\tau) d\tau
\]
\[
+ i \int_0^x (x + 1)g_3(\tau) d\tau. \tag{3.7}
\]
Therefore, for each \( G := (g_0, g_1, g_2, g_3)^T \in H' \), the decomposition (6.1) of [8] is valid with
\[
\Omega_{00}^{-1} G = (\tilde{f}_0(x), ig_0(x), \tilde{f}_2(x), ig_2(x))^T \tag{3.8}
\]
and
\[
T_{z\beta} G = \begin{pmatrix}
 \frac{i}{2} zg_2(0)(x^2 - 1) - i\beta g_0(0)(x + 1), & 0 \\
 \frac{i}{6} zg_2(0)(x^3 - 3x - 2) - \frac{i}{2} \beta g_0(0)(x + 1)^2 \\
 -izg_2(0)(x + 1), & 0
\end{pmatrix}^T. \tag{3.9}
\]
Here, \( \tilde{f}_0(x) \) and \( \tilde{f}_2(x) \) are given by (3.6) and (3.7), respectively. Moreover, \( T_{z\beta} \) is at most a rank-two compact operator on \( H' \) claimed by Theorem 6.1.

However, in order to apply the Keldysh theorem to obtain the completeness of the root subspace of \( \Omega_{z\beta} \) (equivalently \( \Omega_{z\beta}^{-1} \)),
we first need to decompose
\[ Q^{-1}_{2\beta} = \Omega^{-1}_{00}(I + \Omega_{00}T_{2\beta}) \]
and then to show at least that \( S = \Omega_{00}T_{2\beta} \) is compact on \( \mathcal{H} \).

For the above example, there are two cases:

Case 1: \( \beta \neq 0 \). In this case, it follows from (3.9) that the necessary condition for \( G = (g_0, g_1, g_2, g_3)^\top \in D(\Omega_{00}T_{2\beta}) \) is that \( g_0(0) = 0 \). Hence \( D(\Omega_{00}T_{2\beta}) \neq \mathcal{H} \), that is, \( \Omega_{00}T_{2\beta} \) is unbounded in \( \mathcal{H} \).

Case 2: \( \beta = 0, \alpha \neq 0 \). In this case, it follows from (3.9) that the necessary condition for \( G = (g_0, g_1, g_2, g_3)^\top \in D(\Omega_{00}T_{2\beta}) \) is that \( g_3(0) = 0 \). Again \( D(\Omega_{00}T_{2\beta}) \neq \mathcal{H} \) and hence \( \Omega_{00}T_{2\beta} \) is unbounded in \( \mathcal{H} \).

It is seen that in both cases, \( \Omega_{00}T_{2\beta} \) is unbounded in \( \mathcal{H} \). In other words, the Keldysh theorem cannot be applied directly to the operator \( \Omega_{2\beta} \) to get Theorem 6.2 claimed in the paper [8].


Other baffling arguments for the completeness of root subspace can be found in [1]. Let us copy precisely the authors’ statements about their arguments in [1] on p. 355.

**Theorem 7.1.** If the parameters of the problem satisfy (14), then the set of the root vectors of the operator \( \mathcal{H}_{\lambda 0} \) is complete in the energy space \( \mathcal{H} \).

**Proof.** As follows from Theorem 4.2, the operator \( \mathcal{H}_{\lambda 0} \) has purely discrete spectrum. To prove that the set of root vectors is complete, it suffices to show that

\[ \text{Ker}(\mathcal{H}_{\lambda 0}^*) = \{0\}. \tag{135} \]

Indeed, due to the fact that for the energy space \( \mathcal{H} \), the following decomposition is valid:

\[ \mathcal{H} = \overline{R(\mathcal{H}_{\lambda 0}) \oplus \mathcal{N}^*(\mathcal{H}_{\lambda 0})}, \]

where \( R(\mathcal{H}_{\lambda 0}) \) is the range of the operator \( \mathcal{H}_{\lambda 0} \) and \( \mathcal{N}^*(\mathcal{H}_{\lambda 0}) \) is the null-space of the operator \( \mathcal{H}_{\lambda 0}^* \), we obtain that (135) will lead to the fact that \( \mathcal{H} = \overline{R(\mathcal{H}_{\lambda 0})} \). Let us show that (135) is valid, i.e. that the equation

\[ \mathcal{R}_{\lambda 0}^* \psi = 0 \]

has only a trivial solution. ∎

The same arguments occurred again in Theorem 5.1 of [9] on p. 1604.

**References**


