GLOBAL AND BLOW-UP SOLUTIONS FOR NONLINEAR PARABOLIC EQUATIONS WITH A GRADIENT TERM

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ABSTRACT. In this paper, we are concerned with the following nonlinear parabolic equation with a gradient term and Neumann boundary condition:

\[
\begin{aligned}
(b(u))t &= \nabla \cdot (a(u)\nabla u) + f(x, u, |\nabla u|^2, t) \quad \text{in } D \times (0, T), \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial D \times (0, T), \\
u(x, 0) &= u_0(x) > 0 \quad \text{in } \overline{D},
\end{aligned}
\]

where \(D \subset \mathbb{R}^N (N \geq 2)\) is a bounded domain of \(\mathbb{R}^N\) with smooth boundary \(\partial D\). The upper and lower solution technique is adopted in investigations. The sufficient conditions for the existence of global positive solution and an upper estimate of global solution are given. Moreover, under some appropriate assumptions on the functions \(a, b, \) and \(f\), we prove the existence of blow-up positive solution. An upper bound of “blow-up time” is also presented.

1. INTRODUCTION

Consider the following Neumann initial-boundary value problem of nonlinear parabolic equation with a gradient term:

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\begin{align}
&\begin{cases}
  \left( b(u) \right)_t = \nabla \cdot (a(u) \nabla u) + f(x, u, |\nabla u|^2, t) & \text{in } D \times (0, T), \\
  \frac{\partial u}{\partial n} = 0 & \text{on } \partial D \times (0, T), \\
  u(x, 0) = u_0(x) > 0 & \text{in } \overline{D},
\end{cases}
\end{align}

where \( D \subset \mathbb{R}^N (N \geq 2) \) is a bounded domain of \( \mathbb{R}^N \) with smooth boundary \( \partial D \), \( \partial / \partial n \) represents the outward normal derivative on \( \partial D \), \( u_0(x) \) is the initial value, \( T \) is the maximum existence time of \( u(x, t) \), and \( \overline{D} \) is the closure of \( D \). Set \( \mathbb{R}^+ = (0, +\infty) \). We assume, throughout the paper, that \( a(s) \) is a positive \( C^2(\mathbb{R}^+) \) function, \( b(s) \) is a \( C^2(\mathbb{R}^+) \) function; \( b'(s) > 0 \) for any \( s > 0 \), and \( f(x, s, d, t) \) is a positive \( C^1(\overline{D} \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+) \) function. Under these assumptions, the classical parabolic equation theory ([1, Section 3]) assures that there exists a unique classical solution \( u(x, t) \) for the equation (1.1) with some \( T > 0 \) and the solution is positive over \( \overline{D} \times [0, T) \). Moreover, by regularity theorem ([17, Chapter 3]), it follows that \( u(x, t) \in C^3(D \times (0, T)) \cap C^2(\overline{D} \times [0, T)) \).

The problem (1.1) describes many physical phenomena in mechanics, physics and biology, etc. Some typical examples are gas flow in porous media, semiconductor, and the spread of biology populations. We refer to [6, 10, 13] and the reference therein for many other applications. In this paper, several issues are addressed under suitable assumptions on the functions \( a, b, \) and \( f \): The existence of global positive solution; upper estimate of global positive solution; the existence of blow-up positive solution; and upper bound of “blow-up time”. Our approach relies heavily upon the upper and lower solution techniques, the construction of the auxiliary functions, and the Hopf maximum principles.

The problems of global and blow up solution for nonlinear parabolic equations have been investigated extensively by many authors. Souplet et al. [14] dealt with the global and blow-up solutions of initial value problems for parabolic equations with a gradient term. Chen et al. [2, 3, 4], Chipot et al. [5], Fila [8], and Souplet et al. [13, 15, 16] studied the existence of global and blow-up solution for the parabolic equations with a gradient term and Dirichlet initial-boundary value problems. Some special cases of (1.1) were also treated. Lair and Oxley [11] studied the problem following:

\begin{align}
&\begin{cases}
  u_t = \nabla \cdot (a(u) \nabla u) + f(u) & \text{in } D \times (0, T), \\
  \frac{\partial u}{\partial n} = 0 & \text{on } \partial D \times (0, T), \\
  u(x, 0) = u_0(x) > 0 & \text{in } \overline{D},
\end{cases}
\end{align}
where $D$ is a bounded domain of $\mathbb{R}^N$ with smooth boundary. The necessary and sufficient conditions characterized by functions $a$ and $f$ were given for the existence of global and blow-up solutions. Imai and Mochizuki [9] considered the following problem:

\[
\begin{aligned}
\left\{
\begin{array}{ll}
(b(u))_t = \Delta u + f(u) & \text{in } D \times (0, T), \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial D \times (0, T), \\
u(x, 0) = u_0(x) > 0 & \text{in } D,
\end{array}
\right.
\end{aligned}
\]

where $D$ is a bounded domain of $\mathbb{R}^N$ with smooth boundary. The sufficient conditions were developed for the existence of global and blow-up solutions. Ding [7] discussed the problem of the following:

\[
\begin{aligned}
\left\{
\begin{array}{ll}
u_t = \Delta u + f(x, u, |\nabla u|^2, t) & \text{in } D \times (0, T), \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial D \times (0, T), \\
u(x, 0) = u_0(x) > 0 & \text{in } D,
\end{array}
\right.
\end{aligned}
\]

where $D$ is a bounded domain of $\mathbb{R}^N$ with smooth boundary. The sufficient conditions were obtained there for the existence of a blow-up solution. Meanwhile, the upper estimate of “blow-up time” as well as upper estimate of “blow-up rate” were also given. This paper is an extension of the results of [11].

We proceed as follows. In next section, Section 2, we state the main results on the existence of global and blow up solutions for (1.1). Several examples are presented to demonstrate the applications. Section 3 is devoted to the proof of the main results.

2. MAIN RESULTS

Our first result Theorem 2.1 is about the existence of global solution.

**Theorem 2.1.** Let $u$ be a solution of (1.1). Assume that the following conditions (i)-(iii) are fulfilled.

(i). For any $s \in \mathbb{R}^+$,

\[
a(s) > 0, \ b'(s) > 0, \ a(s) + a'(s) \geq 0, \ \left(\frac{a(s)}{b'(s)}\right) \leq 0.
\]
(ii). For any \((x, s, d, t) \in D \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+\),

\[(2.2)\]
\[
f(x, s, d, t) > 0, \ f_s(x, s, d, t) + f(x, s, d, t) \leq 0, \ f_d(x, s, d, t) \geq \frac{a(s)}{2}, \]
\[
f_t(x, s, d, t) \leq 0.\]

(iii).

\[(2.3)\]
\[
\int_{M_0}^{+\infty} \frac{a(s)}{e^{-s}} \, ds = +\infty, \ M_0 = \max_{\overline{D}} u_0(x).\]

Then the solution \(u\) of \((1.1)\) must be a global solution and

\[
u(x, t) \leq H^{-1}(\overline{B}t), \ \forall \ x \in \overline{D}, \ t \geq 0,\]

where

\[(2.4)\]
\[
\overline{B} = \frac{a(M_0)e^{M_0}}{b'(M_0)} \max_{\overline{D}} f(x, M_0, 0, 0), \ H(z) = \int_{M_0}^{z} \frac{a(s)}{e^{-s}} \, ds, \ \forall \ z \geq M_0,\]

and \(H^{-1}\) is the inverse function of \(H\).

The result on blow-up solution is stated as Theorem 2.2 below.

**Theorem 2.2.** Let \(u\) be the solution of \((1.1)\). Assume that the following conditions are satisfied.

(i). For any \(s \in \mathbb{R}^+,\)

\[(2.5)\]
\[
a(s) > 0, \ b'(s) > 0, \ a(s) - a'(s) \geq 0, \ \left(\frac{a(s)}{b'(s)}\right)' \geq 0.\]

(ii). For any \((x, s, d, t) \in \overline{D} \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+,\)

\[(2.6)\]
\[
f(x, s, d, t) > 0, \ f_s(x, s, d, t) - f(x, s, d, t) \geq 0, \ f_d(x, s, d, t) \geq -\frac{a(s)}{2}, \]
\[
f_t(x, s, d, t) \geq 0.\]

(iii).

\[(2.7)\]
\[
\int_{m_0}^{+\infty} \frac{a(s)}{e^s} \, ds < +\infty, \ m_0 = \min_{\overline{D}} u_0(x).\]

Then \(u\) must blow up in the finite time \(T\), and

\[
T \leq \frac{1}{\overline{B}} \int_{m_0}^{+\infty} \frac{a(s)}{e^s} \, ds,\]

where the positive constants

\[(2.8)\]
\[
\alpha = \frac{a(m_0)}{b'(m_0)} e^{-m_0} \min_{\overline{D}} f(x, m_0, 0, 0).\]
When \( b(u) \equiv u, f(x, u, q, t) \equiv f(u) \) or \( a(u) \equiv 1, f(x, u, q, t) \equiv f(u) \), the conclusions of Theorem 2.2 still hold true but Theorem 2.1 fails. While \( a(u) \equiv 1, b(u) \equiv u \), the conclusions of both Theorems 2.1 and 2.2 are valid. In this sense, our results extend and supplement the results of [11].

In what follows, we present several examples to demonstrate the applications of Theorems 2.1 and 2.2.

**Example 2.1.** Let \( u \) be a solution of the following problem

\[
\begin{aligned}
(e^u)_t &= \nabla \cdot (e^{-u} \nabla u) + e^{-u} \left[ |\nabla u|^2 + \exp \left( -t - \sum_{i=1}^{3} x_i^2 \right) \right] \quad \text{in } D \times (0, T), \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial D \times (0, T), \\
u(x, 0) &= u_0(x) = 1 + \left( 1 - \sum_{i=1}^{3} x_i^2 \right)^2 \quad \text{in } \bar{D},
\end{aligned}
\]

where \( D = \{ x = (x_1, x_2, x_3) \mid \sum_{i=1}^{3} x_i^2 < 1 \} \). We now have

\[
a(u) = e^{-u}, \quad b(u) = e^u, \quad f(x, u, q, t) = e^{-u} \left[ q + \exp \left( -t - \sum_{i=1}^{3} x_i^2 \right) \right],
\]

where \( q = |\nabla u|^2 \). It is easily to check that all assumptions (2.1)-(2.3) are satisfied. By (2.4), it is found that

\[
\beta = \frac{a(M_0) e^{M_0}}{b'(M_0)} \max_D f(x, M_0, 0, 0) = \frac{a(2) e^2}{b'(2)} \max_D f(x, 2, 0, 0) = 2e \max_{\bar{D}} \exp \left( -2 - \sum_{i=1}^{3} x_i^2 \right) = \frac{2}{e^3}.
\]

Therefore, Theorem 2.1 can be applied to get that \( u \) is a global solution and

\[
u(x, t) \leq H^{-1} (\beta t) = \frac{2}{e^3} t + 2.
\]
Example 2.2. Let $u$ be a solution of the problem

$$
\begin{align*}
(e^u)_t &= \nabla \cdot (e^u \nabla u) + \exp \left( u + |\nabla u|^2 + t + \sum_{i=1}^{3} x_i^2 \right) \quad \text{in } D \times (0, T), \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial D \times (0, T), \\
u(x, 0) &= u_0(x) = 1 + \left( 1 - \sum_{i=1}^{3} x_i^2 \right)^2 \quad \text{in } \bar{D},
\end{align*}
$$

where $D = \left\{ x = (x_1, x_2, x_3) \mid \sum_{i=1}^{3} x_i^2 < 1 \right\}$. We now have

$$
a(u) = e^u, \quad b(u) = e^u, \quad f(x, u, q, t) = \exp \left( u + q + t + \sum_{i=1}^{3} x_i^2 \right),
$$

where $q = |\nabla u|^2$. Similar to Example 2.1, we can check easily that all assumptions (2.5)-(2.7) are satisfied. By (2.8), we have

$$
\alpha = \frac{a(m_0)}{b'(m_0)} e^{-m_0} \min_D f(x, m_0, 0, 0) = \frac{a(1)}{b'(1)} e^{-1} \min_D f(x, 1, 0, 0) = 4e^{-\frac{3}{4}} \min_D \exp \left( 1 + \sum_{i=1}^{3} x_i^2 \right) = 4e^{\frac{1}{4}}.
$$

The Theorem 2.2 then can be applied to conclude that $u$ blows up in finite time $T$, and

$$
T \leq \frac{1}{\alpha} \int_{m_0}^{+\infty} \frac{a(s)}{e^s} ds = \frac{1}{4e^{\frac{1}{4}}} \int_{1}^{+\infty} \frac{e^{\frac{1}{4}}}{e^s} ds = \frac{1}{2} e^{-\frac{1}{4}}.
$$

3. Proofs of main results

In this section, we give the proofs of Theorems 2.1 and 2.2.

Proof of Theorem 2.1. Let $\bar{u}$ be a solution of the following problem

$$
\begin{align*}
(b(u))_t &= \nabla \cdot (a(u) \nabla u) + f(x, u, q, t) \quad \text{in } D \times (0, T), \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial D \times (0, T), \\
u(x, 0) &= M_0 \quad \text{in } \bar{D},
\end{align*}
$$

where $q = |\nabla u|^2$. Then $\bar{u}$ is an upper solution of (1.1). We show that $\bar{u}$ must be a global solution. Therefore, the solution $u$ of (1.1) must be a global solution.
Consider the function
\[
(3.2) \quad \bar{\Psi} = -a(\bar{u})\bar{u}_t + \beta e^{-\bar{u}}.
\]
We find that
\[
(3.3) \quad \nabla \bar{\Psi} = -a'\bar{u}_t \nabla \bar{u} - a\bar{u}_t \nabla \bar{u} - \beta e^{-\bar{u}} \nabla \bar{u},
\]
\[
\Delta \bar{\Psi} = -a''\bar{u}_t - 2a'\nabla \bar{u} \cdot \nabla \bar{u} - \bar{u}_t \Delta \bar{u} - a\Delta \bar{u}_t + \beta e^{-\bar{u}} \nabla \bar{u} - \beta e^{-\bar{u}} \Delta \bar{u},
\]
\[
\bar{\Psi}_t = -a'(\bar{u}_t)^2 - a(\bar{u}_t)^2 - \beta e^{-\bar{u}} \bar{u}_t
\]
\[
= -a'(\bar{u}_t)^2 - a \left( a' \Delta \bar{u} + a' \beta e^{-\bar{u}} + \frac{f}{b} \right) - \beta e^{-\bar{u}} \bar{u}_t
\]
\[
= -a'(\bar{u}_t)^2 + \left( \frac{a''}{b^2} - \frac{a'}{b} \right) \bar{u}_t \Delta \bar{u} - \frac{a^2}{b^2} \bar{u}_t \bar{u} + \left( \frac{a'a''}{b^3} - \frac{aa''}{b^2} \right) \bar{u}_t \bar{q}
\]
\[
- \left( \frac{2aa'}{b^2} + \frac{2af}{b^2} \right) \nabla \bar{u} \cdot \nabla \bar{u} + \left( \frac{ab'f}{(b')^2} - \frac{af}{b'} e^{-\bar{u}} \right) \bar{u}_t - \frac{af}{b'}
\]
where \( \bar{q} = |\nabla \bar{u}|^2 \).

Hence
\[
\frac{a}{b'} \Delta \bar{\Psi} - \bar{\Psi}_t = -\frac{aa''}{b^2} \bar{u}_t \bar{q} + \frac{2af}{b'} \nabla \bar{u} \cdot \nabla \bar{u}_t - \frac{a'^2}{b^2} \bar{u}_t \Delta \bar{u} + \beta \frac{a}{b'} e^{-\bar{u}} \bar{q}
\]
\[
- \beta \frac{a}{b'} e^{-\bar{u}} \Delta \bar{u} + a'(\bar{u}_t)^2 - \left( \frac{ab'f}{(b')^2} - \frac{af}{b'} e^{-\bar{u}} \right) \bar{u}_t + \frac{af}{b'}
\]
In view of (3.1), we have
\[
(3.5) \quad \Delta \bar{u} = \frac{b'}{a} \bar{u}_t - \frac{a'}{a} \bar{q} - \frac{f}{a}.
\]
Substitute (3.5) into (3.4) to obtain
\[
(3.6) \quad \frac{a}{b'} \Delta \bar{\Psi} - \bar{\Psi}_t = \frac{2af}{b'} \nabla \bar{u} \cdot \nabla \bar{u}_t + b' \left( \frac{a}{b'} \right) \bar{u}_t \bar{q} + \bar{u}_t \nabla \bar{u} + \beta \frac{1}{b'} e^{-\bar{u}}(a + a') + \beta f e^{-\bar{u}} + \frac{af}{b'}.
\]
By (3.3), we have
\[
(3.7) \quad \nabla \bar{u}_t = -\frac{1}{a} \nabla \bar{\Psi} - \frac{a'}{a} \bar{u}_t \nabla \bar{u} - \beta \frac{1}{a} e^{-\bar{u}} \nabla \bar{u}.
Substitute (3.7) into (3.6), to get

\[
\frac{a}{b'} \Delta \Psi + \frac{2f q}{b'} \nabla u \cdot \nabla \Psi - \Psi_t = (af u - 2a'f q) u_t - \beta \left( e^{-\Psi} + \frac{af_t}{b'} \right).
\]

(3.8)

Since (3.9)

\[ u_t = \frac{1}{a} \Psi + \frac{1}{a} e^{-\Psi}, \]

it follows from (3.8) that

\[
\frac{a}{b'} \Delta \Psi + \frac{2f q}{b'} \nabla u \cdot \nabla \Psi + \left( \frac{f q}{b'} - \frac{2a'f q}{ab'} \Psi \right) - \Psi_t = 2\beta \left( e^{-\Psi} + \frac{af_t}{b'} \right).
\]

(3.10)

From the assumptions (2.1) and (2.2), the right-hand side of (3.10) is nonpositive:

\[
\frac{a}{b'} \Delta \Psi + \frac{2f q}{b'} \nabla u \cdot \nabla \Psi + \left( \frac{f q}{b'} - \frac{2a'f q}{ab'} \Psi \right) - \Psi_t \leq 0.
\]

(3.11)

Now, by (2.4) and (3.1), it has

\[
\min_{\overline{\Omega}} \Psi(x, 0) = \min_{\overline{\Omega}} \left\{ -a(M_0) \left[ \frac{1}{b'(M_0)} \nabla \cdot (a(M_0) \nabla M_0) + \frac{1}{b'(M_0)} f(x, M_0, 0, 0) \right] + \beta e^{-M_0} \right\}
\]

(3.12)

\[ = e^{-M_0} \left( \beta_0 - \frac{a(M_0) e^{M_0}}{b'(M_0)} \right) \max f(x, M_0, 0, 0) \leq 0, \]

and

\[
\frac{\partial \Psi}{\partial n} = -a' M_0 \frac{\partial M_0}{\partial n} - a \frac{\partial u_t}{\partial n} - \beta e^{-\Psi} \frac{\partial \Psi}{\partial n} \]

(3.13)

\[ = -a \left( \frac{\partial u_t}{\partial n} \right)_t = 0 \text{ on } \partial \Omega \times (0, T). \]

Combining (3.11)-(3.13), it follows from Hopf’s maximum principles ([12]) that the minimum of \( \Psi \) in \( \overline{\Omega} \times [0, T) \) is zero. Hence

\[ \Psi \geq 0 \text{ in } \overline{\Omega} \times [0, T), \]
i.e.,

\[ \beta \geq \frac{a(\bar{u})}{e^{-\bar{u}}} \bar{u}_t. \]

For each fixed \( x \in \bar{D} \), integrate (3.14) over \([0, t]\) to get

\[ t \geq \frac{1}{\beta} \int_{M_0}^{\bar{u}(x,t)} \frac{a(s)}{e^{-s}} ds. \]

This together with (2.3) shows that \( \bar{u} \) must be a global solution, and so is for \( u \).
Moreover, (3.15) implies that

\[ H(\bar{u}(x,t)) = \int_{M_0}^{\bar{u}(x,t)} \frac{a(s)}{e^{-s}} ds \leq \beta t. \]

Therefore,

\[ \pi(x, t) \leq H^{-1}(\beta t). \]

Since \( \bar{u} \) is an upper solution of (1.1), we finally get

\[ u(x, t) \leq \pi(x, t) \leq H^{-1}(\beta t). \]

The proof is complete. \( \square \)

**Proof of Theorem 2.2.** Let \( \underline{u} \) be a solution of the following equation

\[
\begin{aligned}
(b(u))_t &= \nabla \cdot (a(u) \nabla u) + f(x, u, q, t) \quad \text{in } D \times (0, T), \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial D \times (0, T), \\
u(x, 0) &= m_0 \quad \text{in } \bar{D},
\end{aligned}
\]

where \( q = |\nabla u|^2 \). Then \( \underline{u} \) is a lower solution of (1.1). In order to show that \( u \) blows up in finite time, it suffices to show that \( \underline{u} \) blows up in some finite time moment.

Construct an auxiliary function

\[ \Phi = -a(u)u + \alpha e^u. \]

We find that

\[ \nabla \Phi = -a'u \nabla u - a \nabla u + \alpha e^u \nabla u. \]
\[ \Delta \Phi = -a''u_t q - 2a' \nabla u \cdot \nabla u_t - a' u_t \Delta u - a \Delta u_t + \alpha e_u q + \alpha e_u \Delta u, \]

\[ \Phi_t = -a'(u_t)^2 - a \left( \frac{a}{b'} \Delta u + \frac{a'}{b'} q \right) t + \alpha e_u \]

\[ = -a'(u_t)^2 - a \left( \frac{a}{b'} \Delta u + \frac{a'}{b'} q \right) \]

\[ = -a'(u_t)^2 + \left( \frac{a^2 b'' - a d'}{b'^2} \right) u_t \Delta u - \frac{a^2}{b'} \Delta u_t + \left( \frac{a a' b'' - a''}{b'} \right) u_t q - \left( \frac{2a a'}{b'} + \frac{2 a f q}{b'} \right) \nabla u \cdot \nabla u_t + \left( \frac{a b'' f}{b'} - \frac{a b u}{b'} + \alpha e_u \right) u_t - \frac{a f_t}{b'}, \]

where \( q = |\nabla u|^2 \). Hence

\[ \frac{a}{b'} \Delta \Phi - \Phi_t \]

\[ = \frac{a d b''}{(b')^2} u_t q + \frac{2 a f q}{b'} \nabla u \cdot \nabla u_t - \frac{a^2 b''}{(b')^2} u_t \Delta u + \frac{a}{b'} e_u q + \frac{a}{b'} e_u \Delta u + a' (u_t)^2 - \left( \frac{a b'' f}{b'} - \frac{a b u}{b'} + \alpha e_u \right) u_t + \frac{a f_t}{b'}. \]

By (3.16), we have

\[ \Delta u = \frac{b'}{a} u_t - \frac{a'}{a} q - \frac{f}{a}. \]

Combining (3.19) and (3.20), it follows that

\[ \frac{a}{b'} \Delta \Phi - \Phi_t \]

\[ = \frac{2 a f q}{b'} \nabla u \cdot \nabla u_t + \frac{a}{b'} \left( \frac{a}{b'} \right) u_t (u_t)^2 \]

\[ + \frac{a f}{b'} u_t + \frac{1}{b'} e_u q(a - a') - \frac{f}{b'} e_u + \frac{a f_t}{b'}. \]

Since (3.18) implies that

\[ \nabla u_t = -\frac{1}{a} \nabla \Phi - \frac{a'}{a} u_t \nabla u + \frac{e_u}{a} \nabla u, \]

we have, from (3.21) and (3.22), that

\[ \frac{a}{b'} \Delta \Phi + \frac{2 f q}{b'} \nabla u \cdot \nabla \Phi - \Phi_t \]

\[ = \left( \frac{a f}{b'} - \frac{2 a' f q}{b'} q \right) u_t + \frac{2 f q}{b'} e_u q + \frac{a}{b'} \left( \frac{a}{b'} \right) u_t (u_t)^2 + \frac{1}{b'} e_u q(a - a') \]

\[ - \frac{f}{b'} e_u + \frac{a f_t}{b'}. \]
By (3.17), it has

\[ u_t = -\frac{1}{a} \Phi + \frac{1}{a} e^{u}. \]

Substituting (3.24) into (3.23) gives

\[ \frac{a}{b^\prime} \Delta \Phi + \frac{2f_q}{b^\prime} \nabla u \cdot \nabla \Phi + \left( \frac{f_u}{b^\prime} - \frac{2a'f_q}{ab'q} \right) \Phi - \Phi_t \]

\[ = 2a' e^{u} \frac{a}{b^\prime} \left( \frac{a}{2} + f_q \right) + a \frac{1}{b^\prime} e^{u} (f_u - f) + b' \left( \frac{a}{b} \right)' (u_t)^2 + \frac{a f_t}{b^\prime}. \]

It is seen from (2.5) and (2.6) that the right hand side of (3.25) is nonnegative, i.e.,

\[ \frac{a}{b^\prime} \Delta \Phi + \frac{2f_q}{b^\prime} \nabla u \cdot \nabla \Phi + \left( \frac{f_u}{b^\prime} - \frac{2a'f_q}{ab'q} \right) \Phi - \Phi_t \geq 0. \]

Furthermore, by (2.8) and (3.16), it follows that

\[ \max_{D} \Phi(x,0) \]

\[ = \max_{D} \left\{ -a(m_0) \left[ \frac{1}{b'(m_0)} \nabla \cdot (a(m_0) \nabla m_0) + \frac{1}{b'(m_0)} f(x, m_0, 0,0) \right] + a e^{m_0} \right\} \]

\[ = e^{m_0} \left[ \alpha - \frac{a(m_0)e^{-m_0}}{b'(m_0)} - \min_{D} f(x, m_0, 0,0) \right] = 0, \]

and

\[ \frac{\partial \Phi}{\partial n} = -a' \frac{\partial u}{\partial n} - a \frac{\partial u_t}{\partial n} + \alpha e^{u} \frac{\partial u}{\partial n} \]

\[ = -a \left( \frac{\partial u}{\partial n} \right)_t = 0 \quad \text{on} \ \partial D \times (0,T). \]

Combining (3.26)-(3.28), it follows from Hopf’s maximum principles ([12]) that the maximum of \( \Phi \) in \( \overline{D} \times [0,T) \) is zero. Hence we have

\[ \Phi \leq 0 \quad \text{in} \ \overline{D} \times [0,T), \]

and

\[ \alpha \leq \frac{a(u)}{e^u} u_t. \]

For each fixed \( x \in \overline{D} \), integrate (3.29) over \([0,t]\) to produce

\[ t \leq \frac{1}{\alpha} \int_{m_0}^{u(x,t)} \frac{a(s)}{e^s} ds. \]
This together with assumption (2.7) shows that $u$ must blow up in the finite time $t = T^*$ and
\[
T^* \leq \frac{1}{\alpha} \int_{m_0}^{+\infty} \frac{a(s)}{e^s} ds.
\]
Therefore, $u$ blows up in a finite time $t = T$ and
\[
T \leq T^* \leq \frac{1}{\alpha} \int_{m_0}^{+\infty} \frac{a(s)}{e^s} ds.
\]
The proof is complete. □

REFERENCES


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