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Brief paper

Identification of variable spacial coefficients for a beam equation from boundary measurements ☆

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Abstract

We consider a system described by the Euler–Bernoulli beam equation, with one end clamped and with torque input at the other end. The output function are the displacement and the angle velocity at the non-clamped end of the beam. We study the identification of the spatially variable coefficients in the beam equation, from input–output data. We show that both the density and the flexural rigidity of the beam (which are assumed to be of class C^4) can be uniquely determined if the input and output functions are known for all positive times.

Keywords: Beam equation; Identifiability; Variable coefficients; Inverse problem

1. Introduction

In control engineering, the determination of some physical parameters by means of observed data is recognized as the identification problem which falls actually under the category of inverse problems (Nakano, Ohsumi, & Shintani, 2000). One of the fundamental questions in identification problems is parameter identifiability. The system parameters are said to be identifiable if the input–output map of the system and the input–output data contain sufficient information to uniquely determine these parameters (Kitamura & Nakagiri, 1977). From the inverse problem point of view, the parameter identifiability is equivalent to the uniqueness of the solution of the inverse problem. It is generally considered that the parameter identifiability is the theoretical basis of many identification algorithms, such as the output least-square method, a widely used algorithm in engineering.

Over the last two decades, the parameter identifiability for distributed parameter systems has attracted a great deal of attention. There are numerous works for one-dimensional systems on this aspect. We refer to Giudici (1991), Ito and Nakagiri (1997), Kitamura and Nakagiri (1977), Kravaris and Seinfeld (1986), Lesnic (2000), Nakagiri (1983), Nakagiri (1993), Pierce (1979), Udwadia and Sharma (1985), Yamamoto and Nakagiri (1994) and the references therein. A nice earlier survey can be found in Nakagiri (1993). In these works, three different sorts of methods were used. The first sort of methods reduce the parameter identifiability problem to an inverse Sturm–Liouville problem for which the Gelfand–Levitan result is applicable (Kravaris & Seinfeld, 1986; Pierce, 1979; Udwadia & Sharma, 1985). The second ones, by means of set representations, have developed some conditions under which some of the coefficients can be (or not) uniquely identified (Giudici, 1991; Kitamura & Nakagiri, 1977). Both these methods can be traced back to the work of Kitamura and Nakagiri (1977). The third kind of methods may be considered as the abstract formulation of the first ones. Roughly speaking, the original problem is transformed into the identifiability of some operator in general abstract evolution equation in Banach (or Hilbert) space (Nakagiri, 1983; Yamamoto & Nakagiri, 1994). However, the last two methods require the distributed measurements.

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Although the study of identifiability for second-order systems is quite comprehensive, few results, to our best knowledge, are available for fourth-order systems like beam equations. In Lesnic (2000), the second sort of method was used to identify the flexural rigidity and/or the mass density for an Euler–Bernoulli beam through prescribed deflection and load. In Ito and Nakagiri (1997), the third method was adopted to discuss the identifiability of the flexural rigidity and damping coefficients for an Euler–Bernoulli beam with Kelvin–Voigt damping. In addition, an approach that is different from the methods aforementioned, which is called the boundary control method, was used in Avdonin, Medhin, and Sheronova (2000) to recover an unknown piecewise constant coefficient for an Euler–Bernoulli beam equation through the given Dirichlet-to-Neumann map.

In this paper, we will use the first sort of method to deal with the identifiability of the coefficients for an Euler–Bernoulli beam, which is described by the following equation:

$$\begin{cases} \rho(x)u_{tt}(x,t) + (r(x)u_{xx}(x,t))_{xx} = 0, \\ \quad 0 < x < 1, \ t > 0, \\ u(0,t) = u_x(0,t) = 0, \quad t \geqslant 0, \\ r(x)u_{xx}(x,t)|_{x=1} = 0, \quad t \geqslant 0, \\ (r(x)u_{xx}(x,t))_x|_{x=1} = g(t), \quad t \geqslant 0, \\ u(x,0) = u_t(x,0) = 0, \quad 0 \leqslant x \leqslant 1, \end{cases} \tag{1}$$

where x stands for the position and t the time. $u(x,t)$ is the transverse displacement at x and t, $\rho(\cdot)$ is the mass density and $r(\cdot) = EI(\cdot)$ is the flexural rigidity of the beam, where E is the Young's modulus and I the second moment of inertia of the beam, $g(\cdot)$ is the known boundary input satisfying the compatibility condition $g(0)=0$. The parameters to be identified are $\rho(\cdot)$ and $r(\cdot)$. In practice, the Young's modulus E is not necessarily known exactly even if it is constant, because its true value is occasionally slightly different to the nominal value and cannot be determined until some test is performed (Nakano et al., 2000). The identification of $r(\cdot)$ is necessary.

The aim of this paper is to show that if the boundary input $g(t)$ and observations $u(1,t)$ and $u_x(1,t)$ are available for all $t \geqslant 0$, the unknown parameters $\rho(\cdot)$ and $r(\cdot)$ can be uniquely determined by $\{(g(t), u(1,t), u_x(1,t)),\ t \geqslant 0\}$. From a mathematical point of view, this parameter identifiability problem is an inverse boundary value problem. In the study of inverse problems, the problems in time domain are often reduced to problems in frequency domain (see, e.g., Pierce, 1979; Udwadia & Sharma, 1985). Motivated by this idea, we reduce the inverse boundary value problem under consideration to an corresponding inverse boundary spectral problem (see Section 2). Fortunately, the latter has already been solved by Barcilon (1982, hyperlinkbib3 bib41986). The parameter identifiability problem is thus concluded. As a by-product, we also show that the two inverse problems are equivalent to each other. This equivalence is significant both theoretically and practically. It should be pointed out that the equivalence result obtained in this paper is not contained in Kachalov, Kurylev, Lassas, and Mandache (2004) where the equivalence of several types of boundary inverse problems was discussed.

The remaining parts of the paper are as follows. In Section 2, some necessary facts on the inverse eigenvalue problem for the Euler–Bernoulli equation are introduced. The main results are presented in Section 3.

2. Preliminary

In this section, we list briefly some basic facts on the eigenvalues and eigenfunctions of the Euler–Bernoulli beam equation. A result of an inverse eigenvalue problem for the beam equation originated from Barcilon (1982) is also introduced. Throughout the paper, we always assume that:

(a) $\rho(\cdot), r(\cdot) \in C^4[0,1]$ and $\rho(x), r(x) > 0$ for any $x \in [0,1]$.
(b) $g(\cdot) \in L^2(0,\infty)$ and $g'(\cdot)$ is integrable in any finite interval.

By the eigenvalue problem associated with (1), we mean the following two-points boundary value problem:

$$\begin{cases} (r(x)\phi_n''(x))'' = \omega_n^2 \rho(x)\phi_n(x), \quad 0 < x < 1, \\ \phi_n(0) = \phi_n'(0) = 0, \\ \phi_n''(1) = (r(x)\phi_n''(x))'|_{x=1} = 0, \end{cases} \tag{2}$$

where $\omega_n^2, n = 1, 2, \ldots,$ are eigenvalues with the order of the following:

$$0 < \omega_1 < \omega_2 < \cdots \omega_n < \omega_{n+1} < \cdots.$$

ϕ_n is the eigenfunction corresponding to ω_n. It is well known that $\{\phi_n\}_{n=1}^\infty$, normalized by

$$\int_0^1 \rho(x)\phi_n^2(x)\,dx = 1,$$

form an orthonormal basis for the space $L^2_\rho(0,1)$. Here $L^2_\rho(0,1)$ denotes the space of square integrable functions over $[0,1]$ with weight ρ.

Let $\{\nu_n^2\}$ and $\{\mu_n^2\}$ be the eigenvalues of (2) with boundary conditions replaced by (3) and (4), respectively

$$\phi_n(0) = \phi_n'(0) = \phi_n(1) = (r(x)\phi_n''(x))'|_{x=1} = 0, \tag{3}$$

$$\phi_n(0) = \phi_n'(0) = \phi_n(1) = \phi_n''(1) = 0. \tag{4}$$

Let

$$G_i = \int_0^1 \frac{(1-x)^i}{r(x)}\,dx, \quad i = 1, 2.$$

It is shown in Barcilon (1982) that under the assumption (a), $\rho(\cdot)$ and $r(\cdot)$ can be uniquely determined by $\{\omega_n, \nu_n, \mu_n\}_{n=1}^\infty$ and $G_i, i = 1, 2$. Furthermore, it is shown in Gladwell (2004, pp. 386–389) (see also Barcilon, 1986, pp. 34–35) that $\{\omega_n, \nu_n, \mu_n\}_{n=1}^\infty$ and $G_i,\ i = 1, 2$ can be determined from $\{\omega_n, \phi_n(1), \phi_n'(1)\}_{n=1}^\infty$.We write specifically this result as the following Lemma 2.1.

Lemma 2.1. *$\rho(\cdot)$ and $r(\cdot)$ can be uniquely determined by $\{\omega_n, \phi_n(1), \phi_n'(1)\}_{n=1}^\infty$.*

It is known that the following asymptotic properties hold (see, e.g., Barcilon, 1986, p. 33)

$$\omega_n = \mathcal{O}((n-1/2)^2), \quad \phi_n(1) = \mathcal{O}(1),$$
$$\phi_n'(1) = \mathcal{O}(n-1/2). \tag{5}$$

Moreover (Gladwell, 2004, p. 382), for any n, $\phi_n(1)\phi_n'(1) > 0$, by which we may assume without loss of generality that

$$\phi_n(1) > 0, \quad \phi_n'(1) > 0. \tag{6}$$

To end this section, let us interpret the solution of (1) (see also Lagnese, 1991, Remark 2.1). Let $\mathbf{H} = H_E^2(0,1) \times L_\rho^2(0,1)$, $H_E^2(0,1) = \{p \in H^2(0,1) | p(0) = p'(0) = 0\}$ be the state Hilbert space with the norm induced by the inner product (for any $(p,q) \in \mathbf{H}$):

$$\|(p,q)\|_{\mathbf{H}}^2 = \int_0^1 [\rho(x)|q(x)|^2 + r(x)|p''(x)|^2]\,\mathrm{d}x.$$

It is known that (1) is equivalent to (Rebarber, 1995)

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ u_t \end{pmatrix} = A\begin{pmatrix} u \\ u_t \end{pmatrix} + Bg, \tag{7}$$

where the operators $A : D(A)(\subset \mathbf{H}) \mapsto \mathbf{H}$ and $B : \mathbb{R} \mapsto [D(A)]'$ are defined by

$$A\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \psi \\ -\dfrac{1}{\rho(x)}(r(x)\phi'')'' \end{pmatrix},$$

$$D(A) = \{(\phi,\psi)^\top \in \mathbf{H} | \psi \in H_E^2(0,1), (r(x)\phi'')'' \in L^2(0,1), \\ \phi''(1) = (r(x)\phi''(x))'|_{x=1} = 0\},$$

$$B = \begin{pmatrix} 0 \\ -\dfrac{1}{\rho(x)}\delta(x-1) \end{pmatrix}.$$

A is skew-adjoint in $\mathbf{H} : A^* = -A$. By (5), one can easily show that B is admissible for the C_0-semigroup e^{At} (Ho & Russell, 1983, see also Guo, 2002). Therefore, for any $g \in L^2(0,\infty)$ and initial data $(u(\cdot,0), u_t(\cdot,0)) \in \mathbf{H}$, there exists a unique solution to (7) such that $(u,u_t)^\top \in \mathbf{H}$, and for any $(\phi,\psi)^\top \in D(A)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \begin{pmatrix} u \\ u_t \end{pmatrix}, \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\rangle_{\mathbf{H}} = \left\langle \begin{pmatrix} u \\ u_t \end{pmatrix}, -A\begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\rangle_{\mathbf{H}} + \begin{pmatrix} 0 \\ -\psi(1)g(t) \end{pmatrix}, \quad t \geqslant 0, \text{ a.e.} \tag{8}$$

In particular, since $A(\phi_n, \mathrm{i}\omega_n\phi_n)^\top = \mathrm{i}\omega_n(\phi_n, \mathrm{i}\omega_n\phi_n)^\top$, it follows from (8) that

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\langle u, \phi_n \rangle_{H_E^2(0,1)} = \langle u, -\mathrm{i}\omega_n\phi_n \rangle_{H_E^2(0,1)}, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\langle u_t, \phi_n \rangle_{L_\rho^2(0,1)} = -\left\langle u, \dfrac{1}{\rho}(r\phi_n'')'' \right\rangle_{L_\rho^2(0,1)} - \phi_n(1)g(t), \end{cases} \tag{9}$$

which hold for all n and almost all $t \geqslant 0$.

3. Main results

Due to the basis property of eigenfunctions $\{\phi_n\}_{n=1}^\infty$ in $L_\rho^2(0,1)$, the solution u of (1) can be represented as

$$\begin{cases} u(x,t) = \sum_{n=1}^\infty u_n(t)\phi_n(x), \\ u_n(t) = \int_0^1 \rho(x)u(x,t)\phi_n(x)\,\mathrm{d}x. \end{cases} \tag{10}$$

By virtue of (9), differentiate u_n twice in t, to obtain

$$\begin{aligned} u_n''(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \rho(x)u_t(x,t)\phi_n(x)\,\mathrm{d}x \\ &= -\int_0^1 u(x,t)(r(x)\phi_n''(x))''\,\mathrm{d}x - \phi_n(1)g(t) \\ &= -\omega_n^2 u_n(t) - \phi_n(1)g(t). \end{aligned}$$

This together with the initial conditions $u_n(0) = u_n'(0) = 0$ gives

$$u_n(t) = -\phi_n(1)\int_0^t \frac{\sin\omega_n(t-\tau)}{\omega_n} g(\tau)\,\mathrm{d}\tau.$$

Substitute the above into (10) to yield

$$u(x,t) = -\sum_{n=1}^\infty \phi_n(1)\phi_n(x)\int_0^t q_n(t-\tau)g(\tau)\,\mathrm{d}\tau, \tag{11}$$

where

$$q_n(t) = \frac{\sin\omega_n t}{\omega_n}. \tag{12}$$

In order to obtain $\{\omega_n, \phi_n(1), \phi_n'(1)\}$ from (11), we need the following Lemmas 3.1 and 3.2.

Lemma 3.1. *If $g(\cdot)$ is not identically zero, then $\{\omega_n\}_{n=1}^\infty$ and $\{\phi_n(1)\}_{n=1}^\infty$ are uniquely determined by $\{(g(t), u(1,t)), t \geqslant 0\}$.*

Proof. For any given $t > 0$, by virtue of (5), the series (11) converges uniformly in x over $[0,1]$. In particular,

$$\begin{aligned} u(1,t) &= -\sum_{n=1}^\infty \phi_n^2(1)\int_0^t q_n(t-\tau)g(\tau)\,\mathrm{d}\tau \\ &= -\int_0^t \left[\sum_{n=1}^\infty \phi_n^2(1)q_n(t-\tau)\right] g(\tau)\,\mathrm{d}\tau. \end{aligned} \tag{13}$$

Notice that the change of orders of integration and summation in (13) is guaranteed by the uniform convergence of the series on the right-hand side of (13). Let

$$Q(t) = -\sum_{n=1}^\infty \phi_n^2(1)q_n(t). \tag{14}$$

Again by (5), the series in (14) is uniformly and boundedly convergent over $[0,\infty)$, which implies that Q is a bounded continuous function. This, together with the assumption (b) in the beginning of Section 2, enables us to apply the Laplace transform to (13) to obtain

$$\widehat{u}(1,s)=\widehat{Q}(s)\widehat{g}(s),$$

where ^ denotes the Laplace transform. Since $u(1,\cdot)$ and $g(\cdot)$ are known and $g(\cdot)$ is not null, by the uniqueness of the Laplace transform, Q is uniquely determined by $u(1,\cdot)$ and $g(\cdot)$.

Since $Q(\cdot)$ is uniquely determined by $u(1,\cdot)$ and $g(\cdot)$, the proof will be accomplished if we can show that $\{\omega_n\}_{n=1}^{\infty}$ and $\{\phi_n(1)\}_{n=1}^{\infty}$ can be uniquely determined by $Q(\cdot)$. Indeed, apply the Laplace transform to (14) to give

$$\widehat{Q}(s)=\sum_{n=1}^{\infty}\frac{\phi_n^2(1)}{2\mathrm{i}\omega_n}\left(\frac{1}{s+\mathrm{i}\omega_n}-\frac{1}{s-\mathrm{i}\omega_n}\right). \tag{15}$$

Since $\phi_n(1)>0$, $\{-\mathrm{i}\omega_n\}_{n=1}^{\infty}$ are poles of $\widehat{Q}$ and $\phi_n^2(1)/2\mathrm{i}\omega_n$ is the residue of $\widehat{Q}$ at $\{-\mathrm{i}\omega_n\}$ for any $n\geqslant 1$. Hence $\{\omega_n\}_{n=1}^{\infty}$ and $\{\phi_n(1)\}_{n=1}^{\infty}$can be uniquely determined by $\widehat{Q}$. By the uniqueness of the Laplace transform, $\{\omega_n\}_{n=1}^{\infty}$ and $\{\phi_n(1)\}_{n=1}^{\infty}$ are uniquely determined by $Q(\cdot)$, proving the required result. □

Let us recall that a sequence $\{f_i\}_{i=1}^{\infty}$ is called a basis for a Hilbert space H if any element $w\in H$ has a unique representation

$$w=\sum_{i=1}^{\infty}a_i f_i, \tag{16}$$

and the convergence of the series is in the norm of H. A basis for H is a Riesz basis if it is equivalent to an orthonormal basis, that is, if it is obtained from an orthonormal basis by means of a bounded invertible transform (Young, 2001, p. 26). If the sequence $\{f_i\}_{i=1}^{\infty}$ in (16) is a Riesz basis for H, the series in (16) converges unconditionally in the norm of H.

Let $\Lambda:=\{\lambda_n\mid n\in\mathbb{Z}\}$ be a countable subset of the complex plane. Suppose

$$\inf_n \operatorname{Im}\lambda_n>0,\quad \sup_n \operatorname{Im}\lambda_n<\infty,$$

and λ_n are separable in the sense that

$$\inf_{m\neq n}|\lambda_m-\lambda_n|>0.$$

Then the family $\{\mathrm{e}^{\mathrm{i}\lambda_n t}\}$ forms a Riesz basis for its span in $L^2(0,\infty)$ (see Avdonin & Ivanov, 2002, Remark 1).

Lemma 3.2. *If $g(\cdot)$ is not identical to zero, then $\{\omega_n\}_{n=1}^{\infty}$ and $\{\phi_n(1)\phi_n'(1)\}_{n=1}^{\infty}$ are uniquely determined by $\{(g(t), u_x(1,t)), t\geqslant 0\}$.*

Proof. Since $g'(\cdot)$is integrable in any finite interval, integrate by parts over $[0,t]$ for any $t>0$, to give

$$\int_0^t q_n(t-\tau)g(\tau)\,\mathrm{d}\tau=\mathcal{O}(\omega_n^{-2}). \tag{17}$$

This together with (5) guarantees that we can differentiate (11) in x term by term, which gives

$$u_x(x,t)=-\sum_{n=1}^{\infty}\phi_n(1)\phi_n'(x)\int_0^t q_n(t-\tau)g(\tau)\,\mathrm{d}\tau. \tag{18}$$

By the uniform convergence of the series in (18) that is guaranteed by (17), we have

$$u_x(1,t)=-\sum_{n=1}^{\infty}\phi_n(1)\phi_n'(1)\int_0^t q_n(t-\tau)g(\tau)\,\mathrm{d}\tau. \tag{19}$$

Take $c>0$ and set

$$P(t)=-\sum_{n=1}^{\infty}\phi_n(1)\phi_n'(1)\mathrm{e}^{-ct}q_n(t). \tag{20}$$

We show that

$$\mathrm{e}^{-ct}u_x(1,t)=\int_0^t P(t-\tau)\mathrm{e}^{-c\tau}g(\tau)\,\mathrm{d}\tau. \tag{21}$$

In fact, write (20) to be

$$P(t)=\sum_{n=1}^{\infty}\frac{\phi_n(1)\phi_n'(1)}{2\mathrm{i}\omega_n}[\mathrm{e}^{-(c-\mathrm{i}\omega_n)t}-\mathrm{e}^{-(c+\mathrm{i}\omega_n)t}]. \tag{22}$$

Since $\{\omega_n+\mathrm{i}c,-\omega_n+\mathrm{i}c\}$ are separable and all ω_n are real, $\{\mathrm{e}^{-(c-\mathrm{i}\omega_n)t},\mathrm{e}^{-(c+\mathrm{i}\omega_n)t}\}_{n=1}^{\infty}$ form a Riesz basis for its span in $L^2(0,\infty)$. By (5),

$$\frac{\phi_n(1)\phi_n'(1)}{2\mathrm{i}\omega_n}=\mathcal{O}(n^{-1}).$$

Hence the series in (22) converges unconditionally in $L^2(0,\infty)$. In particular, $P(t)\in L^2(0,\infty)$.

Now set

$$S_n(t)=-\sum_{k=1}^{n}\frac{\phi_k(1)\phi_k'(1)}{2\mathrm{i}\omega_k}[\mathrm{e}^{-(c-\mathrm{i}\omega_k)t}-\mathrm{e}^{-(c+\mathrm{i}\omega_k)t}],$$
$$n=1,2,3\ldots.$$

Then we have, for any $t>0$,

$$\begin{aligned}&\left|\int_0^t S_n(t-\tau)\mathrm{e}^{-c\tau}g(\tau)\,\mathrm{d}\tau-\int_0^t P(t-\tau)\mathrm{e}^{-c\tau}g(\tau)\,\mathrm{d}\tau\right|\\&\leqslant\int_0^t|S_n(t-\tau)-P(t-\tau)|\mathrm{e}^{-c\tau}|g(\tau)|\,\mathrm{d}\tau\\&\leqslant\left(\int_0^t\mathrm{e}^{-2c\tau}|g(\tau)|^2\,\mathrm{d}\tau\right)^{1/2}\left(\int_0^t|S_n(t-\tau)-P(t-\tau)|^2\,\mathrm{d}\tau\right)^{1/2}\\&\leqslant\left(\int_0^t|g(\tau)|^2\,\mathrm{d}\tau\right)^{1/2}\left(\int_0^t|S_n(\tau)-P(\tau)|^2\,\mathrm{d}\tau\right)^{1/2}\\&\leqslant\|g\|_{L^2(0,\infty)}\|S_n(t)-P(t)\|_{L^2(0,\infty)}.\end{aligned} \tag{23}$$

This together with the convergence of $S_n(t)$ to $P(t)$ in $L^2(0,\infty)$ gives (as $n\to\infty$)

$$\left|\int_0^t S_n(t-\tau)\mathrm{e}^{-c\tau}g(\tau)\,\mathrm{d}\tau-\int_0^t P(t-\tau)\mathrm{e}^{-c\tau}g(\tau)\,\mathrm{d}\tau\right|\to 0,$$

proving (21).

Furthermore, since $P(\cdot)\in L^2(0,\infty)$, $\mathrm{e}^{-c\cdot}g(\cdot)\in L^1(0,\infty)$, by Young's inequality, (21) implies that $\mathrm{e}^{-c\cdot}u_x(1,\cdot)\in L^2(0,\infty)$. This enables us to apply the Laplace transform to (21), to obtain

$$\widehat{u_x}(1, s+c) = \widehat{P}(s)\widehat{g}(s+c) \quad \forall \mathrm{Re}(s) \geqslant 0. \tag{24}$$

This together with the uniqueness of the Laplace transform shows that $P(\cdot)$ is uniquely determined by $g(\cdot)$ and $u_x(1,\cdot)$ because $g(\cdot)$ is not null.

Similar to (23), for any $t>0$ and $\mathrm{Re}(s)>0$, we have

$$\begin{aligned}
&\left|\int_0^t \mathrm{e}^{-s\tau} S_n(\tau)\,\mathrm{d}\tau - \int_0^t \mathrm{e}^{-s\tau} P(\tau)\,\mathrm{d}\tau\right| \\
&\quad \leqslant \int_0^t |S_n(\tau) - P(\tau)|\,\mathrm{d}\tau \\
&\quad \leqslant \sqrt{t}\left(\int_0^t |S_n(\tau) - P(\tau)|^2\,\mathrm{d}\tau\right)^{1/2} \\
&\quad \leqslant \sqrt{t}\,\|S_n(\tau) - P(\tau)\|_{L^2(0,\infty)} \to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{25}$$

Therefore, we can integrate the series in (22) over $(0, t)$ to yield

$$\begin{aligned}
&\int_0^t \mathrm{e}^{-s\tau} P(\tau)\,\mathrm{d}\tau \\
&\quad = -\sum_{n=1}^{\infty} \frac{\phi_n(1)\phi_n'(1)}{2\mathrm{i}\omega_n} \\
&\qquad \times \left[\frac{2\mathrm{i}\omega_n}{(s+c)^2+\omega_n^2} - \frac{\mathrm{e}^{-(s+c-\mathrm{i}\omega_n)t}}{s+c-\mathrm{i}\omega_n} + \frac{\mathrm{e}^{-(s+c+\mathrm{i}\omega_n)t}}{s+c+\mathrm{i}\omega_n}\right].
\end{aligned} \tag{26}$$

Passing $t \to \infty$, we obtain

$$\widehat{P}(s) = -\sum_{n=1}^{\infty} \frac{\phi_n(1)\phi_n'(1)}{2\mathrm{i}\omega_n}\left[\frac{1}{s+c-\mathrm{i}\omega_n} - \frac{1}{s+c+\mathrm{i}\omega_n}\right]. \tag{27}$$

This, similar to Lemma 3.1, shows that $\{\omega_n\}_{n=1}^{\infty}$ and $\{\phi_n(1)\phi_n'(1)\}_{n=1}^{\infty}$ are uniquely determined by $P(\cdot)$ and so are by $g(\cdot)$ and $u_x(1,\cdot)$. The proof is complete. □

Now, we are in a position to prove the main result of this paper.

Theorem 3.1. *If $g(\cdot)$ is not identical to zero, then*

(i) $\{\omega_n, \phi_n(1), \phi_n'(1)\}_{n=1}^{\infty}$ *are uniquely determined by* $\{(g(t), u(1,t), u_x(1,t)), t \geqslant 0\}$.
(ii) $\rho(\cdot)$ *and* $r(\cdot)$ *are uniquely determined by* $\{(g(t), u(1,t), u_x(1,t)), t \geqslant 0\}$.

Proof. (i) follows from Lemmas 3.1 and 3.2, and (ii) can be deduced from (i) and Lemma 2.1. □

The following result is a direct consequence of Theorem 3.1, which is much more easily understood for identification problems.

Corollary 3.1. *Let u be a solution of* (1) *and $\widetilde{u}$ be a solution of* (1) *in which $\rho(\cdot)$ and $r(\cdot)$ are replaced by $\widetilde{\rho}(\cdot)$ and $\widetilde{r}(\cdot)$, respectively. If $g(\cdot)$ is not identical to zero, then $\rho(\cdot)=\widetilde{\rho}(\cdot)$ and $r(\cdot)=\widetilde{r}(\cdot)$ follows from $u(1,t)-\widetilde{u}(1,t)=u_x(1,t)-\widetilde{u}_x(1,t)=0,\ t\geqslant 0$.*

Finally, we give a result that reveals the relationship between the inverse boundary value problem and the inverse boundary spectral problem for the beam equation.

Theorem 3.2. *The following two inverse problems are equivalent*:

(i) *Determine $\rho(\cdot)$ and $r(\cdot)$ from the boundary spectral data* $\{\omega_n, \phi_n(1), \phi_n'(1)\}_{n=1}^{\infty}$.
(ii) *Determine $\rho(\cdot)$ and $r(\cdot)$ from the input–output maps*: $g(t) \to u(1,t)$, $g(t) \to u_x(1,t)$, $t \geqslant 0$.

Proof. (i) ⇒ (ii) is just Theorem 3.1.

(ii) ⇒ (i). By (13) and (19), we see that the two maps $g(t) \to u(1,t)$ and $g(t) \to u_x(1,t)$ are completely determined by boundary spectral data $\{\omega_n, \phi_n(1), \phi_n'(1)\}_{n=1}^{\infty}$. The proof is completed. □

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