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AGE-DEPENDENT POPULATION DYNAMICS BASED ON PARITY INTERVAL PROGRESSION

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Abstract—A controlled age-dependent population equation based on the parity interval progression is developed. The asymptotic properties for the stationary case are studied by investigating the spectrum of the corresponding population operator in the semigroup framework.

1. INTRODUCTION

The McKendrick equation of population dynamics [1] is one of the most important age-dependent population models. It has been used to study a great many biological and physical phenomena. Its applications to human populations has recently been reported in [2]. However, being a macrosopic model, it has its limitations. In this article, we shall develop a microscopic population model based on the concepts of birth parity or order (number of births) and birth interval. These ideas can be traced back to Henry [3] and later Whelpton [4] and Feeney [5]. We believe that age structure alone is not adequate to explain the human population. Birth parity and birth interval should play an important role. We are particularly interested in birth control strategies involving both of them and hopefully building up a framework for integrating micro-level birth analysis into macro-level studies of fertility and population growth trends.

2. THE MODEL

Let N be the maximal birth order (number of births) ever attained by females in a closed population. $p_n(r,t,s)$, $n = 0,1,2,\dots,N$ denotes the age distribution of females who have the parity n and whose n^{th} birth has age s. $p_0(r,t,s) = p_0(r,t)$ is the age density of females who have no births at time t. Obviously, the age distribution of females of population $p_f(r,t)$ can be expressed as

$$p_f(r,t) = p_0(r,t) + \sum_{n=1}^{N} \int_0^{r_m - r_1} p_n(r,t,s)\, ds, \tag{1}$$

where r_m is the age limit, $[r_1, r_2]$ denotes the fecundity interval of females, $r_1 > 0$.

Define the age-parity-interval-specific fertility $f_n(r,t,s)$ by

$$\begin{aligned} f_n(r,t,s) &= \frac{\phi_n(r,t,s)}{p_{n-1}(r,t,s)}, \quad n = 1,2,\dots,N, \\ f_1(r,t,s) &= f_1(r,t), \end{aligned} \tag{2}$$

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where $\phi_n(r,t,s)\,dt$ is the number of women aged r who had $n-1$ births previously and whose $(n-1)^{\text{th}}$ birth has age s in time interval $(t, t+dt)$. Of course, $f(r,t,s)$ take zero values outside $[r_1, r_2] \times [0,\infty) \times [0, r_2 - r_1]$.

The absolute birth rate $\phi(t)$ of the population can then be computed as

$$\phi(t) = \int_{r_1}^{r_2} f_1(r,t)\; p_0(r,t)\; dr + \sum_{n=2}^{N} \int_{r_1}^{r_2} \int_0^{r-r_1} f_n(r,t,s)\; p_{n-1}(r,t,s)\; ds\; dr. \tag{3}$$

Denoting the female relative mortality rate by $\mu_f(r,t)$, then the definitions above show that in a small enough time interval Δt and age interval Δr, one has

$$\begin{aligned} p_n(r+\Delta t, t+\Delta t, s+\Delta t)\; \Delta r = {} & [1 - \mu_f(r,t)\; \Delta t] \\ & [p_n(r,t,s)\; \Delta r - f_{n+1}(r,t,s)\; p_n(r,t,s)\; \Delta r \Delta t]. \end{aligned} \tag{4}$$

Taking the limits as $\Delta r \to 0$, $\Delta t \to 0$, one gets the basic partial differential equation

$$\begin{gathered} Dp_n(r,t,s) = -\mu_f(r,t)\; p_n(r,t,s) - f_{n+1}(r,t,s)\; p_n(r,t,s), \\ \text{for all} \quad t > 0 \quad \text{and} \quad 0 < r < r_m, \end{gathered} \tag{5}$$

where

$$Dp_n(r,t,s) = \lim_{\Delta h \to 0} \frac{p_n(r+\Delta h, t+\Delta h, s+\Delta h) - p_n(r,t,s)}{\Delta h}, \tag{6}$$

or

$$Dp_n(r,t,s) = \frac{\partial p_n(r,t,s)}{\partial t} + \frac{\partial p_n(r,t,s)}{\partial r} + \frac{\partial p_n(r,t,s)}{\partial s}, \tag{7}$$

if $p_n(r,t,s)$ is differentiable with respect to r, t, s.

In order to solve Equation (5), some boundary conditions are required. From the definitions, we have

$$\begin{aligned} p_1(r,t,0) &= f_1(r,t)\; p_0(r,t), \\ p_n(r,t,0) &= \int_0^{r_2-r_1} f_n(r,t,s)\; p_{n-1}(r,t,s)\; ds, \quad n \ge 2, \end{aligned} \tag{8}$$

$$\begin{aligned} p_0(0,t) &= k_0(t) \left[\int_{r_1}^{r_2} f_1(r,t)\; p_0(r,t)\; dr + \sum_{n=2}^{N} \int_{r_1}^{r_2} \int_0^{r-r_1} f_n(r,t,s)\; p_{n-1}(r,t,s)\; dr\, ds \right], \\ p_n(0,t,s) &= 0, \quad n \ge 1. \end{aligned} \tag{9}$$

where $k_0(t)$ is the infant's sex ratio.

Let $p_n(r,0,s) = p_{n0}(r,s), p_{00}(r,s) = p_{00}(r)$ be the initial age distribution of parity n and interval s. We then have the population equation of age-parity-interval progression model:

$$\begin{gathered} Dp_n(r,t,s) = -\mu_f(r,t,s)\; p_n(r,t,s) - f_{n+1}(r,t,s)\; p_n(r,t,s), \\ p_n(r,0,s) = p_{n0}(r,s), \quad 0 < r < r_m, \quad 0 \le s \le r_m - r_1, \\ \text{boundary conditions (8) and (9).} \end{gathered} \tag{10}$$

Naturally the mortality rate satisfies the following conditions:

$$\int_0^r \mu_f(\rho,t)\; d\rho < +\infty, \quad \text{for} \quad r < r_m; \quad \int_0^{r_m} \mu_f(\rho,t)\; d\rho = +\infty. \tag{11}$$

Many important indices in birth analysis can be expressed in terms of $f_n(r,t,s)$ and $p_n(r,t,s)$. For example, the n^{th} fertility $\beta_n(r,t)$ of females aged r

$$\beta_n(r,t) = \frac{f_n(r,t)\; p_{n-1}(r,t)}{\sum_{n=0}^{N} p_n(r,t)}, \quad n = 1, 2, \ldots, N, \tag{12}$$

where $p_n(r,t) = \int_0^{r_m-r_1} p_n(r,t,s)\ ds, \quad n \geq 1$, denotes the age distribution density of females with parity n, and

$$f_n(r,t) = \frac{\int_0^{r_m-r_1} f_n(r,t,s)\ p_{n-1}(r,t,s)\ ds}{p_{n-1}(r,t)}, \quad n = 2,3,\ldots,N. \tag{13}$$

Others, include the average fertility $f(r,t)$ of all women aged r at time t

$$f(r,t) = \sum_{n=1}^{N} \beta_n(r,t) \tag{14}$$

and the age-specific fertility $\beta(t)$ of females

$$\beta(t) = \int_{r_1}^{r_2} f(r,t)\ dr. \tag{15}$$

If $\lambda_n(r,t,s)$ denotes the females parity-interval ratio,

$$\lambda_n(r,t,s) = \frac{p_n(r,t,s)}{p_f(r,t)}, \quad n = 0,1,2,\ldots,N, \tag{16}$$

$$\lambda_0(r,t,s) = \lambda_0(r,t), \quad \sum_{n=0}^{N} \int_0^{r_m-r_1} \lambda_n(r,t,s)\ ds = 1, \tag{17}$$

then in view of the fact that for small enough $\Delta r, \Delta t$

$$p_f(r+\Delta t, t+\Delta t, s+\Delta t)\ \Delta r = [1-\mu_f(r,t)\ \Delta t]\ p_f(r,t,s), \tag{18}$$

and combining (4) we get immediately the female parity-interval ratio equation

$$\begin{aligned}
&D\lambda_n(r,t,s) = -f_{n+1}(r,t,s)\cdot\lambda_n(r,t,s),\\
&\lambda_n(r,0,s) = \lambda_{n0}(r,s), \quad n \geq 0,\\
&\lambda_1(r,t,0) = f_1(r,t)\ \lambda_0(r,t),\\
&\lambda_n(r,t,0) = \int_0^{r_2-r_1} f_n(r,t,s)\ \lambda_{n-1}(r,t,s)\ ds, \quad n \geq 2,\\
&\lambda_n(0,t,s) = \begin{cases} 1, & \text{if } n = 0,\\ 0, & \text{if } n \neq 0.\end{cases} \quad n = 0,1,\ldots,N,
\end{aligned} \tag{19}$$

where $\lambda_{n0}(r,s)$, $\lambda_{00}(r,s) = \lambda_{00}(r)$ are the initial conditions, and

$$\lambda_{00}(r,s) + \int_0^{r_m-r_1} \sum_{n=0}^{N} \lambda_{n0}(r,s)\, ds = 1.$$

In the application to population forecasting, Equation (19) is more convenient since the mortality function does not appear in this model.

$$\lambda_n(r,t) = \int_0^{r_2-r_1} \lambda_n(r,t,s)\ ds, \quad n \geq 2, \tag{20}$$

is the female parity ratio and the age-parity fertility

$$f_n(r,t) = \frac{\int_0^{r_m-r_1} f_n(r,t,s)\ \lambda_{n-1}(r,t,s)\ ds}{\lambda_{n-1}(r,t)}, \quad n = 2,3,\ldots,N. \tag{21}$$

Furthermore, integrating with respect to s from 0 to $r_m - r_1$ on both sides of Equation (10), we get the age-dependent parity progression model. On the other hand, if we let

$$x_n(t,s) = \int_0^{r_m} p_n(r,t,s)\, dr, \tag{22}$$

$x_n(t,s)$ denotes the number of females who had n births and whose n^{th} birth has age s, then integrating with respect to r from 0 to r_m on both sides of Equation (10), we have

$$\begin{aligned}
&Dx_n(t,s) = -\mu_f(t)\; x_n(t,s) - f_{n+1}(t,s)\; x_n(t,s), \quad n \geq 1,\\
&\quad Dx_0(t) = -\mu_f(t)\; x_0(t) - (1-k_0(t))\; f_1(t)\; x_0(t) + k_0(t) \int_0^{r_2-r_1} \sum_{n=2}^{N} f_n(t,s)\; x_{n-1}(t,s)\; ds,\\
&x_n(0,s) = x_{n0}(s), \quad n \geq 0,\\
&x_1(t,0) = f_1(t)\; x_0(t),\\
&x_n(t,0) = \int_0^{r_2-r_1} f_n(t,s)\; x_{n-1}(t,s)\; ds, \quad n \geq 2,
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
\mu_f(t)\; x_n(t,s) &= \int_0^{r_m} \mu_f(r,t)\; x_n(r,t,s)\; dr,\\
f_n(t,s)\; x_{n-1}(t,s) &= \int_0^{r_m} f_n(r,t,s)\; x_{n-1}(r,t,s)\; dr, \quad \text{for} \quad n \geq 2,\\
f_1(t)\; x_0(t) &= \int_0^{r_m} f_1(r,t)\; x_0(r,t)\; dr.
\end{aligned}$$

$f_n(t,s)\; x_{n-1}(t,s)$ denotes all the births of the females who had $n-1$ births previously and whose $(n-1)^{\text{th}}$ birth has age s. Equation (23) is very similar to the yeast model discussed in [6]. In fact, starting from Equation (10), we can even deduce the McKendric type population model.

From Equation (10) we see that, when the initial conditions $p_{n0}(r,s)$ are given, the age-parity-interval density $p_n(r,t,s)$ can be determined by the parameters $f_n(r,t,s)$ describing the age distribution of females who have a given birth order and birth interval. The parameters $f_n(r,t,s)$, $n = 1, 2, \ldots, N$, can thus be considered as control variables. A birth control or family-planning program is essentially the regulation of $f_n(r,t,s)$, i.e., control via the birth order and birth interval. For example, in the one-child program one just sets $f_2 = f_3 = \cdots = f_N = 0$. A two-child program is equivalent to the case $f_3 = f_4 = \cdots = f_N = 0$, etc. Any viable birth policy can only be regulated on the basis of the natural birth level of females of the society. For this reason, we consider the "standard age-parity-interval progression fertilities" as the average age-parity-interval progression fertilities of women under the natural fertile state (i.e., no artificial means are imposed). The standard age-parity-interval progression fertilities are denoted by

$$h_n(r,t,s), \quad n = 1, 2, \ldots, N, \tag{24}$$

$h_1(r,t,s) = h_1(r,t)$. In a stationary period we can assume $h_n(r,t,s) = h_n(r,s)$, independent of time t. The programmed age-parity progression fertilities are hence expressed as

$$f_n(r,t,s) = \beta_n(t,s)\; h_n(r,s), \quad n = 1, 2, \ldots, N, \tag{25}$$

$\beta_1(t,s) = \beta_1(t)$. Naturally $h_n(r,s)$ take zero values outside of $[r_1, r_2] \times [0, r_2 - r_1]$.

The parameters $\beta_n(t,s)$ appearing in (25) clearly indicate the relative level of the actual age-parity-interval progression fertilities comparing to the standard age-parity-interval progression fertilities. They form a group of control variables corresponding to various birth policies. For example,

(i) birth order control: that only "one child" is allowed for every fertile woman is just to say $\beta_1 = 1$, and $\beta_2 = \beta_3 = \cdots = \beta_N = 0$; only "two children" is permitted is equivalent to that $\beta_1 = \beta_2 = 1$, $\beta_3 = \beta_4 = \cdots = \beta_N = 0$;

(ii) birth interval control:

$$\beta_1 = 1, \quad \beta_n(t,s) = \begin{cases} 1, & \text{if } s \geq s_0, \\ 0, & \text{if } s < s_0, \end{cases} \quad n = 2,3,\dots,N, \tag{26}$$

correponds to the policy that the first birth is allowed for every fertile women, but another birth is not permitted until the age of her last child is beyond s_0;

(iii) birth order and interval control: first birth is not restricted and the second is permitted only when the age of her first child is beyond s_0, and third birth is forbidden is just to say $\beta_1 = 1$, $\beta_3 = \cdots = \beta_N = 0$, and β_2 take the form of (26).

3. THE STATIONARY CASE

In this section, we will consider the stationary age-parity-interval progression system, i.e., $\beta_n(t,s) = \beta_n(s)$, $\beta_1(s) = \beta_1$, $k_0(t) = k_0$, $\mu_f(r,t) = \mu_f(r)$, $f_n(r,t,s) = \beta_n(s)\, h_n(r,s)$, are all independent of time t for all n, $n = 1,2,\dots,N$, and measurable with respect to their variables in Equation (10). This system describes, to a certain extent, the birth dynamic process of a stationary closed population within a short period of time. The system is as follows:

$$\begin{aligned}
&Dp_n(r,t,s) = -\mu_f(r)\, p_n(r,t,s) - \beta_{n+1}(s)\, h_{n+1}(r,s) \cdot p_n(r,t,s), \quad n \geq 0,\\
&p_n(r,0,s) = p_{n0}(r,s), \quad n \geq 0,\\
&p_1(r,t,0) = \beta_1 h_1(r)\, p_0(r,t),\\
&p_n(r,t,0) = \int_0^{r_2-r_1} \beta_n(s)\, h_n(r,s)\, p_{n-1}(r,t,s)\, ds, \quad n \geq 2,\\
&p_0(0,t) = k_0 \left[\int_{r_1}^{r_2} \beta_1 h_1(r)\, p_0(r,t)\, dr + \sum_{n=2}^{N} \int_{r_1}^{r_2} \int_{r_1}^{r-r_1} \beta_n(s)\, h_n(r,s)\, p_{n-1}(r,t,s)\, ds\, dr\right],\\
&p_n(0,t,s) = 0, \quad n \geq 1,
\end{aligned} \tag{27}$$

where $0 \leq \beta_n(s),\ h_n(r,s) \leq 1$, $\text{meas}\{r \in [r_1,r_2] \times [0,r] \mid h_n(r,s) \neq 0\} > 0$ for $n = 1,2,\dots,N$. For $n = 1,2,\dots,N$, let

$$\mathbf{P}(r,t,s) = \begin{bmatrix} p_0(r,t,s) \\ p_1(r,t,s) \\ \vdots \\ p_N(r,t,s) \end{bmatrix}, \quad \mathbf{P}_0(r,s) = \begin{bmatrix} p_{00}(r,s) \\ p_{01}(r,s) \\ \vdots \\ p_{0N}(r,s) \end{bmatrix},$$

$$\mathbf{A}_n(r,s) = \begin{bmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots & & \vdots \\ \vdots & & & -h_n(r,s) & & \vdots \\ \vdots & & & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}_{(N+1)\times(N+1)}$$

$$\mathbf{B}_0(r,s) = \frac{1}{r_2 - r_1} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \mathbf{B}_1(r,s) = h_1(r)\, \mathbf{B}_0(r,s),$$

$$\mathbf{B}_n(r,s) = -\mathbf{A}_n(r,s), \quad n \geq 2,$$

$$\mathbf{C}_n(r,s) = \begin{bmatrix} 0 & \cdots & h_n(r,s) & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}_{(N+1)\times(N+1)}$$

Then Equation (10) can be written as

$$
\begin{aligned}
&DP(r,t,s) = [-\mu_f(r) - \sum_{n=1}^{N} \beta_n(s)\ \mathbf{A}_n(r,s)]\ \mathbf{P}(r,t,s),\\
&\mathbf{P}(r,0,s) = \mathbf{P}_0(r,s),\\
&\mathbf{P}(r,t,0) = \int_0^{r_2-r_1} \left[\sum_{n=0}^{N} \beta_n \mathbf{B}_n(r,s)\right] \mathbf{P}(r,t,s)\ ds,\\
&\mathbf{P}(0,t,s) = \int_{r_1}^{r_2} \int_0^{r_2-r_1} \left[\sum_{n=1}^{N} \beta_n \mathbf{C}_n(r,s)\right] \mathbf{P}(r,t,s)\ ds\, dr.
\end{aligned}
\tag{28}
$$

Let $\mathbf{T}(r_0, s_0, h)$ be the solution of following matrix ordinary differential Equation

$$
\begin{aligned}
&\frac{d\mathbf{X}(h)}{dh} = -\left[\sum_{n=1}^{N} \beta_n(s)\ \mathbf{A}_n(r_0+h, s_0+h)\right] \mathbf{X}(h),\\
&\mathbf{X}(0) = I,
\end{aligned}
\tag{29}
$$

then integrating along the characteristic curve, we can get the following integrated version of Equation (28):

$$
\mathbf{P}(r,t,s) = \begin{cases}
\mathbf{T}(s-t, r-t, t)\ \mathbf{P}_0(r-t, s-t), \quad r, s \geq t,\\
\mathbf{T}(r-s, t-s, s) \int_0^{r_2-r_1} \left[\sum_{n=0}^{N} \beta_n\ B_n(r-s,\tau)\right] \mathbf{P}(r-s, t-s, \tau)\ d\tau,\\
\qquad\qquad r \geq s, t > s,\\
\int_{r_1}^{r_2} \int_0^{r_2-r_1} \left[\sum_{n=1}^{N} \beta_n \mathbf{C}_n(\vartheta,\tau)\right] \mathbf{P}(\vartheta, t-r, \tau)\ d\vartheta d\tau, \quad t,\ s > r.
\end{cases}
\tag{30}
$$

Now, we discuss the system in the Banach space $\mathbf{X} = L(0, r_m) \times (L(0, r_m) \times (0, r_m - r_1))^N$. In the spirit of [1], we first have:

THEOREM 1. *For every $\mathbf{P}_0(r,s) \in \mathbf{X}$, there exists a unique solution to system (30) which can be expressed as*

$$\mathbf{P}(r,t,s) = \mathbb{T}(t)\ \mathbf{P}_0(r,s) \in C(0,T;\ \mathbf{X}), \quad \text{for all} \quad T > 0,$$

where $\mathbb{T}(t)$ is a strongly continuous semigroup in $\mathbf{X}$, and its infinitesimal generator $\mathbb{A}$ is defined by

$$
\mathbb{A}\Phi(r,s) = -\frac{d}{dr}\left[\frac{d}{ds}\int_0^r \Phi(\vartheta,s)\ d\vartheta + \Phi(r,s)\right] - \mu_f(r)\ \Phi(r,s) - \sum_{n=1}^{N} \beta_n(s)\mathbf{A}_n(r,s)\ \Phi(r,s),
$$

$$
\begin{aligned}
D(\mathbb{A}) = \{&\Phi \in \mathbf{X} : (r,s) \mapsto \textstyle\int_0^r \Phi(\vartheta,s)\, d\vartheta\\
&\quad \textit{is absolutely continuous in } s \textit{ for } r \in [0, r_m],\\
&(r,s) \mapsto \left[\tfrac{d}{ds}\textstyle\int_0^r \Phi(\vartheta,s)\, d\vartheta + \Phi(r,s)\right]\\
&\quad \textit{is absolutely continuous in } r \textit{ for } s \in [0, r_m - r_1], \text{ a.e.},\\
&\lim_{r\to 0} \left[\tfrac{d}{ds}\textstyle\int_0^r \Phi(\vartheta,s)\ d\vartheta + \Phi(r,s)\right] = \textstyle\int_{r_1}^{r_2}\int_0^{r_2-r_1} \left[\sum_{n=1}^{N} \beta_n \mathbf{C}_n(r,s)\right] \Phi(r,s)\, ds\, dr,\\
&\quad \textit{for } s \in [0, r_m - r_1], \text{ a.e.},\\
&(r,s) \mapsto \textstyle\int_0^s \Phi(r,\tau)\ d\tau\\
&\quad \textit{is absolutely continuous in } r \textit{ for } s \in [0, r_m - r_1], \text{ a.e.},\\
&(r,s) \mapsto \left[\tfrac{d}{dr}\textstyle\int_0^s \Phi(r,\tau)\ d\tau + \Phi(r,s)\right]\\
&\quad \textit{is absolutely continuous in } s \textit{ for } r \in [0, r_m], \text{ a.e.},\\
&\lim_{s\to 0} \left[\tfrac{d}{dr}\textstyle\int_0^s \Phi(r,\tau)\ d\tau + \Phi(r,s)\right] = \textstyle\int_0^{r_2-r_1} \left[\sum_{n=0}^{N} \beta_n \mathbf{B}_n(r,s)\right] \Phi(r,s)\, ds,\\
&\quad \textit{for } r \in [0, r_m], \text{ a.e.,}\}
\end{aligned}
\tag{31}
$$

Furthermore, $\pi(t)$ is also a positive semigroup.

On the other hand, starting directly from (27), we have

$$p_n(r,t,s) = \begin{cases} e^{-\int_{s-t}^{s}[\mu_f(\rho+r-s)+\beta_{n+1}(\rho)\, h_{n+1}(\rho+r-s,\rho)]d\rho}\, p_{0n}(r-t,s-t), & r,\ s \ge t, \\ e^{-\int_0^{s}[\mu_f(\rho+r-s)+\beta_{n+1}(\rho)\, h_{n+1}(\rho+r-s,\rho)]d\rho} \int_0^{r_2-r_1} \beta_n(\tau) \\ \quad \times h_n(r-s,\tau)\, p_{n-1}(r-s,t-s,\tau)\, d\tau, & r \ge s, \quad t > s, \\ 0, \quad t,\ s > r, \quad n = 2,3,\dots,N, \end{cases}$$

$$p_1(r,t,s) = \begin{cases} e^{-\int_{s-t}^{s}[\mu_f(\rho+r-s)+\beta_2(\rho)\, h_2(\rho+r-s,\rho)]d\rho}\, p_{01}(r-t,s-t), & r,\ s \ge t, \\ e^{-\int_0^{s}[\mu_f(\rho+r-s)+\beta_2(\rho)\, h_2(\rho)+r-s,\rho)]d\rho} \beta_1 h_1(r-s) \\ \quad \times p_0(r-s,t-s), \quad r \ge s,\ t > s, \\ 0, \quad t,\ s > r, \end{cases}$$

$$p_0(r,t) = \begin{cases} e^{-\int_{r-t}^{r}[\mu_f(\rho)+\beta_1 h_1(\rho)]\, d\rho}\, p_{00}(r-t), \quad r \ge t, \\ k_0 \int_{r_1}^{r_2} \beta_1 h_1(\vartheta)\, p_0(\vartheta,t-r)\, d\vartheta + k_0 \int_{r_1}^{r_2}\int_0^{r_2-r_1} \left[\sum_{n=2}^{N} \beta_n(\tau)\, h_n(\vartheta,\tau)\right] \\ \quad \times p_{n-1}(\vartheta,t-r,\tau)\, d\vartheta d\tau, \quad r < t. \end{cases} \tag{32}$$

Starting from (32), it can be deduced iteratively that

$$\begin{aligned} p_n(r,t,s) &= K_n^{(n)}(r,t,s)\, p_{n0}(r-t,s-t) \\ &\quad + \int_0^{r_2-r_1} K_n^{(n)}(r,t,s) p_{n-1,0}(r-t,s-t+\tau_1)\, d\tau_1 + \dots \\ &\quad + \int_0^{r_2-r_1} \cdots \int_0^{r_2-r_1} K_1^{(n)}(r,t,s,\tau_1,\dots,\tau_{n-1}) \\ &\quad \times p_{10}(r-t,s-t+\tau_1+\tau_2+\cdots+\tau_{n-1})\, d\tau_1\, d\tau_2 \dots d\tau_{n-1} \\ &\quad + \int_0^{r_2-r_1} \cdots \int_0^{r_2-r_1} K_0^{(n)}(r,t,s,\tau_1,\tau_2\dots\tau_{n-1}) \\ &\quad \times p_0(r-s-\tau_1-\tau_2-\cdots-\tau_{n-1},t-s-\tau_1-\tau_2-\cdots-\tau_{n-1})\, d\tau_1\, d\tau_2 \dots d\tau_{n-1}, \\ &= \sum_{m=0}^{n-1} \int_0^{r_2-r_1} \cdots \int_0^{r_2-r_1} K_{n-m}^{(n)}(r,t,s,\tau_1,\tau_2\dots\tau_m) \\ &\quad \times p_{n-m,0}(r-t,s-t+\tau_1+\tau_2+\cdots+\tau_m)\, d\tau_1\, d\tau_2 \dots d\tau_m \\ &\quad + \int_0^{r_2-r_1} \cdots \int_0^{r_2-r_1} K_0^{(n)}(r,t,s,\tau_1,\tau_2\dots\tau_{n-1}) \\ &\quad \times p_0(r-s-\tau_1-\tau_2-\dots\tau_{n-1},t-s-\tau_1-\tau_2-\cdots-\tau_{n-1}) \\ &\quad \times d\tau_1\, d\tau_2 \dots d\tau_{n-1}, \quad n = 1,2,\dots,N, \end{aligned} \tag{33}$$

where $K_m^{(n)}$, $m = 0,1,\dots,n$ are some bounded measurable functions and are continuous with respect to t, and here we assume that all functions take zero values outside their domains of definition.

So $p_0(r,t)$ can be expressed as

$$p_0(r,t) = \begin{cases} e^{-\int_{r-t}^{r}[\mu_f(\rho)+\beta_1 h_1(\rho)\ d\rho}\ p_{00}(r-t), \quad r \geq t, \\ k_0 \int_{r_1}^{r_2} \beta_1 h_1(\vartheta)\ p_0(\vartheta, t-r)\ d\vartheta + k_0 \int_{r_1}^{r_2} \int_0^{r_2-r_1} d\theta\, d\tau \Bigg[\sum_{n=2}^{N} \beta_n(\tau)\ h_n(\vartheta,\tau) \\ \quad \times \sum_{m=0}^{n-2} \int_0^{r_2-r_1} \cdots \int_0^{r_2-r_1} K_{n-1-m}^{(n-1)}(\vartheta, t-r, \tau, \tau_1, \tau_2, \ldots, \tau_m) \\ \quad \times p_{n-1-m,0}(\vartheta - t + r, \tau - t + r + \tau_1 + \tau_2 + \cdots + \tau_m) \Bigg] \\ \quad \times d\tau_1\, d\tau_2 \ldots d\tau_m + k_0 \int_{r_1}^{r_2} \int_0^{r_2-r_1} d\theta\, d\tau \Bigg[\sum_{n=2}^{N} \beta_n(\tau)\, h_n(\vartheta,\tau) \int_0^{r_2-r_1} \cdots \int_0^{r_2-r_1} \\ \quad \times K_0^{(n-1)}(\vartheta, t-r, \tau, \tau_1, \tau_2, \ldots, \tau_{n-2}) \\ \quad \times p_0(\vartheta - \tau - \tau_1 - \tau_2 - \cdots - \tau_{n-2}, t - r - \tau - \tau_1 - \tau_2 - \cdots - \tau_{n-2}) \Bigg] \\ \quad \times d\tau_1\, d\tau_2 \ldots d\tau_{n-2}, \quad r < t. \end{cases} \tag{34}$$

In particular, when $t > r_m$, we have

$$\begin{aligned} p_0(r,t) = {} & k_0 \int_{r_1}^{r_2} \beta_1 h_1(\vartheta)\, p_0(\vartheta, t-r)\, d\vartheta + k_0 \int_{r_1}^{r_2} \int_0^{r_2-r_1} d\theta\, d\tau \\ & \times \Bigg[\sum_{n=2}^{N} \beta_n(\tau)\, h_n(\vartheta,\tau) \int_0^{r_2-r_1} \cdots \int_0^{r_2-r_1} K_0^{(n-2)}(\vartheta, t-r, \tau, \tau_1, \tau_2 \ldots \tau_{n-2}) \\ & \times p_0(\vartheta - \tau - \tau_1 - \tau_2 - \cdots - \tau_{n-2}, t - r - \tau - \tau_1 - \tau_2 - \cdots - \tau_{n-2}) \Bigg] \\ & \times d\tau_1\, d\tau_2 \ldots d\tau_{n-2}, \end{aligned} \tag{35}$$

and when $t \leq r_1$

$$p_0(r,t) = \begin{cases} e^{-\int_{r-t}^{r}[\mu_f(\rho)+\beta_1 h_1(\rho)]\ d\rho}\ p_{00}(r-t), \quad r \geq t, \\ k_0 \int_{r_1}^{r_2} \beta_1 h_1(\vartheta) p_0(\vartheta, t-r)\ d\vartheta + k_0 \int_{r_1}^{r_2} \int_0^{r_2-r_1} d\theta\, d\tau \Bigg[\sum_{n=2}^{N} \beta_n(\tau)\ h_n(\vartheta,\tau) \\ \quad \times \sum_{m=0}^{n-2} \int_0^{r_2-r_1} \cdots \int_0^{r_2-r_1} K_{n-1-m}^{(n-1)}(\vartheta, t-r, \tau, \tau_1, \tau_2 \ldots \tau_m) \\ \quad \times p_{n-1-m,0}(\vartheta - t + r, \tau - t + r + \tau_1 + \tau_2 + \cdots + \tau_m) \Bigg] d\tau_1\ d\tau_2 \ldots d\tau_m \\ \quad + k_0 \int_{r_1}^{r_2} \int_0^{r_2-r_1} \Bigg[\sum_{n=2}^{N} \beta_n(\tau)\ h_n(\vartheta,\tau) \int_0^{r_2-r_1} \cdots \int_0^{r_2-r_1} \\ \quad \times K_0^{(n-1)}(\vartheta, t-r, \tau, \tau_1, \tau_2 \ldots \tau_{n-2})\ e^{\int_{\vartheta-t+r}^{\vartheta-\tau-\tau_1-\ldots\tau_{n-2}}[\mu_f(\rho)+\beta_1 h_1(\rho)]\ d\rho} \\ \quad \times p_{00}(\vartheta - t + r) \Bigg] d\tau_1\ d\tau_2 \ldots d\tau_{n-2}, \quad r < t. \end{cases}$$

$$
\begin{aligned}
p_n(r,t,s) = & \sum_{m=0}^{n-1} \int_0^{r_2-r_1} \cdots \int_0^{r_2-r_1} K_{n-m}^{(n)}(r,t,s,\tau_1,\tau_2,\ldots,\tau_m) \\
& \times p_{n-m,0}(r-t, s-t+\tau_1+\tau_2+\cdots+\tau_m)\, d\tau_1\, d\tau_2 \ldots d\tau_m \\
& + \int_0^{r_2-r_1} \cdots \int_0^{r_2-r_1} K_0^{(n)}(r,t,s,\tau_1,\tau_2 \ldots \tau_{n-1}) \\
& \times e^{-\int_{r-t}^{r-s-\tau_1-\ldots\tau_{n-1}} [\mu_f(\rho)+\beta_1 h_1(\rho)]\, d\rho} p_{00}(r-t)\, d\tau_1\, d\tau_2 \ldots d\tau_{n-1},
\end{aligned} \tag{35'}
$$

$$
p_n(r,t,s) = \mathbb{T}(r_1)^{\left[\frac{t}{r_1}\right]}\, \mathbb{T}\left(t - \left[\frac{t}{r_1}\right]\right), \quad \text{for } t \geq r_1,\ n \geq 1.
$$

Deducing iteratively from (35′), we can determine uniquely $p_0(r,t)$ from $(p_{00}, p_{01}, \ldots, p_{0N})$ and $p_0(r,t)$ form a compact set of $L((0,r_m) \times (r_m,t))$ for all bounded $(p_{00}, p_{01}, \ldots, p_{0N}) \in \mathbf{X}$, when $t > r_m$. This shows that:

THEOREM 2. *$\mathbb{T}(t)$ is compact in $\mathbf{X}$ for $t \geq r_m$, but not for $t < r_m$, and hence does not have an analytic extension.*

By Theorem 2 we know that the spectrum, $\sigma(\mathbb{A})$, of the operator $\mathbb{A}$ consists of distinct eigenvalues of $\mathbb{A}$. There is only a finite number of eigenvalues of A in any finite strip parallel to the imaginary axis. Now we investigate the spectrum of $\mathbb{A}$. If $\lambda \in \sigma(\mathbb{A})$, the point spectrum of $\mathbb{A}$, then there exists a nonzero element $\Phi \in \mathbf{X}$, such that $\mathbb{A}\Phi = \lambda\Phi$, this is equivalent to saying that $\mathbb{T}(t)\Phi = e^{\lambda t}\Phi$, for all $t \geq 0$. By (32), $\Phi = (\phi_0, \phi_1, \ldots, \phi_N)^T$ satisfies

$$
\phi_n(r,s) = \begin{cases} e^{-\lambda s - \int_0^s [\mu_f(\rho+r-s)+\beta_{n+1}(\rho)\, h_{n+1}(\rho+r-s,\rho)]\, d\rho} \\ \quad \times \int_0^{r_2-r_1} \beta_n(\tau)\, h_n(r-s,\tau)\, \phi_{n-1}(r-s,\tau)\, d\tau, & r \geq s, \\ 0, \quad s > r, \quad n = 2,3,\ldots,N, \end{cases}
$$

$$
\phi_1(r,s) = \begin{cases} e^{-\lambda s - \int_0^s [\mu_f(\rho+r-s)+\beta_2(\rho)\, h_2(\rho+r-s,\rho)]\, d\rho} \beta_1 h_1(r-s)\, \phi_0(r-s), & r \geq s, \\ 0, \quad s > r, \end{cases}
$$

$$
\begin{aligned}
\phi_0(r) = & e^{-\lambda r - \int_0^r [\mu_f(\rho)+\beta_1(\rho)]\, d\rho} \\
& \times \left[k_0 \int_{r_1}^{r_2} \beta_1 h_1(\vartheta)\, \phi_0(\vartheta)\, d\vartheta + k_0 \int_{r_1}^{r_2} \int_0^{r-r_1} \sum_{n=2}^{N} \beta_n(\tau)\, h_n(\vartheta,\tau)\, \phi_{n-1}(\vartheta,\tau)\, d\vartheta\, d\tau \right].
\end{aligned} \tag{36}
$$

Iterating, we have

$$
\begin{aligned}
\phi_n(r,s) = & e^{-\lambda s - \int_0^s [\mu_f(\rho+r-s)+\beta_{n+1}(\rho)\ h_{n+1}(\rho+r-s,\rho)]\, d\rho} \\
& \times \int_0^{r_2-r_1} \cdots \int_0^{r_2-r_1} \prod_{m=0}^{n-2} [\beta_{n-m}(\tau_{m+1})\ h_{n-m}(r-s-\tau_1-\cdots-\tau_m, \tau_{m+1}) \\
& \times e^{-\lambda\tau_{m+1} - \int_0^{\tau_{m+1}}} \mu_f(\rho+r-s-\tau_1-\cdots-\tau_{m+1}) \\
& \times e^{-\int_0^{\tau_{m+1}} \beta_{n-m}(\rho)\ h_{n-m}(\rho+r-s-\tau_1-\cdots-\tau_{m+1},\rho)\ d\rho} \\
& \times \beta_1 h_1(r-s-\tau_1-\cdots-\tau_{n-1})\ \phi_0(r-s-\tau_1-\cdots-\tau_{n-1})]\ d\tau_1 \ldots d\tau_{n-1}, \\
& n = 1,2,\ldots,N,
\end{aligned}
$$

$$
\begin{aligned}
\phi_0(r) = & \phi_0(0) e^{-\lambda r - \int_0^r [\mu_f(\rho)+\beta_1(\rho)]\, d\rho}, \\
\phi_0(0) = & k_0 \int_{r_1}^{r_2} \beta_1 h_1(\vartheta)\ \phi_0(\vartheta)\ d\vartheta + k_0 \int_{r_1}^{r_2} \int_0^{r-r_1} \left[\sum_{n=2}^{N} \beta_n(\tau)\ h_n(\vartheta,\tau) \right] \\
& \times \phi_{n-1}(\vartheta,\tau)\ d\vartheta\ d\tau.
\end{aligned} \tag{37}
$$

Hence

$$\phi_n(r,s) = e^{-\lambda r - \int_0^r \mu_f(\rho)\,d\rho}\, e^{-\int_0^s \beta_{n+1}(\rho)\, h_{n+1}(\rho+r-s,\rho)\,d\rho}$$
$$\times \int_0^{r_2-r_1} \cdots \int_0^{r_2-r_1} \prod_{m=0}^{n-2} [\beta_{n-m}(\tau_{m+1})\, h_{n-m}(r-s-\tau_1-\cdots-\tau_m, \tau_{m+1})$$
$$\times e^{-\int_0^{\tau_{m+1}} \beta_{n-m}(\rho)\, h_{n-m}(\rho+r-s-\tau_1-\cdots-\tau_m,\rho)\,d\rho}. \tag{38}$$
$$\times \beta_1 h_1(r-s-\tau_1-\cdots-\tau_{n-1})\, e^{-\int_0^{r-s-\tau_1-\cdots-\tau_{n-1}} \beta_1 h_1(\rho)\,d\rho}\Big]$$
$$\times d\tau_1 \ldots d\tau_{n-1}\, \phi_0(0), \quad n = 1,2,\ldots,N.$$

This says that λ is of geometric multiplicity 1. Let

$$F(\lambda) = 1 - k_0 \int_{r_1}^{r_2} \beta_1 h_1(r)\, e^{-\lambda r - \beta_1 r - \int_0^r \mu_f(\rho)\,d\rho}\,dr - k_0 \int_{r_1}^{r_2} e^{-\lambda r - \int_0^r \mu_f(\rho)\,d\rho}\,dr$$
$$\Bigg\{ \int_0^{r-r_1} ds \left[\sum_{n=2}^{N} \beta_n(s)\, h_n(r,s)\, e^{-\int_0^s \beta_n(\rho)\, h_n(\rho+r-s,\rho)\,d\rho} \right.$$
$$\times \int_0^{r_2-r_1} \cdots \int_0^{r_2-r_1} \left[\prod_{m=0}^{n-3} \beta_{n-1-m}(\tau_{m+1})\, h_{n-1-m}(r-s-\tau_1-\cdots-\tau_m, \tau_{m+1}) \right. \tag{39}$$
$$\times e^{-\int_0^{\tau_{m+1}} \beta_{n-1-m}(\rho)\, h_{n-1-m}(\rho+r-s-\tau_1-\cdots-\tau_m,\rho)\,d\rho}$$
$$\left.\left. \times\ \beta_1 h_1(r-s-\tau_1-\cdots-\tau_{n-2})\, e^{-\int_0^{r-s-\tau_1-\cdots-\tau_{n-2}} \beta_1 h_1(\rho)\,d\rho} \right] d\tau_1 \ldots d\tau_{n-2} \right] \Bigg\},$$

then $F(\lambda) \neq 0$. Conversely, if $F(\lambda) \neq 0$, then $\Phi = (\phi_0, \phi_1, \ldots, \phi_N)^T$, ϕ_n is defined by (37), where $\phi(0) = c$ is any nonzero costant, which satisfies $\mathbb{T}(t)\Phi = \lambda\Phi$, so $\lambda \in \sigma(\mathbb{A})$. Similar to [2] we can prove the following results.

THEOREM 3.

(i) *$\sigma(\mathbb{A}) = \sigma_p(\mathbb{A})$ consists of the zeros of the entire function $F(\lambda)$ defined by (39).*
(ii) *$\mathbb{A}$ has only one real eigenvalue λ_0, its algebraic multiplicity is 1.*
(iii) *$\sigma(\mathbb{A})$ is an infinite set.*
(iv) *The solution $\mathbf{P}(r,t,s)$ of Equation (30) has the following asymptotic expression*

$$\mathbf{P}(r,s,t) = \mathbf{P}_{\lambda_0}\mathbf{P}_0(r,s) \cdot e^{-\lambda_0 t} + o(e^{(-\lambda_0-\varepsilon)t}), \quad \text{as} \quad t \to \infty, \tag{40}$$

where $\varepsilon > 0$ is a small number such that $\sigma(\mathbb{A}) \cap \{\lambda \mid \lambda_0 - \varepsilon < Re\ \lambda < \lambda_0\} = \phi$, λ_0 is the dominant real eigenvalue (the growth index), and

$$\mathbf{P}_{\lambda_0}\mathbf{P}_0(r,s) = \lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)\, R(\lambda, \mathbb{A})\, \mathbf{P}_0(r,s), \tag{41}$$

where $\Phi(r,s) = R(\lambda,\mathbb{A})\,\Psi(r,s)$, it follows from the standard results in semigroup theory that $\Phi = R(\lambda,\mathbb{A})\,\Psi = \int_0^\infty e^{-\lambda t}\,\mathbb{T}(t)\,\Psi dt$, i.e., $\Phi = (\phi_0, \phi_1, \ldots, \phi_N)^T$ satisfies

$$\phi_n(r,s) = \begin{cases} \int_0^s e^{-\lambda t}\, e^{-\int_{s-t}^s [\mu_f(\rho+r-s)+\beta_{n+1}(\rho)\, h_{n+1}(\rho+r-s,\rho)]\,d\rho}\, \Psi_n(r-t, s-t)\,dt \\ \quad + e^{-\lambda s}\, e^{-\int_0^s [\mu_f(\rho+r-s)+\beta_{n+1}(\rho)\, h_{n+1}(\rho+r-s,\rho)]\,d\rho} \\ \quad \times \int_0^{r_2-r_1} \beta_n(\tau)\, h_n(r-s,\tau)\, \phi_{n-1}(r-s,\tau)\,d\tau, \quad r \geq s, \\ 0, \quad s > r, \quad n = 2,3,\ldots,N, \end{cases}$$

$$\phi_1(r,s) = \begin{cases} \int_0^s e^{-\lambda t}\, e^{-\int_{s-t}^s [\mu_f(\rho+r-s)+\beta_2(\rho)\, h_2(\rho+r-s,\rho)]\,d\rho}\, \psi_1(r-t, s-t)\,dt \\ \quad + e^{-\lambda s}\, e^{-\int_0^s [\mu_f(\rho+r-s)+\beta_2(\rho)\, h_2(\rho+r-s,\rho)]\,d\rho}\, \beta_1 h_1(r-s)\,\phi_0(r-s), \quad r \geq s, \\ 0, \quad s > r, \end{cases}$$

$$\phi_0(r) = \int_0^r e^{-\lambda t} e^{-\int_{r-t}^r [\mu_f(\rho)+\beta_1 h_1(\rho)]\,d\rho} \psi_0(r-t) + e^{-\lambda r} k_0$$
$$\times \left[\int_{r_1}^{r_2} \beta_2 h_1(\vartheta)\,\phi_0(\vartheta)\,d\vartheta + \int_{r_1}^{r_2} \int_0^{r_2-r_1} \left[\sum_{n=2}^N \beta_n(\tau)\,h_n(\vartheta,\tau) \right] \psi_{n-1}(\vartheta,\tau)\,d\vartheta\,d\tau \right]. \tag{42}$$

(v) *Let $N_n(t) = \int_0^{r_m} \int_0^{r-r_1} p_n(r,t,s)\,dr\,ds$ be the number of women who have parity n at time t, then when $\lambda_0 = 0$, there is a constant N_n^* such that*

$$\lim_{t\to\infty} N_n(t) = N_n^*, \tag{43}$$

the convergence is in a damped oscillarory fashion [7].

When $\lambda_0 = 0$, Theorem 3 tells us that

$$\lim_{t\to\infty} \mathbf{P}(r,t,s) = \mathbf{P}_0\mathbf{P}_0(r,s),$$

$\Phi_0(r,s) = \mathbf{P}_0\mathbf{P}_0(r,s)$ is a nonnegative equilibrium state of (30).

The distribution of $(\beta_1, \beta_2(s), \dots, \beta_N(s))$ for the case of $F(0) = 0$, i.e., the growth index $\lambda_0 = 0$, seems very important in practice, since the final target of the control of population is to pilot the growth index to zero. Let

$$G(\beta_1, \beta_2(s), \dots, \beta_N(s)] = F(0), \quad \text{for all } (\beta_1, \beta_2(s), \dots, \beta_N(s)) \in \Omega_N, \tag{44}$$

where $\Omega_N = \{(\beta_1, \beta_2(s), \dots, \beta_N(s)) \in \mathbb{R}\times(L^\infty(0,r_2-r_1))^{N-1} \mid \ 0 \le \beta_n(s) \le 1, \text{for } s \in [0, r_2-r_1],$ a.e., $n = 1, 2, \dots, N\}$.

In the following we assume that when

$$(\beta_1, \beta_2(\cdot), \dots, \beta_N(\cdot)) \ge (\bar\beta_1, \bar\beta_2(\cdot), \dots, \bar\beta_N(\cdot)), \tag{45}$$

i.e., $\beta_n(s) \ge \bar\beta_n(s)$, for all $n = 1, 2, \dots, N$ and $s \in [0, r_m - r_1]$, let λ_0 and $\bar\lambda_0$ be the growth index corresponding to $(\beta_1, \beta_2(\cdot), \dots, \beta_N(\cdot))$ and $(\bar\beta_1, \bar\beta_2(\cdot), \dots, \bar\beta_N(\cdot))$, respectively, then $\bar\lambda_0 \ge \lambda_0$. For example, if $N = 1$, then when $\beta_1 > 1/r_1$, λ_0 is increasing with respect to β_1. The general case has some requirements about $h_i(r,s)$ (we omit the details here). The reader can refer to [8].

Furthermore, if $\beta_N(s) = \beta_N$, independent of s, then since $G : \mathbb{R}\times(L^\infty(0, r_m - r_1))^{N-2}\times\mathbb{R} \to \mathbb{R}$ is a continuous function, if for some $(\beta_1, \beta_2(\cdot), \dots, \beta_{N-1}(\cdot), \beta_N) \in \mathring{\Omega}_{Nc}$, such that

$$G(\beta_1, \beta_2(\cdot), \dots, \beta_{N-1}(\cdot), \beta_N)) = 0,$$

(here $\Omega_{Nc} = \{(\beta_1, \beta_2(s), \dots, \beta_{N-1}(s), \beta_N) \in \mathbb{R}\times(L^\infty(0, r_2 - r_1))^{N-2}\times\mathbb{R} \mid 0 \le \beta_n(s) \le 1,$ for $s \in [0, r_2 - r_1]$, a.e., $n = 1, 2, \dots, N\}$), then from (45), we have

$$G'_{\beta_N}(\beta_1, \beta_2(\cdot), \dots, \beta_{N-1}(\cdot), \beta_N)) =$$
$$- k_0 \int_{r_1}^{r_2} d\tau \left[e^{-\int_0^{r_2} \mu_f(\rho)\,d\rho} \int_{r_1}^{r_2-\tau} h_N(\tau+s,\tau)\,e^{-\int_0^s \beta_N h_N(\rho+r_2-s)\,d\rho} g(s)\,ds \right.$$
$$\left. + \int_{\tau-r_1}^{r_2} \mu_f(r)\,e^{-\int_0^r \mu_f(\rho)\,d\rho}\,dr \int_{r_1}^{r-\tau} h_N(\tau+s,\tau)\,e^{-\int_0^{r-s} \beta_N h_N(\rho+s)\,d\rho}\,g(s)\,ds \right],$$

$G'_{\beta_N}(\beta_1, \beta_2(\cdot), \dots, \beta_{N-1}(\cdot), \beta_N)) \ne 0$. By the implicit function theorem, there exists an open neighborhood of $(\beta_1, \beta_2(\cdot), \dots, \beta_{N-1}(\cdot), \beta_N)$ such that

$$\beta_N = g(\beta_1, \beta_2(\cdot), \dots, \beta_{N-1}(\cdot)), \tag{46}$$

where g is a operator from $\mathbb{R}\times(L^\infty(0, r_m - r_1))^{N-2} \to \mathbb{R}$. For $(\beta_1, \beta_2(\cdot), \dots, \beta_{N-1}(\cdot), \beta_N) \in \Omega_{Nc}$ in which $\beta_n(\cdot)$, $n = 1, 2, \dots, N$ are so small that

$$G(\beta_1, \beta_2(\cdot), \dots, \beta_{N-1}(\cdot), \beta_N) > 0,$$

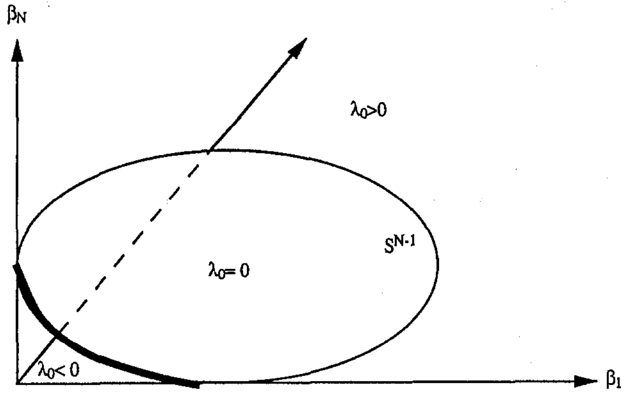

Figure 1. The distribution of critical fertilities of parity interval progression population system.

the corresponding growth index $\lambda_{00} < 0$. For $(\beta_1, \beta_2(\cdot), \dots, \beta_{N-1}(\cdot), \beta_N) \in \Omega_{Nc}$ in which $\beta_n = 1$, $n = 1, 2, \dots, N$, there are two cases:

CASE 1. The corresponding growth index $\lambda_{10} \leq 0$ (e.g., the death rate is very large);

CASE 2. $\lambda_{10} > 0$;

In Case 1, the distribution of $(\beta_1, \beta_2(\cdot), \dots, \beta_{N-1}(\cdot), \beta_N)$ corresponding to the zero growth index may be a piece of an $(N-1)$-dimension curved surface in $\mathbb{R} \times (L^\infty(0, r_m - r_1))^{N-2} \times \mathbb{R}$. Usually, Case 2 is more probable. In such a case, since the growth index λ_0 is a continuous function of $(\beta_1, \beta_2(\cdot), \dots, \beta_{N-1}(\cdot), \beta_N)$ so all $(\beta_1, \beta_2(\cdot), \dots, \beta_{N-1}(\cdot), \beta_N)$ corresponding to the zero growth index forms a $(N-1)$-dimension smooth curved surface in $\mathbb{R} \times (L^\infty(0, r_m - r_1))^{N-2} \times \mathbb{R}$, i.e.,

$$\beta_N = g(\beta_1, \beta_2(\cdot), \dots, \beta_{N-1}(\cdot)), \quad \text{for all} \quad (\beta_1, \beta_2(\cdot), \dots, \beta_{N-1}(\cdot)) \in \Omega_{N-1}, \tag{47}$$

where $\Omega_{N-1} = \{(\beta_1, \beta_2(\cdot), \dots, \beta_{N-1}(\cdot)) \mid 0 \leq \beta_n(s) \leq 1,\ s \in [0, r_2 - r_1]$, a.e., $n = 1, 2, \dots, N-1\}$.

THEOREM 4. *If under the natural fertile state (i.e., $\beta_1 = \beta_2 = \cdots = \beta_N = 1$), the growth index is greater than zero, than there exists a $(N-1)$-dimension smooth curved surface S^{N-1} connected with the axis in $\mathbb{R} \times (L^\infty(0, r_2 - r_1))^{N-2} \times \mathbb{R}$:*

$$S^{N-1}: \quad \beta_N = g(\beta_1, \beta_2(\cdot), \dots, \beta_{N-1}(\cdot)), \tag{48}$$

such that

(i) *When $(\beta_1, \beta_2, (\cdot), \dots, \beta_{N-1}(\cdot), \beta_N)$ lies above S^{N-1}, the corresponding growth index $\lambda_0 > 0$;*

(ii) *When $(\beta_1, \beta_2(\cdot), \dots, \beta_{N-1}(\cdot), \beta_N)$ lies on S^{N-1}, $\lambda_0 = 0$;*

(iii) *When $(\beta_1, \beta_2(\cdot), \dots, \beta_{N-1}(\cdot), \beta_N)$ lies below S^{N-1}, $\lambda_0 < 0$.*

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