Optimal Birth Control of Population Dynamics.  
II. Problems with Free Final Time, Phase Constraints, 
and Mini-Max Costs*  

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Submitted by E. Stanley Lee  

Received August 15, 1988  

We study optimal birth control of population systems of McKendrick type which 
is a distributed parameter system involving first order partial differential equations 
with nonlocal bilinear boundary control. New results on problems with free final 
time, phase constraints, and mini-max costs are presented. © 1990 Academic Press, Inc.

In [1] we discussed optimal birth control of population systems of 
McKendrick type. The present article (which is a direct continuation of 
[1]) presents further new results of current interests. These include 
problems with free final time, of which the minimum time problem is a 
special case (but relaxing many convexity assumptions). Systems with 
phase constraints are also studied. Finally, mini-max control for popula-
tion regulation is characterized. It is assumed that the reader is familiar 
with the terminology and notation of [1].

5. FREE FINAL TIME PROBLEM

Consider the free final time optimal control problem of the population 
control system

\[ \text{Problem (P): Minimize} \quad J(\beta, p) = \int_0^{t_f} \int_0^r L(p(r, t), \beta(t)) \, dr \, dt \]

* This work was carried out with the aid of a grant from the International Development 
Centre, Ottawa, Canada.  
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subject to
\[
\frac{\partial p(r, t)}{\partial t} + \frac{\partial p(r, t)}{\partial r} = -\mu(r) p(r, t), \quad 0 < r < r_m, \quad t > 0,
\]
\[p(r, 0) = p_0(r), \quad 0 < r < r_m,
\]
\[p(0, t) = \beta(t) \int_{r_1}^{r_2} k(r) h(r) p(r, t) \, dr, \quad t \geq 0,
\]
\[p(r, t_1) = p^0(r), \quad t_1 > 0, \quad \beta(t) \in M \subset \mathbb{R}^+,
\] (1)

where \( L \) is a function defined on \( L^2(0, r_m) \times \mathbb{R}^+ \) satisfying

(1) \( L(p(r), \beta) \) is continuous in \( \beta, \)

(2) \( |\partial L(p(r), \beta)/\partial p| \) is bounded for every bounded subset of \( L^2(0, r_m) \times \mathbb{R}^+ \).

For any measurable function \( v(s) \geq 0 \), define the time transformation

\[ t(\tau) = \int_0^\tau v(s) \, ds, \quad t(1) = t_1 \] (2)

and let \( p(r, \tau) = p(r, t(\tau)) \),

\[ \beta(\tau) = \begin{cases} \beta(t(\tau)), & \tau \in S_1, \\ \text{arbitrary}, & \tau \in S_2, \end{cases} \] (3)

then \((p(r, \tau), \beta(\tau))\) satisfies

\[
\frac{\partial p(r, \tau)}{\partial \tau} + v(\tau) \frac{\partial p(r, \tau)}{\partial r} = -\mu(r) v(\tau) p(r, \tau), \quad 0 < r < r_m, \quad 0 \leq \tau \leq 1,
\]
\[p(r, 0) = p_0(r), \quad 0 < r < r_m,
\]
\[v(\tau) p(0, \tau) = v(\tau) \beta(\tau) \int_{r_1}^{r_2} k(r) h(r) p(r, \tau) \, dr, \quad 0 \leq \tau \leq 1,
\]
\[p(r, 1) = p^0(r),
\] (4)

where

\[ S_1 = \{ \tau \mid \tau \in [0, 1], v(\tau) > 0 \}, \]
\[ S_2 = \{ \tau \mid \tau \in [0, 1], v(\tau) = 0 \}. \] (5)

Conversely, if \((p(r, \tau), \beta(\tau))\) solves Eq. (4), define \( p(r, t) = p(r, \tau(t)) \), \( \beta(t) = \beta(\tau(t)) \),

\[ \tau(t) = \inf\{ \tau \mid \tau(\tau) = t \} \] (6)
then \((p(r, t), \beta(t))\) satisfies Eq. (1) for \(t = t(\tau), \ v(\tau) > 0\), but for the monotone function \(t(\tau) = \int_0^\tau v(s) \, ds, \ mes\{t = t(\tau) \mid v(\tau) > 0\} = t_1 = \int_0^1 v(s) \, ds,\) so \((p(r, t), \beta(t))\) satisfies Eq. (1) for \(t \in [0, t_1]\) a.e.

Based on the above arguments, we consider the optimal (fixed final time) problem

\[
\text{Problem (Q): Minimize } J(\beta, p) = \int_0^{t_1} \int_0^{r_m} L(p(r, t), \beta(t)) \, dr \, dt \text{ subject to Eq. (4).}
\]

If \((p^*, \beta^*, t_1)\) solves Problem (P), then for any \(v^*(\tau) \geq 0\) satisfying \(\int_0^1 v^*(s) \, ds = t_1, \ \beta^*(\tau)\) defined similar to (3), \((p^*(r, \tau), \beta^*(\tau), v^*)\) solves Problem (Q). By this \(\beta(\tau)\), we put forward another problem as

\[
\text{Problem (L): Minimize } J(\beta^*, p, v) = \int_0^{t_1} \int_0^{r_m} v(\tau) L(p(r, \tau), \beta^*(\tau)) \, dr \, dt
\]

subject to

\[
\frac{\partial p(r, \tau)}{\partial \tau} + v(\tau) \frac{\partial p(r, \tau)}{\partial r} = -\mu(r) v(\tau) p(r, \tau), \quad 0 < r < r_m, \ 0 \leq \tau \leq 1,
\]

\[
p(r, 0) = p_0(r), \quad 0 \leq r \leq r_m,
\]

\[
v(\tau) p(0, \tau) = v(\tau) \beta^*(\tau) \int_{r_1}^{r_m} k(r) h(r) p(r, \tau) \, dr, \quad 0 \leq \tau \leq 1,
\]

\[
p(r, 1) = p_0^0(r).
\]

and \((p^*, v^*)\) solves Problem (L). Consider the solution of (7) as that of the integral equation

\[
\int_0^\tau p(s, \tau) \, ds - \int_0^\tau p_0(s) \, ds + \int_0^\tau v(\xi) \left[ p(r, \xi) - \beta^*(\xi) \int_{r_1}^{r_m} k(r) h(r) p(r, \xi) \, dr \right] d\xi
\]

\[
+ \int_0^\tau \int_0^\tau v(\xi) \mu(s) p(s, \xi) \, ds \, d\xi
\]

\[
p(r, 1) = p_0^0(r).
\]

Similarly, we consider that the solution of the differential equation is equivalent to that of the corresponding integral equation.

Simple arguments can be found that the Eq (8) has a unique solution on \(C(0, 1; L^2(0, r_m))\) and so we take \(X = C(0, 1; L^2(0, r_m)) \times L^\infty(0, 1)\) as the state space. Define the inequality constraint

\[
\Omega_{1} = \{(p(r, \tau), v(\tau)) \in X \mid v(\tau) \geq 0, \text{ for } \tau \in [0, 1] \text{ a.e.}\}
\]
and the equality constraint
\[ \Omega_2 - \{(p(r, \tau), v(\tau)) \in X \mid (p, v) \text{ satisfies (8) and (9)}\}. \] (11)

Under these notations, we can write problem (L) as
\[
\begin{align*}
\text{Minimize } & J(\beta^*, p, v) = \int_0^1 \int_0^m v(\tau) \left( L(p(r, \tau), \beta^*(\tau)) \right) dr d\tau \\
\text{subject to } & (p(r, \tau), v(\tau)) \in \Omega_1 \cap \Omega_2 \subset X.
\end{align*}
\] (12)

\(J(\beta^*, p, v)\) is Fréchet differentiable at any point \((\hat{p}, \hat{v})\) and
\[
J'(\beta^*, p_0, v_0)(p, \beta) = \int_0^1 \int_0^m v_0(\tau) \frac{\partial L(p_0(r, \tau), \beta^*(\tau))}{\partial p} p(r, \tau) + v(\tau) L(p_0, \beta^*) dr d\tau \] (13)
and so the decreasing direction cone of \(J'(\beta^*, p_0, v_0)\) at \((p^*, v^*)\) is
\[
K_0 = \{(p, v) \mid J'(\beta^*, p^*, v^*)(p, v) < 0\}. \] (14)

If \(K_0 \neq \emptyset\), then for any \(f_0 \in K_0^*\), there exists a constant \(\lambda_0 \geq 0\), such that
\[
f_0(p, v) = -\lambda_0 \int_0^1 \int_0^m v^*(\tau) \left[ \frac{\partial L(p^*(r, \tau), \beta^*(\tau))}{\partial p} p(r, \tau) + L(p^*, \beta^*) v(\tau) \right] dr d\tau. \] (15)

Notice that \(\Omega_1 = C(0, 1; L^2(0, r_m)) \times \hat{\Omega}_1, \hat{\Omega}_1 = \{v(\tau) \in L^\infty(0, 1) \mid v(\tau) \geq 0\}\) is a closed convex subset of \(L^\infty(0, 1)\), \(\hat{\Omega}_1 = C \times \hat{\Omega}_1 \neq \emptyset\), and so the feasible direction cone of \(\Omega_1\) at \((p^*, v^*)\) is
\[
K_1 = \{\lambda(\hat{\Omega}_1 - (p^*, v^*)) \mid \lambda > 0\} \] (16)
for any \(f_1 \in K_1^*\), if \(c(\tau) \in L(0, 1)\) such that
\[
f_1(p, v) = \int_0^1 c(\tau) v(\tau) d\tau \] (17)
then [2]
\[
c(\tau)[v - v^*(\tau)] \geq 0, \quad \forall v \in (0, \infty), \quad \tau \in [0, 1] \text{ a.e.} \] (18)
In order to determine the tangent direction cone of $\Omega_2$ at $(p^*, v^*)$, we define the operator as $G: X \rightarrow X$

$$G(p, v) = \left[ \int_0^r p(s, \tau) \, ds - \int_0^r p_0(s) \, ds + \int_0^r v(\xi) \right]$$

$$\times \left[ \int_0^r (r, \xi) - \beta^*(\xi) \int_{r_1}^{r_0} k(r) h(r) p(r, \xi) \, dr \right] d\xi$$

$$+ \int_0^r \int_0^r v(\xi) \mu(s) p(s, \xi) \, ds \, d\xi, p(r, 1) - p^0(r) \right]$$

(19)

then

$$\Omega_2 = \{(p, v)| G(p, v) = 0\}. \quad (20)$$

Now

$$G'(p^*, v^*)(p, v)$$

$$= \left[ \int_0^r p(s, \tau) \, ds + \int_0^r \left[ v(\xi) p^*(r, \xi) + v^*(\xi) p(r, \xi) \right] \right]$$

$$- \beta^*(\xi) \int_{r_1}^{r_0} k(r) h(r) \left[ v(\xi) p^*(r, \xi) + v^*(\xi) p(r, \xi) \right] dr \right] d\xi$$

$$+ \int_0^r \int_0^r \mu(s) \left[ v(\xi) p^*(s, \xi) + v^*(\xi) p(s, \xi) \right] ds \, d\xi, p(r, 1) \right]$$

(21)

and we solve the equation

$$G'(p^*, v^*)(p, v) = (q, g) \in X;$$

i.e.,

$$\int_0^r p(s, \tau) \, ds + \int_0^r \left[ v(\xi) p^*(r, \xi) + v^*(\xi) p(r, \xi) \right]$$

$$- \beta^*(\xi) \int_{r_1}^{r_0} k(r) h(r) \left[ v(\xi) p^*(r, \xi) + v^*(\xi) p(r, \xi) \right] dr \right] d\xi$$

$$+ \int_0^r \int_0^r \mu(s) \left[ v(\xi) p^*(s, \xi) + v^*(\xi) p(s, \xi) \right] ds \, d\xi = q(r, \tau),$$

$$p(r, 1) = g(r). \quad (22)$$
If the linearized system

\[ \frac{\partial p(r, \tau)}{\partial \tau} + v^*(\tau) \frac{\partial p(r, \tau)}{\partial r} = -\mu(r) [v(\tau) p^*(r, \tau) + v^*(\tau) p(r, \tau)] - v(\tau) \frac{\partial p^*(r, \tau)}{\partial r}, \]

\[ p(r, 0) = 0, \]

\[ v(\tau) p^*(0, \tau) + v^*(0) p(0, \tau)\]

\[ = v(\tau) \beta^*(\tau) \int_{r_1}^{r_2} k(r) h(r) p^*(r, \tau) \, dr \]

\[ + v^*(\tau) \beta^*(\tau) \int_{r_1}^{r_2} k(r) h(r) p(r, \tau) \, dr \]  

(23)

is controllable, then let \( \dot{p}(r, \tau) = p(r, \tau) + d(r, \tau), d(r, \tau) \) be determined by

\[ \int_0^{r'} d(s, \tau) \, ds + \int_0^\tau v^*(\xi) \left[ d(r, \xi) - \beta^*(\xi) \int_{r_1}^{r_2} k(r) h(r) d(r, \xi) \, dr \right] d\xi \]

\[ + \int_0^{r'} \int_0^{r} v^*(\xi) \mu(s) \, ds \, d\xi = q(r, \tau), \]

\((p, \beta), \beta = \tilde{\beta} \) solves Eq. (23) and \( p(r, 1) = g(r) - d(r, 1), \) so \((\dot{p}, \tilde{\beta})\) solves Eq. (22). In this case, the tangent direction cone of \( \Omega_2 \) at \((p^*, v^*)\) is determined by

\[ K_2 = \{ (p, v) \ | \ G'(p^*, v^*)(p, v) = 0 \}, \]

i.e.,

\[ \frac{\partial p(r, \tau)}{\partial \tau} + v^*(\tau) \frac{\partial p(r, \tau)}{\partial r} \]

\[ = -\mu(r) [v(\tau) p^*(r, \tau) + v^*(\tau) p(r, \tau)] - v(\tau) \frac{\partial p^*(r, \tau)}{\partial r}, \]

\[ p(r, 0) = 0, \]

\[ v(\tau) p^*(0, \tau) + v^*(0) p(0, \tau)\]

\[ = v(\tau) \beta^*(\tau) \int_{r_1}^{r_2} k(r) h(r) p^*(r, \tau) \, dr + v^*(\tau) \beta^*(\tau) \int_{r_1}^{r_2} k(r) h(r) p(r, \tau) \, dr \]

\[ p(r, 1) = 0. \]  

(24)
$K_2 = K_{11} \cap K_{12}$, $K_{12} = \{(p, v) \mid p(r, 1) = 0\}$, $K_{11}$ consists of such $(p, v) \in X$ such that
\[
\frac{\partial p(r, \tau)}{\partial \tau} + v^*(\tau) \frac{\partial p(r, \tau)}{\partial r} = -\mu(r)[v(\tau) p^*(r, \tau) + v^*(\tau) p(r, \tau)] - v(\tau) \frac{\partial p^*(r, \tau)}{\partial r},
\]
$p(r, 0) = 0$,
$v(\tau) p^*(0, \tau) + v^*(\tau) p(0, \tau)$
\[
= v(\tau) \beta^*(\tau) \int_{r_1}^{r_2} k(r) h(r) p^*(r, \tau) \, dr
\]
\[
+ v^*(\tau) \beta^*(\tau) \int_{r_1}^{r_2} k(r) h(r) p(r, \tau) \, dr.
\] (25)

For any $f \in K_2^*$, $f = f_{11} + f_{12}$, $f_i \in K_i^*$, $i = 1, 2$,
\[
f_{12}(p, v) = \int_0^m \alpha(r) p(r, 1) \, dr, \quad \alpha(r) \in L^2(0, 1).
\] (26)

By the Dubovitskii–Milyutin Theorem, there exist functionals $f_i \in K_i^*$, $i = 0, 1, 2$, not all identically zero such that
\[
f_0 + f_1 + f_{11} + f_{12} = 0.
\] (27)

In particular for any $(p, v)$ satisfying (25), $f_{11}(p, v) = 0$, and so
\[
f_1(p, v) = -f_0(p, v) - f_{12}(p, v)
\]
\[
= \lambda_0 \int_0^1 \int_0^r \left[ \frac{\partial L(p^*(r, \tau), \beta^*(\tau))}{\partial p} v^*(\tau) p(r, \tau) + \mu^*(r, \tau) v(r, \tau) \right] \, dr \, d\tau - \int_0^m \alpha(r) p(r, 1) \, dr,
\] (28)

where the solution of (25) is considered as that of the integral equation
\[
\int_0^r p(s, \tau) \, ds + \int_0^r \left[ [v(\xi) p^*(r, \xi) + v^*(\xi) p(r, \xi)]
\right.
\]
\[
- \beta^*(\xi) \int_{r_1}^{r_2} k(r) h(r) [v(\xi) p^*(r, \xi) + v^*(\xi) p(r, \xi)] \, dr \] \left[ \right] d\xi
\]
\[+ \int_0^r \int_0^r \mu(s) [v(\xi) p^*(s, \xi) + v^*(\xi) p(s, \xi)] \, ds \, d\xi = 0.
\] (29)
Define the adjoint equation

\[
\frac{\partial q(r, \tau)}{\partial r} + v^*(\tau) \frac{\partial q(r, \tau)}{\partial \tau} = v^*(\tau) \mu(r) q(r, \tau) - \beta^*(\tau) k(r) h(r) q(\tau) + \lambda_0 \frac{\partial L(p^*, \beta^*)}{\partial p}
\]

\[q(r, 1) = \alpha(r)\]

\[v^*(\tau) q(0, \tau) = v^*(\tau) q(\tau)\]  

(30)

and

\[\int_{r}^{r_0} q(s, \tau) \, ds = q(r, \tau).\]  

(31)

As in [1], we have

**Lemma 1.** The solution of Eq. (25) and the solution of Eqs. (30), (31) have the relation

\[
\lambda_0 \int_{0}^{1} \int_{0}^{r_0} \left[ \frac{\partial L(p^*(r, \tau), \beta^*(\tau))}{\partial p} v^*(\tau) p(r, \tau) + L(p^*, \beta^*) v(\tau) \right] \, dr \, d\tau - \int_{0}^{r_0} \alpha(r) p(r, 1) \, dr
\]

\[= \int_{0}^{1} \left[ \int_{0}^{r_0} p^*(r, \tau) \dot{q}(r, \tau) + \beta^*(\tau) \int_{r}^{r_0} k(r) h(r) p^*(r, \tau) q(\tau) \, dr \right. \]

\[\left. + \int_{0}^{r_0} \mu(r) p^*(r, \tau) q(r, \tau) \, dr \right] v(\tau) \, d\tau.\]  

(32)

Lemma 1 together with (28) and (18) implies that

\[
\left[ \int_{0}^{r_0} p^*(r, \tau) \dot{q}(r, \tau) \, dr \, d\tau + \beta^*(\tau) \int_{r_1}^{r_0} k(r) h(r) p^*(r, \tau) q(\tau) \, dr \, d\tau \right.
\]

\[+ \int_{0}^{r_0} \mu(r) p^*(r, \tau) q(r, \tau) \, dr + \lambda_0 \int_{0}^{r_0} L(p^*, \beta^*) \, dr \right] [v - v^*(\tau)] \geq 0 \]

for all \(v \geq 0.\)  

(33)

It follows from (33) that
\[
\int_0^{r_m} p^*(r, \tau) \dot{q}(r, \tau) + \beta^*(\tau) \int_r^{r_2} k(r) h(r) p^*(r, \tau) q(\tau) \, dr \\
+ \int_0^{r_m} \mu(r) p^*(r, \tau) q(r, \tau) \, dr + \lambda_0 \int_0^{r_m} L(p^*, \beta^*) \, dr = 0, \quad \forall \tau \in S_1, \quad (34)
\]

\[
\int_0^{r_m} p^*(r, \tau) \dot{q}(r, \tau) + \beta^*(\tau) \int_r^{r_2} k(r) h(r) p^*(r, \tau) q(\tau) \, dr \\
+ \int_0^{r_m} \mu(r) p^*(r, \tau) q(r, \tau) \, dr + \lambda_0 \int_0^{r_m} L(p^*, \beta^*) \, dr \geq 0, \quad \forall \tau \in S_2. \quad (35)
\]

We say that \( \lambda_0 \) and \( \alpha(r) \) cannot be both zero, since otherwise, \( f_0 = 0, \) \( q(s, \tau) = 0, \) \( f_{12} = 0, \) \( f_1 = 0 \) and hence \( f_{11} = 0. \) This contradicts the Dubovitskii–Milyutin Theorem. Furthermore, if \( K_0 = \emptyset, \) take \( \lambda_0 = 1, \) \( \alpha(r) = 0, \) then (32) implies (33) and hence (34) and (35) are valid. Finally, if Eq. (30) has a nonzero solution \( q(r, \tau) \) such that

\[
\int_0^{r_m} p^*(r, \tau) \dot{q}(r, \tau) + \beta^*(\tau) \int_r^{r_2} k(r) h(r) p^*(r, \tau) q(\tau) \, dr \\
+ \int_0^{r_m} \mu(r) p^*(r, \tau) q(r, \tau) \, dr = 0,
\]

then take \( \lambda_0 = 0, \) and (33) is also valid. On the other hand, for any nonzero solution of (30)

\[
\int_0^{r_m} p^*(r, \tau) \dot{q}(r, \tau) + \beta^*(\tau) \int_r^{r_2} k(r) h(r) p^*(r, \tau) q(\tau) \, dr \\
+ \int_0^{r_m} \mu(r) p^*(r, \tau) q(r, \tau) \, dr \neq 0.
\]

We call this situation the nondegenerate case, since here the linearized system must be controllable. This is because otherwise there exists a \( \alpha(r) \in L^2(0, r_m) \) such that \( \int_0^{r_m} \alpha(r) p(r, 1) \, dr = 0, \) \( \alpha(r) \neq 0, \) and taking \( \lambda_0 = 0, \) we have a contradiction to (36). Hence, no matter what happened, (33) and (35) are always valid.

Define \( q(r, t) = q(r, \tau(t)), \dot{q}(r, t) = \dot{q}(r, \tau(t)), \) \( q(t) = q(0, \tau(t)), \) then (34) can be written as

\[
\int_0^{r_m} p^*(r, t) \dot{q}(r, t) + \beta^*(t) \int_r^{r_2} k(r) h(r) p^*(r, t) q(t) \, dr \\
+ \int_0^{r_m} \mu(r) p^*(r, t) q(r, t) \, dr + \lambda_0 \int_0^{r_m} L(p^*, \beta^*) \, dr = 0,
\]

for all \( t \in [0, t_1] \) a.e. \quad (38)
Choose \( S_1 \) to be a perfect nowhere dense subset of \([0, 1]\) (see [2]) and define

\[
v^*(\tau) = \begin{cases} \frac{t_1}{\mu(s_1)}, & \tau \in S_1 \\ 0, & \tau \in S_2 = [0, 1] \setminus S_1. \end{cases}
\] (39)

Now, analysing the condition (35) as in [2], we can define \( \beta^*(\tau) \) on \( S_2 \) and get (with the same notation as before)

\[
\int_0^m p^*(r, t) q(r, t) + \beta \int_{r_1}^r k(r) h(r) p^*(r, t) q(t) \, dr \\
+ \int_0^m \mu(r) p^*(r, t) q(r, t) \, dr + \lambda_0 \int_0^m L(p^*, \beta) \, dr \geq 0, \quad \forall \beta \in M,
\] (40)

for all \( t \in [0, t_1] \). We have thus proved the following

**Theorem 1 (Maximum Principle).** Under the conditions on \( L \) mentioned in the beginning of this paper, and letting \( (\beta^*, p^*, t_1) \) solve Problem (P), then there exist \( q(r, t), \lambda_0 > 0, \) not both zero, such that

\[
\int_0^m p^*(r, t) q(r, t) + \beta \int_{r_1}^r k(r) h(r) p^*(r, t) q(t) \, dr \\
+ \int_0^m \mu(r) p^*(r, t) q(r, t) \, dr + \lambda_0 \int_0^m L(p^*, \beta) \, dr = 0, \quad \forall t \in [0, t_1] \text{ a.e.}
\]

\[
\int_0^m p^*(r, t) q(r, t) + \beta \int_{r_1}^r k(r) h(r) p^*(r, t) q(t) \, dr \\
+ \int_0^m \mu(r) p^*(r, t) q(r, t) \, dr + \lambda_0 \int_0^m L(p^*, \beta) \, dr \geq 0,
\]

\( \forall \beta \in M, \quad t \in [0, t_1] \text{ a.e.} \),

where

\[
\frac{\partial q(r, t)}{\partial r} + \frac{\partial q(r, t)}{\partial t} = \mu(r) q(r, t) - \beta^*(t) k(r) h(r) q(t) + \lambda_0 \frac{\partial L(p^*, \beta^*)}{\partial p},
\]

\( q(r, t_1) = \alpha(r), \)

\( q(0, t) = q(t), \)

\( q(r, t) - \int_r^m \dot{q}(s, t) \, ds. \) (41)
Note. If the end point condition \( p(r, t_1) = p^0(r) \) is imposed instead of
\[
p(r, t_1) \in \{ p(r) \mid \| p(r) - p^0(r) \| \leq \varepsilon \}
\]
then \( \alpha(r) \) should be taken as \( \alpha(r) = p^*(r, t_1) - p^0(r) \) and \( \lambda_0 \) can be set to 1.

**Corollary 1.** If \( L = 1 \), then problem (P) is the time optimal control problem considered in [1] and the time optimal control satisfies the maximum principle
\[
\beta^*(t) H(t) = \max_{\beta \in \mathcal{M}} \beta H(t), \quad \forall t \in [0, t_1] \text{ a.e.}
\]
\[
H(t) = q(t) \int_{r_1}^{r_2} k(r) h(r) p^*(r, t) dr,
\]
where \( t_1 \) is the minimum time. \( q(t) \) is the solution of adjoint equation (41).

The result is the same as that of [1] but there the convexity assumption on \( \mathcal{M} \) is not assumed.

**6. SYSTEM WITH PHASE CONSTRAINTS**

In this part, we consider the optimal control problem of a population system with phase constraints

**Problem (Q):** Minimize \( \hat{J}(\beta, p) = \int_0^T \int_0^{r_m} Q(p(r, t), \beta(t), t) \, dr \, dt \) \( 44 \)
under the constraints
\[
\frac{\partial p(r, t)}{\partial t} + \frac{\partial p(r, t)}{\partial r} = -\mu(r) \, p(r, t), \quad 0 < r < r_m, \quad t > 0,
\]
\[
p(r, 0) = p_0(r), \quad 0 \leq r \leq r_m,
\]
\[
p(r, T) = p^0(r), \quad 0 \leq r \leq r_m,
\]
\[
p(0, t) = \beta(t) \int_{r_1}^{r_2} k(r) h(r) \, p(r, t) \, dr, \quad t \geq 0,
\]
\[
\beta(t) \in [\beta_0, \beta_1] \quad \text{for} \quad t \in [0, T] \text{ a.e.}
\]
\[
\int_0^{r_m} G(p(r, t), t) \, dr \leq 0, \quad t \geq 0,
\]
(45)
in the class of
\[ (p(r, t), \beta(t)) \in X = C(0, T, L^2(0, r_m)) \times L^\infty(0, T). \] (46)

The time \( T \) is fixed.

Define
\[ Q_1 = \{(p(r, t), \beta(t)) \in X | \beta(t) \in [\beta_0, \beta_1], t \in [0, T] \text{ a.e.}\} \]
(47)

\[ Q_2 = \left\{ (p(r, t), \beta(t)) \in X | p_t = -\mu p, \right\} \]
\[ p(0, t) = \beta(t) \int_{t_1}^{t_2} k(r) h(r) \frac{r^2}{t_1} \frac{r^2}{t_2} p(r, t) \, dr, \quad p(r, 0) = p_0(r), \quad p(r, T) = p^0(r) \right\} \]
(48)

\[ Q_3 = \left\{ (p(r, t), \beta(t)) \in X | \int_0^{\infty} G(p(r, t), t) \, dr \leq 0 \right\}. \]
(49)

Then Problem (Q) is equivalent to finding \((p^*, \beta^*) \in Q_1 \cap Q_2 \cap Q_3\) such that
\[ \tilde{J}(\beta^*, p^*) = \min_{(p, \beta) \in Q_1 \cap Q_2 \cap Q_3} \tilde{J}(\beta, p). \] (50)

This is a minimum problem formed by the inequality constraints \(Q_1, Q_3\) and the equality \(Q_2\). We can use again the general theory of Dubovitskii-Milyutin for the extremum problem.

We had already investigated the corresponding cones of \(Q_1\) and \(Q_2\) of the Dubovitskii-Milyutin Theorem. Now we need only to consider constraint \(Q_3\). Notice that \(Q_3\) can be written as
\[ Q_3 = \{(p(r, t), \beta(t)) \in X | F(p) \leq 0\}, \] (51)
where \(F(p) = \max_{0 \leq t \leq 1} \int_0^{\infty} G(p(r, t), t) \, dr\) and assume

1. \(\int_0^{\infty} G(p(r), t) \, dr\) is a continuous functional on \(L^2(0, r_m) \times [0, \infty]\);
2. \(\int_0^{\infty} G(p_0(r), 0) \, dr < 0, \int_0^{\infty} G(p^0(r), T) \, dr < 0\);
3. \(\int_0^{\infty} G_p(p(r), t) \, dr\) is also continuous on \(L^2(0, r_m) \times [0, \infty]\) and \(\int_0^{\infty} G_p(p(r), t) \, dr \neq 0\) if \(\int_0^{\infty} G(p(r), t) \, dr = 0\).

Let \((\beta^*, p^*)\) solve Problem (Q), then we consider \(F(p^*) = 0\), since otherwise, the feasible direction cone \(K_3\) of \(Q_3\) at \((\beta^*, p^*)\) is the whole space, i.e., \(K_3 = X\). So \(Q_3 = \{(p(r, t), \beta(t)) \in X | F(p) \leq F(p^*)\}\). Applying arguments as in [2] we can prove that
LEMMA 2. $F(p)$ is differentiable at any point in any direction and

$$F'(\hat{p}, p) = \max_{t \in S} \int_0^{r_m} G'_p(\hat{p}(r, t), t) p(r, t) \, dr,$$

where $S = \{ t \in [0, T] \mid \int_0^{r_m} (\hat{p}(r, t), t) \, dr = F(\hat{p}) \}$. Furthermore $F(p)$ satisfies a Lipschitz condition in any ball.

Notice that $F'(p^*, G'_p(p^*, t)) < 0$, we know that [2]

$$K_3 = \{ (p, \beta) \in X \mid F'(p^*, p) < 0 \}. \quad (53)$$

Define the linear operator $A : X \to C[0, T]$ by

$$Ap(r, t) = -\int_0^{r_m} G'_p(p^*(r, t), t) p(r, t) \, dr \quad (54)$$

and

$$K = \{ y(t) \in C[0, T] \mid y(t) \geq 0, \forall t \in S \}$$

then $K_3 = \{ p(r, t) \in X \mid Ap \in K \}$. Since $A(-G'_p(p^*(r, t), t)) \in \hat{K}$, so $K_3^* = A^*K^*$; i.e., for any $f \in K_3^*$, there exists a measure $dm(t)$, nonnegative and with support on $S$, such that

$$f(p(r, t)) = \int_0^T Ap(r, t) \, dm(t) = \int_0^T Ap(r, t) \, dm(t)$$

\[= -\int_S \int_0^{r_m} G'_p(p^*(r, t), t) p(r, t) \, dr \, dm(t). \quad (55)\]

Based on the previous results, there exist $\lambda_0 \geq 0$, $\alpha(r) \in L^2(0, r_m)$ such that

$$f_1(p, \beta) = \lambda_0 \int_0^T \int_0^{r_m} \left[ \frac{\partial Q(p^*, \beta^*, t)}{\partial p} p(r, t) + \frac{\partial Q(p^*, \beta^*, t)}{\partial \beta} \beta(t) \right] \, dr \, dt$$

\[-\int_0^{r_m} p(r, T) \alpha(r) \, dr + \int_0^T \int_0^{r_m} G'_p(p^*(r, t), t) p(t, t) \, dr \, dm(t), \quad (56)\]

where $(p, \beta)$ satisfies
with the assumption that the decreasing direction cone of \( J \) at \((p^*, \beta^*)\) is not empty and system (57) is controllable.

Define the adjoint system

\[
\dot{q}(r, t) + \frac{\partial q(r, t)}{\partial r} = \mu(r) q(r, t) - \beta^*(t) k(r) h(r) q(t) + \kappa_0 \frac{\partial Q(p^*, \beta^*, t)}{\partial p} + G'_p(p^*(r, t), t) \frac{dm(t)}{dt},
\]

\[
q(r, T) = \alpha(r),
\]

\[
q(0, t) = q(t).
\] \tag{58}

The solution of Eq. (58) should be considered as that of the integral equation

\[
- \int_0^r q(s, t) \, ds = - \int_0^r \alpha(s) \, ds - \int_0^T \int_s^T \left[ q(s, \tau) - q(\tau) \right] \, ds \, d\tau + \int_0^r \int_s^T \mu(s) q(s, \tau) \, ds \, d\tau
\]

\[
= - \int_0^r k(s) h(s) \, ds \int_s^T \beta^*(\tau) q(\tau) \, d\tau + \kappa_0 \int_0^T \int_s^T \frac{\partial Q}{\partial p} \, ds \, d\tau
\]

\[
+ \int_0^T \int_s^T G'_p(p^*(s, \tau), \tau) \, dm(\tau) \, dr.
\] \tag{59}

As before, we have

**Lemma 3.** The solution of Eq. (57) and the adjoint equation has the relation
\[
\lambda_0 \int_0^T \int_0^{r_m} \left[ \frac{\partial Q(p^*, \beta^*, t)}{\partial p} p(r, t) + \frac{\partial Q(p^*, \beta^*, t)}{\partial \beta} \beta(t) \right] \, dr \, dt \\
- \int_0^{r_m} p(r, T) \varphi(r) \, dr + \int_0^T \int_0^{r_m} G_p(p^*(r, t), t) p(r, t) \, dr \, dm(t) \\
= \int_0^T \left[ \int_0^{r_m} \left( \lambda_0 \frac{\partial Q(p^*, \beta^*, t)}{\partial \beta} - q(t) \int_{r_1}^{r_2} k(r) h(r) p^*(r, t) \, dr \right) \beta(t) \, dt \right].
\]

(60)

Same reason as before, whether or not the decreasing direction cone of \( J \) at \((p^*, \beta^*)\) is empty and the system (57) is controllable, we always have

**Theorem 2 (Maximum Principle).** Let \((p^*, \beta^*)\) solve Problem (Q), then there exist \( \lambda_0 \geq 0 \), \( q(t) \) not both zero, such that

\[
\int_0^{r_m} \left[ \lambda_0 \frac{\partial Q(p^*, \beta^*, t)}{\partial \beta} - q(t) \int_{r_1}^{r_2} k(r) h(r) p^*(r, t) \, dr \right] \beta(t) \, dt \geq 0,
\]

\( \forall t \in [0, T] \) a.e. \hspace{1cm} (61)

We can also consider the free final time problem with phase constraints

**Problem (W):** Minimize \( \tilde{J}(\beta, p) = \int_0^{r_1} \int_0^{r_m} W(p(r, t), \beta(t), t) \, dr \, dt \)

under the constraints

\[
\frac{\partial p(r, t)}{\partial t} + \frac{\partial p(r, t)}{\partial r} = -\mu(r) \, p(r, t), \quad 0 < r < r_m, \quad t > 0, \\
p(r, 0) = p_0(r), \quad 0 \leq r \leq r_m, \\
p(r, t_1) = p_0^0(r), \quad 0 \leq r \leq r_m, \\
p(0, t) = \beta(t) \int_{r_1}^{r_2} k(r) h(r) p(r, t) \, dr, \quad t \geq 0, \\
\beta(t) \in M, \quad \text{for} \quad t \in [0, t_1] \text{ a.e.} \\
\int_0^{r_m} G(p(r, t), t) \, dr \leq 0, \quad t \geq 0,
\]

(62)
in the class of

\[
(p(r, t), \beta(t)) \in X = C(0, t_1; L^2(0, r_m)) \times L^\infty(0, t_1).
\]

(63)
The time \( t_1 \) is free.

Following the same lines of reasoning in Section 5, we can prove
**Theorem 3.** Let \((p^*, \beta^*, t_1)\) solve Problem \((W)\), then there exist \(\lambda_0, \alpha(r) \in L^2(0, r_m)\) with support on \(S = \{ t \in [0, T] \mid \int_0^T G(\dot{p}(r, t), t) \, dt = F(\dot{p}) \}\) and a nonnegative measure \(dm(t)\) such that

\[
\int_0^{r_m} p^*(r, t) \, q(r, t) \, + \int_0^{r_m} \lambda_0 W(p^*, \beta^*) \, dr = 0, \quad \forall t \in [0, t_1] \text{ a.e.}
\]

\[
\int_0^{r_m} p^*(r, t) \, q(r, t) \, + \int_0^{r_m} \lambda_0 W(p^*, \beta^*) \, dr \geq 0, \quad \forall t \in [0, t_1] \text{ a.e.},
\]

where

\[
\frac{\partial q(r, t)}{\partial r} + \frac{\partial q(r, t)}{\partial t} = \mu(r) q(r, t) - \beta^*(t) k(r) h(r) q(t) \]

\[
+ \lambda_0 \frac{\partial W(p^*, \beta^*)}{\partial p} \left. \, G_p(p^*(r, t), t) \, \frac{dm(t)}{dt} \right) \]

\[q(r, t_1) = \alpha(r),\]

\[q(0, t) = q(t)\]

\[q(r, t) = \int_0^{r_m} \dot{q}(s, t) \, ds.\]

7. **Mini-Max Problems**

The mini-max control problem of a population control system can be stated as

**Problem (Y):** Minimize \(F(p) = \max_{0 \leq t \leq t_1} \int_0^{r_m} G(p(r, t), t) \, dr \, dt\) \(67\)

with respect to \((p(r, t), \beta(t)) \in X\) and \(t_1\) under the constraints

\[
\frac{\partial p(r, t)}{\partial t} + \frac{\partial p(r, t)}{\partial r} = -\mu(r) p(r, t), \quad 0 < r < r_m, t > 0,
\]

\[p(r, 0) = p_0(r), \quad 0 \leq r \leq r_m,\]
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\[ p(r, t_1) = p_0(r), \quad 0 \leq r \leq r_m, \]

\[ p(0, t) = \beta(t) \int_{r_1}^{r_m} k(r) h(r) p(r, t) \, dr, \quad t \geq 0, \]

\[ \beta(t) \in M, \quad \text{for } t \notin [0, t_1] \text{ a.e.} \] (68)

We only state the results since the proof is similar.

**THEOREM 4.** Let \( \int_0^m G(p(r), t) \, dr \) be continuously differentiable with respect to \( p(r) \), \( \int_0^m G'(p(r), t) \neq 0 \) when \( G(p(r), t) \neq 0 \). Let \( (p^*, \beta^*, t_1) \) solve Problem (Y), then there exist \( q(r, t), \alpha(r) \in L^2(0, r_m) \) and a nonnegative measure \( dm(t) \) with support on the set

\[ S = \left\{ t \in [0, t_1] \mid \int_0^m G(p^*(r, t), t) \, dr = \max_{0 \leq t \leq t_1} \int_0^m G(p^*(r, t), t) \, dr \, dt \right\} \]

such that

\[ \int_0^m p^*(r, t) \frac{\partial q(r, t)}{\partial t} + \beta^*(t) \int_{r_1}^{r_m} k(r) h(r) p^*(r, t) q(t) \, dr \\
+ \int_0^m \mu(r) p^*(r, t) q(r, t) \, dr, \quad \forall t \in [0, t_1] \text{ a.e.} \] (69)

\[ \int_0^m p^*(r, t) \frac{\partial q(r, t)}{\partial t} + \beta \int_{r_1}^{r_m} k(r) h(r) p^*(r, t) q(t) \, dr \\
+ \int_0^m \mu(r) p^*(r, t) q(r, t) \, dr \geq 0, \quad \forall \beta \in M, t \in [0, t_1] \text{ a.e.,} \] (70)

where \( q(r, t) \) is the solution of the adjoint equation

\[ \frac{\partial q(r, t)}{\partial r} + \frac{\partial q(r, t)}{\partial t} = \mu(r) q(r, t) - \beta^*(t) k(r) h(r) q(t) + G'(p^*) \frac{dm(t)}{dt} \]

\[ q(r, t_1) = \alpha(r), \]

\[ q(0, t) = q(t) \]

\[ q(r, t) = \int_0^t \hat{q}(s, t) \, ds. \] (71)

**REFERENCES**
