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Optimal Birth Control of Population Dynamics*

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We study optimal birth control policies for an age-structured population of McKendrick type which is a distributed parameter system involving first order partial differential equations with nonlocal bilinear boundary control. The functional analytic approach of Dubovitskii and Milyutin is adopted in the investigation. Maximum principles for problems with a free end condition a fixed final horizon are developed, and the time optimal control problem, the problem with target sets, and the infinite planning horizon case are investigated. © 1989 Academic Press, Inc.

1. INTRODUCTION

A wide variety of problems dealing with biological populations and resource management have been formulated in an optimal control setting [1, 2]. Much work is on models described by ordinary differential equations. On the other hand, age-structured population models involving partial, differential equations are becoming increasingly emphasized [3–5]. Analysis of such distributed systems in the optimal control theory framework has only recently been reported [3, 6]. In this paper we shall work in the spirit of [3] on optimal birth control policies of the human population using the McKendrick type model. We adopt Dubovitskii and Milyutin's functional analytical approach [7] in the optimization yielding more transparent results. We first study the "standard" problem with a free end condition and fixed final horizon (time). Other aspects, not treated in [6], such as the time optimal control problem, the problem with target sets, and the infinite planning horizon case are investigated. The role of controllability [9, 10] is also discussed.

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2. Fixed Horizon and Free End Point Problem

Consider the control of the following population distributed parameter system [3]

$$\begin{aligned} \frac{\partial p(r,t)}{\partial t}+\frac{\partial p(r,t)}{\partial r} &= -\mu(r)\, p(r,t), \qquad 0<r<r_m,\ t>0, \\ p(r,0) &= p_0(r), \qquad 0\leqslant r\leqslant r_m, \\ p(0,t) &= \beta(t)\int_{r_1}^{r_2} k(r)\, h(r)\, p(r,t)\, dr, \qquad t\geqslant 0 \end{aligned} \tag{1}$$

in which $p(r, t)$ is the population density, r denotes age, t represents time, r_m is the maximum age, $\beta(t)$ is the specific fertility rate of females at time $t, k(r)$ and $h(r)$ denote, respectively, the female ratio and the fertility pattern; $[r_1, r_2]$ is the fertility interval with $\int_{r_1}^{r_2} h(r)\, dr = 1$. The initial population density $p_0(r)$ and the mortality rate $\mu(r)$ satisfy

$$\int_0^r \mu(\rho)\, d\rho < +\infty, \qquad r<r_m,$$

$$\int_0^{r_m} \mu(\rho)\, d\rho = +\infty.$$

Generally speaking, the population parameters $\mu(r), k(r)$, and $h(r)$ are time dependent. Here, we assume that they are time independent functions, in order to simplify arguments. However, under suitable smoothness assumptions, the results obtained for the optimal control problems continue to hold.

For the population dynamical system there are two independent controlling variables $\beta(t)$ and $h(r)$ (may be $h(r, t)$ and can be combined into one). The latter reflects the fertility pattern of the female such as late marriage and fertility. $\beta(t)$ reflects an average birth rate. We study here, under certain demands of the society, what the optimal birth policy is. This is an optimal control problem in control theory. We determine necessary conditions for the optimal control, extending Pontryagin's maximum principle to population systems with distributed parameters.

Assume that the population parameters in Eq. (1) are nonnegative and measurable functions. Furthermore, let β, h, and k be bounded functions whose values outside their domain of definition are zero.

By the method of characteristics, the solution of Eq. (1) can be written (formally) as

$$p(r,t)=\begin{cases} p_0(r-t)\, e^{-\int_{r-t}^{r}\mu(\rho)\,d\rho}, & r\geqslant t, \\ \beta(t-r)\int_{r_1}^{r_2} k(s)\, h(s)\, p(s,t-r)\, ds\, e^{-\int_0^r \mu(\rho)\,d\rho}, & r<t, \end{cases} \tag{2}$$

The classical solution of (1) is a solution of (2). Under certain smoothness conditions on the population parameters, the two are equivalent. For a detailed discussion, see [3].

For an arbitriay $p_0(r) \in L^2(0, r_m)$, Eq. (2) in $L^2(0, r_m)$ has a unique solution $p(r, t) \in C(0, \infty; L^2(0, r_m))$; moreover,

$$p(r, t) = p_0(r-t)\, e^{-\int_{r-t}^{r} \mu(\rho)\, d\rho} + \sum_{k=0}^{\infty} \phi_k(t-r)\, e^{-\int_0^r \mu(\rho)\, d\rho}$$

$$\phi_0(t) = \beta(t) \int_{r_1}^{r_2} k(s)\, h(s)\, p_0(s-t)\, e^{-\int_{s-t}^{s} \mu(\rho)\, d\rho}\, ds, \tag{3}$$

$$\phi_k(t) = \beta(t) \int_{r_1}^{r_2} k(s)\, h(s)\, \phi_{k-1}(t-s)\, e^{-\int_0^s \mu(\rho)\, d\rho}\, ds, \qquad k = 1, 2 \ldots$$

and $\phi_k(t)$ does not vanish only in $[kr_1, (k+1)r_2]$.

Because of the above reasons, we call the solution of Eq. (2) a weak solution of Eq. (1). Unless otherwise stated, in what follows when we speak of a solution of Eq. (1) we shall mean the weak solution.

Consider now the optimal control problem: Determine (β^*, p^*), $\beta^*(\cdot) \in U_{\text{ad}}$, such that

$$J(\beta^*, p^*) = \min_{\beta(\cdot) \in U_{\text{ad}}} J(\beta, p)$$

$$J(\beta, p) = \int_0^T \int_0^{r_m} L(p(r, t), \beta(t), r, t)\, dr\, dt \tag{4}$$

$$+ \frac{1}{2} \int_0^{r_m} [p(r, T) - \bar{p}(r)]^2\, dr,$$

where $p(r, t)$ is the trajectory of the control $\beta(t)$, $\bar{p}(r) \in L^2(0, r_m)$ is an arbitrary fixed function, and L is a functional defined on $L^2(0, r_m) \times [\beta_0, \beta_1] \times [0, r_m] \times [0, T]$ satisfying the following conditions:

(1) $\partial L(p(r), \beta, r, t)/\partial p$, $\partial L(p(r), \beta, r, t)/\partial \beta$ exist for every $(p(r), \beta, r, t) \in L^2(0, r_m) \times [\beta_0, \beta_1] \times [0, r_m] \times [0, T]$ and L is continuous about its variables;

(2) $\int_0^{r_m} |\partial L(p(r), \beta, r, t)/\partial p|\, dr$, $\int_0^{r_m} |\partial L(p(r), \beta, r, t)/\partial \beta|\, dr$ are bounded for $t \in [0, T]$ and any bounded subset of $L_2(0, r_m) \times [\beta_0, \beta_1] \times [0, r_m] \times [0, T]$

$$U_{\text{ad}} = \{ \beta(t) \mid 0 \leqslant \beta_0 \leqslant \beta(t) \leqslant \beta_1,\ t \in [0, T] \text{ a.e.},$$
$$\beta(t) \text{ is measurable on } [0, T] \}. \tag{5}$$

Let (β^*, p^*) be an optimal solution of problem (4) and define the adjoint equation of Eq. (1) to be

$$\begin{aligned} \frac{\partial q(r,t)}{\partial r} + \frac{\partial q(r,t)}{\partial t} &= \mu(r)\, q(r,t) - \beta^*(t)\, k(r)\, h(r) q(t) \\ &\quad + \frac{\partial L(p^*, \beta^*, r, t)}{\partial p} \\ q(r,T) &= \bar{p}(r) - p^*(r,T), \\ q(0,t) &= q(t). \end{aligned} \tag{6}$$

As with Eq. (1), we call solutions (weak solutions) of Eq. (6) to be the solutions of the following equation:

$$\begin{aligned} q(t) &= e^{-\int_0^{T-t} \mu(\rho)\, d\rho}\, q(T-t, T) \\ &\quad + \int_t^T e^{-\int_0^{s-t} \mu(\rho)\, d\rho}\, \beta^*(s)\, k(s-t)\, h(s-t)\, q(s)\, ds \\ &\quad - \int_t^T e^{-\int_0^{s-t} \mu(\rho)\, d\rho} \left.\frac{\partial L(p^*, \beta^*, \cdot, \cdot)}{\partial p}\right|_{(s-t,s)} ds \\ q(r,t) &= e^{-\int_0^{r+T-t} \mu(\rho)\, d\rho}\, q(r+T-t, T) \\ &\quad + \int_t^T e^{-\int_r^{r+s-t} \mu(\rho)\, d\rho}\, \beta^*(s)\, k(r+s-t)\, h(r+s-t)\, q(s)\, ds \\ &\quad - \int_t^T e^{-\int_r^{r+s-t} \mu(\rho)\, d\rho} \left.\frac{\partial L(p^*, \beta^*, \cdot, \cdot)}{\partial p}\right|_{(r+s-t,s)} ds, \\ &\quad 0 \leqslant t \leqslant T,\ 0 \leqslant r \leqslant r_m. \end{aligned} \tag{7}$$

In $L^2(0, r_m)$, Eq. (6) has a unique solution.

First, we have the following

LEMMA 1. *The solutions of Eq.* (1) *and its adjoint Eq.* (6) *satisfy the following relation*:

$$\begin{aligned} &\int_0^{r_m} q(r,T)[p(r,T) - p^*(r,T)]\, dr \\ &\quad = \int_0^T \int_{r_1}^{r_2} q(t)\, k(r)\, h(r)\, p(r,t)[\beta(t) - \beta^*(t)]\, dr\, dt \\ &\qquad + \int_0^T \int_0^{r_m} \frac{\partial L(p^*, \beta^*, r, t)}{\partial p} [p(r,t) - p^*(r,t)]\, dr\, dt. \end{aligned} \tag{8}$$

Proof.

$$\int_0^{r_m} q(r, T)\, p(r, t)\, dr \qquad (T > r)$$

$$= \int_0^{r_m} q(r, T)\, \beta(T-r) \int_{r_1}^{r_2} k(\tau)\, h(\tau)\, p(\tau, T-r)\, d\tau\, e^{-\int_0^{r} \mu(\rho)\, d\rho}$$

$$= \int_{T-r}^{T} q(T-r, T)\, \beta(t) \int_{r_1}^{r_2} k(\tau)\, h(\tau)\, p(\tau, t)\, d\tau\, e^{-\int_0^{T-t} \mu(\rho) d\rho}\, dt$$

$$= \int_0^{T} q(T-r, T)\, \beta(t) \int_{r_1}^{r_2} k(\tau)\, h(\tau)\, p(\tau, t)\, d\tau\, e^{-\int_0^{T-t} \mu(\rho)\, d\rho}\, dt$$

$$= \int_0^T \left[q(t) - \int_t^T e^{-\int_0^{s-t} \mu(\rho)\, d\rho}\, \beta^*(s)\, k(s-t)\, h(s-t)\, q(s)\, ds \right.$$
$$\left. + \int_t^T e^{-\int_0^{s-t} \mu(\rho)\, d\rho} \left.\frac{\partial L(p^*, \beta^*, \cdot, \cdot)}{\partial p}\right|_{(s-t, s)} ds \right]$$
$$\cdot \beta(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p(r, t)\, dr$$

$$= \int_0^T q(t)\, \beta(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p(r, t)\, dr$$
$$+ \int_0^T \int_t^T e^{-\int_0^{s-t} \mu(\rho)\, d\rho} \left.\frac{\partial L(p^*, \beta^*, \cdot, \cdot)}{\partial p}\right|_{(s-t, s)} ds$$
$$\cdot \beta(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p(r, t)\, dr$$
$$- \int_0^T \int_t^T e^{-\int_0^{s-t} \mu(\rho)\, d\rho}\, \beta^*(s)\, k(s-t)\, h(s-t)\, q(s)\, ds$$
$$\cdot \beta(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p(r, t)\, dr\, dt$$

$$= \int_0^T q(t)\, \beta(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p(r, t)\, dr$$
$$- \int_0^T \beta^*(s)\, q(s)\, ds \int_{r_1}^{r_2} k(r)\, h(r)\, dr$$
$$\cdot \int_0^S e^{-\int_0^{s-t} \mu(\rho)\, d\rho}\, \beta(t)\, k(s-t)\, h(s-t)\, p(r, t)\, dt$$

$$+\int_0^T ds \int_{r_1}^{r_2} k(r)\, h(r)\, dr$$

$$\cdot \int_0^S e^{-\int_0^{s-t} \mu(\rho)\, d\rho} \left.\frac{\partial L(p^*, \beta^*, \cdot, \cdot)}{\partial p}\right|_{(s-t, s)}$$

$$\cdot \beta(t)\, k(s-t)\, h(s-t)\, p(r, t)\, dt$$

$$= \int_0^T q(t)\, \beta(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p(r, t)\, dr\, dt$$

$$-\int_0^T \beta^*(s)\, q(s)\, ds \int_0^s e^{-\int_0^t \mu(\rho)\, d\rho}\, k(t)\, h(t)\, dt$$

$$\cdot \beta(s-t) \int_{r_1}^{r_2} k(r)\, h(r)\, p(r, s-t)\, dr$$

$$+\int_0^T ds \int_0^s e^{-\int_0^t \mu(\rho)\, d\rho} \left.\frac{\partial L(p^*, \beta^*, \cdot, \cdot)}{\partial p}\right|_{(t, s)}$$

$$\cdot \beta(s-t) \int_{r_1}^{r_2} k(r)\, h(r)\, p(s-t)\, dt$$

$$= \int_0^T q(t)\, \beta(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p(r, t)\, dr\, dt$$

$$-\int_0^T q(s)\, \beta^*(s) \int_0^s k(r)\, h(r)\, p(r, s)\, dr$$

$$+\int_0^T ds \int_0^s \left.\frac{\partial L(p^*, \beta^*, \cdot, \cdot)}{\partial p}\right|_{(t, s)} p(r, s)\, dr$$

so

$$\int_0^{r_m} q(r, T)[p(r, T) - p^*(r, T)]\, dr$$

$$= \int_0^T q(t)\, \beta(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p(r, t)\, dr\, dt$$

$$-\int_0^T q(s)\, \beta^*(s) \int_{r_1}^s k(r)\, h(r)\, p(r, s)\, dr$$

$$-\int_0^T q(t)\, \beta^*(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p(r, t)\, dr\, dt$$

$$+\int_0^T ds \int_0^s \left.\frac{\partial L(p^*, \beta^*, \cdot, \cdot)}{\partial p}\right|_{(r, s)} p(r, s)\, dr$$

$$+\int_0^T q(s)\,\beta^*(s)\int_{r_1}^{s} k(r)\,h(r)\,p^*(r,s)\,dr$$

$$-\int_0^T ds\int_0^s \left.\frac{\partial L(p^*,\beta^*,\cdot,\cdot)}{\partial p}\right|_{(r,s)} p^*(r,s)\,dr$$

$$=\int_0^T q(t)\,\beta(t)\int_{r_1}^{r_2} k(r)\,h(r)\,p(r,t)\,dr\,dt$$

$$-\int_0^T q(t)\,\beta^*(t)\int_{r_1}^{r_2} k(r)\,h(r)\,p^*(r,t)\,dr\,dt$$

$$-\int_0^T q(t)\,\beta^*(t)\int_{r_1}^{r_2} k(r)\,h(r)[p(r,t)-p^*(r,t)\,dr\,dt$$

$$+\int_0^T\int_0^{r_m}\left.\frac{\partial L(p^*,\beta^*,\cdot,\cdot)}{\partial p}\right|_{(r,t)}[p(r,t)-p^*(r,t)]\,dr$$

$$=\int_0^T\int_{r_1}^{r_2} q(t)\,k(r)\,h(r)\,p(r,t)[\beta(t)-\beta^*(t)]\,dr\,dt$$

$$+\int_0^T\int_0^{r_m}\left.\frac{\partial L(p^*,\beta^*,\cdot,\cdot)}{\partial p}\right|_{(r,t)}[p(r,t)-p^*(r,t)]\,dr.$$

This is Lemma 1.

It can be easily deduced from Lemma 1 that

$$\begin{aligned}
&J(\beta,p)-J(\beta^*,P^*)\\
&\quad=\int_0^T\left[\int_0^{r_m}\left.\frac{\partial L(p^*,\beta^*,\cdot,\cdot)}{\partial p}\right|_{(r,t)}dr\right][\beta(t)-\beta_\varepsilon(t)]\,dt\\
&\quad-\int_0^T q(t)\int_{r_1}^{r_2} k(r)\,h(r)\,p^*(r,t)\,dr\,[\beta(t)-\beta^*(t)]\,dt\\
&\quad-\int_0^T q(t)\int_{r_1}^{r_2} k(r)\,h(r)[p(r,t)-p^*(r,t)]\,dr\,[\beta(t)-\beta^*(t)]\,dt\\
&\quad+\frac{1}{2}\int_0^{r_m}[p(r,T)-p^*(r,T)]^2\,dr\\
&\quad+\int_0^T\int_0^{r_m}[o(p(r,t)-p^*(r,t))+o(\beta(t)-\beta^*(t))]\,dr\,dt. \qquad (9)
\end{aligned}$$

From Eq. (3), we can show that for $T>0$ there exists $M_1>0$ such that

$$\int_0^{r_m}[p(r,t)-p^*(r,t)]^2\,dr\leqslant M_1\int_0^T[\beta(t)-\beta^*(t)]^2\,dt \qquad \forall(\beta,p)\in U_{\mathrm{ad}},\ t\in[0,T]. \qquad (10)$$

In (9), substitute $\theta\beta(t)+(1-\theta)\,\beta^*(t)$, $\theta\in(0,1)$, for $\beta(t)$; paying attention to (10), we obtain immediately (note that the integrand is bounded and measurable)

$$\left[q(t)\int_{r_1}^{r_2} k(r)\,h(r)\,p^*(r,t)\,dr-\int_0^{r_m}\left.\frac{\partial L(p^*,\beta^*,\cdot,\cdot)}{\partial p}\right|_{(r,t)}dr\right]$$
$$\cdot\,[\beta-\beta^*(t)]\leqslant 0,\qquad \forall\beta\in[\beta_0,\beta_1],\ t\in[0,T]\ \text{a.e.} \tag{11}$$

THEOREM 1. *The solution of problem* (4) *satisfies the maximum principle*

$$\beta^*(t)\,H(t)=\max_{\beta_0\leqslant\beta\leqslant\beta_1}\beta H(t),\qquad \forall t\in[0,T]\ a.e.$$

$$H(t)=q(t)\int_{r_1}^{r_2} k(r)\,h(r)\,p^*(r,t)\,dr-\int_0^{r_m}\left.\frac{\partial L(p^*,\beta^*,\cdot,\cdot)}{\partial p}\right|_{(r,t)}dr, \tag{12}$$

from which we have

$$\beta(t)=\begin{cases}\beta_0, & H(t)<0,\\ \beta_1, & H(t)>0,\\ \textit{indeterminate}, & H(t)=0.\end{cases}$$

$H(t)$ is the switching function.

3. Time Optimal Control Problem

We consider the time optimal control problem for system (1); that is, determine $T^*>0$ and $(\beta^*,p^*)\in\bar U_{\mathrm{ad}}$ such that

$$\begin{aligned}
&T^*=\min\{T\mid p(r,T)\cap V\neq\varnothing,\ \forall(\beta,p)\in\bar U_{\mathrm{ad}}\}\\
&p^*(r,T^*)\cap V\neq\varnothing\\
&V=\{\phi(r)\mid \|\phi(r)-\bar p(r)\|\leqslant M,\ \phi,\bar p\in L^2(0,r_m)\}\\
&\bar U_{\mathrm{ad}}=\{(\beta,p)\mid 0\leqslant\beta_0\leqslant\beta(t)\leqslant\beta_1,\ (\beta,p)\ \text{satisfies (1)}\}.
\end{aligned}\tag{13}$$

If (β^*,p^*,T^*) is the solution of the time optimal control problem (13), then in [9] it is shown that

$$\int_0^{r_m}[p^*(r,T^*]-\bar p(r)][p^*(r,T^*)-p(r,T^*)]\,dr\geqslant 0\qquad \forall(\beta,p)\in\bar U_{\mathrm{ad}}. \tag{14}$$

Define the adjoint equation

$$\frac{\partial q(r,t)}{\partial r}+\frac{\partial q(r,t)}{\partial t}=\mu(r)\,q(r,t)-\beta^*(t)\,k(r)\,h(r)q(t)$$
$$q(r,T)=p^*(r,T)-\bar{p}(r), \tag{15}$$
$$q(0,t)=q(t).$$

Its solution is understood to be as in (7). Combining (13) and Lemma 1, we obtain

THEOREM 2 (Maximum Principle for Time Optimal Control). *The time optimal control satisfies the following maximum principle*:

$$\beta^*(t)\,H(t)=\max_{\beta_0\leqslant\beta\leqslant\beta_1}\beta H(t), \qquad \forall t\in[0,T]\ a.e.$$
$$H(t)=q(t)\int_{r_1}^{r_2}k(r)\,h(r)\,p^*(r,t)\,dr. \tag{16}$$

4. INFINITE HORIZON PROBLEM

We consider further the optimal control problem on an infinite time interval

$$\min_{\beta(\cdot)\in U_{\mathrm{ad}}}J(\beta,p)=\min_{\beta(\cdot)\in U_{\mathrm{ad}}}\int_0^\infty\int_0^{r_m}L(p(r,t),\beta(t),r,t)\,dr\,dt, \tag{17}$$

with other conditions similar to (4). We will assume that L is continuously differentiable with respect to its arguments. Moreover, for each admissible (β, p), the integral in (17) is convergent.

LEMMA 2. *Let (β^*, p^*) be the solution of the optimal control problem (17). Then for each arbitrary $T>0$, (β^*, p^*) is a solution of the following optimal control problem*:

$$J_T(\beta,p)=\min_{\beta(\cdot)\in U_{\mathrm{ad}}}\int_0^T\int_0^{r_m}L(p(r,t),\beta(t),r,t)\,dr\,dt,$$
$$\frac{\partial p(r,t)}{\partial t}+\frac{\partial p(r,t)}{\partial r}=-\mu(r)\,p(r,t), \qquad 0<r<r_m,\ t>0,$$
$$p(r,0)=p_0(r), \qquad 0\leqslant r\leqslant r_m,$$
$$p(r,T)=p^*(r,T),$$
$$p(0,t)=\beta(t)\int_{r_1}^{r_2}k(r)\,h(r)\,p(r,t)\,dr, \qquad t\geqslant 0,$$
$$U_{\mathrm{ad}}=\{\beta(t)\,|\,0\leqslant\beta_0\leqslant\beta(t)\leqslant\beta_1,\ t\in[0,T]\ a.e.\}. \tag{18}$$

Proof. If not, let $(\hat{\beta}, \hat{p})$ satisfy Eq. (18), and

$$\int_0^T \int_0^{r_m} L(\hat{p}(r, t), \hat{\beta}(t), r, t)\, dr\, dt < \int_0^T \int_0^{r_m} L(p^*(r, t), \beta^*(t), r, t)\, dr\, dt.$$

Then define

$$\hat{\beta}^*(t) = \begin{cases} \hat{\beta}(t), & 0 \leqslant t \leqslant T, \\ \beta^*(t), & t > T, \end{cases}$$

$$\hat{p}^*(r, T) = \begin{cases} \hat{p}(r, t), & 0 \leqslant t \leqslant T, \\ p^*(r, t), & t > T. \end{cases}$$

$(\hat{\beta}^*(t), \hat{p}^*(r, t))$ is admissible and

$$J(\hat{\beta}^*(\cdot), \hat{p}^*(\cdot, \cdot)) < J(\beta^*(\cdot), p^*(\cdot, \cdot)).$$

This is a contradiction. So, Lemma 2 holds.

Let $X = C(0, T; L^2(0, r_m)) \times L^\infty(0, T)$. We consider the necessary conditions that must be satisfied for the optimal control problem (17). From the definition of solution (2), each admissible control $(p, \beta) \in X$. Define

$$\Omega_1 = \{p(r, t), \beta(t) \in X \mid \beta_0 \leqslant \beta(t) \leqslant \beta_1,\, t \in [0, T] \text{ a.e.}\}$$

$$\Omega_2 = \{(p(r, t), \beta(t)) \in X \mid p_t + p_r = -\mu p,\, p(0, t) = \beta(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p(r, t)\, dr,\, p(r, 0) = p_0(r), p(r, T) = p^*(r, T)\}.$$

Then probnlem (17) is equivalent to finding $(p^*, \beta^*) \in \Omega_1 \cap \Omega_2$ such that

$$J_T(\beta^*, p^*) = \min_{(p, \beta) \in \Omega_1 \cap \Omega_2} J_T(\beta, p). \tag{19}$$

This is a minimum problem formed by the inequality constraint Ω_1 and the equality Ω_2. We can use the general theory of Dubovitskii and Milyutin for extremum problems.

THEOREM 3 [7]. *Let the functional $J_T(\beta, p)$ assume a local minimum at the point (p^*, β^*) in $\Omega_1 \cap \Omega_2$. Assume that $J_T(\beta, p)$ is regularly decreasing at (p^*, β^*) with directions of decrease cone K_0; assume that the inequality* *constraint is regular at (p^*, β^*) with feasible directions cone K_1, and that the equality constraint is also regular at (p^*, β^*) with tangent directions cone K_2. Then there exist continuous linear functionals f_0, f_1, f_2, not all identically zero, such that $f_i \in K_i$, $i = 0, 1, 2$, and such that they satisfy the condition*

$$f_0 + f_1 + f_2 = 0. \tag{20}$$

We will now determine systematically the corresponding cones in problem (19). Under the assumptions for $J(\beta, p)$, the functional $J_T(\beta, p)$ is differentiable at any point (β_0, p_0) and

$$J'_T(\beta_0, p_0)(p, \beta) = \int_0^T \int_0^{r_m} \left[\frac{\partial L(p_0(r,t), \beta_0(t), r, t)}{\partial p} p(r,t) + \int_0^T \int_0^{r_m} \frac{\partial L(p_0(r,t), \beta_0(t), r, t)}{\partial \beta} \beta(t) \right] dr\, dt.$$

Since $J_T(p, \beta)$ is regularly decreasing at (p^*, β^*), its directions of decrease cone is

$$K_0 = \{(p, \beta) \in X \mid J'_T(\beta^*, p^*)(p, \beta) < 0\}, \tag{21}$$

If $K_0 \neq \emptyset$, then for arbitrary $f_0 \in K_0^*$, there exists $\lambda_0 \geqslant 0$ such that

$$f_0(p, \beta) = -\lambda_0 \int_0^T \int_0^{r_m} \left[\frac{\partial L(p^*(r,t), \beta^*(t), r, t)}{\partial p} p(r,t) + \int_0^T \int_0^{r_m} \frac{\partial L(p^*(r,t), \beta^*(t), r, t)}{\partial \beta} \beta(t) \right] dr\, dt. \tag{22}$$

Note that $\Omega_1 = C(0, T; L^2(0, r_m)) \times \hat{\Omega}_1$, $\hat{\Omega}_1 = \{\beta(t) \in L^\infty(0, T) \mid \beta_0 \leqslant \beta(t) \leqslant \beta_1\}$ is a closed convex subset of $L^\infty(0, T)$. Thus, $\mathring{\Omega}_1 = C \times \mathring{\Omega}_1 \neq \emptyset$, and for Ω, at the point (p^*, β^*) the feasible directions cone is

$$K_1 = \{\lambda(\mathring{\Omega}_1 - (p^*, \beta^*)) \mid \lambda > 0\}.$$

For an arbitrary $f_1 \in K^*$, if there exists $a(t) \in L(0, T)$ such that [7]

$$f_1(p, \beta) = \int_0^T a(t)\, \beta(t)\, dt \tag{23}$$

then

$$a(t)[\beta - \beta^*(t)] \geqslant 0, \qquad \forall \beta \in [\beta_0, \beta_1],\ t \in [0, T] \text{ a.e.} \tag{24}$$

In order to determine the tangent directions cone, we define the operator

$$G\colon X \to C(0, T; L^2(0, r_m) \times L^2(0, r_m)),$$

by

$$G(p, \beta) = \left[p(r, t) - \begin{cases} p_0(r-t)\, e^{-\int_{r-t}^{r} \mu(\rho)\, d\rho}, & r \geq t, \\ \beta(t-r) \displaystyle\int_{r_1}^{r_2} k(s)\, h(s)\, p(s, t-r)\, ds\, e^{-\int_0^r \mu(\rho)\, d\rho}, & \\ \qquad r < t; & p(r, T) - p^*(r, T) \end{cases} \right].$$

Thus, for $G(p, \beta)$, we can write

$$\Omega_2 = \{(p, \beta) \in X \mid G(p, \beta) = 0\} \tag{25}$$

$G(p_0 + \hat{p}, \beta_0 + \hat{\beta}) - G(p_0, \beta_0)$

$$= \left[\hat{p}(r, t) - \begin{cases} 0, & r \geq t, \\ \beta_0(t-r) \displaystyle\int_{r_1}^{r_2} k(s)\, h(s)\, \hat{p}(s, t-r)\, ds\, e^{-\int_0^r \mu(\rho)\, d\rho} & \\ \quad + \hat{\beta}(t-r) \displaystyle\int_{r_1}^{r_2} k(s)\, h(s)\, p_0(s, t-r)\, ds\, e^{-\int_0^r \mu(\rho)\, d\rho}, & \\ \qquad r < t; & \hat{p}(r, T) \end{cases} \right]$$

$$+ \left[- \begin{cases} 0, & r \geq t, \\ \hat{\beta}(r-r) \displaystyle\int_{r_1}^{r_2} k(s)\, h(s)\, \hat{p}(s, t-r)\, ds\, e^{-\int_0^r \mu(\rho)\, d\rho}, & \\ \qquad r < t; & 0 \end{cases} \right],$$

from which $G'(p_0, \beta_0)$ exists and

$G'(p_0, \beta_0)(\hat{p}, \hat{\beta})$

$$= \left[\hat{p}(r, t) - \begin{cases} 0, & r \geq t, \\ \beta_0(t-r) \displaystyle\int_{r_1}^{r_2} k(s)\, h(s)\, \hat{p}(s, t-r)\, ds\, e^{-\int_0^r \mu(\rho)\, d\rho} & \\ \quad + \hat{\beta}(t-r) \displaystyle\int_{r_1}^{r_2} k(s)\, h(s)\, p_0(s, t-r)\, ds\, e^{-\int_0^r \mu(\rho)\, d\rho}, & \\ \qquad r < t; & \hat{p}(r, T) \end{cases} \right].$$

Let (p^*, β^*) be the solution of (19). Then $G(p^*, \beta^*) = 0$. Choose arbitrary $(q(r, t), g(r)) \in C(0, T; L^2(0, r_m) \times L^2(0, r_m))$, and solve the equation

$$G'(p^*, \beta^*)(\hat{p}, \hat{\beta}) = (q, g).$$

Then

$$\hat{p}(r, t) = q(r, t), \qquad r \geqslant t,$$

$$\hat{p}(r, t) - \beta^*(t-r) \int_{r_1}^{r_2} k(s)\, h(s)\, \hat{p}(s, t-r)\, ds\, e^{-\int_0^r \mu(\rho)\, d\rho}$$

$$\times \hat{\beta}(t-r) \int_{r_1}^{r_2} k(s)\, h(s)\, p^*(s, t-r)\, ds\, e^{-\int_0^r \mu(\rho)\, d\rho}$$

$$= q(r, t), \qquad r < t,$$

$$\hat{p}(r, t) = g(r). \tag{26}$$

Assume that the linearized system

$$\begin{gathered}
\frac{\partial p(r, t)}{\partial t} + \frac{\partial p(r, t)}{\partial r} = -\mu(r)\, p(r, t), \qquad 0 < r < r_m,\ t > 0, \\
p(r, 0) = 0, \qquad 0 \leqslant r \leqslant r_m, \\
p(0, t) = \beta^*(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p(r, t)\, dr \\
+ \beta(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p^*(r, t)\, dr
\end{gathered} \tag{27}$$

is controllable. Then choose $\beta(t) = \hat{\beta}(t) \in L^\infty$ such that $p(r, t) = g(r) - f(r, T)$, and let $\hat{p}(r, t) = p(r, t) + f(r, t)$; here $f(r, t) = q(r, t)$ for $r \geqslant t$, and $\beta^*(t-r) \int_{r_1}^{r_2} k(r)\, h(r)\, f(r, t-r)\, dr$ for $r < t$, $(\hat{p}, \hat{\beta})$ satisfies Eq. (23). Now the tangent directions cone K_2 is formed by the kernal of $G'(p, \beta)$. In other words (p, β), satisfying the following equation, belongs to X

$$\begin{gathered}
\frac{\partial p(r, t)}{\partial t} + \frac{\partial p(r, t)}{\partial r} = -\mu(r)\, p(r, t), \qquad 0 < r < r_m,\ t > 0, \\
p(r, 0) = 0, \qquad 0 \leqslant r \leqslant r_m, \\
p(0, t) = \beta^*(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p(r, t)\, dr \\
+ \beta(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p^*(r, t)\, dr
\end{gathered} \tag{27}$$

$$p(r, T) = 0. \tag{28}$$

Define

$$K_{11} = \{(p(r, t), \beta(t)) \in X \mid (p, \beta) \text{ satisfies Eq. (25)}\}$$

$$K_{12} = \{(p(r, t), \beta(t)) \in X \mid (p, \beta) \text{ satisfies Eq. (26)}\}.$$

Then the tangent directions cone

$$K_2 = K_{11} + K_{12}, \tag{29}$$

and K_{11}, K_{12} are linear subspaces of X. Because

$$K_2^* = K_{11}^* + K_{12}^*,$$

for arbitrary $f_2 \in K_2^*$, $f_2 = f_{11} + f_{12}$, $f_{1i} \in K_{12}^*$, $i = 1, 2$, $f_{12} = (f_{12}', 0)$, $f_{12}'(p(r, t)) = 0$, and for all $p(r, t) \in C(0, T; L^2(0, r_m))$ satisfying $p(r, t) = 0$, there exists $\alpha(r) \in L^2(0, r_m)$ such that

$$f_{12}(p, \beta) = \int_0^{r_m} p(r, T)\, \alpha(r)\, dr. \tag{30}$$

From Theorem 3, there exist in X not identically zero linear functionals f_0, f_1, f_{11}, f_{12} such that

$$f_0 + f_1 + f_{11} + f_{12} = 0.$$

In particular, for $\beta(t) \in L^\infty(0, T)$, select p such that (p, β) satisfies (27). Then $(p, \beta) \in K_{11}$ and $f_{11}(p, \beta) = 0$, from which

$$\begin{aligned} f_1(p, \beta) &= -f_0(p, \beta) - f_{12}(p, \beta) \\ &= \lambda_0 \int_0^T \int_0^{r_m} \left[\frac{\partial L(p^*(r, t), \beta^*(t), r, t)}{\partial p}\, p(r, t) \right. \\ &\quad \left. + \int_0^T \int_0^{r_m} \frac{\partial L(p^*(r, t), \beta^*(t), r, t)}{\partial \beta}\, \beta(t) \right] dr\, dt \\ &\quad - \int_0^{r_m} p(r, T)\, \alpha(r)\, dr. \end{aligned} \tag{31}$$

Define the adjoint system

$$\begin{aligned} \frac{\partial q(r, t)}{\partial r} + \frac{\partial q(r, t)}{\partial t} &= \mu(r)\, q(r, t) - \beta^*(t)\, k(r)\, h(r)\, q(t) \\ &\quad + \lambda_0 \frac{\partial L(p^*, \beta^*, r, t)}{\partial p} \\ q(r, T) &= \alpha(t), \\ q(0, t) &= q(t). \end{aligned} \tag{32}$$

As in Lemma 1, we can prove

LEMMA 3. *The following relation holds between the solution* (p, β) *of Eq.* (18) *and the solution of* (29):

$$\lambda_0 \int_0^T \int_0^{r_m} \frac{\partial L(p^*(r, t), \beta^*(t), r, t)}{\partial p} p(r, t)\, dr\, dt - \int_0^{r_m} p(r, t)\, \alpha(r)\, dr$$

$$= -\int_0^T q(t)\, \beta(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p^*(r, t)\, dr\, dt. \tag{33}$$

From Lemma 3, (32) can be written as

$$f_1(p, \beta) = \lambda_0 \int_0^T \left[\int_0^{r_m} \frac{\partial L(p^*(r, t), \beta^*(t), r, t)}{\partial p} dr \right.$$

$$\left. - q(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p^*(r, t) \right] \beta(t)\, dt. \tag{34}$$

Then Eq. (15) states that

$$\int_0^{r_m} \left[\lambda_0 \frac{\partial L(p^*(r, t), \beta^*(t), r, t)}{\partial \beta} \right.$$

$$\left. - q(t)\, k(t)\, h(t)\, p^*(r, t) \right] dr\, [\beta - \beta^*(t)] \geqslant 0 \qquad \forall t \in [0, T] \text{ a.e.} \tag{35}$$

We claim that there cannot exist situations in which $\lambda_0, \alpha(r)$ are simultaneously zero. For then $f_1 \equiv 0, q(r, t) \equiv 0, f_{12} = 0, f_0 = 0$, from which $f_{11} = 0$. This contradicts the fact that f_0, f_1, f_{11}, f_{12} are not all identically zero.

On the other hand, if $K_0 = \varnothing$, then for arbitrary $(p, \beta) \in X$,

$$\lambda_0 \int_0^T \int_0^{r_m} \left[\frac{\partial L(p^*(r, t), \beta^*(t), r, t)}{\partial p} p(r, t) \right.$$

$$\left. + \frac{\partial L(p^*(r, t), \beta^*(t), r, t)}{\partial \beta} \beta(t) \right] dr\, dt = 0.$$

In particular choose $\lambda_0 = 1, \alpha(r) = 0$; then (30) shows that

$$\lambda_0 \int_0^T \int_0^{r_m} \frac{\partial L(p^*(r, t), \beta^*(t), r, t)}{\partial p} p(r, t)\, dr\, dt$$

$$= -\int_0^T q(t)\, \beta(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p^*(r, t)\, dr\, dt.$$

Thus,

$$\int_0^T \left[\int_0^{r_m} \frac{\partial L(p^*(r,t), \beta^*(t), r, t)}{\partial \beta} \right.$$
$$\left. - q(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p^*(r,t)\, dr \right] \beta(t)\, dt = 0, \qquad \forall \beta(t) \in L^\infty(0,T),$$

from which

$$\int_0^{r_m} \left[\frac{\partial L(p^*(r,t), \beta^*(t), r, t)}{\partial \beta} \right.$$
$$\left. - q(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p^*(r,t) \right] dr = 0, \qquad \forall t \in [0,T] \text{ a.e.}$$

Therefore, (33) still holds.

Finally, if the adjoint system

$$\begin{aligned} \frac{\partial \hat{q}(r,t)}{\partial t} + \frac{\partial \hat{q}(r,t)}{\partial r} &= \mu(r)\, \hat{q}(r,t) - \beta^*(t)\, k(r)\, h(r)\, \hat{q}(t) \\ \hat{q}(0,t) &= \hat{q}(t) \end{aligned} \tag{36}$$

has a nonzero solution $\hat{q}(r,t)$ (in which case $\hat{q}(r,T) \not\equiv 0$) such that

$$\hat{q}(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p^*(r,t)\, dr = 0, \qquad \forall t \in [0,T] \text{ a.e.}$$

then choose $\lambda_0 = 0$, $\alpha(r) = \hat{q}(r,T)$; we know (34) is still valid. Otherwise, if for an arbitray nonzero solution $\hat{q}(r,t)$ of Eq. (36) we always have

$$\hat{q}(r,t) \int_{r_1}^{r_2} k(r)\, h(r)\, p^*(r,t)\, dr \not\equiv 0, \tag{37}$$

in which case we say the situation is nondegenerate, then the linearized system (27) is controllable. For if not there exist $\alpha(t) \in L^2(0, r_m)$ such that

$$\int_0^{r_m} \alpha(r)\, p(r,T)\, dr = 0, \qquad \alpha(r) \not\equiv 0,$$

Then selecting $\lambda_0 = 0$ and the solution $\hat{q}(r,t)$ of Eq. (32) corresponding to $\alpha(r)$ (note that it is also a solution of (36)), we have from Lemma 3

$$\int_0^T \hat{q}(r,t)\, \beta(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p^*(r,t)\, dr = 0, \qquad \forall t \in L^\infty(0,T),$$

from which

$$\hat{q}(t)\int_{r_1}^{r_2} k(r)\,h(r)\,p^*(r,t)\,dr = 0, \qquad \forall t \in [0, T] \text{ a.e.}$$

This is a contradiction. So, under assumption (37) the linearized system is controllable.

Under all circumstances, we obtain

THEOREM 4. *Assume that* (p^*, β^*) *is the solution of the optimal control problem. Then there exist* $\lambda_0 \geqslant 0$, $\alpha(r) \in L^2(0, r)$, *not identically zero, such that the following maximum principle holds*:

$$\beta^*(t)\,H_\beta(p^*, \beta^*) = \max_{\beta_0 \leqslant \beta \leqslant \beta_1} \beta H_\beta(p^*, \beta^*)$$

$$H(p, \beta) = q(t)\,\beta(t)\int_{r_1}^{r_2} k(r)\,h(r)\,p(r,t)\,dr - \lambda_0 L(p, \beta, r, t), \tag{38}$$

$$H_\beta(p^*, \beta^*) = \frac{\partial H(p^*, \beta^*)}{\partial \beta}.$$

Note. In Ref. [9], it is shown that for an arbitrarily given ideal situation $p^*(r)$ and $\varepsilon > 0$, if the initial function $p_0(r)$ satisfies suitable conditions and provided $\beta_1 > \beta_{\mathrm{cr}} = [\int_{r_1}^{r_2} k(r)\,h(r)\,e^{-\int_0^r \mu(\rho)\,d\rho}\,dr]^{-1}$, there exist a control $\beta(t) \in U_{\mathrm{ad}}$ and a time $T > 0$ such that $\|p(r, T) - p^*(r)\| \leqslant \varepsilon$. This suggests to us the following optimal control problem.

Determine the optimal control $(p^*, \beta^*) \in X$ such that

$$J(p^*, \beta^*) = \min_{\beta(\cdot) \in U_{\mathrm{ad}}} \int_0^T \int_0^{r_m} L(p(r,t), \beta(t), r, t)\,dr\,dt,$$

$$\left.\begin{aligned}
&\frac{\partial p(r,t)}{\partial t} + \frac{\partial p(r,t)}{\partial r} = -\mu(r)\,p(r,t), \qquad 0 < r < r_m,\ t > 0\\
&p(r, 0) = p_0(r), \qquad 0 \leqslant r \leqslant r_m,\\
&p(r, T) \in V = \{p(r) \mid \|p(r) - p^0(r)\| \leqslant M\}\\
&p(0, t) = \beta(t)\int_{r_1}^{r_2} k(r)\,h(r)\,p(r,t)\,dr, \qquad t \geqslant 0,\\
&U_{\mathrm{ad}} = \{\beta(t) \mid 0 \leqslant \beta_0 \leqslant \beta(t) \leqslant \beta_1,\ t \in [0, T] \text{ a.e.}\}.
\end{aligned}\right. \tag{39}$$

The assumptions on L are as before. Let

$$\Omega_1 = \{p(r, t), \beta(t) \in X \mid \beta_0 \leqslant \beta(t) \leqslant \beta_1, t \in [0, T] \text{ a.e.}\}$$

$$\Omega_2 = \{(p(r, t), \beta(t)) \in X \mid p(r, T) \in V\}$$

$$\Omega_3 = \{(p(r, t), \beta(t)) \in X \mid p_t + p_r = -\mu p, p(r, t) = p_0(r),$$

$$p(0, t) = \beta(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p(r, t)\, dr\}. \tag{40}$$

Now, the directions of decrease cone and its adjoint are as in (21) and (22). Corresponding to Ω_1 the feasible directions cone and its adjoint are as in (20) and (23). Because Ω_2 is a closed convex set and $\mathring{\Omega}_2 \neq \varnothing$, the dual f_2 corresponding to its feasible directions is a supporting functional; that is,

$$f_2(p, \beta) \geqslant f_2(p^*, \beta^*), \qquad \forall p(r, T) \in V.$$

Thus, there exists $\alpha(r) \in L^2(0, r_m)$ such that

$$f_2(p, \beta) = \int_0^{r_m} \alpha(r)\, p(r, T)\, dr. \tag{41}$$

Therefore,

$$\alpha(r) = \hat{\lambda}_0[p^0(r) - p^*(r, T)], \qquad \hat{\lambda}_0 \geqslant 0.$$

For Ω_3, define the operator $G: X \to C(0, T; L^2(0, r_m))$ by

$$G(p, \beta) = p(r, t) - \begin{cases} p_0(r-t)\, e^{-\int_{r-t}^{r} \mu(\rho)\, d\rho}, & r \geqslant t, \\ \beta(t-r) \int_{r_1}^{r_2} k(s)\, h(s)\, p(s, t-r)\, ds\, e^{-\int_0^r \mu(\rho)\, d\rho}, & r < t. \end{cases} \tag{42}$$

Then $\Omega_3 = \{(p, \beta) \mid G(p, \beta) = 0\}$. As before,

$$G'(p^*, \beta^*)(\hat{p}, \hat{\beta})$$

$$= \hat{p}(r, t) - \begin{cases} 0, & r \geqslant t, \\ \beta^*(t-r) \int_{r_1}^{r_2} k(s)\, h(s)\, \hat{p}(s, t-r)\, ds\, e^{-\int_0^r \mu(\rho)\, d\rho} & \\ \quad + \hat{\beta}(t-r) \int_{r_1}^{r_2} k(s)\, h(s)\, p^*(s, t-r)\, ds\, e^{-\int_0^r \mu(\rho)\, d\rho}, & r < t. \end{cases}$$

For an arbitrary $g(r, t) \in C(0, T; L^2(0, r_m))$, the equation

$$G'(p^*, \beta^*)(\hat{p}, 0) = g(r, t)$$

has a unique solution. Thus, $G'(p^*, \beta^*)X = C(0, T; L^2(0, r_m))$. Therefore, for Ω_3 the tangent directions cone $K_3 = \{(p, \beta) \mid G'(p^*, \beta^*)(p, \beta) = 0\}$, from which we have

THEOREM 5. *Assume that* (p^*, β^*) *is the solution of the optimal control problem* (39). *Then there exist* $\lambda_0 \geqslant 0$, $\hat{\lambda}_0 \geqslant 0$, *not identically zero, such that*

$$\begin{aligned}
\frac{\partial q(r, t)}{\partial r} + \frac{\partial q(r, t)}{\partial t} &= \mu(r)\, q(r, t) - \beta^*(t)\, k(r)\, h(r)\, q(t) \\
&\quad + \lambda_0 \frac{\partial L(p^*, \beta^*, r, t)}{\partial p} \\
q(r, T) &= \hat{\lambda}_0[p^0(r) - p^*(r, T)], \\
q(0, t) &= q(t).
\end{aligned} \tag{43}$$

(p, β^*) *satisfies the maximum principle*

$$\beta^*(t)\, H_\beta(p^*, \beta^*) = \max_{\beta_0 \leqslant \beta \leqslant \beta_1} \beta H_\beta(p^*, \beta^*). \tag{44}$$

Here, H_β *is as shown in* (38).

Return to the infinite time problem (15). Assume that

$$\lambda_{OT} + \|q_T(t)\|_{L^2(0, T)} = 1. \tag{45}$$

The subscripts show the relationship to T. Take $T_N \to \infty$ such that $\lambda_{OT_N} \to \lambda_\infty$. From (6) and for fixed t and T_N sufficiently large t,

$$\begin{aligned}
q_{T_N} &= \int_t^{T_N} e^{-\int_0^{s-t} \mu(\rho)\, d\rho}\, \beta^*(s)\, k(s-t)\, h(s-t)\, q_{T_N}(s)\, ds \\
&\quad - \lambda_{TN} \int_t^T e^{-\int_0^{s-t} \mu(\rho)\, d\rho} \left.\frac{\partial L(p^*, \beta^*, \cdot, \cdot)}{\partial p}\right|_{(s-t, s)} ds.
\end{aligned}$$

It is easily proven that

$$q_{T_N} \to q(t), \qquad \text{as} \quad N \to \infty$$

and $q(t)$ satisfies

$$q(t) = \int_t^{\infty} e^{-\int_0^{s-t} \mu(\rho)\, d\rho}\, \beta^*(s)\, k(s-t)\, h(s-t)\, q(s)\, ds$$
$$-\lambda_\infty \int_t^{T} e^{-\int_0^{s-t} \mu(\rho)\, d\rho}\, \frac{\partial L(p^*, \beta^*, \cdot, \cdot)}{\partial p}\bigg|_{(s-t, s)} ds.$$

THEOREM 6. *For the optimal control problem on an infinite time interval, there exist* $\lambda_\infty \geqslant 0$ *and* $q(t)$, *both not identically zero, such that*

$$\beta^*(t)\, H_\beta(p^*, \beta^*) = \max_{\beta_0 \leqslant \beta \leqslant \beta_1} \beta H_\beta(p^*, \beta^*), \qquad \forall t \in [0, T] \text{ a.e.},$$

$$H(p, \beta) = q(t)\, \beta(t) \int_{r_1}^{r_2} k(r)\, h(r)\, p(r, t)\, dr - \lambda_\infty L(p, \beta, r, t)$$

$$H_\beta(p^*, \beta^*) = \frac{\partial H(p^*, \beta^*)}{\partial \beta}.$$

The function $q(r, t)$ *satisfies*

$$\begin{aligned} \frac{\partial q(r,t)}{\partial r} + \frac{\partial q(r,t)}{\partial t} &= \mu(r)\, q(r,t) - \beta^*(t)\, k(r)\, h(r)\, q(t) \\ &\quad + \lambda_\infty \frac{\partial L(p^*, \beta^*, r, t)}{\partial p} \\ q(r, \infty) &= 0, \\ q(0, t) &= q(t). \end{aligned} \tag{46}$$

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