Well-posedness and regularity of Euler–Bernoulli equation with variable coefficient and Dirichlet boundary control and collocated observation

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Two types of open-loop systems of an Euler–Bernoulli equation with variable coefficient and Dirichlet boundary control and collocated observation are considered. The uncontrolled boundary is either hinged or clamped. It is shown, with the help of multiplier method on Riemannian manifold, that in both cases, systems are well-posed in the sense of D. Salamon and regular in the sense of G. Weiss. In addition, the feedthrough operators are found to be zero. The result implies that the exact controllability of open-loop is equivalent to the exponential stability of closed-loop under a proportional output feedback for these systems. Copyright © 2013 John Wiley & Sons, Ltd.

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1. Introduction and main results

The well-posed systems belong to a wide class of linear infinite-dimensional systems, which cover many partial differential equations (PDEs) with actuators and sensors on isolated points, on sub-domains, or on a part of the boundary of the spatial regions. This class of infinite-dimensional systems, although the input and output operators are allowed to be unbounded, possess many properties that parallel in many ways to the finite-dimensional counterparts. Many PDEs have been proved to be well-posed. We refer to [1–15] and the references therein.

The objective of this paper is to generalize the well-posedness and regularity for Euler–Bernoulli plate equation with Dirichlet boundary control and collocated observation to the cases where the boundary condition is either hinged or clamped, and the coefficients are spatial variable dependent ([16]). The systems that we are concerned with in this paper are described by the following PDEs (1) and (2), respectively:

\begin{align}
\dot{w}_t(x, t) + P^2 w(x, t) &= 0, \quad x \in \Omega, \ t > 0, \\
\dot{w}(x, t) &= u(x, t), \quad x \in \partial \Omega, \ t \geq 0, \\
Pw(x, t) &= 0, \quad x \in \partial \Omega, \ t \geq 0, \\
y(x, t) &= \frac{\partial (A^{-2} w(x, t))}{\partial n_A}, \quad x \in \partial \Omega, \ t \geq 0,
\end{align}

(hinged B.C.)

\begin{align}
\dot{v}_t(x, t) + P^2 v(x, t) &= 0, \quad x \in \Omega, \ t > 0, \\
v(x, t) &= u(x, t), \quad x \in \partial \Omega, \ t \geq 0, \\
\frac{\partial v(x, t)}{\partial n_A} &= 0, \quad x \in \partial \Omega, \ t \geq 0, \\
y(x, t) &= \frac{\partial (A_1^{-3/2} v_t(x, t))}{\partial n_A}, \quad x \in \partial \Omega, \ t \geq 0,
\end{align}

(clamped B.C.)

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where \( \Omega \subset \mathbb{R}^n \) \((n \geq 2)\) is an open-bounded domain with smooth \( C^3 \)-boundary \( \partial \Omega = \Gamma \). \( A_1 \) is defined in (9) later, and \( P \) is a second-order partial differential operator:

\[
P = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial}{\partial x_j} \right)
\]

which satisfies, for some constants \( a, b > 0 \), that

\[
a \sum_{i=1}^{n} |\xi_i|^2 \leq b \sum_{i=1}^{n} |\eta_i|^2, \quad \forall \, x \in \overline{\Omega}, \, \xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{C}^n,
\]

\[
a_{ij} = a_{ij} \in C^\infty(\mathbb{R}^n), \quad \forall \, i, j = 1, 2, \cdots, n.
\]

Define operators \( A \) and \( A_1 \) as follows:

\[
\begin{align*}
A \varphi &= P \varphi, \quad D(A) = H^2(\Omega) \cap H^1_0(\Omega), \\
A_1 \varphi &= P \varphi, \quad D(A_1) = H^2(\Omega),
\end{align*}
\]

and define

\[
\nu_{v, A} = \left( \sum_{k=1}^{n} v_k a_{k1}(x), \sum_{k=1}^{n} v_k a_{k2}(x), \cdots, \sum_{k=1}^{n} v_k a_{kn}(x) \right), \quad \nu_{v, A} \partial = \sum_{i,j=1}^{n} a_{ij}(x) v_j \frac{\partial}{\partial x_i}
\]

where \( v = (v_1, v_2, \cdots, v_n) \) is the unit normal vector of \( \partial \Omega \) pointing to the exterior of \( \Omega \), \( u \) and \( y \) are, by abuse of the notation which is clear from the context, the boundary control and observation for systems (1) and (2), respectively.

Now, let \( A \) be the positive self-adjoint operator in \( L^2(\Omega) \), which is defined by

\[
\begin{align*}
A \varphi &= P^2 \varphi, \\
D(A) &= \{ \varphi \in H^4(\Omega) : \varphi|_{\Gamma} = P \varphi|_{\Gamma} = 0 \}.
\end{align*}
\]

Similar to [14], one can show that

\[
A^{1/2} = -A.
\]

The following space identifications with equivalent norms can be obtained from the interpolations (see, e.g., [17]):

\[
\begin{align*}
D(A^\theta) &= \{ \varphi \in H^m(\Omega) : \varphi|_{\Gamma} = 0 \} = H^m(\Omega) \cap H^1_0(\Omega), \quad 1/8 < \theta < 5/8, \\
D(A^\theta) &= \{ \varphi \in H^m(\Omega) : \varphi|_{\Gamma} = P \varphi|_{\Gamma} = 0 \}, \quad 5/8 < \theta \leq 1.
\end{align*}
\]

Therefore,

\[
V \equiv D(A^{3/4}) = \{ \varphi \in H^2(\Omega) : \varphi|_{\Gamma} = P \varphi|_{\Gamma} = 0 \}.
\]

Let \( \mathcal{H} = H^{-1}(\Omega) \times V' \) and \( U = L^2(\Gamma) \), where \( V' \) is the dual of space \( V \) with respect to the pivot space \( L^2(\Omega) \) in the sense of the Gelfand triple inclusions \( V \hookrightarrow L^2(\Omega) \hookrightarrow V' \). Theorem 1.1 is the generalization of Theorem 4.35 of [16].

**Theorem 1.1**

System (1) is well-posed. More precisely, for any \( T > 0 \), initial value \( (w_0, w_1) \in \mathcal{H} \), and \( u \in L^2(0, T; U) \), there exists a unique solution \( (w, w_t) \in C(0, T; \mathcal{H}) \) to system (1), which satisfies \( w(., 0) = w_0 \) and \( w_t(., 0) = w_1 \). In addition, there exists a constant \( C_T > 0 \) that is independent of \( (w_0, w_1, u) \) such that

\[
\begin{equation}
\left\| (w(., T), w_t(., T)) \right\|^2_{\mathcal{H}} + \left\| y \right\|^2_{L^2(0, T; U)} \leq C_T \left[ \left\| (w_0, w_1) \right\|^2_{\mathcal{H}} + \left\| u \right\|^2_{L^2(0, T; U)} \right].
\end{equation}
\]

Next, let \( A_1 \) be the positive self-adjoint operator in \( L^2(\Omega) \), which is defined by

\[
\begin{align*}
A_1 \varphi &= P^2 \varphi, \\
D(A_1) &= \{ \varphi \in H^4(\Omega) : \varphi|_{\Gamma} = \frac{\partial \varphi}{\partial \nu_{v, A}}|_{\Gamma} = 0 \}.
\end{align*}
\]

It can be shown that ([14])

\[
A_1^{1/2} = -A_1.
\]
Similarly, by virtue of the interpolation results ([17]), we have the following space identifications with the equivalent norms.

\[
\begin{align*}
(D(\mathcal{A}^3))^0 = & \{ \varphi \in H^3(\Omega) : \varphi|_\Gamma = 0 \} = H^\theta(\Omega) \cap H^\theta_0(\Omega), \\
(D(\mathcal{A}^3))^1 = & \{ \varphi \in H^3(\Omega) : \varphi|_\Gamma = \frac{\partial \varphi}{\partial n}\|_\Gamma = 0 \}.
\end{align*}
\]

Therefore,

\[
V_1 \Delta D(\mathcal{A}^{3/4}_1) = \{ \varphi \in H^3(\Omega) : \varphi|_\Gamma = \frac{\partial \varphi}{\partial n}\|_\Gamma = 0 \}.
\]

Let \( \mathcal{H}_1 = H^{-1}(\Omega) \times V'_1 \) and \( U = L^2(\Gamma) \), where \( V'_1 \) is the dual of space \( V_1 \) with respect to the pivot space \( L^2(\Omega) \) in the sense of the Gelfand triple inclusions \( V_1 \hookrightarrow L^2(\Omega) \hookrightarrow V'_1 \). Theorem 1.2 is the generalization of Theorem 4.21 of [16].

**Theorem 1.2**

System (2) is well-posed. More precisely, for any \( T > 0 \), initial value \( (v_0, v_1) \in \mathcal{H}_1 \), and \( u \in L^2(0, T; U) \), there exists a unique solution \( (v, v_t) \in C(0, T; \mathcal{H}_1) \) to system (2) that satisfies \( v(0) = v_0 \) and \( v_t(0) = v_1 \). Moreover, there exists a constant \( C_T > 0 \) that is independent of \( (v_0, v_1, u) \) such that

\[
\begin{align*}
\|v(\cdot, T), v_t(\cdot, T)\|_{\mathcal{H}_1}^2 + \|v_t(\cdot, T)\|_{L^2(0, T; U)}^2 & \leq C_T \left[ \|v(0, v_1)\|_{\mathcal{H}_1}^2 + \|u(0, T, U)\|_{L^2(0, T; U)}^2 \right].
\end{align*}
\]

(11)

Theorems 1.1 and 1.2 imply that both the open-loop systems (1) and (2) are well-posed in the sense of D. Salamon with the state spaces \( \mathcal{H} \) and \( \mathcal{H}_1 \) and the input and output spaces \( U \), respectively. From these well-posedness results and Theorems 6.9 and 6.10 of [18] (see also Theorems 5.3 and 5.4 of [19] for the first order systems), we have the following result.

**Corollary 1.1**

The systems (1) or (2) is exactly controllable in some time interval \([0, T]\) if and only if the corresponding closed-loop system under the proportional output feedback \( u = -ky, k > 0 \) is exponentially stable.

**Theorem 1.3**

The system (1) is regular with zero feedthrough operator. More precisely, if \( w(\cdot, 0) = w_t(\cdot, 0) = 0 \) and \( u(\cdot, t) = u(\cdot) \in U \) is a step input, then the corresponding output \( y \) satisfies

\[
\lim_{\sigma \to 0^+} \int_{\Gamma} \left| \frac{1}{\sigma} \int_0^\sigma y(x, t) \, dt \right|^2 \, d\Gamma = 0.
\]

(12)

**Theorem 1.4**

The system (2) is regular with zero feedthrough operator. More precisely, if \( v(\cdot, 0) = v_t(\cdot, 0) = 0 \) and \( u(\cdot, t) = u(\cdot) \in U \) is a step input, then the corresponding output \( y \) satisfies

\[
\lim_{\sigma \to 0^+} \int_{\Gamma} \left| \frac{1}{\sigma} \int_0^\sigma y(x, t) \, dt \right|^2 \, d\Gamma = 0.
\]

(13)

We point out that the variable coefficients occur often for the plate when the mass of plate is not uniformly distributed with respect to spatial position. Theorems 1.1 and 1.2 are generalizations of [16] where the constant coefficients are considered. However, these generalizations are not direct. As it was shown in [20, 21] that the classical multiplier in Euclidean space is not efficient to deal with the case of variable coefficients. Actually, to deal with the variable coefficients, some computations on Riemannian manifold are required.

The paper is organized as follows. In section 2, we formulate systems (1) and (2) into collocated abstract second-order systems. Some basic knowledge on Riemannian geometry is stated. Sections 3 and 4 are devoted to the proof of Theorems 1.1 and 1.2, respectively. The proofs of Theorems 1.3 and 1.4 are presented in Section 5.

### 2. Collocated formulations and preliminary results

In this section, we list some notation and facts in Riemannian geometry that we need in the following sections. By the ellipticity condition (3), we denote by \( A(x) \) and \( G(x) \), respectively, the coefficient matrix and its inverse and by \( \rho(x) \) the determinant of \( G(x) \),

\[
A(x) = [a_{ij}(x)]_{n \times n}, \quad G(x) = [g_{ij}(x)]_{n \times n} = [a_{ij}(x)]_{n \times n}^{-1} = A(x)^{-1}, \quad \rho(x) = \det[g_{ij}(x)]_{n \times n}, \quad \forall \, x \in \mathbb{R}^n.
\]

(14)

Let \( \mathbb{R}^n \) be the usual Euclidean space. For each \( x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \), define the inner product and norm over the tangent space \( R^t_x \) of \( \mathbb{R}^n \) by

\[
g(X, Y) := \langle X, Y \rangle_g = \sum_{i,j=1}^n g_{ij}x_iy_j,
\]

\[
|X|_g := \langle X, X \rangle_g^{1/2}, \quad \forall \, X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i} \in \mathbb{R}^t_x.
\]

(15)
Then, \((\mathbb{R}^n, g)\) becomes a Riemannian manifold with Riemannian metric \(g\) ([20,21]). Denote by \(D\) the Levi-Civita connection with respect to \(g\). Let \(N\) be a smooth vector field on \((\mathbb{R}^n, g)\). For each \(x \in \mathbb{R}^n\), the covariant differential \(DN\) of \(N\) determines a bilinear form on \(\mathbb{R}^n \times \mathbb{R}^n:\)

\[
DN(X, Y) = (D_Y N) g, \quad X, Y \in \mathbb{R}^n,
\]

where \(D_Y N\) stands for the covariant derivative of the vector field \(N\) with respect to \(Y\). For any \(\varphi \in C^2(\mathbb{R}^n)\) and \(N = \sum_{i=1}^{n} h_i(x) \frac{\partial}{\partial x_i}\), denote

\[
div g(N) = \sum_{i=1}^{n} \frac{1}{\sqrt{\rho(x)}} \frac{\partial}{\partial x_i} \left( \sqrt{\rho(x)} h_i(x) \right),
\]

Then, \(\frac{\partial}{\partial x_i}\) is the divergence operator in Euclidean space \(\mathbb{R}^n\) and \(\nabla g\), \(\text{div}_g\), and \(\Delta g\) are the gradient operator, the divergence operator, and the Beltrami–Laplace operator in \((\mathbb{R}^n, g)\), respectively.

Let \(\mu = \frac{\nu}{|A|}\) be the unit outward-pointing normal vector to \(\partial \Omega\) in terms of the Riemannian metric \(g\). The following Lemma 2.1 ([22]) provides some useful identities.

**Lemma 2.1**

Let \(\varphi, \psi \in C^2(\overline{\Omega})\) and let \(N\) be a vector field on \((\mathbb{R}^n, g)\), then,

1. Divergence formulae and theorems:

\[
div g(N) = \varphi \text{div}_g(N) + N(\varphi), \quad \text{div}_g(\varphi N) = \varphi \text{div}_g(N) + N(\varphi),
\]

\[
\int_{\Omega} \text{div}_g(N) dx = \int_{\Gamma} N \cdot v d\Gamma, \quad \int_{\Omega} \text{div}_g(N) dx = \int_{\Gamma} (N, \mu)_g d\Gamma.
\]

2. Green’s formulae:

\[
\int_{\Omega} \psi \Delta g \varphi dx = \int_{\Gamma} \psi \frac{\partial \varphi}{\partial \nu} A d\Gamma - \int_{\Omega} \langle \nabla g \varphi, \nabla g \psi \rangle_g dx,
\]

\[
\int_{\Omega} \psi \Delta g \varphi dx = \int_{\Gamma} \psi \frac{\partial \varphi}{\partial \nu} d\Gamma - \int_{\Omega} \langle \nabla g \varphi, \nabla g \psi \rangle_g dx.
\]

Similar to Lemma 2.1 (4) of [20], we have the following multiplier identity.

**Lemma 2.2**

Let \(N\) be a smooth real vector field on \((\mathbb{R}^n, g)\). For \(\varphi \in C^2(\overline{\Omega})\), the following formula holds

\[
\text{Re} \langle \nabla g \varphi, \nabla g(N(\varphi)) \rangle_g = \text{Re} DN(\nabla g \varphi, \nabla g \varphi) + \frac{1}{2} \text{div}_g(\langle \nabla g \varphi, \nabla g \varphi \rangle_g) + \frac{1}{2} \|\nabla g \varphi\|_g^2 \text{div}_g(N).
\]

(18)

Denote by \(T^2(\mathbb{R}^n)_g\) the set of all covariant tensors of order 2 on \(\mathbb{R}^n\). \(T^2(\mathbb{R}^n)_g\) is an inner product space of dimension \(n^2\) with inner product of the following:

\[
\langle E, F \rangle_{T^2(\mathbb{R}^n)_g} = \sum_{i=1}^{n} E(e_i, e_j) F(e_i, e_j), \quad \forall E, F \in T^2(\mathbb{R}^n)_g,
\]

where \(\{e_1, e_2, \cdots, e_n\}\) is an arbitrarily chosen orthonormal basis for \((\mathbb{R}^n, g)\).

Let \(\mathcal{X}(\mathbb{R}^n)\) be the set of all vector fields on \(\mathbb{R}^n\). Denote by \(\Delta : \mathcal{X}(\mathbb{R}^n) \to \mathcal{X}(\mathbb{R}^n)\) the Hodge-Laplace operator. Then ([21]):

\[
\Delta g(N(\varphi)) = (\Delta N(\varphi)) + 2(DN, D^2 \varphi)_g + N(\Delta g \varphi) + \text{Ric}(N, D\varphi), \quad \forall \varphi \in C^2(\mathbb{R}^n),
\]

where \(\text{Ric}(\cdot, \cdot)\) is the Ricci curvature tensor of the Riemannian metric \(g\), \(D^2 \varphi\) is the Hessian of \(\varphi\) in terms of the Riemannian metric \(g\).

**Lemma 2.3**

Let \(\varphi\) be a complex function defined on \(\overline{\Omega}\) with suitable regularity. Then, there exists constant \(C\), possibly depending on \(g, N\) and \(\Omega\), such that
1. Define an extension

\[ \sup_{x \in \Omega} |N|_g \leq C, \sup_{x \in \Omega} |DN|_g \leq C, \sup_{x \in \Omega} |\text{div}_g(N)| \leq C, \sup_{x \in \Omega} |Dp|_g \leq C, \sup_{x \in \Omega} |\nabla_g(\text{div}_g N)|_g \leq C, \]

2. Define the map

\[ |N(\varphi)| \leq C|\nabla_g \varphi|_g, |Dp(\varphi)| \leq C|\nabla_g \varphi|_g, |DN(\nabla_g \varphi, \nabla_g \varphi)| \leq C|\nabla_g \varphi|_g^2, \]

\[ |(\nabla_g \varphi, \nabla_g (\text{div}_g N))_g| \leq C|\nabla_g \varphi|_g, |(\Delta N)\varphi|_g \leq C|\Delta N|_g |\nabla_g \varphi|_g \leq C|\nabla_g \varphi|_g, \]

\[ |(DN, D^2 \varphi)_{\Omega'_{\{g\}}}| \leq C|DN|_g |D^2 \varphi|_g \leq C|D^2 \varphi|_g, \]

\[ |D^2 p(N, Dp)| \leq |D^2 p|_g |N|_g |Dp|_g \leq |Dp|_g, |D^2 p(N, Dp)| \leq |D^2 \varphi|_g |N|_g |Dp|_g \leq |D^2 \varphi|_g, \]

\[ |\text{Ric}(N, Dp)| \leq |\text{Ric}|_g |N|_g |Dp|_g \leq |Dp|_g, \]

where \( p(x) = \frac{1}{2} \text{ln}(|\text{det}(a_g(x))|) \).

3. \[ \int_{\Omega} |\varphi|^2 dx \leq C|\varphi|_{H^2(\Omega)}^2, \int_{\Omega} |D\varphi|_g^2 dx \leq C|\varphi|_{H^2(\Omega)}^2, \int_{\Omega} |D^2 \varphi|_g^2 dx \leq C|\varphi|_{H^2(\Omega)}^2. \]

The following Lemma 2.4 follows from (3),(5), and (15).

**Lemma 2.4**

\[ a \leq |\varphi|_{\Omega}^2 \leq b. \]  

Now, we formulate the system (1) as an abstract second-order system in the state space \( \mathcal{H} = H^{-1}(\Omega) \times H \) with \( H = V' \). Extend the operator \( \mathcal{A} \) of \( A \) into the space \( V \) by

\[ \langle \mathcal{A} \varphi, \psi \rangle' = \langle A^{1/2} \varphi, A^{1/2} \psi \rangle', \ \forall \ \varphi, \psi \in V. \]  

Then, \( \mathcal{A} \) is a positive self-adjoint operator in \( V' \). In fact,

\[ \langle \mathcal{A} \varphi, \psi \rangle' = \langle A^{1/2} \varphi, A^{1/2} \psi \rangle' = \langle A^{-1/2} \varphi, A^{-1/2} \psi \rangle_{L^2(\Omega)} \]

\[ \geq C|\varphi|_{L^2(\Omega)}^2 \geq C' |A^{-3/4} \varphi|_{L^2(\Omega)}^2 = C' |\varphi|_{V'}^2, \ \forall \ \varphi \in V, \]

where \( C, C' \) are positive constants. We identify \( H \) with its dual \( H' \). Then the following Gelfand triple of continuous and dense inclusions hold:

\[ D \left( \mathcal{A}^{1/2} \right) \hookrightarrow H = H' \hookrightarrow D \left( \mathcal{A}^{1/2} \right)' . \]

Define an extension \( \mathcal{A} \in \mathcal{L} \left( D \left( \mathcal{A}^{1/2} \right), D \left( \mathcal{A}^{1/2} \right)' \right) \) of \( \mathcal{A} \) by

\[ \langle \mathcal{A} \varphi, \psi \rangle_{D \left( \mathcal{A}^{1/2} \right)' , D \left( \mathcal{A}^{1/2} \right)'} = \langle \mathcal{A}^{1/2} \varphi, \mathcal{A}^{1/2} \psi \rangle'_{V', V}, \ \forall \ \varphi, \psi \in D \left( \mathcal{A}^{1/2} \right)'. \]

Define the map \( G \in \mathcal{L}(L^2(\Gamma), H^{1/2}(\Omega)) \) (17, p.188-189) so that \( Gu = \phi \) if and only if

\[ \begin{cases} P^2 \phi(x) = 0, & x \in \Omega, \\ P\phi |_{\Gamma} = 0, & \phi(x) |_{\Gamma} = u(x). \end{cases} \]

In terms of operators \( \mathcal{A} \) and \( G \), the system (1) is written in \( D \left( \mathcal{A}^{1/2} \right)' \) as

\[ \tilde{w} + \mathcal{A}w = Bu, \]

where \( B \in \mathcal{L} \left( U, D \left( \mathcal{A}^{1/2} \right)' \right) \) is given by

\[ Bu = \mathcal{A}Gu, \ \forall \ u \in U. \]
Define $B^* \in \mathcal{L}(A^{1/2}, U)$ by
\begin{equation}
(B^*f, u)_U = (f, Bu)_{D(A^{1/2}), D(A^{1/2})'}, \quad \forall \ f \in \text{Dom}(A^{1/2}) = H_0^1(\Omega), \ u \in U.
\end{equation}
(27)

Then, for any $f \in D\left(\overline{A}^{1/2}\right)$ and $u \in C_0^\infty(\Gamma)$, we have
\begin{equation}
(Bu, f)_{D(\overline{A}^{1/2})', D(\overline{A}^{1/2})} = (\overline{A}Gu, f)_{D(\overline{A}^{1/2})', D(\overline{A}^{1/2})} = (\overline{A}A^{1/2}Gu, f)_{D(\overline{A}^{1/2})'} = (\overline{A}Gu, A^{1/2}(A^{-2}f))_{D(\overline{A}^{1/2})'} = (u, -\frac{\partial(A^{-2}f)}{\partial V_A})_U.
\end{equation}
(28)

Because $C_0^\infty(\Gamma)$ is dense in $L^2(\Gamma)$, we obtain
\begin{equation}
B^*f = -\frac{\partial(A^{-2}f)}{\partial V_A}, \quad \forall \ f \in D(\overline{A}^{1/2}) = H_0^1(\Omega).
\end{equation}
(29)

We have thus formulated system (1) into a second-order abstract form in $\mathcal{H}$:
\begin{equation}
\begin{cases}
\dot{\psi} + \overline{A}\psi = Bu, \\
y = B^*\dot{\psi},
\end{cases}
\end{equation}
(30)
where $\overline{A}$, $B$, and $B^*$ are defined by (23), (26), and (29), respectively.

Next, we formulate the system (2) as an abstract second-order system in $\mathcal{H}_1 = H^{-1}(\Omega) \times H_1$ with $H_1 = \mathcal{V}_1$. Extend the operator $\overline{A}_1$ of $A_1$ to the space $\mathcal{V}_1$ by
\begin{equation}
(\overline{A}_1\psi, \psi)_{\mathcal{V}_1} = (A_1^{1/2}\psi, A_1^{1/2}\psi)_{\mathcal{V}_1}, \quad \forall \, \psi, \varphi \in \mathcal{V}_1.
\end{equation}
(31)

Then, $\overline{A}_1$ is a positive self-adjoint operator in $\mathcal{V}_1$. Actually,
\begin{equation}
\begin{aligned}
&(\overline{A}_1\psi, \varphi)_{\mathcal{V}_1} = (A_1^{1/2}\psi, A_1^{1/2}\varphi)_{\mathcal{V}_1} = (A_1^{-1/4}\psi, A_1^{-1/4}\varphi)_{L^2(\Omega)} \\
&\geq C\|\psi\|_{L^2(\Omega)} \geq C'\|A_1^{-3/4}\psi\|_{L^2(\Omega)} = C'\|\psi\|_{\mathcal{V}_1}^2, \quad \forall \, \psi, \varphi \in \mathcal{V}_1,
\end{aligned}
\end{equation}
(32)
where $C, C'$ are positive constants. We identify $H_1$ with its dual $H_1'$. Then the following Gelfand triple inclusions hold:
\begin{equation}
D\left(A_1^{1/2}\right) \hookrightarrow H_1 = H_1' \hookrightarrow D\left(A_1^{1/2}\right)'.
\end{equation}
(33)

Define an extension $\tilde{\overline{A}}_1 \in \mathcal{L}\left(D\left(A_1^{1/2}\right), D\left(A_1^{1/2}\right)'ight)$ of $\overline{A}_1$ by
\begin{equation}
(\tilde{\overline{A}}_1\psi, \varphi)_{D(\overline{A}_1^{1/2})', D(\overline{A}_1^{1/2})} = (A_1^{1/2}\psi, A_1^{1/2}\varphi)_{D(\overline{A}_1^{1/2})'} = (A_1^{1/2}\psi, A_1^{1/2}\varphi)_{D(\overline{A}_1^{1/2})'}, \quad \forall \, \psi, \varphi \in D\left(A_1^{1/2}\right).
\end{equation}
(34)

Define the map $G_1 \in \mathcal{L}(L^2(\Gamma), H^{-1/2}(\Omega))$ so that $G_1u = \phi$ if and only if
\begin{equation}
\begin{cases}
p^2\phi(x) = 0, \quad x \in \Omega, \\
\frac{\partial\phi(x)}{\partial\nu_A} |_{\Gamma} = 0, \quad \phi(x) |_{\Gamma} = u(x).
\end{cases}
\end{equation}
(35)

In terms of the operators $\tilde{\overline{A}}_1$ and $G_1$, the system (2) can be written in $D\left(A_1^{1/2}\right)'$ as
\begin{equation}
\dot{\psi} + \tilde{\overline{A}}_1\psi = B_1u,
\end{equation}
(36)
where $B_1 \in \mathcal{L}\left(U, D\left(A_1^{1/2}\right)'\right)$ is given by
\begin{equation}
B_1u = \tilde{\overline{A}}_1G_1u, \quad \forall \, u \in U.
\end{equation}
(37)

Define $B_1^* \in \mathcal{L}\left(D\left(A_1^{1/2}\right), U\right)$ by
\begin{equation}
(B_1^*f, u)_U = (f, B_1u)_{D(A_1^{1/2}), D(A_1^{1/2})'}, \quad \forall \, f \in D\left(A_1^{1/2}\right) = H_0^1(\Omega), \ u \in U.
\end{equation}
(38)
Then, for any \( f \in D(\widetilde{A}^{1/2}_1) \) and \( u \in C_0^\infty(\Gamma) \), we have

\[
(B_1u, f)_D(\widetilde{A}^{1/2}_1) = (\widetilde{A}_1G_1u, f)_D(\widetilde{A}^{1/2}_1) = (\widetilde{A}_1G_1u, \widetilde{A}^{1/2}_1f)_V = (\widetilde{A}_1G_1u, \widetilde{A}^{1/2}_1(\widetilde{A}^{-3/2}_1f))_V.
\]

(39)

Because \( C_0^\infty(\Gamma) \) is dense in \( L^2(\Omega) \), we obtain

\[
B_1^*f = \frac{\partial (\widetilde{A}_1^{1/2}f)}{\partial V_A}, \quad \forall f \in D(\widetilde{A}^{1/2}_1) = H^1_0(\Omega).
\]

(40)

We have once again formulated system (2) into a second-order abstract form in \( \mathcal{H}_1 \):

\[
\begin{align*}
\dot{y} + \widetilde{A}_1v &= B_1u, \\
y &= B_1^*\dot{v}.
\end{align*}
\]

(41)

where \( \widetilde{A}_1, B_1, \) and \( B_1^* \) are defined by (34), (37), and (40), respectively.

3. Proof of Theorem 1.1

In order to prove Theorem 1.1, we need the following Lemma 3.1 that comes from Theorem A.1 [6].

**Lemma 3.1**

If there exist constants \( T > 0 \) and \( C_T > 0 \) such that the input and output of system (1) satisfy

\[
\int_0^T \|y(t)\|^2_{L^2} dt \leq C_T \int_0^T \|u(t)\|^2_{L^2} dt,
\]

(42)

with zero initial data, then the system (1) is well-posed.

Introduce the transform \( z(t) = \mathcal{A}^{-1}w(t) \in C(0, T; V) \). Instead of (1), we consider the following system in \( W \times H^1_0(\Omega) \):

\[
\left\{
\begin{align*}
z_t(t, x) + P^2 z(t, x) &= (Gu)(x, t), & (x, t) \in \Omega \times (0, T) \\
z(x, 0) &= 0, & x \in \Omega, \\
z(x, t) &= Pz(x, t) = 0, & (x, t) \in \partial \Omega \times (0, T) \\
y(x, t) &= \frac{\partial z}{\partial V_A}(x, t), & (x, t) \in \Sigma.
\end{align*}
\right.
\]

(43)

It is seen that Theorem 1.1 holds true if and only if for some (and hence for all) \( T > 0 \), there exists a \( C_T > 0 \) such that the solution to (43) satisfies

\[
\int_\Sigma |\frac{\partial z}{\partial V_A}(x, t)|^2 d\Sigma \leq C_T \int_\Omega |u(x, t)|^2 d\Omega.
\]

(44)

**Proof of Theorem 1.1.**

The proof of (44) will be split into three steps.

Step 1. Energy identity. By Lemma 4.1 of [14], there exists a \( C^2 \) vector field \( N \) on \( \overline{\Omega} \) such that

\[
N(x) = \mu(x), \quad x \in \Gamma; \quad |N(x)|_g \leq 1, \quad x \in \Omega.
\]

(45)

Multiply both sides of the first equation of (43) by \( N(Pz) \) and perform the integration over \( Q = \Omega \times (0, T) \) to obtain

\[
\int_Q Z_t N(Pz) dQ + \int_Q P^2 z N(Pz) dQ = \int_Q Gu N(Pz) dQ.
\]

(46)

Compute the first term on the left-hand side of (46) to give

\[
\int_Q Z_t N(Pz) dQ = \int_\Omega Z_t N(Pz) dx \left|_0^T \right. - \int_Q Z_t N(Pz_t) dQ
\]

\[
= \int_\Omega Z_t N(Pz) dx \left|_0^T \right. - \int_Q \text{div}_g(z_t N(Pz_t) - (Pz_t) N(z_t) - z_t (Pz_t) \text{div}_g N) dQ
\]

\[
= \int_\Omega Z_t N(Pz) dx \left|_0^T \right. + \int_Q (Pz_t) N(z_t) dQ + \int_Q z_t (Pz_t) \text{div}_g N dQ.
\]

(47)
Furthermore, by virtue of Lemma 2.2, we have

\[
\text{Re} \int_Q \left( \frac{\partial z_i}{\partial \mu} N(z_i) dQ \right) = \int_Q \left( \Delta g(z_i) N(z_i) dQ + \text{Re} \int_Q Dp(z_i) N(z_i) dQ \right)
\]

\[
= \int_Q \sum_{\mu} \left[ \frac{\partial z_i}{\partial \mu} N(z_i) dQ - \int_Q (\nabla g(z_i), \nabla g(z_i)) dQ + \int_Q Dp(z_i) N(z_i) dQ \right]
\]

\[
= \int_Q \sum_{\mu} \left[ \frac{\partial z_i}{\partial \mu}^2 dQ - \int_Q (\nabla g(z_i), \nabla g(z_i)) dQ - \frac{1}{2} \int_Q \text{div}_g(\nabla g(z_i)^2 N) dQ \right]
\]

\[
+ \frac{1}{2} \int_Q |\nabla g(z_i)|^2 dQ + \int_Q Dp(z_i) N(z_i) dQ
\]

\[
= \int_Q \sum_{\mu} \left[ \frac{\partial z_i}{\partial \mu}^2 dQ - \int_Q (\nabla g(z_i), \nabla g(z_i)) dQ - \frac{1}{2} \int_Q \text{div}_g(\nabla g(z_i)^2 N) dQ \right]
\]

\[
+ \frac{1}{2} \int_Q |\nabla g(z_i)|^2 dQ + \int_Q Dp(z_i) N(z_i) dQ
\]

(48)

Substitute (48) and (49) into (47) to yield

\[
\text{Re} \int_Q z_i(Pz_i) \text{div}_g N dQ = \text{Re} \int_Q z_i(\Delta g(z_i)) \text{div}_g N dQ + \text{Re} \int_Q z_i Dp(z_i) \text{div}_g N dQ
\]

\[
= \text{Re} \int_Q \sum_{\mu} \left[ \frac{\partial z_i}{\partial \mu} (\Delta g(z_i)) \text{div}_g N dQ - \int_Q (\nabla g(z_i), \nabla g(z_i)) dQ + \int_Q z_i Dp(z_i) \text{div}_g N dQ \right]
\]

\[
= - \int_Q \text{div}_g N |\nabla g(z_i)|^2 dQ - \int_Q \text{div}_g N (\nabla g(z_i) \nabla g(z_i)) dQ + \int_Q z_i Dp(z_i) \text{div}_g N dQ
\]

\[
= - \int_Q \text{div}_g N |\nabla g(z_i)|^2 dQ - \frac{1}{2} \int_Q \text{div}_g N |z_i|^2 dQ + \frac{1}{2} \int_Q Dp(z_i) |z_i|^2 dQ
\]

(49)

50

Comput the second term on the left-hand side of (46) to obtain

\[
\text{Re} \int_Q p^2 N(Pz) dQ = \text{Re} \int_Q \Delta g(Pz) N(Pz) dQ + \text{Re} \int_Q Dp(Pz) N(Pz) dQ
\]

\[
= \text{Re} \int_Q \sum_{\mu} \left[ \frac{\partial (Pz)}{\partial \mu} N(Pz) dQ - \int_Q (\nabla g(Pz), \nabla g(N(Pz))) dQ + \int_Q Dp(Pz) N(Pz) dQ \right]
\]

\[
= \int_Q \sum_{\mu} \left[ \frac{\partial (Pz)}{\partial \mu}^2 dQ - \int_Q (\nabla g(Pz), \nabla g(Pz)) dQ - \frac{1}{2} \int_Q \text{div}_g(\nabla g(Pz)^2 N) dQ \right]
\]

\[
+ \frac{1}{2} \int_Q |\nabla g(Pz)|^2 dQ + \int_Q Dp(Pz) N(Pz) dQ
\]

\[
= \int_Q \sum_{\mu} \left[ \frac{\partial (Pz)}{\partial \mu}^2 dQ - \int_Q (\nabla g(Pz), \nabla g(Pz)) dQ - \frac{1}{2} \int_Q \text{div}_g(\nabla g(Pz)^2 N) dQ \right]
\]

\[
+ \frac{1}{2} \int_Q |\nabla g(Pz)|^2 dQ + \int_Q Dp(Pz) N(Pz) dQ
\]

(51)

Combine (46),(50), and (51), to obtain

\[
\frac{1}{2} \left( \int_Q \frac{\partial z_i}{\partial \mu}^2 dQ + \int_Q \frac{\partial (Pz)}{\partial \mu}^2 dQ \right)
\]

\[
= \left( \text{Re} \int_Q (Dn g(Pz), \nabla g(Pz)) dQ - \frac{1}{2} \int_Q |\nabla g(Pz)|^2 dQ + \int_Q Dp(Pz) N(Pz) dQ \right)
\]

(52)
Proof of Theorem 1.2.

By Lemma 3.1, Theorem 1.2 is equivalent to that the solution to system (2) with zero initial value satisfies

\[
\|y\|_{L^2(0,T;L^2(\Gamma))} \leq C_T \|u\|_{L^2(0,T;L^2(\Gamma))}, \ \forall \ u \in L^2(0,T;L^2(\Gamma)).
\]  

\[\text{Proof of Theorem 1.2.}\]

Let \( z = A^{-3/2}v \in C(0,T;V_1) \). Instead of (2), we consider the following system in \( V_1 \times H^0_0(\Omega) \):

\[
\begin{align*}
\frac{1}{2} \int_\Omega \div_g N(\nabla z^2) dz + \frac{1}{2} \int_\Omega \nabla_g (\div_g N) (|z|^2) dQ - \frac{1}{2} \int_\Omega \div(N(z) dQ) + \int_\Omega D(N(z_1) dQ) - \int_\Omega D(N(z_1) dQ) \\
+ \text{Re} \int_\Omega D(\nabla z_1, \nabla z_1) dQ - \int_\Omega D(\nabla z_1, \nabla z_1) dQ - \frac{1}{2} \int_\Omega \div(N(z_1) dQ) & \triangleq R_1 \\
+ \text{Re} \int_\Omega G u(Pz) dQ & \triangleq R_2 \\
- \text{Re} \int_\Omega z_1 N(Pz) dx & \triangleq b_{0,T}
\end{align*}
\]

Step 2. Evaluation of \( R_1 \). Firstly, consider the dual system of (30):

\[
\begin{align*}
& \left\{ \begin{array}{l}
\Delta t z_1(x,t) + A^2 z_1(x,t) = 0, \\
z(0) = z_0, \ z_1(0) = z_1,
\end{array} \right.
\end{align*}
\]

(53)

where \( \overline{\Delta} \) and \( A \) are given by (20) and (4), respectively. It is noted that the solution to (53) generates a \( C_0 \)-group on space \( V \times H^0_0(\Omega) \).

That is to say, for any \((z_0, z_1) \in V \times H^0_0(\Omega) \), the corresponding solution to (53) satisfies \((z, z_1) \in V \times H^0_0(\Omega) \) and depends continuously on \((z_0, z_1) \). From this fact, setting \( Gu = 0 \) in (52) and making use of Lemmas 2.4 and 2.3, we obtain from (52) that

\[ \int_\Sigma \left| \frac{\partial z_1}{\partial \nu_A} \right|^2 d\Sigma \leq C_T \| (z_0, z_1) \|^2_{V \times H^0_0(\Omega)}, \]  

(54)

that is, for any initial value \((w_0, w_1) \in H^{-1}(\Omega) \times V' \), the solution to system (1) with \( u = 0 \) satisfies

\[ \int_\Sigma \left| \frac{\partial(A^{-2}w_1)}{\partial \nu_A} \right|^2 d\Sigma \leq C_T \| (w_0, w_1) \|^2_{H^{-1}(\Omega) \times V'}. \]  

(55)

This shows that \( B^* \) is admissible and so is \( B \) ([5]). In other words,

\[ u \rightarrow (w, w_1) \text{is continuous from} \ L^2(\Sigma) \rightarrow C(0,T;H^{-1}(\Omega) \times V'), \]  

and hence,

\[ (z, z_1) \in C(0,T;V \times H^0_0(\Omega)) \text{depends continuously on} \ u \in L^2(\Sigma). \]  

(57)

It then follows from (52) that

\[ R_1 \leq C_T \| u \|^2_{L^2(\Sigma)}, \ \forall \ u \in L^2(\Sigma), \]  

(58)

where we used Lemma 2.3.

Step 3. Evaluation of \( R_2 \) and \( b_{0,T} \). Because \( Gu \in L^2(0, T; H^{1/2}(\Omega)) \) depends continuously on \( u \in L^2(\Sigma) \), by (57), it follows

\[ R_2 + b_{0,T} \leq C_T \| u \|^2_{L^2(\Sigma)}, \ \forall \ u \in L^2(\Sigma). \]  

(59)

Finally, from (52), (58), (59), and Lemma 2.4, we can obtain

\[ \int_\Sigma \left| \frac{\partial z_1}{\partial \nu_A} \right|^2 d\Sigma + \int_\Sigma \left| \frac{\partial(Pz)}{\partial \nu_A} \right|^2 d\Sigma \leq C_T \int_\Sigma |u(x,t)|^2 d\Sigma. \]  

(60)

(44) then follows.

\[ \square \]

4. Proof of Theorem 1.2

By Lemma 3.1, Theorem 1.2 is equivalent to that the solution to system (2) with zero initial value satisfies

\[ \|y\|_{L^2(0,T;L^2(\Gamma))} \leq C_T \|u\|_{L^2(0,T;L^2(\Gamma))}, \ \forall \ u \in L^2(0,T;L^2(\Gamma)). \]  

(61)
\[
\begin{align*}
&z_t(x, t) + \nabla^2 z(x, t) = (A_1^{-1/2}G_1u_1(\cdot, t))(x, t), \\
&(x, t) \in \Omega \times (0, T) \supseteq Q, \\
&z(x, 0) = 0, \quad z_t(x, 0) = 0, \\
&z(x, t) = \frac{\partial z(x, t)}{\partial v_A} = 0, \\
&y(x, t) = \frac{\partial (A_1z(x, t))}{\partial v_A}, \\
&(x, t) \in \partial \Omega \times (0, T) \supseteq \Sigma, \\
&x \in \Omega.
\end{align*}
\]

(62)

Therefore, Theorem 1.2 is valid if and only if for some (and hence for all) \( T > 0 \), there exists a constant \( C_T > 0 \) such that the solution to (62) satisfies

\[
\int_{\Sigma} \left[ \frac{\partial (A_1z(x, t))}{\partial v_A} \right]^2 d\Sigma \leq C_T \int_{\Sigma} |u(x, t)|^2 d\Sigma.
\]

(63)

The proof of (63) will be split into five steps.

**Step 1. Energy identity.** Multiply both sides of the first equation of (62) by \( N(Pz) \) and perform integration over \( Q = \Omega \times (0, T) \), where \( N \) is the vector field given by (45), to obtain

\[
\int_Q z_t N(Pz) dQ + \int_Q \nabla^2 z N(Pz) dQ = \int_Q A_1^{-1/2} G_1 u_1 N(Pz) dQ.
\]

(64)

Now, we compute the first term on the left-hand side of (64) and perform integration by parts to obtain

\[
\begin{align*}
\int_Q z_t N(Pz) dQ &= \int_Q z_t N(Pz) dx \bigg|_0^T - \int_Q z_t N(Pz_t) dQ \\
&= \int_Q z_t N(Pz) dx \bigg|_0^T - \int_Q \nabla g (z_t Pz_t N) - Pz_t N(z_t) - z_t Pz_t \nabla g N dQ \\
&= \int_Q z_t N(Pz) dx \bigg|_0^T + \int_Q \Delta g z_t N(z_t) dQ + \int_Q (Dp) (z_t) N(z_t) dQ \\
&\quad + \int_Q z_t \Delta g z_t \nabla g N + \int_Q z_t (Dp) (z_t) \nabla g N dQ \\
&= \int_Q z_t N(Pz) dx \bigg|_0^T + \int_Q \frac{\partial z_t}{\partial \mu} N(z_t) d\Sigma - \int_Q (\nabla g Z_t, \nabla g N(z_t))_g dQ + \int_Q \frac{\partial z_t}{\partial \mu} z_t \nabla g N d\Sigma \\
&\quad - \int_Q (\nabla g Z_t, \nabla g (z_t \nabla g N))_g dQ + \int_Q (Dp) (z_t) N(z_t) dQ + \int_Q z_t (Dp) (z_t) \nabla g N dQ.
\end{align*}
\]

(65)

By Lemma 2.2,

\[
\text{Re}(\nabla g Z_t, \nabla g (N(z_t)))_g = \text{Re}DN(\nabla g Z_t, \nabla g z_t) + \frac{1}{2} \text{div}_g (|\nabla g Z_t|^2 N) - \frac{1}{2} |\nabla g Z_t|^2 \text{div}_g N.
\]

(66)

Substituting (66) into (65) gives

\[
\text{Re} \int_Q z_t N(Pz) dQ
\]

\[
= \text{Re} \int_Q z_t N(Pz) dx \bigg|_0^T - \text{Re} \int_Q DN(\nabla g Z_t, \nabla g z_t) dQ - \frac{1}{2} \int_Q |\nabla g Z_t|^2 \text{div}_g N dQ \\
&\quad - \text{Re} \int_Q z_t (\nabla g Z_t, \nabla g (z_t \nabla g N))_g dQ + \text{Re} \int_Q (Dp) (z_t) N(z_t) dQ + \text{Re} \int_Q z_t (Dp) (z_t) \nabla g N dQ.
\]

(67)

Compute the second term on the left-hand side of the equality (64) and make use of (66), to give

\[
\begin{align*}
\text{Re} \int_Q \nabla^2 z N(Pz) dQ &= \text{Re} \int_Q \Delta g (Pz) N(Pz) dQ + \text{Re} \int_Q Dp (Pz) N(Pz) dQ \\
&= \int_{\Sigma} \left| \frac{\partial (Pz)}{\partial \mu} \right|^2 d\Sigma - \text{Re} \int_Q DN(\nabla g (Pz), \nabla g (Pz))_g dQ - \frac{1}{2} \int_Q |\nabla g (Pz)|^2 \text{div}_g N dQ \\
&\quad + \frac{1}{2} \int_Q |\nabla g (Pz)|^2 \text{div}_g N dQ + \text{Re} \int_Q (Dp) (Pz) N(Pz) dQ.
\end{align*}
\]

(68)
It then follows from (64), (67), and (68) that

$$
\frac{1}{2} \int_{\Sigma} \left( \frac{\partial (P)}{\partial x} \right)^2 d\Sigma = \frac{1}{2} \int_{\Sigma} \left( \frac{\partial (P)}{\partial x} \right)^2 d\Sigma \triangleq R_1
$$

$$
+ \left( \text{Re} \int_{Q} \sum_{i=1}^{3} (\nabla_{g}(g_{ij}))_{g} dQ + \frac{1}{2} \int_{\Sigma} |\nabla_{g}(P)|_{g}^2 d\Sigma + \text{Re} \int_{Q} \sum_{i=1}^{3} (\nabla_{g}(g_{ij}))_{g} dQ \right)
$$

$$
\text{Re} \int_{Q} \sum_{i=1}^{3} (\nabla_{g}(g_{ij}))_{g} dQ
$$

$$
\text{Re} \int_{Q} \sum_{i=1}^{3} (\nabla_{g}(g_{ij}))_{g} dQ
$$

$$
\text{Re} \int_{Q} \sum_{i=1}^{3} (\nabla_{g}(g_{ij}))_{g} dQ
$$

where

$$
\triangleq R_1
$$

Step 2. Computation of $R_1$. To estimate $R_1$, we confine $u$ in the following class, which is dense in $L^2((0, T) \times \Gamma)$:

$$
u \in C^2((0, T) \times \Gamma), \ u(0, T) = u(T, T) = 0.
$$

First, we introduce the operator: $B$ = first order differential operator on $\mathbb{R}$, tangent to $\Gamma$ (i.e., without transversal derivatives to $\Gamma$ in local coordinates), and with smooth coefficients on $\mathbb{R}$. Next, we define a new variable:

$$
\eta = Bz \in C(0, T; H^2(\Omega)) \text{ and } \eta_t = Bz_t \in C(0, T; L^2(\Omega)).
$$

Apply $B$ to system (62) to obtain

$$
\begin{align*}
\eta_t(x, t) + P^2 \eta(x, t) &= S, \quad (x, t) \in Q, \\
\eta(x, 0) &= 0, \quad \eta_t(x, 0) = 0, \quad x \in \Omega, \\
\eta(x, t) &= \frac{\partial \eta}{\partial n_A} = 0, \quad (x, t) \in \Sigma,
\end{align*}
$$

where

$$
S = [P^2, B]z + BA^{-1/2} G_1 u_t, \quad Kz = [P^2, B]z \in C(0, T; H^{-1}(\Omega)).
$$

Because

$$
\frac{\partial \eta}{\partial n_A} = \left[ \frac{\partial}{\partial n_A}, B \right]z \text{ and } \left[ \frac{\partial}{\partial n_A}, B \right]z \text{ is smooth, we can replace it with the homogeneous boundary value without loss of generality to obtain}
$$

$$
\int_{\Gamma} \left| \frac{\partial (P)}{\partial t} \right|^2 d\Gamma = \int_{\Gamma} \left| B(P) \right|^2 d\Gamma = \int_{\Gamma} \left| P(B) \right|^2 d\Gamma + h = \int_{\Gamma} \left| \eta \right|^2 d\Gamma + h,
$$

where $h$ is used to denote the lower order terms of $z$. Hence, we need only to evaluate $\int_{\Gamma} \left| \eta \right|^2 d\Gamma$ for system (72) in order to evaluate $R_1$.

Now, multiply both sides of the first equation of (72) by $N(\eta)$ and integrate over $Q$ and make use of Equation (23) of [13], with $N$ given by (45), to obtain

$$
\frac{1}{2} \int_{\Gamma} \left| \eta \right|^2 d\Gamma = R_1' + R_2' + R_3' + b_{0, T},
$$

where

$$
R_1' = \text{Re} \int_{Q} P|\eta| (D_1 g, D(div_g(N)))_{g} dQ + \frac{1}{2} \text{Re} \int_{Q} \eta P (div_g(N))_{g} dQ
$$

$$
+ \text{Re} \int_{Q} P\eta (|\triangle N(\eta)| + 2(DN, D^2 \eta) dQ + D^2 p(N, D\eta) - D^2 \eta(N, Dp) + Ric(N, D\eta)) dQ
$$

$$
R_2' = - \frac{1}{2} \text{Re} \int_{Q} P\eta Dp(\eta div_g(N))_{g} dQ + \frac{1}{2} \text{Re} \int_{Q} \eta div_g N Dp(P\eta) dQ + \text{Re} \int_{Q} N(\eta) Dp(P\eta) dQ
$$

$$
R_3' = - \frac{1}{2} \text{Re} \int_{Q} SN(\eta) dQ
$$

$$
b_{0, T} = \text{Re} \int_{\Omega} \eta N(\eta) dx \bigg|_0^T + \frac{1}{2} \text{Re} \int_{\Omega} \eta \eta div_g N dx \bigg|_0^T.
$$

By divergence formulae,

$$
div_0 (\eta div_g N P\eta) = \eta div_g (N) Dp(P\eta) + P\eta Dp(\eta div_g(N)) + \eta div_g(N) P\eta div_0(Dp),
$$

and

$$
\eta(p^2 + \frac{1}{2} P^2) = \eta p^2 + \frac{1}{2} \eta P^2.
$$

Therefore,

$$
\frac{1}{2} \int_{\Gamma} \left| \eta \right|^2 d\Gamma = R_1' + R_2' + R_3' + b_{0, T}.
$$

and
\[
\text{div}_0(N(\overline{\eta})P_\eta Dp) = N(\overline{\eta})Dp(P_\eta) + P_\eta Dp(N(\overline{\eta})) + N(\overline{\eta})P_\eta \text{div}_0(Dp),
\]
we have
\[
R'_2 = - \text{Re} \int_Q P_\eta Dp(\eta \text{div}_0(N))dQ - \frac{1}{2} \text{Re} \int_Q \eta \text{div}_0 NP_\eta \text{div}_0(Dp)dQ
- \text{Re} \int_Q P_\eta Dp(N(\overline{\eta}))dQ - \text{Re} \int_Q N(\overline{\eta})P_\eta \text{div}_0(Dp)dQ.
\] (77)

In what follows, we compute the two terms of $R'_2$, respectively:
\[
- \frac{1}{2} \text{Re} \int_Q \text{div}_0 NdQ = - \frac{1}{2} \text{Re} \int_Q ([P^2, B]z) N(\overline{\eta})dQ - \frac{1}{2} \text{Re} \int_Q (\text{div}_0(B_1^{-1/2}G_1 \overline{\eta})) \text{div}_0(N)dQ
= - \frac{1}{2} \text{Re} \int_Q \text{div}_0(N)dQ - \frac{1}{2} \text{Re} \int_Q (\text{div}_0(B_1^{-1/2}G_1 \overline{\eta})) \text{div}_0(N)dQ,
\] (78)
and
\[
- \text{Re} \int_Q \text{div}_0 N(\overline{\eta})dQ = - \text{Re} \int_Q \text{div}_0 N(\overline{\eta})dQ - \text{Re} \int_Q \text{div}_0(N(\overline{\eta}))dQ
= - \text{Re} \int_Q \text{div}_0(N)dQ - \text{Re} \int_Q \text{div}_0(N(\overline{\eta}))dQ
= - \text{Re} \int_Q \text{div}_0 N(\overline{\eta})dQ - \text{Re} \int_Q \text{div}_0(N(\overline{\eta}))dQ.
\] (79)

Step 3. Evaluation of $R_1$ and $R_2$. Let $G_1 u_t = 0$ in (62). It is known that under the transformation $z = A_1^{-3/2} v_t \in C(0, T; V_1)$, $z_t = A_1^{-3/2} v_t = -A_1^{-1/2} v_t \in H_0^1(\Omega)$. Thus, the solution to (62) generates a $C_0^1$-group in the space $V_1 \times H_0^1(\Omega)$, that is to say, for any $(z_0, z_1) \in V_1 \times H_0^1(\Omega)$, the corresponding solution to (62) satisfies $(z, z_t) \in V_1 \times H_0^1(\Omega)$ and depends continuously on $(z_0, z_1)$.

From these facts, (69), the computation of $R_1$, and Lemma 2.4, there exists a constant $C_T > 0$ such that
\[
\frac{1}{2} \int_\Sigma \left| \frac{\partial (Pz)}{\partial v} \right|^2 d\Sigma \leq C_T \| (z_0, z_t) \|^2_{V \times H_0^1(\Omega)}.
\]
This shows that the operator $B^*_1$ is admissible and so is $B_1$ ([5]). In other words,
\[
u \rightarrow (v, v_t) \text{ is continuous from } L^2(\Sigma) \rightarrow C(0, T; H^{-1} \times V_1).
\] (80)

By (80), $z = A_1^{-3/2} v_t \in C(0, T; V_1)$, and $z_t = -A_1^{-1/2} v_t \in H_0^1(\Omega)$, it follows
\[
(z, z_t) \in C(0, T; V_1 \times H_0^1(\Omega)) \text{ depends continuously on } u \in L^2(\Sigma)
\] (81)
and hence,
\[
R_1 + R_2 \leq C_T \| u \|_{L^2(\Sigma)}, \quad \forall \ u \in L^2(\Sigma),
\] (82)
where we applied Lemma 2.3.

Step 4. Evaluation of $R_3$ for smooth $u$. To evaluate $R_3$, we still confine $u$ within the smooth class (70). For the $R_3$ with $u$ in the class (70), we perform integration by parts with respect to $t$ to obtain
\[
Re \int_Q A_1^{-1/2} G_1 u N(Pz)dQ = Re \int_Q A_1^{-1/2} G_1 u N(Pz)dx \bigg|_0^T - Re \int_Q A_1^{-1/2} G_1 u N(Pz)dQ
= -Re \int_Q \text{div}_0(A_1^{-1/2} G_1 u Pz)N (\overline{\eta})dQ + Re \int_Q Pz N(A_1^{-1/2} G_1 u)N (\overline{\eta})dQ
+ Re \int_Q A_1^{-1/2} G_1 u N(Pz) \text{div}_0 N (\overline{\eta})dQ
= Re \int_Q \Delta z N(A_1^{-1/2} G_1 u)dQ + Re \int_Q A_1^{-1/2} G_1 u \Delta z \text{div}_0 N (\overline{\eta})dQ
+ Re \int_Q (Dp)(\overline{\eta})N (A_1^{-1/2} G_1 u)dQ
\] (83)
and
\[
Re \int_Q (\nabla z)(\overline{\eta})N(A_1^{-1/2} G_1 u)dQ = -Re \int_Q (\nabla z)(\overline{\eta})N(A_1^{-1/2} G_1 u)dQ
+ Re \int_Q (Dp)(\overline{\eta})N(A_1^{-1/2} G_1 u d \text{div}_0 N (\overline{\eta})dQ
\]
By (81) and the representation of (83), it is easy to obtain
\[ R_3 \leq C_T \|u\|_{L^2(\Sigma)}, \quad \forall \, u \in L^2(\Sigma). \tag{84} \]
Step 5. Evaluation of \( b_0, T \). By virtue of (81), we can readily obtain
\[ b_0, T = \int_Q \zeta^2 N(Pz) dQ \leq C_T \|u\|_{L^2(\Sigma)}. \tag{85} \]
From (69),(82),(84), (85), and Lemma 2.4, it is seen that (63) holds true.

\[ \square \]

5. Proofs of Theorems 1.3 and 1.4

From the Appendix of [23], the transfer functions of system (30) and (41) are, respectively
\[ H(\lambda) = \lambda B^* (\lambda^2 + \tilde{A})^{-1} B, \tag{86} \]
and
\[ H_1(\lambda) = \lambda B_1^* (\lambda^2 + \tilde{A}_1)^{-1} B_1, \tag{87} \]
where \( \tilde{A}, B \), and \( B^* \) are defined by (23),(26), and (29); \( \tilde{A}_1, B_1 \), and \( B_1^* \) are defined by (34),(37), and (40), respectively. Moreover, the well-posedness claimed by Theorems 1.1 and 1.2 implies that there exist positive constants \( M, M_1, \alpha, \alpha_1 > 0 \) such that ([5])
\[ \sup_{\mathbb{R} \geq \alpha} \|H(\lambda)\|_{L(U)} = M < +\infty, \tag{88} \]
and
\[ \sup_{\mathbb{R} \geq \alpha_1} \|H_1(\lambda)\|_{L(U)} = M_1 < +\infty. \tag{89} \]

**Proposition 5.1**

The Theorem 1.3 is valid if for any \( u \in C_0^\infty (\Gamma) \) and \( \varepsilon > 0 \), the solution \( w_\varepsilon \) to the following system
\[
\begin{cases}
\varepsilon^2 P^2 w_\varepsilon(x) = 0, & x \in \Omega, \\
w_\varepsilon(x) = u(x), P w_\varepsilon(x) = 0, & x \in \Gamma,
\end{cases}
\tag{90}
\]
satisfies
\[ \lim_{\varepsilon \to 0^+} \int_\Gamma \left| \frac{\partial w_\varepsilon}{\partial n} \right|^2 d\Gamma = 0. \tag{91} \]

**Proof**

We need only to show that \( H(\lambda)u \) converges to zero in the strong topology of \( U \) along the positive real axis ([15]), that is,
\[ \lim_{\lambda \to +\infty} H(\lambda)u = 0, \tag{92} \]
for any \( u \in L^2(\Gamma) = U \). By density argument and (88), it suffices to show that (92) holds for all \( u \in C_0^\infty (\Gamma) \). To this purpose, let
\[ w_\lambda(x) = ((\lambda^2 + \tilde{A})^{-1} B u)(x). \]
Then, \( w_\lambda \) satisfies
\[
\begin{cases}
\lambda^2 w_\lambda(x) + P^2 w_\lambda(x) = 0, & x \in \Omega, \\
w_\lambda(x) = u(x), P w_\lambda(x) = 0, & x \in \Gamma,
\end{cases}
\tag{93}
\]
and
\[ (H(\lambda)u)(x) = -\lambda \frac{\partial (A^{-2} w_\lambda(x))}{\partial n}, \quad \forall \, x \in \Gamma. \tag{94} \]
Because \( u \in C_0^\infty (\Gamma) \), there exists a unique classical solution to (93). Let \( \tilde{w}(x) \in H^4(\Omega) \) be the unique solution to the following boundary value problem
\[
\begin{cases}
P^2 \tilde{w}(x) = 0, & x \in \Omega, \\
\tilde{w}(x) = u(x), P \tilde{w}(x) = 0, & x \in \Gamma.
\end{cases}
\]
Then, (93) becomes

\[
\begin{align*}
\lambda^2 w_\lambda(x) + P^2 (w_\lambda(x) - \tilde{w}(x)) &= 0, \quad x \in \Omega, \\
w_\lambda(x) - \tilde{w}(x) &= 0, \quad P(w_\lambda(x) - \tilde{w}(x)) = 0, \quad x \in \Gamma,
\end{align*}
\]

or

\[
\lambda^2 (A^{-2} w_\lambda(x)) = -w_\lambda(x) + \tilde{w}(x).
\]

Therefore, (94) is found to be

\[
(H(\lambda)u)(x) = \frac{1}{\lambda} \frac{\partial w_\lambda(x)}{\partial v_4} - \frac{1}{\lambda} \frac{\partial \tilde{w}(x)}{\partial v_4}.
\] 

(95)

If we set \(w_\varepsilon(x) = w_\lambda(x)\) with \(\varepsilon = 1/\lambda\), we obtain the required result immediately.

**Proof of Theorem 1.3.**

Multiply both sides of the first equation of (90) by \(\overline{w_\varepsilon}\) and \(\overline{P w_\varepsilon}\), respectively, and perform integration over \(\Omega\), to obtain

\[
\int_\Omega |w_\varepsilon|^2 \, dx + \varepsilon^2 \int_{\partial \Omega} \frac{\partial (P w_\varepsilon)}{\partial v} \, d\Gamma + \varepsilon^2 \int_\Omega |\overline{P w_\varepsilon}|^2 \, dx = 0,
\] 

(96)

\[
\int_{\partial \Omega} \frac{\partial \overline{w_\varepsilon}}{\partial v} u \, d\Gamma = -\int_\Omega |\nabla g w_\varepsilon|^2 \, dx - \varepsilon^2 \int_\Omega |\nabla g (\overline{P w_\varepsilon})|^2 \, dx = 0.
\] 

(97)

Equality (96) implies that

\[
\int_\Omega |w_\varepsilon|^2 \, dx \leq \frac{1}{8} \frac{\varepsilon^2}{2^5/2} \int_{\partial \Omega} \left( \frac{\partial (P w_\varepsilon)}{\partial v} \right)^2 \, d\Gamma + \frac{1}{2^{1/2}} \int_{\partial \Omega} |u|^2 \, d\Gamma
\]

\[
\leq C \varepsilon^{9/2} \| P w_\varepsilon \|^2_{L^2(\Omega)} + \frac{1}{2^{1/2}} \| u \|^2_{L^2(\Gamma)},
\] 

(98)

where \(C, C' > 0\) are constants independent of \(\varepsilon\). Note that in obtaining (98), we used the inequalities

\[
\left\| \frac{\partial (P w_\varepsilon)}{\partial v} \right\|_{L^2(\Gamma)} \leq C \| P w_\varepsilon \|_{H^1(\Omega)}, \quad \| P w_\varepsilon \|_{H^1(\Omega)} \leq C \| \varepsilon^{-2} w_\varepsilon \|_{L^2(\Omega)}, \quad (99)
\]

for some constants \(C > 0\) independent of \(\varepsilon\). The first inequality comes from the trace theorem in the Sobolev spaces, and the second inequality is from the elliptic boundary regularity. Choose \(\varepsilon\) small enough so that \(1 - C \varepsilon^{1/2} \geq 1/2\). Then,

\[
\| w_\varepsilon \|^2_{L^2(\Omega)} \leq \frac{1}{2^{1/2}} \| u \|^2_{L^2(\Gamma)}.
\] 

(100)

Now, multiply both sides of the first equation of (90) by \(N(\overline{P w_\varepsilon})\), perform integration over \(\Omega\), and make use of divergence formulae and Green's formulae, to obtain

\[
\int_{\partial \Omega} \frac{\partial \mu_\varepsilon}{\partial v} \, d\Gamma = \int_\Omega \nabla g (\overline{P w_\varepsilon}) \cdot \nabla g (N(\overline{P w_\varepsilon})) \, dx + \int_\Omega \overline{P w_\varepsilon} \div g (N) \, d\Gamma + \int_\Omega \div g (N) \nabla g w_\varepsilon \, dx
\]

\[
+ \int_\Omega w_\varepsilon (\nabla g \overline{P w_\varepsilon}) \cdot \nabla g (N) \, dx - \int_\Omega (D p)(\overline{P w_\varepsilon}) N(w_\varepsilon) \, dx
\]

\[
- \int_\Omega w_\varepsilon (D p)(\overline{P w_\varepsilon}) \div g (N) \, dx + \varepsilon^2 \int_\Omega P^2 w_\varepsilon N(\overline{P w_\varepsilon}) \, dx.
\] 

(101)
In addition, by (18), it follows
\[
\frac{1}{2} \int_{\Omega} \left| \frac{\partial w_k}{\partial \mu} \right|^2 d\Gamma = \text{Re} \int_{\Omega} DN(\nabla_y w, \nabla_y w) dx + \frac{1}{2} \int_{\Omega} |\nabla_y w|^2 d\Gamma - \text{Re} \int_{\partial \Omega} \frac{\partial w}{\partial \mu} u\text{div}_y (N) d\Gamma \\
+ \frac{1}{2} \int_{\Omega} \text{div}_y (N) |\nabla_y w|^2 dx + \text{Re} \int_{\Omega} w \nabla_y (\nabla_y (\nabla_y (N)y)) dx - \text{Re} \int_{\partial \Omega} (Dp) (\nabla_y w) N(w) dx \\
- \text{Re} \int_{\Omega} w (Dp w)\text{div}_y (N) dx - \text{Re} \int_{\Omega} w e N(\nabla w) dx
\]
\[
\leq \frac{1}{2} \int_{\partial \Omega} \left| \frac{\partial w_k}{\partial \mu} \right|^2 d\Gamma + \frac{1}{2} \int_{\Omega} \left| \frac{\partial u}{\partial \tau} \right|^2 d\Gamma + C_1 \|\nabla_y w\|_{L^2(\Omega)}^2 \\
+ C_2 \left\| \frac{\partial w_e}{\partial \mu} \right\|_{L^2(\Gamma)} \|u\|_{L^2(\Omega)} + C_3 \|w_e\|_{L^2(\Omega)} + C_4 \|w_e\|_{L^2(\Omega)} \|N(\nabla w)\|_{L^2(\Omega)} \\
= \frac{1}{2} \int_{\partial \Omega} \left| \frac{\partial w_k}{\partial \mu} \right|^2 d\Gamma + \frac{1}{2} \int_{\Omega} \left| \frac{\partial u}{\partial \tau} \right|^2 d\Gamma + (C_1 b_1^{1/2} + C_2) \left\| \frac{\partial w_e}{\partial \mu} \right\|_{L^2(\Gamma)} \|u\|_{L^2(\Gamma)}^2 \\
+ C_3 \left\| w_e \right\|_{L^2(\Gamma)} + 2C_5 \left\| u \right\|_{L^2(\Gamma)}^2 + 2C_7 \|w_e\|_{L^2(\Omega)} \|w_e\|_{L^2(\Omega)} \|N(\nabla w)\|_{L^2(\Omega)}
\]
where in the last step, we used (97), (100), and Lemmas 2.3 and 2.4. And $C_i > 0$ ($i = 1, 2, 3, 4, 5$) are constants, and $\tau = \tau(x)$ is the tangential vector at $x \in \partial \Omega$. Hence,
\[
\int_{\partial \Omega} \left| \frac{\partial w_k}{\partial \mu} \right|^2 d\Gamma \leq \frac{1}{2} \int_{\partial \Omega} \left| \frac{\partial w_k}{\partial \mu} \right|^2 d\Gamma + 2(C_1 b_1^{1/2} + C_2) \varepsilon \left\| \frac{\partial w_e}{\partial \mu} \right\|_{L^2(\Gamma)} \|u\|_{L^2(\Gamma)}^2 \\
+ 2C_3 \|w_e\|_{L^2(\Gamma)} + 2C_5 \|w_e\|_{L^2(\Omega)} \|w_e\|_{L^2(\Omega)} \|N(\nabla w)\|_{L^2(\Omega)}
\]
which shows that $\lim_{\varepsilon \to 0^+} \left\| \frac{\partial w_k}{\partial \mu} \right\|_{L^2(\Gamma)} < +\infty$, and hence,
\[
\lim_{\varepsilon \to 0^+} \int_{\partial \Omega} \left| \frac{\partial w_k}{\partial \mu} \right|^2 d\Gamma = 0.
\]
The proof of Theorem 1.3 is then completed by (19). \hfill \Box

**Proposition 5.2**

Theorem 1.4 is valid if for any $u \in C_0^\infty (\Gamma)$ and $\varepsilon > 0$, the solution $v_\varepsilon$ to the following system
\[
\left\{ \begin{array}{ll}
\varepsilon v_\varepsilon(x) + c_1^2 \varepsilon v_\varepsilon(x) = 0, & x \in \Omega, \\
v_\varepsilon(x) = u(x), & x \in \Gamma,
\end{array} \right.
\]
(104)
satisfies
\[
\lim_{\varepsilon \to 0^+} \int_{\Gamma} \left| \frac{\partial v_\varepsilon}{\partial \nu} \right|^2 d\Gamma = 0.
\]
(105)

**Proof**

As in the beginning of the proof of Proposition 5.1, we need only to show that $H_1(\lambda)u$ converges to zero in the strong topology of $U$ along the positive real axis ([15]). In other words,
\[
\lim_{\lambda \to +\infty} H_1(\lambda)u = 0,
\]
for any $u \in C_0^\infty (\Gamma)$. To do this, set
\[
v_\lambda(x) = ((\lambda^2 + A_1)^{-1} B_1 u)(x).
\]
(107)
Then, $v_\lambda$ satisfies
\[
\left\{ \begin{array}{ll}
\lambda^2 v_\lambda(x) + c_1^2 v_\lambda(x) = 0, & x \in \Omega, \\
v_\lambda(x) = u(x), & x \in \Gamma,
\end{array} \right.
\]
(108)
and
\[
(H_1(\lambda)u)(x) = \lambda \frac{\partial (A_1^{-3/2}v_\lambda(x))}{\partial \nu}, \quad \forall \ x \in \Gamma.
\] (109)

Because \( u \in C^0_0(\Gamma) \), there exists a unique classical solution to (108). Let \( \widetilde{v}(x) \in H^4(\Omega) \) be the unique solution to the following boundary value problem
\[
\begin{align*}
\rho^2 \widetilde{v}(x) &= 0, \quad x \in \Omega, \\
\widetilde{v}(x) &= u(x), \quad \frac{\partial \widetilde{v}(x)}{\partial \nu} = 0, \quad x \in \Gamma.
\end{align*}
\]

Then, (108) becomes
\[
\begin{align*}
\lambda^2 v_\lambda(x) + \rho^2 (v_\lambda(x) - \widetilde{v}(x)) &= 0, \quad x \in \Omega, \\
v_\lambda(x) - \widetilde{v}(x) &= 0, \quad \frac{\partial (v_\lambda(x) - \widetilde{v}(x))}{\partial \nu} = 0, \quad x \in \Gamma,
\end{align*}
\]
or
\[
\lambda^2 A_1^{-3/2}v_\lambda(x) = v_\lambda(x) - \widetilde{v}(x).
\]

Therefore, (109) is found to be
\[
(H_1(\lambda)u)(x) = \frac{1}{\lambda} \frac{\partial v_\lambda(x)}{\partial \nu} - \frac{1}{\lambda} \frac{\partial \widetilde{v}(x)}{\partial \nu},
\] (110)

If we set \( v_\varepsilon(x) = v_\lambda(x) \) with \( \varepsilon = 1/\lambda \), we obtain the required result immediately. \( \square \)

**Proof of Theorem 1.4.**
From the boundary condition of (104), it is easy to know that (105) holds. This completes the proof of Theorem 1.4. \( \square \)

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