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Observers and observability for uncertain nonlinear systems: A necessary and sufficient condition

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Summary

In this paper, observers and observability for uncertain nonlinear systems are systematically discussed. It is shown that for the convergence of a large class of observers, featured with the augment state to estimate the uncertainty, it requires not only the observability condition for the augment matrix pair but, more importantly, requires a structural condition first proposed in this paper. Furthermore, it is demonstrated that the combination of this structural condition and the observability of the augment matrix pair is a necessary and sufficient condition for the convergence of the observers and the observability of the original uncertain nonlinear systems. This implies that both the structural condition and the observability condition of the augment matrix pair reveal essential feature of the observing problems for uncertain nonlinear systems. In addition, for unobservable uncertain nonlinear systems, which do not satisfy this necessary and sufficient condition, the biased estimation error is explicitly presented, which can be used to evaluate the estimation performance of this class of observers. The numerical simulations for three typical examples are carried out to validate our theoretical analysis.

KEYWORDS

convergence of the observer, extended state observer, observers, uncertain nonlinear systems

1 | INTRODUCTION

Uncertainties, including but not limited to external disturbances, unmodeled dynamics, and parameter perturbations, are ubiquitous in industrial systems and often bring adverse effects on performance and even stability of engineering systems.[1-5] For this reason, ensuring normal operation for a control system under various uncertainties becomes a central issue in control theory. In recent years, the control approaches with estimation/compensation of uncertainty to force the system to satisfy desired performance have been substantially developed for the control of uncertain systems.[6-16] The effectiveness of such control strategies lies in large part in the observer design, which is used not only to recover the state but also the uncertainty. In this sense, the observer design has become a bedrock for the control of uncertain systems.

During the past decades, several classes of observers, including but no limited to extended state observer (ESO),[17,18] extended high-gain observer (EHGO),[19,20] unknown input observer (UIO),[21] generalized extended state observer (GESO),[22] proportional-integral observer (PIO)[23] and disturbance observer (DO)[6] have been proposed to estimate state or uncertainty. Although using different names, these observers are based on the similar idea: expanding the state in the observer to include the uncertainty estimation. This state augment design has been shown as an effective way for estimating uncertainties in practice. Substantial researches have been produced to analyze these classes of observers theoretically. The convergence of nonlinear ESO with matched uncertainty has been studied in the work of Guo and Zhao.[11,18] Under the assumption that the nonlinear system is minimum phase, Freidovich and Khalil[19] prove that the closed-loop system based on the EHGO-based controller is stable. Hammouri and Tmar[24] study a necessary and sufficient condition for the existence of UIO, and the stability condition of UIO is also presented. For bounded external disturbance, stability of GESO for systems with mismatched disturbances has been studied in the work of Li et al.[22] Considering discrete-time nonminimum phase systems, it is shown that the estimation errors of the system state and unknown disturbance from PIO can be constrained in a small bounded region.[25] The global exponential stability of the estimation error from nonlinear DO has been investigated in the work of Chen.[6]

Although some observers have been studied thoroughly from different perspectives, the theoretical analyses may have the following two limitations:

- The structure of uncertain systems is limited to a special form. For instance, ESO and EHGO are constructed for those uncertain systems that have cascade form and control matched uncertainty[17,19];
- The assumptions on large-time behavior of uncertainties are conservative. For instance, the asymptotic stability of the estimation errors of GESO and DO are proved under the assumption that the uncertainty converges to a constant as time goes to infinity.[6,22]

However, in real physical plants, the uncertainties may affect the system from each channel, which can be mismatched with the control input. Moreover, the uncertainties might be composed of the time-varying external disturbances and linear/nonlinear unknown dynamics with respect to system states. To meet the requirement of engineering applications, the detailed analysis of observers for uncertain nonlinear systems under a more general framework becomes significant both theoretically and practically.

A fundamental problem motivates this investigation is: what kinds of uncertainties can be estimated by observer for general uncertain nonlinear systems?

We are therefore focused ourselves to uncertainty observing problem for general uncertain nonlinear systems in this paper. Firstly, we analyze estimation performance of observers for a large class of uncertain nonlinear systems. The observability for uncertain nonlinear systems is thus defined, which was neglected in previous studies. Secondly, we try to reveal the essential nature of the observability for uncertain nonlinear systems. The main contributions of this paper consist of four main points: a) The properties of the observers for a general class of uncertain nonlinear systems are rigorously analyzed and a structural condition, which is proved to be the essential condition to ensure the convergence of the observers, is proposed in this paper; Specially, the performance of observers for uncertain nonlinear systems is quantitatively analyzed; b) The combination of the structural condition proposed in this paper and the observability of the augment matrix pair is proved to be a necessary and sufficient condition for the observers to be convergent; c) The observability for uncertain nonlinear systems is defined. The combination of the structural condition and the observability of the augment matrix pair is proved to be a necessary and sufficient condition for the uncertain systems to be observable. In particular, an algebraic criterion of the observability for uncertain nonlinear systems is presented; d) For unobservable uncertain nonlinear systems, the biased estimation error of the observers is quantitatively given in this paper, which can be used to evaluate the estimation performance of the observers.

The paper is organized as follows. In Section 2, the observation problem and the observers for a large class of uncertain nonlinear systems are introduced. In Section 3, the properties of the observers are analyzed and a necessary and sufficient condition for convergence of the observers is presented. In Section 4, the condition in Section 3 is also proved to be necessary and sufficient for observability of uncertain nonlinear systems. Finally, in Section 5, some numerical experiments are performed to validate the theoretical results.

The following notations are used throughout the paper. The $\mathbb{R}^n$ represents the n-dimensional Euclidean space and $\mathbb{R}^{n\times m}$ stands for the space of real $n \times m$-matrices; $\mathbb{C}^-$ denotes the left-half complex plane; for a vector or matrix Y, Y^{T} denotes its transpose; for a square matrix Y, Y^{-1} denotes its inverse matrix; for a square matrix Y, $\sigma(Y)$ denotes the set of eigenvalues of Y; $I_{n\times n}$ denotes the $n \times n$ unit matrix; $\lambda_{\min}(Y)$ and $\lambda_{\max}(Y)$ represent the minimal and maximal eigenvalues of the symmetric real matrix Y, respectively; $\|Y\|$ denotes the Euclidean norm of the vector Y and the corresponding induced norm when Y is a matrix; $C^s(\mathbb{R}^n;\mathbb{R})$ denotes the set of all continuous differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$ up to s-order; for any given function g, $g^{(j)}$ denotes its j-order total differentiation with respect to t, and specially $f^{(j)}$ means $\frac{d^j f(X(t),d(t),t)}{dt^j}$ for the unknown function $f(\cdot)$ in system (1) in the beginning of next section for simplicity.

2 | PROBLEM FORMULATION

In this paper, we consider a class of uncertain nonlinear systems described as follows:

$$\begin{cases} \dot{X}(t) = A_o X(t) + b_u u(t) + b_d f(X(t), d(t), t), \\ y(t) = cX(t), \end{cases} \tag{1}$$

where $X(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}$ is the measured output, $u(t) \in \mathbb{R}$ is the control input, $d(t) \in \mathbb{R}$ is the external disturbance, and $f \in C^n(\mathbb{R}^{n+1} \times [0, \infty); \mathbb{R})$ is an unknown function representing the uncertainty, which contains external disturbance $d(t)$, unmodeled dynamics, and parameter perturbations. $A_o \in \mathbb{R}^{n\times n}$, $b_u \in \mathbb{R}^n$, $b_d \in \mathbb{R}^n$, $c \in \mathbb{R}^{1\times n}$ are known constant matrix and vectors. It is suppose without loss of generality that all b_u, b_d, c are nonzero.

We first specify some assumptions about system (1) in what follows. The following Assumption 1 is about the state $X(t)$, the input $u(t)$, and the external disturbance $d(t)$.

Assumption 1. The state $X(t)$ and the input $u(t)$ of system (1) are supposed to be bounded

$$\sup_{t\geq 0} \|X(t)\| + \sup_{t\geq 0} |u(t)| \leq M_1; \tag{2}$$

and the external disturbance $d(t)$ satisfies

$$\sup_{t\geq 0} \|(d(t), \dot{d}(t), \dots, d^{(n)}(t))\| \leq M_2, \tag{3}$$

where $M_i (i = 1, 2)$ are some positive constants.

The following Assumption 2 is about the unknown function $f(\cdot)$.

Assumption 2. There exists a nonnegative continuous function $\varsigma \in C(\mathbb{R}^{n+1}; [0, \infty))$, such that for any function $g_j \in \nabla^{(j)} f (j = 2, \dots, n)$,

$$|f(X, d, t)| + \|\nabla f(X, d, t)\| + \sum_{j=2}^{n} |g_j(X, d, t)| \leq \varsigma(X, d), \forall t \in [0, \infty), X \in \mathbb{R}^n, d \in \mathbb{R}, \tag{4}$$

where ∇f and $\nabla^{(j)} f$ represent the gradient of f and the finite set of all j-order partial derivatives of f with respect to its arguments, respectively.

Remark 1. It is significant to emphasize that both Assumption 1 and Assumption 2 are essentially to ensure the boundedness of $f^{(j)} (j = 1, 2, \dots, n)$. It also should be noticed that the boundedness requirement for the state $X(t)$ in Assumption 1 aims for estimating the state-dependent uncertainty $f(X(t), d(t), t)$, so that this boundedness assumption can be removed in the case that the uncertainty function is state independent. Moreover, the state could be bounded in many practical control systems such as those for faults diagnosis.[26] Finally, since the observer is usually designed for feedback purpose, we can use observer-based feedback to make the state be bounded.[9,11,12,18]

Remark 2. Assumption 2 is about the "growth" requirement of the time-varying unknown function $f(X, d, t)$ with respect to (X, d), which is reasonable as stated in Remark 1. However, if the unknown function $f(X, d, t)$ is time independent, the Assumption 2 can be removed since in this case the left-hand side of (4) is always a nonnegative continuous function with respect to (X, d) and $\nabla^{(j)} f$ is just a finite set for each $j = 1, 2, \dots, n$.

We point out that many physical systems can be modeled by system (1). Examples can be found from robotic systems,[27] DC-DC converter,[28] flight systems,[29] among many others (see, eg, the work of Xue et al[30]). For the control of system (1), there is a widely used method that performs two steps.[15] One step is to estimate the uncertainty through observe and the other is to compensate for the uncertainty in the feedback loop. The central strategy of this approach is the disturbance/observer/estimator, which provides an online estimation of the disturbance or uncertainty such as ESO, EHGO, PIO, and GESO.

General observers of variable augment design to estimate the uncertainty are of the following form:

$$\begin{bmatrix} \dot{\hat{X}}(t) \\ \dot{\hat{f}}(t) \end{bmatrix} = A \begin{bmatrix} \hat{X}(t) \\ \hat{f}(t) \end{bmatrix} + B_u u(t) + L \left(y(t) - C \begin{bmatrix} \hat{X}(t) \\ \hat{f}(t) \end{bmatrix} \right), \tag{5}$$

where $\hat{X}(t) \in \mathbb{R}^n$ and $\hat{f}(t) \in \mathbb{R}$ are the estimates of the state $X(t)$ and the lumped uncertainty $f(X(t), d(t), t)$, respectively; A, B_u, and C are the following extended matrices

$$A = \begin{bmatrix} A_o & b_d \\ 0 & 0 \end{bmatrix}, \quad B_u = \begin{bmatrix} b_u \\ 0 \end{bmatrix}, \quad C = [\, c \ 0 \,], \tag{6}$$

and $L \in \mathbb{R}^{n+1}$ is the observer gain vector to be designed.

In fact, the observer (5) is a general form for uncertain systems in the sense that all ESO, EHGO, PIO, and GESO are special cases of (5), which can be summarized from the following two aspects.

a) Both ESO and EHGO[17,19] are designed for special integral cascade systems, where

$$A_o = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \\ 0 & \dots & \dots & 0 \end{bmatrix}, b_d = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, b_u = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, c = [\, 1 \ 0 \ \dots \ 0 \,], \tag{7}$$

which assumes that the relative degree from the control input to the controlled output is n and the lumped uncertainty is matched with the control input;

b) The GESO and PIO[22,23] are for general A_o, b_d, b_u, c, the lumped uncertainty is, however, commonly assumed to be ultimately steady, ie, $\lim_{t\to\infty} f^{(1)} = 0$, which ensures asymptotically convergence of the estimation error. This paper will further study the transient performance of the observer (5) without this ultimately steady assumption.

To analyze the performance of observer (5), we start from the estimation error:

$$\eta(t) = \begin{bmatrix} X(t) \\ f(X(t), d(t), t) \end{bmatrix} - \begin{bmatrix} \hat{X}(t) \\ \hat{f}(t) \end{bmatrix}. \tag{8}$$

It is seen that the estimation error is governed by

$$\dot{\eta}(t) = (A - LC)\eta(t) + B_d f^{(1)}, \tag{9}$$

where $B_d = \begin{bmatrix} 0_{n\times1} \\ 1 \end{bmatrix}$. To achieve asymptotically unbiased estimation $\eta(\infty) = 0$, the observability of (A, C) is a basic condition, which is commonly used in previous studies.[17,19,22,23] This is because if (A, C) is observable, $A - LC$ can be designed to be Hurwitz by tuning L. Solve the error Equation (9) to obtain

$$\eta(t) = e^{(A-LC)t}\eta(0) + \int_0^t e^{(A-LC)(t-s)} B_d f^{(1)} ds. \tag{10}$$

It is seen that when (A, C) is observable, $e^{(A-LC)t}$ can decay as fast as desired by choosing the appropriate gain matrix L from the pole assignment. Thus, if the uncertainty $f(\cdot)$ is a constant function, ie, $f^{(1)} \equiv 0$, it follows from (7) and (10) that $\eta(\infty) = 0$. Therefore, it is often assumed that the uncertainty is ultimately steady, ie, $\lim_{t\to\infty} f^{(1)} = 0$ to ensure asymptotically unbiased estimation.[22,23] However, if $f(\cdot)$ is not a constant function, ie, $f^{(1)} \not\equiv 0$, there is not necessarily existing a gain vector L to achieve asymptotically unbiased estimation even if (A, C) is observable, which can be easily seen from some simple counter examples, such as example 2 in Section 5.1 later. It is natural that some additional conditions about b_d in system (1), which demonstrates the position of the uncertainty f, should be required to ensure the asymptotically unbiased estimation of the observer (5).

Actually, for general uncertainty $f(\cdot)$ satisfying Assumption 2, if (A,C) is observable and Assumptions 1 and 2 are satisfied, the following proposed structural condition:

$$cb_d = 0, cA_0 b_d = 0, \dots, cA_0^{n-2} b_d = 0, cA_0^{n-1} b_d \neq 0 \tag{11}$$

will be proved to be not only a sufficient condition but also a necessary one to guarantee that there always exists a gain vector L such that the observer (5) is convergent, which is presented in Theorem 2 in the next section. Specially, if we only consider the sufficient condition for the convergence of the observer (5), we can replace Assumptions 1 and 2 by the following weaker Assumption 3, then (A, C) is observable and condition (11) can insure that there always exists a gain vector L such that the observer (5) is convergent, which is shown in Proposition 1 in the next section.

3 | CONVERGENCE OF THE OBSERVER

Before analyzing the performance of the observer (5), we first give two essential properties, which make the observer feasible. The observer (5) is said to be convergent if the following conditions (P1) and (P2) are satisfied.

(P1). The observer is interiorly stable, ie, $\sigma(A - LC) \subset \mathbb{C}^-$.

(P2). The estimation accuracy of the observer can be adjusted by $\sigma(A - LC)$, that is to say, for any $T > 0$ and any $\varepsilon \in (0, 1]$, there exists L' such that $\sigma(A - L'C) = \frac{1}{\varepsilon}\sigma(A - LC)$ and

$$\|\eta(t)\| \leq \Gamma\varepsilon, \forall\, t > T, \tag{12}$$

where $\eta(t)$ is defined as in (8) and Γ is a constant independent of ε.

As discussed in Section 2, (P1) is a basic property for the observer to be practicable. (P2) means that the estimation error of the observer (5) can be eventually unbiased : $\lim_{\varepsilon\to 0}\|\eta(t)\| = 0$ by tuning the observer gain L' in every time. Actually, unlike properties analyzed in general observers, (P2) ensures not only the ultimate performance but also, more importantly, the transient performance. Since (P1) and (P2) are preconditions for the observer (5) to be convergent, it is significant to discuss under what conditions or for what kind of uncertainties, the observer can satisfy both (P1) and (P2).

According to the dynamics of the estimation error (9), it is difficult or even impossible to find analytically the estimation error for general A, C, and f. Thus, we will first transform (9) into a special form that is more easy to be analyzed. The key steps of this transform can be summarized as follows. First, since the observability of (A, C) is the prime condition for the feasibility of observer, we always assume that (A, C) is observable. In this case, there is an $(n + 1) \times (n + 1)$ invertible matrix

$$S \triangleq \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^n \end{bmatrix} = \begin{bmatrix} c & 0 \\ cA_0 & cb_d \\ cA_0^2 & cA_0 b_d \\ \vdots & \vdots \\ cA_0^n & cA_0^{n-1} b_d \end{bmatrix} \tag{13}$$

such that (A, C) can be changed into the observable canonical form

$$SAS^{-1} = \bar{A}, \quad CS^{-1} = \bar{C}, \tag{14}$$

where

$$\bar{A} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ \multicolumn{3}{c}{\begin{bmatrix} cA_0^{n+1} & cA_0^n b_d \end{bmatrix} S^{-1}} \end{bmatrix}_{(n+1)\times(n+1)}, \quad \bar{C} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}_{1\times(n+1)}. \tag{15}$$

Then, the estimation error (9) can be reformulated as

$$\dot{\bar{\eta}}(t) = (\bar{A} - \bar{L}\bar{C})\bar{\eta}(t) + \bar{B}_d f^{(1)}, \tag{16}$$

where we set

$$\bar{\eta}(t) = S\eta(t),\ \bar{L} = SL,\ \bar{B}_d = SB_d. \tag{17}$$

We can see that the if we only consider the first term of the right-hand side of (16), it has been the estimation error of the observable canonical form. We focus only on the second term $\bar{B}_d f^{(1)}$. Let

$$\Delta = \begin{bmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ cb_d & \dots & 0 \\ \vdots & \ddots & \vdots \\ cA_0^{n-2} b_d & \dots & cb_d \end{bmatrix} \begin{bmatrix} f^{(1)} \\ \vdots \\ f^{(n-1)} \end{bmatrix}. \tag{18}$$

It is verified that

$$\dot{\Delta} = \bar{A}\Delta - \bar{B}_d f^{(1)} + F, \tag{19}$$

where

$$F = -\begin{bmatrix} 0_{n\times 1} \\ 1 \end{bmatrix} \begin{bmatrix} cA_0^{n-1} b_d & \dots & cb_d \end{bmatrix} \begin{bmatrix} f^{(1)} \\ \vdots \\ f^{(n)} \end{bmatrix} + \begin{bmatrix} 0_{n\times 1} \\ 1 \end{bmatrix} \begin{bmatrix} cA_0^{n+1} & cA_0^n b_d \end{bmatrix} S^{-1}\Delta. \tag{20}$$

Since $\bar{C}\Delta = 0$, (19) becomes

$$\dot{\Delta} = (\bar{A} - \bar{L}\bar{C})\Delta - \bar{B}_d f^{(1)} + F. \tag{21}$$

Denote

$$\eta_o = \bar{\eta} + \Delta. \tag{22}$$

It then follows from (16) and (21) that

$$\dot{\eta}_o = (\bar{A} - \bar{L}\bar{C})\eta_o + F. \tag{23}$$

From the discussion above, we can see that the error Equation (9) is reformulated as (23) in the observable canonical form. The convergence of system (23) is presented in the succeeding Lemma 1.

Lemma 1. *Suppose that F defined in (20) is bounded. Then, there exists an $(n+1)$-dimensional vector $\bar{L}$ such that system (23) satisfies both (P1) and (P2).*

Proof. Since $(\bar{A}, \bar{C})$ is observable with both $\bar{A}$ and $\bar{C}$ defined in (15), for any $\{\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_n\} \subset \mathbb{C}^{-1}$, there exists an $(n+1)$-dimensional vector $\bar{L}_0$ such that

$$\bar{A} - \bar{L}_0\bar{C} = P_0\bar{\Lambda}_0 P_0^{-1}, \tag{24}$$

where

$$\bar{\Lambda}_0 = \begin{bmatrix} \bar{\lambda}_1 & & \\ & \ddots & \\ & & \bar{\lambda}_{n+1} \end{bmatrix},$$

$$P_0^{-1} = \begin{bmatrix} \bar{\lambda}_1^n - \sum_{i=2}^{n+1} a_i\bar{\lambda}_1^{i-2} & \bar{\lambda}_1^{n-1} - \sum_{i=3}^{n+1} a_i\bar{\lambda}_1^{i-3} & \dots & \bar{\lambda}_1 - a_{n+1} & 1 \\ \bar{\lambda}_2^n - \sum_{i=2}^{n+1} a_i\bar{\lambda}_2^{i-2} & \bar{\lambda}_2^{n-1} - \sum_{i=3}^{n+1} a_i\bar{\lambda}_2^{i-3} & \dots & \bar{\lambda}_2 - a_{n+1} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \bar{\lambda}_{n+1}^n - \sum_{i=2}^{n+1} a_i\bar{\lambda}_{n+1}^{i-2} & \bar{\lambda}_{n+1}^{n-1} - \sum_{i=3}^{n+1} a_i\bar{\lambda}_{n+1}^{i-3} & \dots & \bar{\lambda}_{n+1} - a_{n+1} & 1 \end{bmatrix},$$

$$[a_1, \dots, a_{n+1}] \triangleq \begin{bmatrix} cA_0^{n+1} & cA_0^n b_d \end{bmatrix} S^{-1}.$$

When we replace $\bar{\Lambda}_0$ with $\frac{1}{\varepsilon}\bar{\Lambda}_0$ ($\varepsilon \in (0,1]$) in (24), there also exists an $(n+1)$-dimensional vector $\bar{L}$ such that

$$\bar{A} - \bar{L}\bar{C} = P\left[\frac{1}{\varepsilon}\bar{\Lambda}_0\right]P^{-1}, \tag{25}$$

where $P^{-1} = P_0^{-1}T$ with

$$T = T_a^{-1}\begin{bmatrix} \frac{1}{\varepsilon^{n+1}} & & & \\ & \frac{1}{\varepsilon^n} & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} T_a \tag{26}$$

and

$$T_a = \begin{bmatrix} 1 & 0 & \dots & 0 \\ -a_{n+1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 \\ -a_2 & \dots & -a_{n+1} & 1 \end{bmatrix}. \tag{27}$$

The observe error Equation (23) becomes therefore

$$\dot{\eta}_0 = T^{-1}P_0\begin{bmatrix} \frac{\bar{\lambda}_1}{\varepsilon} & & \\ & \ddots & \\ & & \frac{\bar{\lambda}_{n+1}}{\varepsilon} \end{bmatrix} P_0^{-1}T\eta_0 + F. \tag{28}$$

Denote$\vartheta = T\eta_0$. It follows from (24)-(28) that

$$\dot{\vartheta} = \frac{1}{\varepsilon}(\bar{A} - \bar{L}_0\bar{C})\vartheta + TF. \tag{29}$$

Since $TF = F$, (29) becomes

$$\dot{\vartheta} = \frac{1}{\varepsilon}(\bar{A} - \bar{L}_0\bar{C})\vartheta + F. \tag{30}$$

Since $\bar{A} - \bar{L}_0\bar{C}$ is Hurwitz, there is an $(n+1) \times (n+1)$ positive definite matrix Q such that

$$(\bar{A} - \bar{L}_0\bar{C})^\top Q + Q(\bar{A} - \bar{L}_0\bar{C}) = -I_{(n+1)\times(n+1)}. \tag{31}$$

Let $V = \vartheta^\top Q\vartheta$, which satisfies

$$\lambda_{\min}(Q)\|\vartheta\|^2 \le V \le \lambda_{\max}(Q)\|\vartheta\|^2,$$

and hence

$$\begin{aligned}\dot{V} &= \left(\frac{1}{\varepsilon}(\bar{A} - \bar{L}_0\bar{C})\vartheta + F\right)^\top Q\vartheta + \vartheta^\top Q\left(\frac{1}{\varepsilon}(\bar{A} - \bar{L}_0\bar{C})\vartheta + F\right)\\ &\le -\frac{1}{\varepsilon}\vartheta^\top\vartheta + \frac{1}{2}\vartheta^\top\vartheta + 2\|QF\|^2\\ &\le -\frac{\frac{1}{\varepsilon} - \frac{1}{2}}{\lambda_{\max}(Q)}V + 2\|Q\|^2\|F\|^2.\end{aligned} \tag{32}$$

Furthermore, since F is bounded, there exists a positive constant C_1 such that $2\|Q\|^2\|F\|^2 \le C_1$. This, together with (32), yields

$$V(t) \le e^{\frac{-\left(\frac{1}{\varepsilon} - \frac{1}{2}\right)}{\lambda_{\max}(Q)}t}V(0) + C_1\int_0^t e^{\frac{-\left(\frac{1}{\varepsilon} - \frac{1}{2}\right)}{\lambda_{\max}(Q)}(t-s)}ds. \tag{33}$$

Since $0 < \varepsilon \le 1$, we then have

$$\begin{aligned}V(t) &\le e^{\frac{-1}{2\lambda_{\max}(Q)\varepsilon}t}V(0) + C_1\int_0^t e^{\frac{-1}{2\lambda_{\max}(Q)\varepsilon}(t-s)}ds\\ &\le e^{\frac{-1}{2\lambda_{\max}(Q)\varepsilon}t}V(0) + 2\lambda_{\max}(Q)C_1\varepsilon.\end{aligned} \tag{34}$$

In addition, for any given $T > 0$, there exists a positive constant C_2 independent of ε such that

$$e^{\frac{-1}{2\lambda_{\max}(Q)\varepsilon}t}V(0) \le C_2\varepsilon,\ \forall\, t \in [T, \infty). \tag{35}$$

Let $\gamma = C_2 + 2\lambda_{\max}(Q)C_1$. Then, for any $t \ge T$,

$$V \le \gamma\varepsilon. \tag{36}$$

This follows that

$$\|\eta_0\|^2 = \|T^{-1}\vartheta\|^2 \le \|T^{-1}\|^2\frac{1}{\lambda_{\min}(Q)}V \le \Gamma\varepsilon,\ \forall\, t \in [T, \infty), \tag{37}$$

where $\Gamma \triangleq \frac{\|T^{-1}\|^2}{\lambda_{\min}(Q)}\gamma$ is a constant independent of ε. □

By Lemma 1, the capability of observer (5) is demonstrated in the succeeding Theorem 1.

Theorem 1. *For the observer (5), if (A, C) is observable and Assumptions 1 and 2 are satisfied, then there exists an $(n+1)$-dimensional vector L such that the property (P1) is satisfied; in addition, for any $\varepsilon \in (0, 1]$ and $T > 0$, when the gain vector L^* is chosen such that $\sigma(A - L^*C) = \frac{1}{\varepsilon}\sigma(A - LC)$, it holds*

$$\|\eta(t) + S^{-1}\Delta\|^2 \le \Gamma\varepsilon, \forall\, t > T, \tag{38}$$

where Γ is a constant independent of ε and $\eta(t)$, S, Δ are defined as those in (8), (13), (18), respectively.

Proof. Since Assumptions 1 and 2 are satisfied, we can easily show by a computation that $f^{(j)}(j = 1, 2, \dots, n)$ are bounded. It follows from (20) that F is bounded. By Lemma 1, there exists an $(n+1)$-dimensional gain vector $\bar{L}$ such that the estimation error system (23) satisfies both (P1) and (P2). Furthermore, since

$$SAS^{-1} = \bar{A}, CS^{-1} = \bar{C}, \bar{L} = SL, \tag{39}$$

which is specified in (14) and (17), we have

$$S(A - LC)S^{-1} = \bar{A} - \bar{L}\bar{C}, \tag{40}$$

which means that

$$\sigma(A - LC) = \sigma(\bar{A} - \bar{L}\bar{C}). \tag{41}$$

It then follows from Lemma 1 that for any $\varepsilon \in (0, 1]$ and $T > 0$, there exists a gain vector L such that (P1) is satisfied, and

$$\|\eta(t) + S^{-1}\Delta\|^2 \le \Gamma\varepsilon, \forall\, t > T, \tag{42}$$

for some ε-independent constant Γ when we choose the gain vector L^* such that $\sigma(A - L^*C) = \frac{1}{\varepsilon}\sigma(A - LC)$. This completes the proof of the theorem. □

Theorem 1 shows that the estimation error $(\eta(t) + S^{-1}\Delta)$ can be tuned as small as possible by the observer gain L^*. Moreover, $(\eta(t) + S^{-1}\Delta)$ will converge to zero with the eigenvalues of $\sigma(A - L^*C)$ tending to negative infinity. Since $S^{-1}\Delta$ depends on the uncertainty $f(\cdot)$ and the structure of the original system (1), it cannot be adjusted by the observer gain L^*, which is therefore regarded as a biased estimation of the observer (5).

Theorem 1 is significant in the sense that it gives a quantitative estimation error of the observer (5). To achieve unbiased estimations, $S^{-1}\Delta$ should be a zero vector. From this fact, condition (11) specified in last section is considered as not only a sufficient condition but also a necessary one for the convergence of the observer (5), which is shown in the following Theorem 2.

Theorem 2. *For the observer (5), suppose that (A, C) is observable and Assumptions 1 and 2 are satisfied. Then, there always exists an $(n + 1)$-dimensional gain vector L such that the observer (5) is convergent if and only if condition (11) is satisfied.*

Proof. **Sufficiency:** Since (A, C) is observable, there exists an $(n + 1)$-dimensional gain vector L such that (P1) is satisfied. If condition (11) is satisfied, from (18), we have $\Delta \equiv 0$. It then follows from Theorem 1 that (P2) holds.

Necessity: Suppose that (P1) and (P2) are satisfied. Then, for any $\varepsilon \in (0, 1]$ and $T > 0$, there exists an $(n + 1)$-dimensional gain vector L such that $\sigma(A - LC) \subset \mathbb{C}^-$, and when we choose the gain L' such that $\sigma(A - L'C) = \frac{1}{\varepsilon}\sigma(A - LC)$, we have

$$\|\eta(t)\| \leq \Gamma_1\varepsilon, \forall\, t > T, \tag{43}$$

for some ε-independent constant Γ_1. Suppose that there is an integer $0 \leq i \leq n - 2$ such that

$$cA_0^i b_d \neq 0,$$

which yields $\Delta \not\equiv 0$ for the uncertainty f satisfying $f^{(1)} \not\equiv 0$.

Since (A, C) is observable and Assumptions 1 and 2 are satisfied, it follows from Theorem 1 that

$$\|\eta(t) + S^{-1}\Delta\| \leq \Gamma_2\varepsilon, \forall\, t > T, \tag{44}$$

for some ε-independent constant Γ_2 with the choice of the gain L'. Now,

$$\|S^{-1}\Delta\| - \|\eta(t)\| \leq \|\eta(t) + S^{-1}\Delta\| \leq \Gamma_2\varepsilon, \tag{45}$$

which yields

$$\|\eta(t)\| \geq \|S^{-1}\Delta\| - \Gamma_2\varepsilon. \tag{46}$$

It is easy to see that there exists $\gamma > 0$ such that $\theta^* \triangleq \min_{t\in[T,T+\gamma]}\|S^{-1}\Delta\| > 0$, and we suppose that $0 < \varepsilon < \min\{\frac{\theta^*}{2\Gamma_2}, \frac{\theta^*}{3\Gamma_1}, 1\}$. This, together with (46), yields

$$\|\eta(t)\| \geq \frac{\theta^*}{2}, \forall\, t \in [T, T + \gamma], \tag{47}$$

which contradicts with (43). Therefore,

$$cA_0^i b_d = 0, 0 \leq i \leq n - 2. \tag{48}$$

In addition, since (A, C) is observable, S is nonsingular. Thus, we have $cA_0^{n-1}b_d \neq 0$. This shows condition (11) finally. □

Theorem 2 gives the necessary and sufficient condition (11), which is about the system structure for convergence of the observer (5). It should be noted that the integral cascade system, which is widely discussed in ESO and EHGO, definitely satisfies (11). This is the reason why ESO and EHGO show good estimation performance for uncertain systems.

The Assumptions 1 and 2 in Theorem 2 can be replaced by the following weaker Assumption 3 as a sufficient condition about the unknown function f to guarantee the convergence of the observer (5), which is substantially required for the boundedness of $f^{(1)}$.

Assumption 3. The state $X(t)$ and the input $u(t)$ of system (1) are supposed to be bounded in the sense that

$$\sup_{t\geq 0}\|X(t)\| + \sup_{t\geq 0}|u(t)| \leq M_1, \tag{49}$$

and the external disturbance $d(t)$ satisfies

$$\sup_{t\geq 0} \|(d(t), \dot{d}(t))\| \leq M_2, \tag{50}$$

where $M_i (i = 1, 2)$ are some positive constants;

There exists a nonnegative continuous function $\varsigma \in C(\mathbb{R}^{n+1}; [0, \infty))$, such that

$$|f(X, d, t)| + \|\nabla f(X, d, t)\| \leq \varsigma(X, d), \forall\, t \in [0, \infty), X \in \mathbb{R}^n, d \in \mathbb{R}. \tag{51}$$

Proposition 1. *Suppose that (A, C) is observable, condition (11) and Assumption 3 are satisfied. Then, there exists an $(n + 1)$-dimensional gain vector L such that the observer (5) is convergent.*

Proof. Since condition (11) is satisfied, it follows from (20) that

$$F = -\begin{bmatrix} 0_{n\times 1} \\ 1 \end{bmatrix} cA_0^{n-1} b_d f^{(1)}. \tag{52}$$

Since Assumption 3 is satisfied, a direct computation easily shows that $f^{(1)}$ is bounded and then F is bounded. Similar to the proof of Theorem 1, we can conclude that there exists an $(n + 1)$-dimensional vector $\bar{L}$ such that (P1) is satisfied. Besides, for any $\varepsilon \in (0, 1]$ and $T > 0$,

$$\|\eta(t) + S^{-1}\Delta\| \leq \Gamma\varepsilon, \forall\, t > T, \tag{53}$$

provided we choose the gain matrix L so that $\sigma(A - LC) = \frac{1}{\varepsilon}\sigma(A - \bar{L}C)$. Since $\Delta \equiv 0$, there exists an $(n + 1)$-dimensional gain vector L such that (P1) and (P2) are satisfied. This completes the proof of the proposition. □

Remark 3. Theorem 2 and Proposition 1 indicate that for general uncertainty $f(\cdot)$, condition (11) is essentially the structural condition to ensure the convergence of the observer (5). Nevertheless, condition (11) is often neglected in previous observer studies. It seems that as long as (A, C) is observable and $f^{(1)}$ is bounded, the estimation errors can always be tuned small enough through $\sigma(A - LC)$. However, it turns out that the convergence of the observer (5) is not only related to the observability of (A, C) and the boundedness of $f^{(1)}$ but also depends on which channel the uncertainty exists in as stated in condition (11).

In the next section, it will be further revealed that condition (11) is the fundamental one to ensure the observability of uncertain systems.

4 | OBSERVABILITY FOR UNCERTAIN SYSTEMS

Up to present, we have studied the convergence of the observer (5). However, a basic question may be ignored: does the output contains the information of uncertainty? If it does, whether the output uniquely determines the state and uncertainty? Motivated by this question, we discuss the observability of the uncertain systems.

For exactly known linear/nonlinear systems, observability of the state means that the continuous dependence of the output with respect to the initial value, ie,

$$y(t) \equiv 0, u(t) \equiv 0, \forall\, t \in [0, \infty) \Rightarrow X(0) = 0. \tag{54}$$

This is because with the initial value and the exactly known model, the state $X(t)$ will be uniquely determined by the output. However, this may not hold true for uncertain systems. With uncertain dynamics, the state of an uncertain system may not be uniquely determined by the output even if (54) is satisfied. Hence, observability for uncertain nonlinear systems not only requires the continuous dependence of the output with respect to the initial value but also with respect to the uncertainty. For this, we give a definition.

Definition 1 (Observability for uncertain nonlinear systems).
The state $X(t)$ and uncertainty $f(\cdot)$ of system (1) are said to be observable, if for any unknown initial state value $X(0)$ and $T > 0$, the state $X(t)$ and uncertainty $f(X(t), d(t), t) (t \in [0, T])$ can be uniquely determined by the output $y(t)$ and the control input $u(t) (t \in [0, T])$.

Observability for uncertain nonlinear systems is well defined by Definition 1, which means that the output not only contains all information of the state but also the uncertainty as well. In particular, if $f \equiv 0$ or $b_d = 0$, system (1) is reduced to be linear time-invariant one, and Definition 1 is line with the observability of linear time-invariant systems.[31,32] The necessary and sufficient condition for observability of system (1) is presented in the following Theorem 3.

Theorem 3. *For $n \geq 2$, system (1) is observable under Definition 1 if and only if (A, C) is observable and condition (11) is satisfied.*

Proof. The first n-order derivatives of y satisfy

$$\begin{bmatrix} y \\ y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(n)} \end{bmatrix} = \begin{bmatrix} c & 0 & 0 & \dots & 0 \\ cA_0 & cb_d & 0 & \dots & 0 \\ cA_0^2 & cA_0 b_d & cb_d & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ cA_0^n & cA_0^{n-1}b_d & cA_0^{n-2}b_d & \dots & cb_d \end{bmatrix} \begin{bmatrix} X \\ f \\ f^{(1)} \\ \vdots \\ f^{(l-1)} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \dots & 0 \\ cb_u & 0 & \dots & 0 \\ cA_0 b_u & cb_u & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ cA_0^{n-1}b_u & \dots & \dots & cb_u \end{bmatrix} \begin{bmatrix} u \\ u^{(1)} \\ u^{(2)} \\ \vdots \\ u^{(n-1)} \end{bmatrix}. \tag{55}$$

Sufficiency: Since (A, C) is observable, S is nonsingular. This, together with condition (11), gives

$$\begin{bmatrix} X \\ f \end{bmatrix} = S^{-1} \left\{ \begin{bmatrix} y \\ y^{(1)} \\ \vdots \\ y^{(n)} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ cA_0^{n-1} b_u u. \end{bmatrix} \right\}. \tag{56}$$

It follows from (56) that the state $X(t)$ and the uncertainty $f(X(t), d(t), t)$ on $[0, T]$ can be uniquely determined by the output $y(t)$ and the control input $u(t)$ on $[0, T]$.

Necessity: Consider a constant disturbance: $f \equiv D$ for some constant D. In this case, $f^{(i)} \equiv 0 (i \geq 1)$ and system (1) is reduced to the following linear time-invariant one:

$$\begin{cases} \begin{bmatrix} \dot{X}(t) \\ f^{(1)} \end{bmatrix} = A \begin{bmatrix} X(t) \\ f \end{bmatrix} + \begin{bmatrix} b_u \\ 0 \end{bmatrix} u(t), \\ y(t) = C \begin{bmatrix} X(t) \\ f \end{bmatrix}. \end{cases} \tag{57}$$

By observability for linear time-invariant systems, the observability condition for this special case is that the pair (A, C) is observable. For general disturbance f, assuming

$$\bar{i} = \min_{i \geq 0} \left\{ i | cA^i b_d \neq 0, i \in Z \right\}, l > 0, \tag{58}$$

we can get

$$\begin{bmatrix} y \\ y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(\bar{i}+l)} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \dots & 0 \\ cb_u & 0 & \dots & 0 \\ cAb_u & cb_u & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ cA^{\bar{i}+l-1}b_u & \dots & \dots & cb_u \end{bmatrix} \begin{bmatrix} u \\ u^{(1)} \\ u^{(2)} \\ \vdots \\ u^{(\bar{i}+l-1)} \end{bmatrix} = \begin{bmatrix} c & 0 & 0 & \dots & 0 \\ cA & \vdots & \vdots & \vdots & \vdots \\ \vdots & 0 & 0 & \vdots & \vdots \\ cA^{\bar{i}+1} & cA^{\bar{i}}b_d & 0 & & \\ cA^{\bar{i}+2} & cA^{\bar{i}+1}b_d & cA^{\bar{i}}b_d & & \\ \vdots & \vdots & \vdots & \ddots & \\ cA^{\bar{i}+l} & cA^{\bar{i}+l-1}b_d & cA^{\bar{i}+l-2}b_d & \dots & cA^{\bar{i}}b_d \end{bmatrix} \begin{bmatrix} X \\ f \\ f^{(1)} \\ \vdots \\ f^{(l-1)} \end{bmatrix}, \tag{59}$$

where $\{X, f, f^{(1)}, \dots, f^{(l-1)}\}$ are unknown variables to be determined. To get $\{X, f, f^{(1)}, \dots, f^{(l-1)}\}$ from the Equation (59), it is necessary to require $\bar{i} \geq n - 1$. Moreover, since (A, C) is observable, S is nonsingular, and thus we can get $\bar{i} \leq n - 1$. Hence $\bar{i} = n - 1$. In conclusion, the pair (A, C) is observable and condition (11) is satisfied. This completes the proof of the theorem. □

Remark 4. If $n = 1$, system (1) is definitely observable.

Theorem 3 gives a necessary and sufficient condition for system (1) to be observable. Moreover, according to Theorems 2 and 3, we can get the following Theorem 4.

Theorem 4. *Suppose that Assumptions 1 and 2 are satisfied. Then, the following three assertions are equivalent:*

(i). The observer (5) is convergent.
(ii). System (1) is observable.
(iii). (A, C) *is observable and condition (11) is satisfied.*

Remark 5. Condition (11) is often neglected in the literature, which essentially reveals the nature of observability for the uncertain system (1). More importantly, the combination of observability of (A, C) and condition (11) is an explicit observability criterion that is easily to be verified. With the observability criterion for uncertain nonlinear systems, it is easy to check whether the uncertain nonlinear systems are observable before we design observers. On the other hand, we can also use the observability criterion to choose the least feasible output to ensure the convergence. If the observability criterion is not satisfied, one may make the uncertain nonlinear systems to be observable by seeking more output information according to the observability criterion (iii) of Theorem 4.

Remark 6. For unobservable uncertain nonlinear systems, the estimation error for each state and uncertainty, which actually reflects the degree of observability for each state and uncertainty, is quantitatively presented in (38) of Theorem 1. According to the estimation error, the estimation accuracy for the state and uncertainty can be evaluated through the structure of the uncertain nonlinear systems and the bound of the derivatives of uncertainty. For the state or uncertainty with high estimation accuracy, the estimations of the state and the uncertainty are almost "reliable," which can be treated as the approximation of the corresponding state and uncertainty although the uncertain systems are unobservable. As a result, (38) in Theorem 1 can be used as an evaluation of the observability degree, or in other words the estimation accuracy, of each state and uncertainty for unobservable uncertain systems, which will be shown in details in example 5.2 of Section 5 later. The above theorem will be verified by three benchmark examples in the next section.

5 | APPLICATION EXAMPLE AND SIMULATIONS

5.1 | Numerical simulation examples

1. Observable uncertain nonlinear systems. Consider the following observable uncertain system:

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = d(t), \\ y(t) = x_1(t), \end{cases} \tag{60}$$

where $(x_1, x_2)^T \in \mathbb{R}^2$, $y \in \mathbb{R}$, and $f(\cdot, t) = d(t) \in \mathbb{R}$ are the state, the measurement output, and the external disturbance, respectively. The initial states are chosen as

$$x_1(0) = 0, \quad x_2(0) = -1. \tag{61}$$

The observer (5) is used to estimate the state (x_1, x_2) and the external disturbance $d(t) = \sin(t)$ and the parameters of observer are selected as

$$\begin{bmatrix} \hat{X}_x(0) \\ \hat{f}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad L = \begin{bmatrix} 3\omega_o \\ 3{\omega_o}^2 \\ {\omega_o}^3 \end{bmatrix}, \quad \omega_o = 80, \tag{62}$$

which satisfies the bandwidth design of observers presented in the work of Gao.[33] The estimates $\hat{X}_x = (\hat{x}_1, \hat{x}_2)$ and $\hat{f}(t)$ of the state (x_1, x_2) and the uncertainty $d(t)$ are depicted in Figure 1. The results indicate that the observer (5) estimates both state and uncertainty accurately. Notice that system (60) is observable according to the observability conditions given by Theorem 4. The simulation results in Figure 1 validate that the observability for uncertain systems is a sufficient condition for observer to achieve satisfactory estimation performance.

2. Unobservable uncertain nonlinear systems. Consider the following unobservable uncertain system:

$$\begin{cases} \dot{x}_1(t) = x_2(t) + d(t), \\ \dot{x}_2(t) = d(t), \\ y(t) = x_1(t), \end{cases} \tag{63}$$

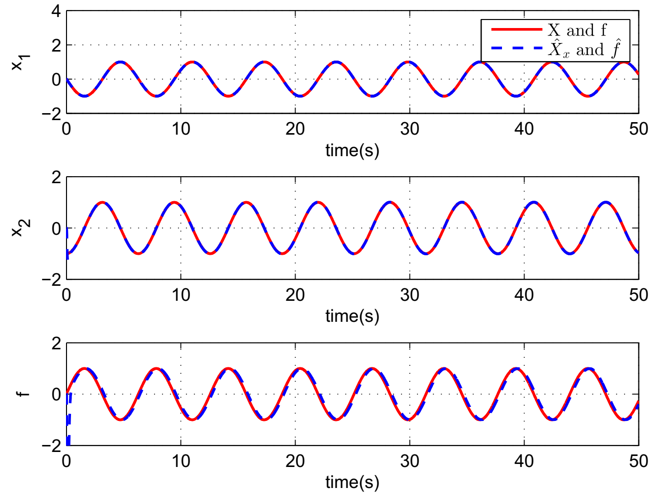

FIGURE 1 The states, disturbance, and their estimates from the example one [Colour figure can be viewed at wileyonlinelibrary.com]

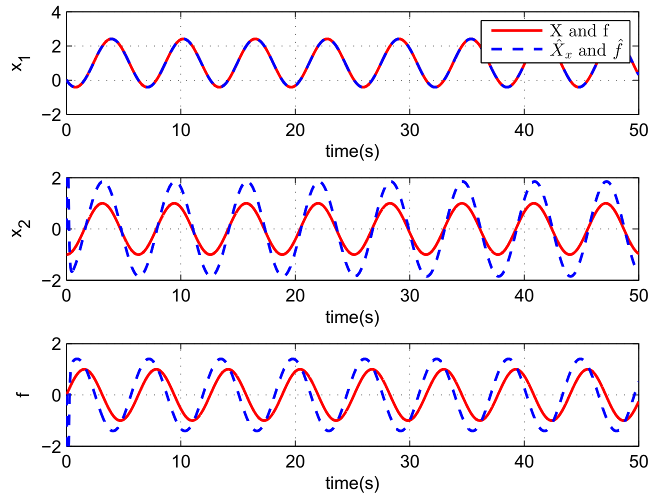

FIGURE 2 The states, disturbance, and their estimates from example two [Colour figure can be viewed at wileyonlinelibrary.com]

where $(x_1, x_2)^T \in \mathbb{R}^2$, $y \in \mathbb{R}$, and $f(\cdot, t) = d(t) \in \mathbb{R}$ are the state, the measurement output, and the external disturbance, respectively. The initial state $(x_1(0), x_2(0))$ and the external disturbance $d(t)$ are chosen the same as (61) and $d(t) = \sin(t)$, respectively.

The observer (5) is designed with the same initial values and bandwidth such that

$$L = \begin{bmatrix} 3\omega_o \\ 3{\omega_o}^2 - {\omega_o}^3 \\ {\omega_o}^3 \end{bmatrix}, \tag{64}$$

and both $\hat{X}_x(0)$ and ω_o are the same as (62). The simulation results are depicted in Figure 2. Figure 2 shows that the observer (5) fails to estimate the state component x_2 and the uncertainty $f(\cdot, t) = d(t)$ accurately no matter how to tune the gain matrix L. The estimations are always biased.

From Theorem 1, in the second example, the observer (5) actually gives the estimations of the following observable group of state and uncertainty, in other words,

$$\begin{bmatrix} \hat{X}_x \\ \hat{f} \end{bmatrix} \xrightarrow[|\omega_o| \to \infty]{} \tilde{X} \triangleq \begin{bmatrix} x_1 \\ x_2 - f^{(1)} \\ f + f^{(1)} \end{bmatrix}, \ \forall\, t > 0. \tag{65}$$

The estimates $\begin{bmatrix} \hat{X}_x \\ \hat{f} \end{bmatrix}$ and $\tilde{X}$ are plotted in Figure 3, which shows that each component of $\begin{bmatrix} \hat{X}_x \\ \hat{f} \end{bmatrix}$ is almost coincident with

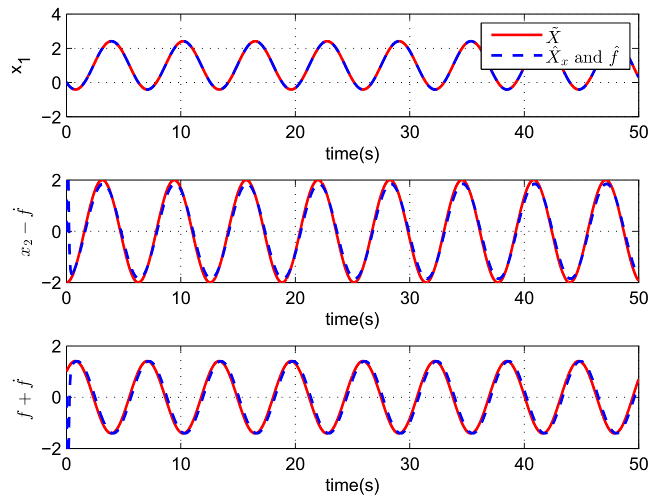

FIGURE 3 The observable group $\tilde{X}$ and their estimates from example two [Colour figure can be viewed at wileyonlinelibrary.com]

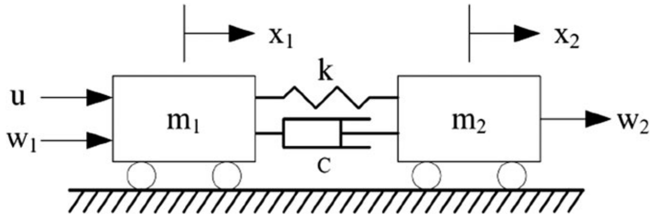

FIGURE 4 Two-mass-spring system with uncertain parameters

corresponding component of $\tilde{X}$. This validates that only the observable group of states and uncertainty can be estimated accurately by the designed observer.

These two examples show that the observability condition for uncertain systems is a sufficient and necessary condition for the observer to achieve satisfactory estimations of state and uncertainty.

5.2 | Application Example

A benchmark two-mass-spring system was considered in the work of Zhang et al[34] and its schematic is given in Figure 4. The system consists of two masses m_1 and m_2 that are free to slide over a frictionless horizontal surface. The masses are attached to one and another by means of a light horizontal spring of spring constant k. The states of the system are defined as the displacements and velocities of the two masses, respectively, where $x_1(t)$ and $x_3(t)$ denote the displacement and velocity of mass m_1, and $x_2(t)$ and $x_4(t)$ denote those of the mass m_2. The control signal $u(t)$ is the force applied to the object one. By Newton's second law and Hooke's law, the two-mass-spring system dynamics can be described by the following equation:

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{m_1} & \frac{k}{m_1} & 0 & 0 \\ \frac{k}{m_2} & -\frac{k}{m_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} cx_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix} [u(t) + b_1 w_1(t)] + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2} \end{bmatrix} b_2 w_2(t), \\ y(t) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \end{cases} \tag{66}$$

where $w_1(t)$ and $w_2(t)$ are two external disturbance forces applied to the masses m_1 and m_2, respectively. When $w_1(t)$ is considered only, $b_1 = 1$ and $b_2 = 0$, and when $w_2(t)$ is considered only, $b_1 = 0$ and $b_2 = 1$. In the same way, $c_1 = 1$ and $c_2 = 0$ means that the position of the mass that m_1 is measured, and $c_1 = 0$ and $c_2 = 1$ means that the m_2 is measured.

We mainly consider the uncertainty observing problem for multiple uncertainties $w_1(t)$ and $w_2(t)$ with the measurements $y(t) = x_1(t)$ or $y(t) = x_2(t)$. For this system, the general observer is designed as (5) with the control input to be set as $u(t) = -k_1x_1(t) - k_2x_2(t) - k_3x_3(t) - k_4x_4(t)$ to stabilize.

When both $w_1(t)$ and $w_2(t)$ are considered,

$$b_d = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{b_1}{m_1} & 0 \\ 0 & \frac{b_2}{m_2} \end{bmatrix}.$$

A calculation shows that (A, C) is unobservable no matter $x_1(t)$ or $x_2(t)$ is measured. The following four cases are considered:

- *Case 1:* $b_1 = 1, b_2 = 0, c_1 = 1, c_2 = 0$;
- *Case 2:* $b_1 = 0, b_2 = 1, c_1 = 1, c_2 = 0$;
- *Case 3:* $b_1 = 1, b_2 = 0, c_1 = 0, c_2 = 1$; and
- *Case 4:* $b_1 = 0, b_2 = 1, c_1 = 0, c_2 = 1$.

According to observability conditions for uncertain systems in Theorem 4, the observability for uncertain system (66) for these different four cases is listed in Table 1. It follows from Table 1 that although (A, C) for four cases are all observable, the uncertain system for Cases 1 and 4 is unobservable for condition (11) is not satisfied. Next, we investigate the estimation performance of the observer (5) for four cases. For the sake of simplicity and easy tuning, the observer gain $L = \begin{bmatrix} l_1 & l_2 & l_3 & l_4 & l_5 \end{bmatrix}^\top$ for four cases are designed so that all eigenvalues of $A - LC$ would be placed at the observer bandwidth $-\omega_o$. The coefficients of L for four cases are listed in Table 2, where $a = (m_1 + m_2)/(m_1m_2)$. The coefficients of the controller gain can be chosen as

$$k_1 = 6\omega_c^2 m_1 - (m_1 + m_2)/m_2,\ k_2 = m_1m_2\omega_c^4 - k_1,\ k_3 = 4\omega_c m_1,\ k_4 = 4\omega_c^3 m_1m_2 - k_3,$$

to place all closed-loop poles at $-\omega_c$. The observability for uncertain system is tested by the two-mass-spring system with the parameters chosen to be

$$m_1 = 0.1kg, m_1 = 0.1kg, k = 1N/m, \omega_o = 100, \omega_c = 10, \omega_1 = \sin(t), \omega_2 = \sin(0.5t).$$

The real and estimated states and uncertainty for four cases are depicted in Figures 5 to 8.

It is observed from Figures 6 to 7 that the estimates of both state and the uncertainty exactly track the real values in Cases 2 and 3 while certain deviations exist between the real and estimated values of state and uncertainty in Cases 1 and 4, which are shown in Figures 5 and 8. The estimation performances of Figures 5 to 8 are rightly consistent with the observability results of Table 2, which means that the observability for uncertain systems is a sufficient and necessary condition for observer (5) to achieve unbiased estimations.

TABLE 1 The observability of two-mass-spring system with four cases of uncertainty and measurements

	(*A*,*C*) is observable	**(11) is satisfied**	**The uncertain system is observable**
Case 1	Yes	No	No
Case 2	Yes	Yes	Yes
Case 3	Yes	Yes	Yes
Case 4	Yes	No	No

TABLE 2 Coefficients in L in four cases of uncertainty and measurements

	Case 1	**Case 2**	**Case 3**	**Case 4**
l_1	$5\omega_o$	$5\omega_o$	$(10\omega_o^3 m_1m_2 - l_2m_2)/m_1$	$(10\omega_o^3 m_1m_2 - l_5m_1 - l_2m_2)/m_1$
l_2	$(10\omega_o^3 m_1m_2 - l_5m_2 - l_1m_1)/m_2$	$(10\omega_o^3 m_1m_2 - l_1m_1)/m_2$	$5\omega_o$	$5\omega_o$
l_3	$10\omega_o^2 - a$	$10\omega_o^2 - a$	$(5\omega_o^4 m_1m_2 - l_4m_2)/m_1$	$(5\omega_o^4 m_1m_2 - l_4m_2)/m_1$
l_4	$(5\omega_o^4 m_1m_2 - l_3m_1)/m_2$	$(5\omega_o^4 m_1m_2 - l_3m_1)/m_2$	$10\omega_o^2 - a$	$10\omega_o^2 - a$
l_5	$\omega_o^5 m_1m_2$	$\omega_o^5 m_1m_2$	$\omega_o^5 m_1m_2$	$\omega_o^5 m_1m_2$

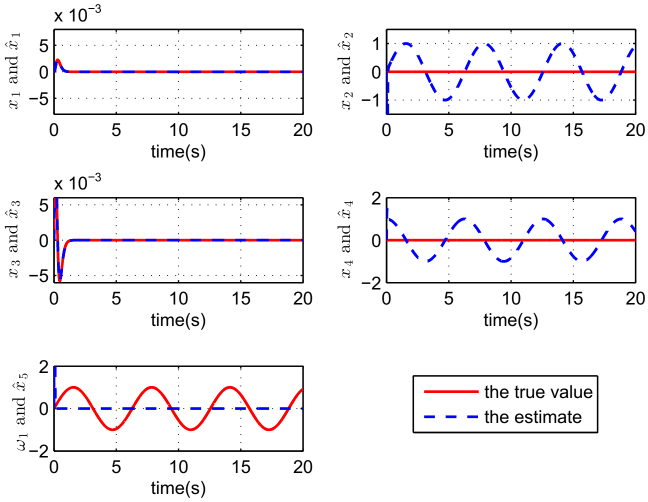

FIGURE 5 The real and estimated values of states and uncertainty for Case 1 (example three) [Colour figure can be viewed at wileyonlinelibrary.com]

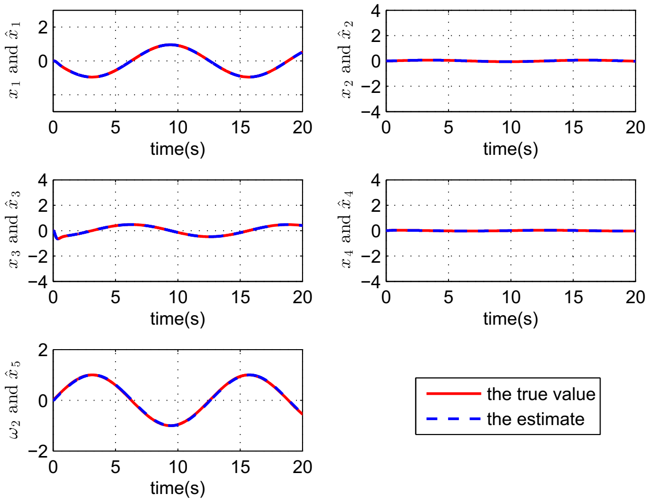

FIGURE 6 The real and estimated values of states and uncertainty for Case 2 (example three) [Colour figure can be viewed at wileyonlinelibrary.com]

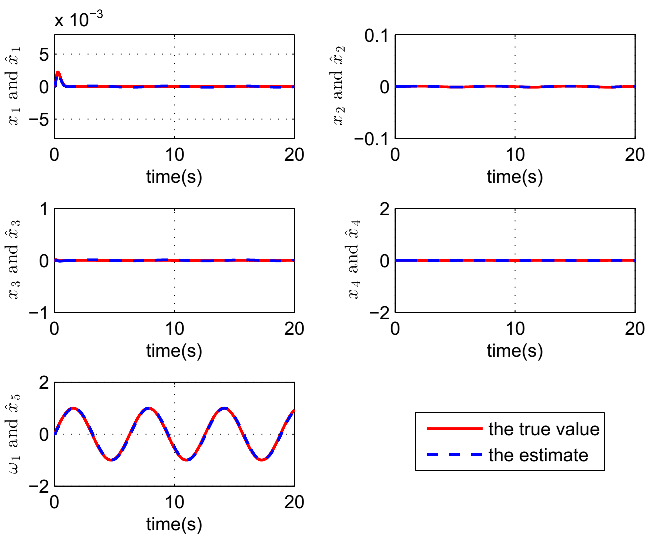

FIGURE 7 The real and estimated values of states and uncertainty for Case 3 (example three) [Colour figure can be viewed at wileyonlinelibrary.com]

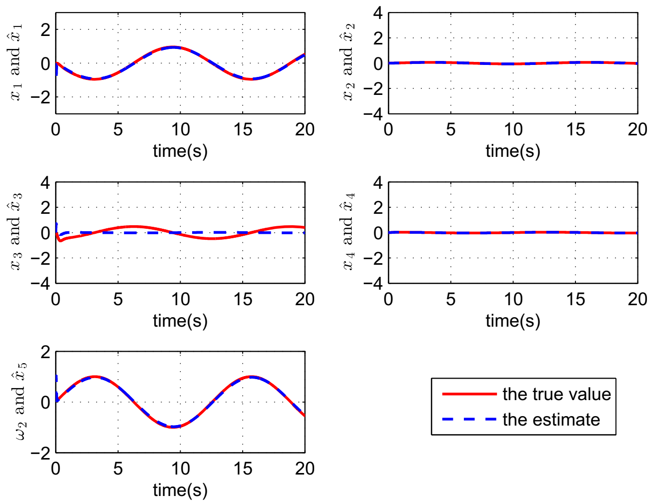

FIGURE 8 The real and estimated values of states and uncertainty for Case 4 (example three) [Colour figure can be viewed at wileyonlinelibrary.com]

Moreover, it follows from Table 2 and Figures 5 to 8 that when $x_1(t)$ is measured, the uncertain system with $w_1(t)$ is observable, while the uncertain system with $w_2(t)$ is unobservable. Conversely, when $x_2(t)$ is measured, the uncertain system with $w_2(t)$ is observable whereas the uncertain system with $w_1(t)$ is unobservable. Thus, for given measurement, observability corresponding to different uncertainties may be opposite. Fortunately, the observability criterion (iii) in Theorem 4 clearly tells us the case of uncertainty that can be accurately estimated before we try to design an observer to estimate the uncertainty. This is of great significance both theoretically and practically. On the other hand, for certain uncertainty, different measurements have different observability properties. How to choose the least "feasible" output to achieve unbiased estimation of uncertain systems is an important problem to be solved. The observability criterion (iii) rightly gives the intuitional answer to this problem. For instance, Table 1 indicates that if we want to estimate $w_1(t)$, $x_2(t)$ needs to be measured, and similarly, if we want to estimate $w_2(t)$, $x_1(t)$ needs to be measured, which is verified by Figures 6 to 7.

A comparison of Figures 5 and 8 shows that although the estimations for Cases 1 and 4 are both biased, the deviation for Case 4 is much smaller than that for Case 1. According to (53) in Theorem 1, the estimation error of unobservable system is $S^{-1}\Delta$, which is calculated as the following values for two cases:

Case 1:

$$S^{-1}\Delta = \begin{bmatrix} 0 & -\ddot{w}_1(t) & 0 & \dot{w}_1(t) & \dddot{w}_1(t) \end{bmatrix}^\top = \begin{bmatrix} 0 & \sin(t) & 0 & \cos(t) & -\sin(t) \end{bmatrix}^\top;$$

Case 4:

$$S^{-1}\Delta = \begin{bmatrix} -0.1\ddot{w}_2(t) & 0 & \dot{w}_2(t) & 0 & 0.1\dddot{w}_2(t) \end{bmatrix}\top = \begin{bmatrix} 0.025\sin(0.5t) & 0 & 0.5\cos(0.5t) & 0 & -0.025\sin(0.5t) \end{bmatrix}^\top.$$

The (53) in Theorem 1 quantitatively gives an estimation error of unobservable Cases 1 and 4, which helps to explain why the deviation for Case 4 is smaller than that of Case 1. Since the estimation value is almost coincident with the true value of $w_2(t)$ for Case 4, the estimate of $w_2(t)$ can be treated as "reliable" one, which can be used to approximate the true value of $w_2(t)$ although the uncertain system in Case 4 is unobservable.

6 | CONCLUSIONS

This paper analyzes the performance of observer for a large class of uncertain nonlinear systems and proposes a structural condition, which is proved to be essential to ensure the convergence of the observer. It is shown that the combination of the structural condition and the observability for the augment matrix pair is a necessary and sufficient condition for the observer to be convergent. By defining observability for uncertain nonlinear systems, it is further proved that the uncertain nonlinear systems are observable if and only if both the structural condition and the observability of the augment matrix pair are satisfied. In addition, for unobservable uncertain nonlinear systems, which do not satisfy this necessary and sufficient condition, it presents explicitly a biased estimation error, which can be used to evaluate the estimation performance of the proposed observer. The numerical simulations for three typical examples are carried out to validate the theoretical analysis.

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