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Performance output tracking for one-dimensional wave equation subject to unmatched general disturbance and non-collocated control☆

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ABSTRACT

In this paper, we consider performance output tracking for a boundary controlled one-dimensional wave equation with possibly unknown internal nonlinear uncertainty and external disturbance. We first show that the open-loop system is well-posed and then propose a disturbance estimator. It is shown that the disturbance estimator can estimate successfully the total disturbance that consists of internal uncertainty and external disturbance. An servomechanism based on the estimated total disturbance is then designed. It is shown that the closed-loop system is well-posed. Three control objectives are achieved: (a) the output is tracking the reference signal; (b) all the internal signals are uniformly bounded; (c) the closed-loop system is internally asymptotically stable if both the reference signal and the disturbance vanish or belong to the space $H^2(0,\infty)$ and $L^2(0,\infty)$, respectively. The unmatched performance output tracking control is first time applied to a system described by the partial differential equation for complete general disturbance rejection and reference tracking purpose. Another key feature of this paper is that we do not use the high-gain to estimate total disturbance for unmatched system. The numerical experiments are carried out to illustrate effectiveness of the proposed control law.

1. Introduction

In this paper, we consider performance output tracking for a one-dimensional wave equation with Neumann boundary control and unknown internal nonlinear uncertainty and external disturbance. The system is governed by the following partial differential equation:

$$\begin{cases} w_{tt}(x,t)=w_{xx}(x,t), \quad x\in(0,1), & t>0,\\ w_x(0,t)=qw_t(0,t)+f(w(\cdot,t))+d(t), & t\geq 0,\\ w_x(1,t)=u(t), & t\geq 0,\\ w(x,0)=w_0(x), \quad w_t(x,0)=w_1(x), & x\in(0,1),\\ y_m(t)=\{w(0,t),w_t(1,t)\}, & t\geq 0,\\ y_o(t)=w(1,t), & t\geq 0, \end{cases} \tag{1.1}$$

where we denote by $w'(x,t)$ or $w_x(x,t)$ the derivative of $w(x,t)$ with respect to x and by $\dot{w}(x,t)$ or $w_t(x,t)$ the derivative of $w(x,t)$ with respect to t. The $u(t)$ is the control input, $y_m(t)$ the measured output, $y_o(t)$ the performance output signal to be regulated, $f: H^1(0,1)\to\mathbb{R}$ an unknown possibly nonlinear mapping that represents internal uncertainty. Examples include like $f(w)=\sin(w^2(0,t))$, $f(w)=\sin(\int_{1/3}^{1/2} w(x,t)dx)$. These nonlocal boundary condition arises mainly when the data on the boundary cannot be measured directly [1]. The $d(t)$ is the unknown external disturbance which is only supposed to satisfy $d\in L^\infty(0,\infty)$, and $q>0(\neq 1)$ is a constant.

For a given reference signal $r(t)$, we are expected to design an output feedback control for uncertain system (1.1) to reject the external disturbance so that

$$e(t)=y_o(t)-r(t)\to 0 \text{ as } t\to\infty. \tag{1.2}$$

System (1.1) is a typical non-collocated Neumann control problem: Control is on the one end and the disturbance is on the other end, which was first investigated in [12] where no internal uncertainty is concerned and the external disturbance is of harmonic disturbance only, i.e., $d(t)=\sum_{j=1}^m(\theta_j\sin(\alpha_j t)+\vartheta_j\cos(\alpha_j t))$.

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Except the general external disturbance, another point that is different from [12] is that for asymptotic stability of the closed-loop system, we only assume that the disturbance and reference signal belong to $L^2(0, \infty)$ and $H^2(0, \infty)$, respectively, which does not necessarily vanish at infinity required in [12]. This paper is a non-trivial generalization of the results from [12]. The boundary condition in (1.1) implies that system (1.1) suffers from the external disturbance and the internal uncertainty and system (1.1) is exponentially stable when there are no disturbance and control input involved.

The above problem is a special kind of output regulation problem where the disturbance is not limited to special class. It is well known that the output regulation is one of the fundamental issues in control theory. There are many works dedicated to the output regulation for finite-dimensional systems since from [3,4,9], and the related results have been generalized to the infinite-dimensional systems, see, for instance, [2,5,6,14,17–20], among many others. These results are based on the internal model principle where reference signal and disturbance are generated by exosystem. For finite harmonic signals, this requires the frequencies to be known. A few surveys are contributed to estimate the frequencies [16] where the order of parameter update law is huge as the number of frequencies increase. A first attempt on an infinite-dimensional signal is [13] where a general periodic signal that has infinitely many frequencies is considered. However, it does not cover general disturbance considered in this paper. A recent interesting work is [15] where output tracking problem is considered for a general 2×2 system of first order linear hyperbolic PDEs but no uncertainty and disturbance are taken into consideration.

To the best of our knowledge, the general disturbance signals are not considered in output tracking problems for PDEs. In this paper, we solve the performance output tracking problem with general disturbance by designing a new disturbance estimator that can estimate the total disturbance yet does not use high-gain in estimation. For external disturbance estimation/cancellation only, we refer to [10,11] via active disturbance rejection control.

We consider system (1.1) in the state Hilbert space $\mathbb{H}=H^1(0,1)\times L^2(0,1)$ with the inner product given by

$$\langle(\phi_1,\psi_1)^\top,(\phi_2,\psi_2)^\top\rangle_{\mathbb{H}}=\int_0^1[\phi_1'(x)\overline{\phi_2'(x)}+\psi_1(x)\overline{\psi_2(x)}]dx +\phi_1(0)\overline{\phi_2(0)},\quad \forall\ (\phi_i,\psi_i)^\top\in\mathbb{H},\quad i=1,2. \tag{1.3}$$

Define the operators $\mathbb{A}:D(\mathbb{A})(\subset\mathbb{H})\to\mathbb{H}$ and $\mathbb{B}_1,\mathbb{B}_2:\mathbb{R}\to\mathbb{H}$ as

$$\begin{cases}\mathbb{A}(\phi,\psi)^\top=(\psi,\phi'')^\top,\quad \forall\ (\phi,\psi)^\top\in D(\mathbb{A}),\\ D(\mathbb{A})=\big\{(\phi,\psi)^\top\in H^2(0,1)\times H^1(0,1)\\ \qquad :\phi'(0)=q\psi(0)+\phi(0),\quad \phi'(1)=0\big\},\\ \mathbb{B}_1=(0,\delta_0)^\top,\quad \mathbb{B}_2=(0,\delta_1)^\top,\quad D(\mathbb{B}_1)=D(\mathbb{B}_2)=\mathbb{R},\end{cases} \tag{1.4}$$

where δ_a is the Dirac distribution which satisfies $\delta_a(\phi)=\phi(a)$ for all $\phi\in H^1(0,1)$. It is readily found that

$$\begin{cases}\mathbb{A}^*(\phi,\psi)^\top=(-\psi,-\phi'')^\top,\quad \forall\ (\phi,\psi)^\top\in D(\mathbb{A}^*),\\ D(\mathbb{A}^*)=\big\{(\phi,\psi)^\top\in H^2(0,1)\times H^1(0,1)\\ \qquad :\phi'(0)=-q\psi(0)+\phi(0),\quad \phi'(1)=0\big\}.\end{cases} \tag{1.5}$$

The system (1.1) can be rewritten as

$$\frac{d}{dt}\begin{pmatrix}w(\cdot,t)\\ w_t(\cdot,t)\end{pmatrix}=\mathbb{A}\begin{pmatrix}w(\cdot,t)\\ w_t(\cdot,t)\end{pmatrix}+\mathbb{B}_1(f(w(\cdot,t)) -w(0,t)+d(t))+\mathbb{B}_2u(t). \tag{1.6}$$

Before stating Proposition 1.1, we introduce some terminology. Let X and U be Hilbert spaces. Suppose that A generate a C_0-semigroup e^{At} on X. Let X_{-1} be the completion of X with respect to the norm $\|x\|_{-1}=\|(\beta I-A)^{-1}x\|$, where β is some element in the resolvent set $\rho(A)$. An operator $B\in L(U,X_{-1})$ is an admissible control operator to e^{At} if for some (hence for any) $\tau>0$ and for every $u\in L^2([0,\infty);U)$,

$$\Phi_\tau u=\int_0^\tau e^{A(\tau-\sigma)}Bu(\sigma)d\sigma\in X$$

(the integral is computed in X_{-1}). Then, this integral gives the strong solution of $\dot z(t)=Az(t)+Bu(t)$ in the space X, corresponding to $z(0)=0$, evaluated at the time τ. B is called bounded if $B\in L(U,X)$, and unbounded otherwise. If B is admissible and $\alpha>0$, then there exists $M_\alpha\ge 0$ such that

$$\|(sI-A)^{-1}B\|_{L(U,X)}\le\frac{M_\alpha}{\sqrt{\mathrm{Re}s-\alpha}}\qquad\text{for}\quad \mathrm{Re}s>\alpha.$$

For more admissibility and its properties, we refer the reader to [21, Chapter 4].

Proposition 1.1. *The operator $\mathbb{A}$ defined by (1.4) generates a C_0-group $e^{\mathbb{A}t}$ on $\mathbb{H}$ and $\mathbb{B}_1$ and $\mathbb{B}_2$ are admissible to $e^{\mathbb{A}t}$. Suppose that $f:H^1(0,1)\to\mathbb{R}$ is continuous and satisfies local Lipschitz condition in $H^1(0,1)$. Then, for any $(w_0,w_1)^\top\in\mathbb{H}$, $u\in L^2_{loc}(0,\infty)$, and $d\in L^2_{loc}(0,\infty)$, there exists a unique local solution to (1.1) such that $(w(\cdot,t),\dot w(\cdot,t))^\top\in C(0,T;\mathbb{H})$ for some $T>0$ and for $t\in[0,T)$,*

$$\begin{pmatrix}w(\cdot,t)\\ w_t(\cdot,t)\end{pmatrix}=e^{\mathbb{A}t}\begin{pmatrix}w_0(\cdot)\\ w_1(\cdot)\end{pmatrix}+\int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}_1[f(w(\cdot,s))-w(0,t)+d(s)]ds+\int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}_2u(s)ds. \tag{1.7}$$

Moreover, if $f:H^1(0,1)\to\mathbb{R}$ satisfies the uniform Lipschitz condition

$$|f(w_1)-f(w_2)|\le L\|w_1-w_2\|_{H^1(0,1)},\quad \forall\ w_1,w_2\in H^1(0,1),$$

for some $L>0$ or $f:H^1(0,1)\to\mathbb{R}$ is bounded. Then, there exists a unique global solution $(w,\dot w)^\top\in C(0,\infty;\mathbb{H})$ to (1.1) satisfying (1.7) with $T=+\infty$.

Proof. By Lemma 6.1 in Appendix, $\mathbb{A}$ generates a C_0-group $e^{\mathbb{A}t}$ on $\mathbb{H}$. We now show that $\mathbb{B}_1$ and $\mathbb{B}_2$ are admissible to $e^{\mathbb{A}t}$. By [21, Theorem 4.4.3], it suffices to show that $\mathbb{B}_1^*$ and $\mathbb{B}_2^*$ are admissible observation operators for the adjoint semigroup $e^{\mathbb{A}^*t}$. This is equivalent to showing that a) $\mathbb{B}_1^*\mathbb{A}^{*-1}$ and $\mathbb{B}_2^*\mathbb{A}^{*-1}$ are bounded from $\mathbb{H}$ to $\mathbb{C}$, and b) for every $T_*>0$, there exists $M_{T_*}>0$ depending on T_* only such that the system of the following:

$$\begin{cases}w^*_{tt}(x,t)=w^*_{xx}(x,t),\quad x\in(0,1), & t>0,\\ w^*_x(0,t)=-qw^*_t(0,t)+w^*(0,t), & t\ge 0,\\ w^*_x(1,t)=0, & t\ge 0,\\ w^*(x,0)=w^*_0(x),\quad w^*_t(x,0)=w^*_1(x), & x\in(0,1),\\ y_w=(w^*_t(0,t),w^*_t(1,t)), & t\ge 0,\end{cases} \tag{1.8}$$

satisfies

$$\int_0^{T_*}[(w^*_t(0,t))^2+(w^*_t(1,t))^2]dt\le M_{T_*}\|(w^*_0,w^*_1)^\top\|_{\mathbb{H}}. \tag{1.9}$$

By Lemma 6.1 in Appendix, it is easy to see that (1.8) admits a unique solution $(w^*(\cdot,t),w^*_t(\cdot,t))^\top\in C(0,\infty;\mathbb{H})$ and there exist two constants $M_*>0$, and $\omega_*\in\mathbb{R}$ such that for all $t\ge 0$,

$$\int_0^1[(w^*_x(x,t))^2+(w^*_t(x,t))^2]dx+(w^*(0,t))^2\le M_*e^{\omega_*t}\|(w^*_0,w^*_1)^\top\|^2_{\mathbb{H}}. \tag{1.10}$$

Let

$$\rho_*(t)=\int_0^1(2x-1)w^*_x(x,t)w^*_t(x,t)dx. \tag{1.11}$$

Then

$$|\rho_*(t)| \le \frac{1}{2}\int_0^1 [(w_x^*(x,t))^2 + (w_t^*(x,t))^2]dx. \tag{1.12}$$

Finding the derivative of $\rho_*(t)$ along the solution of (1.8), we obtain

$$\begin{aligned}\dot\rho_*(t) &= (w_t^*(0,t))^2 + (w_t^*(1,t))^2 + (w_x^*(0,t))^2 + (w_x^*(1,t))^2 \\ &\quad - \int_0^1 [(w_x^*(x,t))^2 + (w_t^*(x,t))^2]dx,\end{aligned} \tag{1.13}$$

which, together with (1.10) and (1.12), gives

$$\begin{aligned}&\int_0^{T_*} [(w_t^*(0,t))^2 + (w_t^*(1,t))^2]dt \le \rho_*(t) - \rho_*(0) \\ &\quad + \int_0^{T_*}\int_0^1 [(w_x^*(x,t))^2 + (w_t^*(x,t))^2]dx \\ &\le \left(\frac{1}{2} + \left(\frac{1}{2} + T_*\right)\right) M_* e^{\omega_* T_*} \|(w_0^*, w_1^*)^\top\|_{\mathbb{H}}^2.\end{aligned} \tag{1.14}$$

A direct computation shows that

$$\begin{cases}\mathbb{A}^{*-1}(\phi,\psi)^\top = \Big(q\phi(0) + (1+x)\int_0^1 \psi(x)dx \\ \qquad - \int_0^x (x-\xi)\psi(\xi)d\xi, -\phi(x)\Big)^\top, \\ \mathbb{B}_1^*\mathbb{A}^{*-1}(\phi,\psi)^\top = -\phi(0), \quad \mathbb{B}_2^*\mathbb{A}^{*-1}(\phi,\psi)^\top = -\phi(1)\end{cases} \tag{1.15}$$

By the Sobolev embedding theorem, $H^1(0,\ 1)\hookrightarrow C(0,\ 1)$. By $\phi \in H^1(0,\ 1)$ and (1.15), we know that $\mathbb{B}_1^*\mathbb{A}^{*-1}$ and $\mathbb{B}_2^*\mathbb{A}^{*-1}$ are bounded from $\mathbb{H}$ to $\mathbb{C}$, which, together with (1.14), implies that $\mathbb{B}_1$ and $\mathbb{B}_2$ are admissible to $e^{\mathbb{A}t}$. Therefore, for any fixed $T>0$, and for any given $u, d \in L^2_{loc}(0,\infty)$, we have

$$\begin{aligned}&\int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}_1 d(s)ds \in C(0,T;\mathbb{H}), \quad \text{and} \\ &\int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}_2 u(s)ds \in C(0,T;\mathbb{H}).\end{aligned} \tag{1.16}$$

Denote by $F(w(\cdot,t)) = f(w(\cdot,t)) - w(0,t)$. Since $f : H^1(0,1) \to \mathbb{R}$ satisfies the local Lipschitz condition, so does F. For any initial value $(w_0, w_1)^\top \in \mathbb{H}$, let $(\eta_1(t), \eta_2(t))^\top = e^{\mathbb{A}t}(w_0(\cdot), w_1(\cdot))^\top$. For any given $\sigma > \max_{0\le t\le 1}\|\eta_1(t)\|_{H^1(0,1)} > 0$ and for $t \in [0,\ 1]$, define a set Λ_t given by

$$\Lambda_t = \{z : (z, z_t) \in \mathbb{H}, \quad \|z - \eta_1(t)\|_{H^1(0,1)} \le \sigma\}.$$

Then there exists a constant $L_\sigma > 0$ independent of t such that

$$|F(z_1) - F(z_2)| \le L_\sigma \|z_1 - z_2\|_{H^1(0,1)}, \quad \forall\ z_1, z_2 \in \Lambda_t. \tag{1.17}$$

The admissibility of $\mathbb{B}_1$ implies that for all $t>0$,

$$\left\|\int_0^t e^{\mathbb{A}(\tau-s)}\mathbb{B}_1\zeta(s)ds\right\|_{\mathbb{H}} \le C_t\|\zeta\|_{L^2(0,t)} \le C_t t\|\zeta\|_{L^\infty(0,t)} \tag{1.18}$$

for some constant C_t which is independent of ζ. By [22, Proposition 2.3], we know that C_t is nondecreasing with respect to t. Let $\tau \le 1$. Then $C_\tau \le C_1$. Choose $\tau > 0$ so that $C_1\tau L_\sigma < 1$ and

$$\begin{aligned}C_1\tau L_\sigma\Bigg(\sigma + \Bigg\| e^{\mathbb{A}t}\begin{pmatrix} w_0(\cdot) \\ w_1(\cdot)\end{pmatrix} + \int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}_1 d(s)ds \\ + \int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}_2 u(s)ds\Bigg\|_{C(0,1;\mathbb{H})}\Bigg) < \sigma.\end{aligned} \tag{1.19}$$

Let

$$\Theta = \left\{\begin{array}{l}(\varphi(\cdot,t), \varphi_t(\cdot,t))^\top \in C(0,\tau;\mathbb{H}) : \varphi(\cdot,0) = w_0(\cdot), \\ \qquad \varphi_t(\cdot,0) = w_1(\cdot) \\ \left\|\begin{pmatrix}\varphi_1(\cdot,t) \\ \varphi_{1t}(\cdot,t)\end{pmatrix} - e^{\mathbb{A}t}\begin{pmatrix} w_0(\cdot) \\ w_1(\cdot)\end{pmatrix} - \int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}_1 d(s)ds \right. \\ \qquad \left. - \int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}_2 u(s)ds\right\|_{\mathbb{H}} \le \sigma\end{array}\right\}$$

be a closed subset of $C(0,\tau;\mathbb{H})$. Define the nonlinear map $\mathbb{F}$ from Θ to $C(0,T;\mathbb{H})$ by

$$\begin{aligned}\mathbb{F}\begin{pmatrix}\varphi(\cdot,t) \\ \varphi_t(\cdot,t)\end{pmatrix} &= e^{\mathbb{A}t}\begin{pmatrix} w_0(\cdot) \\ w_1(\cdot)\end{pmatrix} + \int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}_1 d(s)ds \\ &\quad + \int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}_2 u(s)ds \\ &\quad + \int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}_1 F(\varphi(\cdot,s))ds.\end{aligned} \tag{1.20}$$

It follows from (1.17), (1.18), and (1.20) that for any $(\varphi_1, \varphi_{1t})^\top, (\varphi_2, \varphi_{2t})^\top \in \Theta$,

$$\begin{aligned}&\left\|\mathbb{F}\begin{pmatrix}\varphi_1(\cdot,t) \\ \varphi_{1t}(\cdot,t)\end{pmatrix} - \mathbb{F}\begin{pmatrix}\varphi_2(\cdot,t) \\ \varphi_{2t}(\cdot,t)\end{pmatrix}\right\|_{\mathbb{H}} \\ &= \left\|\int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}_1[F(\varphi_1(\cdot,s)) - F(\varphi_2(\cdot,s))]ds\right\|_{\mathbb{H}} \\ &\le C_t t\|F(\varphi_1(\cdot,s)) - F(\varphi_2(\cdot,s))\|_{L^\infty(0,t)} \\ &\le C_1\tau\|F(\varphi_1(\cdot,s)) - F(\varphi_2(\cdot,s))\|_{L^\infty(0,\tau)} \\ &\le C_1\tau L_\sigma\|\varphi_1(\cdot,s) - \varphi_2(\cdot,s)\|_{L^\infty(0,\tau;H^1(0,1))} \\ &\le C_1\tau L_\sigma\left\|\begin{pmatrix}\varphi_1(\cdot,s) \\ \varphi_{1t}(\cdot,s)\end{pmatrix} - \begin{pmatrix}\varphi_2(\cdot,s) \\ \varphi_{2t}(\cdot,s)\end{pmatrix}\right\|_{C(0,\tau;\mathbb{H})},\end{aligned} \tag{1.21}$$

which, together with $C_1\tau L_\sigma < 1$, implies that $\mathbb{F}$ is a strict contraction on Θ. Letting $(\varphi_2, \varphi_{2t})^\top = (0,0)^\top$ in (1.21), by (1.19), we can see that $\mathbb{F}\Theta \subset \Theta$. By the contraction mapping theorem, (1.20) has a unique fixed point $(w,\dot w)^\top \in C(0,T;\mathbb{H})$, which is a solution of (1.7).

Now, we claim the second assertion. Suppose that $f(\cdot)$ satisfies the uniform Lipschitz condition with constant L. So does $F(\cdot)$ with constant $L+1$. Let $[0,\ T)$ be the maximal interval of existence of the solution of (1.1). Obviously, it suffices to show that $T=+\infty$. Assuming $T<\infty$, it follows from (1.18) that for $t \in [0,\ T)$,

$$\begin{aligned}\left\|\int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}_1 F(w(\cdot,s))ds\right\|_{\mathcal{H}}^2 &\le C_t^2\|F(w(\cdot,s))\|_{L^2(0,t)}^2 \\ &\le C_T^2\|F(w(\cdot,s))\|_{L^2(0,t)}^2 \\ &\le C_T^2(L+1)^2\|w(\cdot,s)\|_{L^2(0,t;H^1(0,1))}^2 \\ &= C_T^2(L+1)^2\int_0^t \|w(\cdot,s)\|_{H^1(0,1)}^2 ds \\ &\le C_T^2(L+1)^2\int_0^t \left\|\begin{pmatrix}w(\cdot,s) \\ w_t(\cdot,s)\end{pmatrix}\right\|_{\mathbb{H}}^2 ds.\end{aligned} \tag{1.22}$$

Since the solution on $[0,\ T)$ satisfies (1.7), by (1.22), we have

$$\begin{aligned}\left\|\begin{pmatrix}w(\cdot,t) \\ w_t(\cdot,t)\end{pmatrix}\right\|_{\mathbb{H}}^2 &\le 2\Bigg\|e^{\mathbb{A}t}\begin{pmatrix} w_0(\cdot) \\ w_1(\cdot)\end{pmatrix} + \int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}_1 d(s)ds \\ &\quad + \int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}_2 u(s)ds\Bigg\|_{\mathbb{H}}^2 \\ &\quad + 2\left\|\int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}_1 F(w(\cdot,s))ds\right\|_{\mathbb{H}}^2 \\ &\le 2\max_{t\in[0,T]}\Bigg\|e^{\mathbb{A}t}\begin{pmatrix} w_0(\cdot) \\ w_1(\cdot)\end{pmatrix} + \int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}_1 d(s)ds \\ &\quad + \int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}_2 u(s)ds\Bigg\|_{\mathbb{H}}^2 \\ &\quad + 2C_T^2(L+1)^2\int_0^t \left\|\begin{pmatrix}w(\cdot,s) \\ w_t(\cdot,s)\end{pmatrix}\right\|_{\mathbb{H}}^2 ds,\end{aligned}$$

which, by using Gronwall's inequality, yields

$$\left\|\begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t)\end{pmatrix}\right\|_{\mathbb{H}}^2 \le 2\max_{t\in[0,T]}\left\|e^{\mathbb{A}t}\begin{pmatrix} w_0(\cdot) \\ w_1(\cdot)\end{pmatrix} + \int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}_1 d(s)ds + \int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}_2 u(s)ds\right\|_{\mathbb{H}}^2 e^{2C_T^2(L+1)^2T},$$

that is, $(w,\dot{w})^\top$ is uniformly bounded on $\mathbb{H}$ over $[0, T]$. Thus, if $T<\infty$, similar to the proof of the existence of local solution, we can prove that (1.1) has a unique solution on $[0, T+\sigma_0)$ for some $\sigma_0>0$. This is a contradiction. When $f: H^1(0,1)\to\mathbb{R}$ is bounded, similar contradiction also happens. Therefore, (1.1) admits a unique global solution. □

Remark 1.1. By Proposition 1.1, we can assume that $f: H^1(0,1)\to\mathbb{R}$ is continuous, and system (1.1) admits a unique solution $(w,\dot{w})^\top\in C(0,\infty;\mathbb{H})$.

The next Lemma 1.1 is well-known and is not difficult to prove by using the results of [21] and [22]. For the reader's convenience, we give a simple proof.

Lemma 1.1. *Let A be the generator of exponentially stable C_0-semigroup e^{At} on the Hilbert space X. Assume that $B\in\mathcal{L}(U,X_{-1})$ is an admissible control operator for e^{At}. Then, the initial value problem*

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0,$$

admits a unique solution $x\in C(0,\infty;X)$, which tends to zeros as $t\to\infty$ if $u\in L^2(0,\infty;U)$ or $\lim_{t\to\infty}\|u(t)\|_U = 0$, and is bounded if $u\in L^\infty(0,\infty;U)$.

Proof. Since $B\in\mathcal{L}(U,X_{-1})$ is an admissible control operator, by [21, Proposition 4.2.5.], the solution can be represented as

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)ds \in X. \tag{1.23}$$

Since A generates an exponentially stable C_0-semigroup e^{At}, there exist two constants M, $\mu>0$ such that $\|e^{At}\|\le Me^{-\mu t}$. Suppose that $u\in L^\infty(0,\infty;U)$. By [22, Remark 4.7], the admissibility of B implies that there exists a constant $L_1>0$ independent of $u(s)$ such that

$$\left\|\int_0^t e^{A(t-s)}Bu(s)ds\right\|_X \le L_1\|u\|_{L^\infty(0,\infty;U)}, \tag{1.24}$$

which, together with (1.23) and the exponential stability of e^{At}, implies that the solution $x(t)$ is bounded. Next, suppose that $u\in L^2(0,\infty;U)$. For any given $\sigma>0$, there exists $t_0>0$ such that

$$\|u\|_{L^2(t_0,\infty;U)} \le \sigma. \tag{1.25}$$

It follows from the admissibility of B and [22, Remark 2.6] that

$$\left\|\int_{t_0}^t e^{A(t-s)}Bu(s)ds\right\|_X \le \left\|\int_0^t e^{A(t-s)}B(0\underset{t_0}{\diamond}u(s))ds\right\|_X \le L_2\|u\|_{L^2(t_0,\infty;U)} \le L_2\sigma, \tag{1.26}$$

where L_2 is a constant that is independent of $u(s)$, and

$$(u\underset{\tau}{\diamond}v)(t) = \begin{cases} u(t), & 0\le t\le\tau, \\ v(t), & t>\tau. \end{cases}$$

Using the exponential stability of e^{At} again, we have

$$\left\|e^{A(t-t_0)}\int_0^{t_0} e^{A(t_0-s)}Bu(s)ds\right\|_X \le \|e^{A(t-t_0)}\|\left\|\int_0^{t_0} e^{A(t_0-s)}Bu(s)ds\right\|_X \le Me^{-\mu(t-t_0)}\left\|\int_0^{t_0} e^{A(t_0-s)}Bu(s)ds\right\|_X. \tag{1.27}$$

Rewriting (1.23) as

$$x(t) = e^{At}x_0 + e^{A(t-t_0)}\int_0^{t_0} e^{A(t_0-s)}Bu(s)ds + \int_{t_0}^t e^{A(t-s)}Bu(s)ds, \tag{1.28}$$

it follows from (1.26) and (1.27) that

$$\|x(t)\| \le Me^{-\mu t}\|x_0\| + Me^{-\mu(t-t_0)}\left\|\int_0^{t_0} e^{A(t_0-s)}Bu(s)ds\right\|_X + L_2\sigma. \tag{1.29}$$

This shows that $\lim_{t\to\infty}\|x(t)\|\le L_2\sigma$. By the arbitrariness of σ, $x(t)\to 0$ as $t\to\infty$.

Finally, suppose that $\lim_{t\to\infty}\|u(t)\|_U = 0$. For any given $\sigma>0$, there exists $t_0>0$ such that

$$\|u\|_{L^\infty(t_0,\infty;U)} \le \sigma. \tag{1.30}$$

It follows from the admissibility of B and [22, Remark 4.7] that

$$\left\|\int_{t_0}^t e^{A(t-s)}Bu(s)ds\right\|_X \le \left\|\int_0^t e^{A(t-s)}B(0\underset{t_0}{\diamond}u(s))ds\right\|_X \le L_1\|u\|_{L^\infty(t_0,\infty;U)} \le L_1\sigma. \tag{1.31}$$

The rest of the proof is similar to (1.27)–(1.29). This ends the proof of the lemma. □

Remark 1.2. Consider the control problem: $\dot{x}(t) = Ax(t) + B(u(t) + d(t))$, $x(0) = x_0$, where $d(t)$ is the disturbance. By Lemma 1.1, if $d\in L^2(0,\infty;U)$, it does not affect the asymptotic stability of the system. When $d\notin L^2(0,\infty;U)$, if we can find estimate $\widehat{d}(t)$ of $d(t)$ so that $(\widehat{d}-d)\in L^2(0,\infty;U)$, we do not need to construct disturbance estimator likewise [10,11] such that $\|\widehat{d}(t)-d(t)\|_U\to 0$ as $t\to\infty$ which is usually stronger than $(\widehat{d}-d)\in L^2(0,\infty;U)$.

We proceed as follows. In Section 2, we design a total disturbance estimator which estimates the total disturbance that consists of internal uncertainty and the external disturbance. Section 3 is devoted to design of servomechanism and output tracking of the closed-loop system. Some numerical simulations are presented in Section 4 for illustration.

2. Estimator design

In this section, we design a disturbance estimator in terms of input and output $(u(t), y_m(t))$ for system (1.1):

$$\begin{cases} z_{tt}(x,t) = z_{xx}(x,t), \quad x\in(0,1), & t>0, \\ z_x(0,t) = qz_t(0,t) + c_1[z(0,t) - w(0,t)], & t\ge 0, \\ z_x(1,t) = u(t), & t\ge 0, \\ \widehat{d}_{tt}(x,t) = \widehat{d}_{xx}(x,t), \quad x\in(0,1), & t>0, \\ \widehat{d}(0,t) = z(0,t) - w(0,t), & t\ge 0, \\ \widehat{d}_x(1,t) = -c_2[\widehat{d}_t(1,t) - z_t(1,t) + w_t(1,t)], & t\ge 0, \\ z(x,0) = z_0(x), \quad z_t(x,0) = z_1(x), & x\in[0,1], \\ \widehat{d}(x,0) = \widehat{d}_0(x), \quad \widehat{d}_t(x,0) = \widehat{d}_1(x), & x\in[0,1], \end{cases} \tag{2.1}$$

where $c_1, c_2>0$ are the design parameters. We will show that this estimator can estimate the total disturbance $f(w(\cdot,t)) + d(t)$. It is seen that system (2.1) is completely determined by the measured output $y_m(t)$ of system (1.1) and the control input $u(t)$.

Let $\widehat{z}(x,t) = z(x,t) - w(x,t)$. Then $\widehat{z}(x,t)$ is governed by

$$\begin{cases} \widehat{z}_{tt}(x,t) = \widehat{z}_{xx}(x,t), \quad x\in(0,1), & t>0, \\ \widehat{z}_x(0,t) = c_1\widehat{z}(0,t) + q\widehat{z}_t(0,t) - f(w(\cdot,t)) - d(t), & t\ge 0, \\ \widehat{z}_x(1,t) = 0, & t\ge 0, \\ \widehat{z}(x,0) = \widehat{z}_0(x), \quad \widehat{z}_t(x,0) = \widehat{z}_1(x), & x\in[0,1]. \end{cases} \tag{2.2}$$

We consider system (2.2) in the state space $\mathbb{H} = H^1(0,1)\times L^2(0,1)$ with the inner product given by

$$\langle(\phi_1,\psi_1)^\top,(\phi_2,\psi_2)^\top\rangle = \int_0^1[\phi_1'(x)\overline{\phi_2'(x)}+\psi_1(x)\overline{\psi_2(x)}]dx + c_1\phi_1(0)\overline{\phi_2(0)},\quad \forall\ (\phi_i,\psi_i)^\top\in\mathbb{H},\quad i=1,2.$$

System (2.2) can be written as an evolutionary equation in $\mathbb{H}$:

$$\frac{d}{dt}\begin{pmatrix}\widehat{z}(\cdot,t)\\ \widehat{z}_t(\cdot,t)\end{pmatrix} = \mathbb{A}_z\begin{pmatrix}\widehat{z}(\cdot,t)\\ \widehat{z}_t(\cdot,t)\end{pmatrix}+\mathbb{B}_1(-f(w(\cdot,t))-d(t)), \tag{2.3}$$

where the operator $\mathbb{B}_1$ is given by (1.4) and $\mathbb{A}_z$ given by

$$\begin{cases}\mathbb{A}_z(\phi,\psi)^\top = (\psi,\phi'')^\top,\quad \forall\ (\phi,\psi)^\top\in D(\mathbb{A}_z),\\ D(\mathbb{A}_z)=\Big\{(\phi,\psi)^\top\in H^2(0,1)\times H^1(0,1)\\ \qquad :\phi'(0)=c_1\phi(0)+q\psi(0),\quad \phi'(1)=0\Big\}.\end{cases} \tag{2.4}$$

Lemma 2.1. *Suppose that $d\in L^\infty(0,\infty)$, $f: H^1(0,1)\to\mathbb{R}$ is continuous and that (1.1) admits a unique solution $(w,\dot{w})^\top\in C(0,\infty;\mathbb{H})$ which is bounded. For any initial value $(\widehat{z}_0,\widehat{z}_1)^\top\in H^1(0,1)\times L^2(0,1)$, there exists a unique solution $(\widehat{z},\widehat{z}_t)^\top\in C(0,\infty;H^1(0,1)\times L^2(0,1))$ to (2.2) such that*

$$\sup_{t\ge 0}\|(\widehat{z}(\cdot,t),\widehat{z}_t(\cdot,t))^\top\|_{H^1(0,1)\times L^2(0,1)}<+\infty. \tag{2.5}$$

Proof. By Lemma 6.1 in Appendix, $\mathbb{A}_z$ generates an exponentially stable C_0-semigroup $e^{\mathbb{A}_z t}$. By Proposition 1.1, $\mathbb{B}_1$ is admissible to $e^{\mathbb{A}_z t}$. Since $f: H^1(0,1)\to\mathbb{R}$ is continuous and $(w,\dot{w})^\top\in C(0,\infty;\mathbb{H})$ is bounded, $f(w)\in L^\infty(0,\infty)$ and hence $(f(w)+d)\in L^\infty(0,\infty)$. Thus, it follows from Lemma 1.1 that system (2.2) admits a unique bounded solution. □

Let $\widetilde{d}(x,t)=\widehat{d}(x,t)-\widehat{z}(x,t)=\widehat{d}(x,t)-z(x,t)+w(x,t)$. We can see that $\widetilde{d}(x,t)$ satisfies

$$\begin{cases}\widetilde{d}_{tt}(x,t)=\widetilde{d}_{xx}(x,t),\quad x\in(0,1), & t>0,\\ \widetilde{d}(0,t)=0, & t\ge 0,\\ \widetilde{d}_x(1,t)=-c_2\widetilde{d}_t(1,t), & t\ge 0,\\ \widetilde{d}(x,0)=\widetilde{d}_0(x),\quad \widetilde{d}_t(x,0)=\widetilde{d}_1(x), & x\in[0,1].\end{cases} \tag{2.6}$$

We consider (2.6) in the energy space $H_L^1(0,1)\times L^2(0,1)$, where $H_L^1(0,1)=\{\phi\in H^1(0,1):\phi(0)=0\}$. It is well known that for any initial value $(\widetilde{d}_0,\widetilde{d}_1)\in H_L^1(0,1)\times L^2(0,1)$, system (2.6) admits a unique solution and there exist two constants $M_d,\mu_d>0$ such that

$$\int_0^1[(\widetilde{d}_x(x,t))^2+(\widetilde{d}_t(x,t))^2]dx\le M_d\|(\widetilde{d}_0,\widetilde{d}_1)\|^2_{H_L^1(0,1)\times L^2(0,1)}e^{-\mu_d t}. \tag{2.7}$$

Lemma 2.2. *For any initial value $(\widetilde{d}_0,\widetilde{d}_1)\in H_L^1(0,1)\times L^2(0,1)$, the solution of (2.6) satisfies*

$$\int_0^\infty \widetilde{d}_x^2(0,t)dt<+\infty. \tag{2.8}$$

Proof. Let

$$\varrho(t)=\int_0^1(x-1)\widetilde{d}_t(x,t)\widetilde{d}_x(x,t)dx. \tag{2.9}$$

Then,

$$|\varrho(t)|\le\frac{1}{2}\int_0^1[\widetilde{d}_x^2(x,t)+\widetilde{d}_t^2(x,t)]dx \le\frac{1}{2}M_d\|(\widetilde{d}_0,\widetilde{d}_1)\|^2_{H_L^1(0,1)\times L^2(0,1)}e^{-\mu_d t}. \tag{2.10}$$

Finding the derivative of $\varrho(t)$ along the solution of (2.6) yields

$$\dot{\varrho}(t)=\frac{1}{2}\widetilde{d}_x^2(0,t)-\frac{1}{2}\int_0^1[\widetilde{d}_t^2(x,t)+\widetilde{d}_x^2(x,t)]dx. \tag{2.11}$$

Integrating from 0 to T with respect to t for (2.11), we have

$$\begin{aligned}\frac{1}{2}\int_0^T\widetilde{d}_x^2(0,t)dt &= \frac{1}{2}\int_0^T\int_0^1[\widetilde{d}_t^2(x,t)+\widetilde{d}_x^2(x,t)]dxdt+\varrho(T)-\varrho(0)\\ &\le\frac{1}{2\mu_d}M_d\|(\widetilde{d}_0,\widetilde{d}_1)\|_{H_L^1(0,1)\times L^2(0,1)}(1-e^{-\mu_d T})\\ &\quad+\frac{1}{2}M_d\|(\widetilde{d}_0,\widetilde{d}_1)\|^2_{H_L^1(0,1)\times L^2(0,1)}e^{-\mu_d T}-\varrho(0),\end{aligned} \tag{2.12}$$

which, passing to the limit as $T\to\infty$, yields $\widetilde{d}_x(0,\cdot)\in L^2(0,\infty)$. □

Remark 2.1. If we take $c_2=1$, then $(\widetilde{d}(x,t),\widetilde{d}_t(x,t))=(0,0)$ for $t\ge 2$. Thus, $\widetilde{d}_x(0,t)\equiv 0$ for $t\ge 2$. In this case, the total disturbance is exactly estimated.

Remark 2.2. By (2.2) and (2.6), a simple computation shows that

$$\widetilde{d}_x(0,t)=f(w(\cdot,t))+d(t)-(-\widehat{d}_x(0,t)+c_1\widehat{d}(0,t)+q\widehat{d}_t(0,t)). \tag{2.13}$$

By Lemma 2.2, $-\widehat{d}_x(0,t)+c_1\widehat{d}(0,t)+q\widehat{d}_t(0,t)$ can be regarded as an estimate of the total disturbance $f(w(\cdot,t))+d(t)$.

3. Servomechanism design

Denote $W^{2,\infty}(0,\infty)=\{\phi:\phi\in L^\infty(0,\infty),\ \ \phi'\in L^\infty(0,\infty),\ \ \phi''\in L^\infty(0,\infty)\}$. For the reference signal $r(t)$, we design the following reference model:

$$\begin{cases}v_{tt}(x,t)=v_{xx}(x,t),\quad x\in(0,1), & t>0,\\ v_x(0,t)=qv_t(0,t)-\widehat{d}_x(0,t)+c_1\widehat{d}(0,t)+q\widehat{d}_t(0,t), & t\ge 0,\\ v(1,t)=r(t), & t\ge 0,\\ v(x,0)=v_0(x),\quad v_t(x,0)=v_1(x), & x\in[0,1].\end{cases} \tag{3.1}$$

It is seen that system (3.1) is completely determined by the measured output of system (1.1), the output of estimator (2.1), and the reference signal $r(t)$ only.

Let $\varepsilon(x,t)=w(x,t)-v(x,t)$ be the error between system (1.1) and state reference system (3.1). Then $\varepsilon(x,t)$ is governed by

$$\begin{cases}\varepsilon_{tt}(x,t)=\varepsilon_{xx}(x,t),\quad x\in(0,1), & t>0,\\ \varepsilon_x(0,t)=q\varepsilon_t(0,t)+\widetilde{d}_x(0,t), & t\ge 0,\\ \varepsilon_x(1,t)=u(t)-v_x(1,t), & t\ge 0,\\ \varepsilon(x,0)=w_0(x)-v_0(x),\quad \varepsilon_t(x,0)=w_1(x)-v_1(x), & x\in[0,1],\end{cases} \tag{3.2}$$

where $\varepsilon(1,t)=w(1,t)-r(t)=e(t)$ is the performance output tracking error. We propose the following output feedback control:

$$\begin{aligned}u(t)&=v_x(1,t)-c_3\varepsilon(1,t)=v_x(1,t)-c_3(w(1,t)-r(t))\\ &=v_x(1,t)-c_3e(t),\end{aligned} \tag{3.3}$$

where $c_3>0$ is a design parameter.

Under control (3.3), the resulting closed-loop of (3.2) is

$$\begin{cases}\varepsilon_{tt}(x,t)=\varepsilon_{xx}(x,t),\quad x\in(0,1), & t>0,\\ \varepsilon_x(0,t)=q\varepsilon_t(0,t)+\widetilde{d}_x(0,t), & t\ge 0,\\ \varepsilon_x(1,t)=-c_3\varepsilon(1,t), & t\ge 0,\\ \varepsilon(x,0)=\varepsilon_0(x),\quad \varepsilon_t(x,0)=\varepsilon_1(x), & x\in[0,1].\end{cases} \tag{3.4}$$

We consider system (3.4) in the state space $\mathbb{H}=H^1(0,1)\times L^2(0,1)$ with the inner product given by

$$\langle(\phi_1,\psi_1)^\top,(\phi_2,\psi_2)^\top\rangle=\int_0^1[\phi_1'(x)\overline{\phi_2'(x)}+\psi_1(x)\overline{\psi_2(x)}]dx+c_3\phi_1(1)\overline{\phi_2(1)},\quad\forall\ (\phi_i,\psi_i)^\top\in\mathbb{H},\quad i=1,2.$$

Theorem 3.1. *Suppose that $\widetilde{d}_x(0,\cdot)\in L^2(0,\infty)$. For any initial value $(\varepsilon(\cdot,0),\varepsilon_t(\cdot,0))^\top\in\mathbb{H}$, there exists a unique solution to (3.4) such that $(\varepsilon,\varepsilon_t)^\top\in C(0,\infty;\mathbb{H})$ satisfying*

$$\lim_{t\to\infty}\left(\int_0^1[\varepsilon_x^2(x,t)+\varepsilon_t^2(x,t)]dx+\varepsilon^2(1,t)\right)=0. \tag{3.5}$$

Proof. Define an operator $A_\varepsilon:D(A_\varepsilon)(\subset\mathbb{H})\to\mathbb{H}$ by

$$\begin{cases}A_\varepsilon(\phi,\psi)^\top=(\psi,\phi'')^\top,\quad\forall(\phi,\psi)^\top\in D(A_\varepsilon),\\ D(A_\varepsilon)=\{(\phi,\psi)^\top\in\mathbb{H}\cap H^2(0,1)\times H^1(0,1)\\ \quad:\phi'(0)=q\psi(0),\quad\phi'(1)=-c_3\phi(1)\}.\end{cases} \tag{3.6}$$

We can write (3.4) into operator form of the following:

$$\frac{d}{dt}(\varepsilon(\cdot,t),\varepsilon_t(\cdot,t))^\top=A_\varepsilon(\varepsilon(\cdot,t),\varepsilon_t(\cdot,t))^\top+\mathbb{B}_1\widetilde{d}_x(0,t), \tag{3.7}$$

where $\mathbb{B}_1$ is given by (1.4). It is readily found that

$$\begin{cases}A_\varepsilon^*(\phi,\psi)^\top=(-\psi,-\phi'')^\top,\quad\forall(\phi,\psi)^\top\in D(A_\varepsilon^*),\\ D(A_\varepsilon^*)=\Big\{(\phi,\psi)^\top\in\mathbb{H}\cap(H^2(0,1)\times H^1(0,1))\\ \qquad:\phi'(0)=-q\psi(0),\quad\phi'(1)=-c_3\phi(1)\Big\}.\end{cases} \tag{3.8}$$

By Lemma 6.2 in Appendix, A_ε generates an exponentially stable C_0-semigroup $e^{A_\varepsilon t}$ on $\mathbb{H}$. Now we show that $\mathbb{B}_1$ is admissible to $e^{A_\varepsilon t}$. Once again, this is equivalent to showing that a) $\mathbb{B}_1^*A_\varepsilon^{*-1}$ is bounded from $\mathbb{H}$ to $\mathbb{C}$, and b) for every $T_*>0$, there exists $M_{T_*}>0$ depending on T_* only such that the system of the following:

$$\begin{cases}\varepsilon_{tt}^*(x,t)=\varepsilon_{xx}^*(x,t),\quad x\in(0,1), & t>0,\\ \varepsilon_x^*(0,t)=-qv_t^*(0,t), & t\ge0,\\ \varepsilon_x^*(1,t)=-c_3\varepsilon^*(1,t), & t\ge0,\\ \varepsilon^*(x,0)=\varepsilon_0^*(x),\quad\varepsilon_t^*(x,0)=\varepsilon_1^*(x), & x\in[0,1],\\ y_\varepsilon=\varepsilon_t^*(0,t),\end{cases} \tag{3.9}$$

satisfies

$$\int_0^{T_*}(\varepsilon_t^*(0,t))^2dt\le M_{T_*}\|(\varepsilon_0^*,\varepsilon_1^*)^\top\|_{\mathbb{H}}^2. \tag{3.10}$$

A direct computation shows that

$$\begin{cases}A_\varepsilon^{*-1}(\phi,\psi)^\top=\left(\frac{1}{c_3}\left(-q\phi(0)(c_3+1)+\int_0^1\psi(\xi)d\xi\right.\right.\\ \qquad\left.+\ c_3\int_0^1(1-\xi)\psi(\xi)d\xi\right)\\ \qquad\left.+\ q\phi(0)x-\int_0^x(x-\xi)\psi(\xi)d\xi,-\phi(x)\right)^\top,\\ \mathbb{B}_1^*A_\varepsilon^{*-1}(\phi,\psi)^\top=-\phi(0).\end{cases} \tag{3.11}$$

Thus, $\mathbb{B}_1^*A_\varepsilon^{*-1}$ is bounded from $\mathbb{H}$ to $\mathbb{C}$. By Lemma 6.2 in Appendix, it is easy to see that (3.9) admits a unique solution $(\varepsilon^*,\varepsilon_t^*)^\top\in C(0,\infty;\mathbb{H})$ and there exist two constants $M_*>0$ and $\omega_*\in\mathbb{R}$ such that for all $t\ge0$,

$$E_\varepsilon^*(t):=\int_0^1[(\varepsilon_x^*(x,t))^2+(\varepsilon_t^*(x,t))^2]dx+c_3(\varepsilon^*(1,t))^2\le M_*e^{\omega_*t}\|(\varepsilon_0^*,\varepsilon_1^*)^\top\|_{\mathbb{H}}^2. \tag{3.12}$$

On the other hand, differentiating $E_v^*(t)$ with respect to t along the solution to (3.9) gives

$$\dot{E}_\varepsilon^*(t)=2q(\varepsilon_t^*(0,t))^2. \tag{3.13}$$

Integrating from 0 to T_* with respect to t and by (3.12), we obtain

$$2q\int_0^{T_*}(\varepsilon_t^*(0,t))^2dt=E_\varepsilon^*(T_*)-E_\varepsilon^*(0)\le M_*(e^{\omega_*T_*}+1)\|(\varepsilon_0^*,\varepsilon_1^*)^\top\|_{\mathbb{H}}^2. \tag{3.14}$$

This, together with (3.11), implies that $\mathbb{B}_1$ is admissible to $e^{A_\varepsilon t}$. Therefore, (3.7) admits a unique solution. Since $\widetilde{d}_x(0,\cdot)\in L^2(0,\infty)$, it follows from Lemma 1.1 that system (3.7) admits a unique solution that tends to zeros as t goes to infinity, i.e., (3.5) holds. □

Remark 3.1. By Remark 2.1, if we take $c_2=1$ in (2.1), then system (3.4) is exponentially stable. In this case, we can see that the performance output tracking is exponentially convergent.

Now we consider the reference model (3.1) in the state space $\mathbb{H}=H^1(0,1)\times L^2(0,1)$ with the inner product given by

$$\langle(\phi_1,\psi_1)^\top,(\phi_2,\psi_2)^\top\rangle=\int_0^1[\phi_1'(x)\overline{\phi_2'(x)}+\psi_1(x)\overline{\psi_2(x)}]dx+\phi_1(1)\overline{\phi_2(1)},\quad\forall\ (\phi_i,\psi_i)^\top\in\mathbb{H},\quad i=1,2.$$

Lemma 3.1. *Suppose that $d\in L^\infty(0,+\infty)$, $r\in W^{2,\infty}(0,\ \infty)$, $f:H^1(0,1)\to\mathbb{R}$ is continuous and bounded, and system (1.1) admits a unique solution $(w,w_t)\in C(0,\infty;\mathbb{H})$. For any initial value $(v_0,v_1)^\top\in\mathbb{H}$, there exists a unique solution to (3.1) such that $(v,v_t)^\top\in C(0,\infty;\mathbb{H})$. Moreover, there exists a constant $M>0$ such that*

$$\sup_{t\ge0}\left(\int_0^1[v_x^2(x,t)+v_t^2(x,t)]dx+v^2(1,t)\right)\le M. \tag{3.15}$$

Suppose additionally that $f(w)+d\in L^2(0,\infty)$ and $r\in H^2(0,\ \infty)$. Then,

$$\lim_{t\to\infty}\left(\int_0^1[v_x^2(x,t)+v_t^2(x,t)]dx+v^2(1,t)\right)=0. \tag{3.16}$$

Proof. Introducing the transformation $\widehat{v}(x,t)=v(x,t)-x^2r(t)$, we can transform system (3.1) into an equivalent system:

$$\begin{cases}\widehat{v}_{tt}(x,t)=\widehat{v}_{xx}(x,t)-x^2\ddot{r}(t)+2r(t),\quad x\in(0,1), & t>0,\\ \widehat{v}_x(0,t)=q\widehat{v}_t(0,t)-\widehat{d}_x(0,t)+c_1\widehat{d}(0,t)+q\widehat{d}_t(0,t), & t\ge0,\\ \widehat{v}(1,t)=0, & t\ge0,\\ \widehat{v}(x,0)=v_0(x)-x^2r(0),\quad\widehat{v}_t(x,0)=v_1(x)-x^2\dot{r}(0), & x\in[0,1].\end{cases} \tag{3.17}$$

Since $r\in W^{2,\infty}(0,\ \infty)$, it suffices to prove that there exists a constant $M>0$ such that

$$\sup_{t\ge0}\int_0^1[\widehat{v}_x^2(x,t)+\widehat{v}_t^2(x,t)]dx\le M. \tag{3.18}$$

We consider system (3.17) in the state space $\mathcal{H}=H_R^1(0,1)\times L^2(0,1)$, where $H_R^1(0,1)=\{\phi\in H^1(0,1):\phi(1)=0\}$, with the inner product given by

$$\langle(\phi_1,\psi_1)^\top,(\phi_2,\psi_2)^\top\rangle=\int_0^1[\phi_1'(x)\overline{\phi_2'(x)}+\psi_1(x)\overline{\psi_2(x)}]dx,\quad\forall\ (\phi_i,\psi_i)^\top\in\mathbb{H},\quad i=1,2. \tag{3.19}$$

Define the operator $A_v:D(A_v)(\subset\mathbb{H})\to\mathbb{H}$ by

$$\begin{cases}A_v(\phi,\psi)^\top=(\psi,\phi'')^\top,\quad\forall\ (\phi,\psi)^\top\in D(A_v),\\ D(A_v)=\{(\phi,\psi)^\top\in H^2(0,1)\times H^1(0,1)\\ \quad:\phi'(0)=q\psi(0),\quad\phi(1)=0\}.\end{cases} \tag{3.20}$$

We can write (3.17) into an operator form of the following:

$$\frac{d}{dt}(\widehat{v}(\cdot,t),\widehat{v}_t(\cdot,t))^\top = A_v(\widehat{v}(\cdot,t),\widehat{v}_t(\cdot,t))^\top + I(0,-x^2\ddot{r}(t) + 2r(t))^\top + \mathbb{B}_1 g(t), \tag{3.21}$$

where I is an identity operator and $g(t) = -\widehat{d}_x(0,t) + c_2\widehat{d}(0,t) + q\widehat{d}_t(0,t)$. It is readily found that

$$\begin{cases} A_v^*(\phi,\psi)^\top = (-\psi,-\phi'')^\top, \quad \forall\ (\phi,\psi)^\top \in D(A_v^*),\\ D(A_v^*) = \{(\phi,\psi)^\top \in H^2(0,1)\times H^1(0,1) \\ \quad : \phi'(0) = -q\psi(0), \quad \phi(1) = 0\}. \end{cases} \tag{3.22}$$

By Lemma 6.3 in Appendix, A_v generates an exponentially stable C_0-semigroup $e^{A_v t}$ on $\mathbb{H}$. Since I is a bounded operator, I is admissible to $e^{A_v t}$. Now we show that $\mathbb{B}_1$ is admissible to $e^{A_v t}$. This is equivalent to showing that a) $\mathbb{B}_1^* A_v^{*-1}$ is bounded from $\mathbb{H}$ to $\mathbb{C}$, and b) for any $T_*>0$, there exists $M_{T_*}>0$ depending on T_* only such that the system of the following:

$$\begin{cases} v_{tt}^*(x,t) = v_{xx}^*(x,t), \quad x\in(0,1), & t>0,\\ v_x^*(0,t) = -qv_t^*(0,t), & t\ge 0,\\ v^*(1,t)=0, & t\ge 0,\\ v^*(x,0) = v_0^*(x), \quad v_t(x,0) = v_1^*(x), & x\in[0,1],\\ y_v = v_t^*(0,t), & t\ge 0, \end{cases} \tag{3.23}$$

satisfies

$$\int_0^{T_*} (v_t^*(0,t))^2 dt \le M_{T_*}\|(v_0^*,v_1^*)^\top\|_{\mathcal{H}}^2. \tag{3.24}$$

A direct computation shows that

$$\begin{cases} A_v^{*-1}(\phi,\psi)^\top = \left(q\phi(0)(x-1) + \int_0^1 (1-\xi)\psi(\xi)d\xi \right. \\ \qquad \left. - \int_0^x (x-\xi)\psi(\xi)d\xi, -\phi(x)\right)^\top,\\ \mathbb{B}_1^* A_v^{*-1}(\phi,\psi)^\top = -\phi(0). \end{cases} \tag{3.25}$$

Thus, $\mathbb{B}_1^* A_v^{*-1}$ is bounded from $\mathbb{H}$ to $\mathbb{C}$. By Lemma 6.3 in Appendix, it is easy to see that (3.23) admits a unique solution $(v^*,v_t^*)^\top \in C(0,\infty;\mathcal{H})$ and there exist two constants $M_*>0$, and $\omega_*\in\mathbb{R}$ such that for all $t\ge 0$,

$$E_v^*(t) := \int_0^1 [(v_x^*(x,t))^2 + (v_t^*(x,t))^2]dx \le M_* e^{\omega_* t}\|(v_0^*,v_1^*)^\top\|_{\mathcal{H}}^2. \tag{3.26}$$

On the other hand, differentiating $E_v^*(t)$ with respect to t along the solution to (3.23), we obtain

$$\dot{E}_v^*(t) = 2q(v_t^*(0,t))^2. \tag{3.27}$$

Integrating from 0 to T_* with respect to t and by (3.26), we have

$$2q\int_0^{T_*} (v_t^*(0,t))^2 dt = E_v^*(T_*) - E_v^*(0) \le M_*(e^{\omega_* T_*}+1)\|(v_0^*,v_1^*)^\top\|_{\mathcal{H}}^2. \tag{3.28}$$

This, together with (3.25), implies that $\mathbb{B}_1$ is admissible to $e^{A_v t}$. Therefore, the solution of (3.17) can be written as

$$(\widehat{v}(\cdot,t),\widehat{v}_t(\cdot,t))^\top = e^{A_v t}(\widehat{v}(\cdot,0),\widehat{v}_t(\cdot,0))^\top + \int_0^t e^{A_v(t-s)} I(0,-x^2\ddot{r}(s)+2r(s))^\top ds + \int_0^t e^{A_v(t-s)}\mathbb{B}_1 g(s)ds \tag{3.29}$$

Since $r\in W^{2,\infty}(0,\infty)$, we have $(-x^2\ddot{r}+2r)\in L^\infty(0,\infty)$ and since I is admissible control operator to $e^{A_v t}$, by Lemma 1.1,

$$\sup_{t\ge 0}\left\|\int_0^t e^{A_v(t-s)} I(0,-x^2\ddot{r}(s)+2r(s))^\top ds\right\| < +\infty. \tag{3.30}$$

By assumption $d\in L^\infty(0,+\infty)$ and $f(w)$ being bounded on $[0,\infty)$, it follows from the admissibility of $\mathbb{B}_1$ and Lemma 1.1 that

$$\sup_{t\ge 0}\left\|\int_0^t e^{A_v(t-s)}\mathbb{B}_1(f(w(\cdot,s))+d(s))ds\right\| < +\infty. \tag{3.31}$$

From Lemma 2.2, $\widetilde{d}_x(0,\cdot)\in L^2(0,\infty)$, and by Lemma 1.1,

$$\lim_{t\to\infty}\left\|\int_0^t e^{A_v(t-s)}\mathbb{B}_1(-\widetilde{d}_x(0,\cdot))ds\right\| = 0. \tag{3.32}$$

On the other hand, by Remark 2.2, $g(t) = f(w(\cdot,t)) + d(t) - \widetilde{d}_x(0,t)$. This, together with (3.31) and (3.32), yields

$$\sup_{t\ge 0}\left\|\int_0^t e^{A_v(t-s)}\mathbb{B}_1 g(s)ds\right\| < +\infty. \tag{3.33}$$

It then follows from (3.29), (3.30), and (3.33) that $(\widehat{v}(\cdot,t),\widehat{v}_t(\cdot,t))^\top$ is bounded on $\mathbb{H}$, that is, (3.15) holds.

Suppose that $(f(w)+d)\in L^2(0,\infty)$ and $r\in H^2(0,\infty)$. It is observed that $(-x^2\ddot{r}+2r)\in L^2(0,\infty)$. By Lemma 1.1,

$$\lim_{t\to\infty}\left(\int_0^1 [\widehat{v}_x^2(x,t)+\widehat{v}_t^2(x,t)]dx + \widehat{v}^2(1,t)\right) = 0. \tag{3.34}$$

Now, we show that (3.16) can be reduced to (3.34). To do this, we claim that $\dot{r}\in H^1(0,\infty)$ will lead to $\lim_{t\to\infty}|\dot{r}(t)|=0$. Actually, since $\dot{r}\in L^2(0,\infty)$, $\ddot{r}\in L^2(0,\infty)$, for any $t_1,t_2\ge 0$,

$$|\dot{r}^2(t_1)-\dot{r}^2(t_2)| = 2\left|\int_{t_1}^{t_2}\dot{r}(t)\ddot{r}(t)dt\right| \le 2\sqrt{\int_{t_1}^{t_2}\dot{r}^2(t)dt}\sqrt{\int_{t_1}^{t_2}\ddot{r}^2(t)dt} \to 0 \text{ as } t_1,t_2\to\infty.$$

So for any given $\varepsilon>0$, there exists a $T>0$ such that for all t_1, $t_2>T$,

$$|\dot{r}^2(t_1)-\dot{r}^2(t_2)|\le\varepsilon.$$

This shows that $\lim_{t\to\infty}\dot{r}^2(t)$ exists. But since $\dot{r}\in L^2(0,\infty)$, this limit must be zero. Hence, $\lim_{t\to\infty}\dot{r}^2(t)=0$ and so $\lim_{t\to\infty}|\dot{r}(t)|=0$. In an analogue way, we can obtain $\lim_{t\to\infty}|r(t)|=0$. Since $v(x,t)=\widehat{v}(x,t)+x^2 r(t)$,

$$\lim_{t\to\infty}\left(\int_0^1 [v_x^2(x,t)+v_t^2(x,t)]dx + v^2(1,t)\right) \le 2\lim_{t\to\infty}\left(\int_0^1 [\widehat{v}_x^2(x,t)+\widehat{v}_t^2(x,t)]dx+\widehat{v}^2(1,t)\right) + 2\lim_{t\to\infty}\left(\int_0^1 [4x^2r^2(t)+x^4\dot{r}^2(t)]dx + r^2(1,t)\right) = 0.$$

i.e., (3.16) holds. This ends the proof of the lemma. □

Now we turn to the closed-loop which is composed of (1.1), (2.1), and (3.1) as follows:

$$\begin{cases} w_{tt}(x,t)=w_{xx}(x,t), \quad x\in(0,1), & t>0,\\ w_x(0,t)=qw_t(0,t)+f(w(x,t))+d(t), & t\ge 0,\\ w_x(1,t)=v_x(1,t)-c_3(w(1,t)-r(t)), & t\ge 0,\\ z_{tt}(x,t)=z_{xx}(x,t), \quad x\in(0,1), & t>0,\\ z_x(0,t)=qz_t(0,t)+c_2[z(0,t)-w(0,t)], & t\ge 0,\\ z_x(1,t)=v_x(1,t)-c_3(w(1,t)-r(t)), & t\ge 0,\\ \widehat{d}_{tt}(x,t)=\widehat{d}_{xx}(x,t), \quad x\in(0,1), & t>0,\\ \widehat{d}(0,t)=z(0,t)-w(0,t), & t\ge 0,\\ \widehat{d}_x(1,t)=-c_3[\widehat{d}_t(1,t)-z_t(1,t)+w_t(1,t)], & t\ge 0,\\ v_{tt}(x,t)=v_{xx}(x,t), \quad x\in(0,1), & t>0,\\ v_x(0,t)=qv_t(0,t)-\widehat{d}_x(0,t)+c_1\widehat{d}(0,t)+q\widehat{d}_t(0,t), & t\ge 0,\\ v(1,t)=r(t), & t\ge 0,\\ w(x,0)=w_0(x), \quad w_t(x,0)=w_1(x), & x\in[0,1],\\ z(x,0)=z_0(x), \quad z_t(x,0)=z_1(x), & x\in[0,1],\\ \widehat{d}(x,0)=\widehat{d}_0(x), \quad \widehat{d}_t(x,0)=\widehat{d}_1(x), & x\in[0,1],\\ v(x,0)=v_0(x), \quad v_t(x,0)=v_1(x), & x\in[0,1]. \end{cases} \tag{3.35}$$

We consider system (3.35) in the state space $\mathcal{X}=(H^1(0,1)\times L^2(0,1))^4$.

Theorem 3.2. *Let $c_1,\ c_3>0$ and $c_2=1$. Suppose that $d\in L^\infty(0,+\infty)$, $r\in W^{2,\infty}(0,\ \infty)$, $f: H^1(0,1)\to\mathbb{R}$ is continuous, bounded, and satisfies the local Lipschitz condition in $H^1(0,\ 1)$. Then, for any initial value $(w_0,w_1,z_0,z_1,\widehat{d}_0,\widehat{d}_1,v_0,v_1)\in\mathcal{X}$ with compatible boundary conditions*

$$\widehat{d}_0(0)-z_0(0)+w_0(0)=0, \quad v_0(1)=r(0), \tag{3.36}$$

There exists a unique solution to (3.35) such that $(w,w_t,z,z_t,\widehat{d},\widehat{d}_t,v,v_t)\in C(0,\infty;\mathcal{X})$. Moreover, the closed-loop system solution has the following properties:

(i)

$$\sup_{t\ge 0}\left(\int_0^1[w_x^2(x,t)+w_t^2(x,t)+z_x^2(x,t)+z_t^2(x,t)+\widehat{d}_x^2(x,t) + \widehat{d}_t^2(x,t)]dx+\int_0^1[v_x^2(x,t)+v_t^2(x,t)]dx+w^2(0,t) + z^2(0,t)+\widehat{d}^2(0,t)+v^2(1,t)\right)<+\infty;$$

(ii) There exist two constants $M,\ \mu>0$ such that

$$\int_0^1\left([v_x(x,t)-w_x(x,t)]^2+[v_t(x,t)-w_t(x,t)]^2\right)dx +[v(0,t)-w(0,t)]^2\le Me^{-\mu t}, \quad \forall t\ge 0;$$

(iii) There exist two constants $M,\ \mu>0$ such that

$$|e(t)|=|w(1,t)-r(t)|\le Me^{-\mu t}, \quad \forall t\ge 0;$$

(iv) When $f\equiv 0$, $d\in L^2(0,\ \infty)$, $r\in H^2(0,\ \infty)$, especially when $d(t)\equiv 0$, $r(t)\equiv 0$,

$$\lim_{t\to\infty}\left(\int_0^1[w_x^2(x,t)+w_t^2(x,t)+z_x^2(x,t)+z_t^2(x,t)+\widehat{d}_x^2(x,t) +\widehat{d}_t^2(x,t)]dx+\int_0^1[v_x^2(x,t)+v_t^2(x,t)]dx+w^2(0,t)+z^2(0,t) +\widehat{d}^2(0,t)+v^2(1,t)\right)=0.$$

Proof. Let $\widehat{z}(x,t)=z(x,t)-w(x,t)$, $\widetilde{d}(x,t)=\widehat{d}(x,t)-z(x,t)+w(x,t)$, $\varepsilon(x,t)=w(x,t)-v(x,t)$, $\widehat{v}(x,t)=v(x,t)-x^2r(t)$, and $p(x,t)=\widehat{v}(x,t)-w(x,t)$. It is easy to verify that $(\widehat{z}(x,t),\widetilde{d}(x,t),\varepsilon(x,t))$ satisfies

$$\begin{cases} \widehat{z}_{tt}(x,t)=\widehat{z}_{xx}(x,t), \quad x\in(0,1), & t>0,\\ \widehat{z}_x(0,t)=c_1\widehat{z}(0,t)+q\widehat{z}_t(0,t)-f(w(x,t))-d(t), & t\ge 0,\\ \widehat{z}_x(1,t)=0, & t\ge 0,\\ \widetilde{d}_{tt}(x,t)=\widetilde{d}_{xx}(x,t), \quad x\in(0,1), & t>0,\\ \widetilde{d}(0,t)=0, & t\ge 0,\\ \widetilde{d}_x(1,t)=-c_2\widetilde{d}_t(1,t), & t\ge 0,\\ \varepsilon_{tt}(x,t)=\varepsilon_{xx}(x,t), \quad x\in(0,1), & t>0,\\ \varepsilon_x(0,t)=q\varepsilon_t(0,t)+\widetilde{d}_x(0,t), & t\ge 0,\\ \varepsilon_x(1,t)=-c_3\varepsilon(1,t), & t\ge 0, \end{cases} \tag{3.37}$$

and $(p(x,t),\widehat{v}(x,t))$ is governed by

$$\begin{cases} p_{tt}(x,t)=p_{xx}(x,t)-x^2\ddot{r}(t)+2r(t), \quad x\in(0,1), & t>0,\\ p_x(0,t)=qp_t(0,t)-\widetilde{d}_x(0,t), & t\ge 0,\\ p_x(1,t)=-c_3p(1,t)-(c_3+2)r(t), & t\ge 0,\\ \widehat{v}_{tt}(x,t)=\widehat{v}_{xx}(x,t)-x^2\ddot{r}(t)+2r(t), \quad x\in(0,1), & t>0,\\ \widehat{v}_x(0,t)=q\widehat{v}_t(0,t)-\widetilde{d}_x(0,t)+f(\widehat{v}(\cdot,t)-p(\cdot,t))+d(t), & t\ge 0,\\ \widehat{v}(1,t)=0, & t\ge 0. \end{cases} \tag{3.38}$$

Note that $c_2=1$. By the compatible boundary condition (3.36), (2.7), Lemma 2.2, Theorem 3.1 and Remark 3.1, the "$(\widetilde{d},\varepsilon)$-part" of (3.37) admits a unique solution such that

$$\int_0^1[(\widetilde{d}_x(x,t))^2+(\widetilde{d}_t(x,t))^2+\varepsilon_x^2(x,t)+\varepsilon_t^2(x,t)]dx +\varepsilon^2(1,t)\le M_0e^{-\mu_0 t}, \quad \forall\ t\ge 0, \tag{3.39}$$

with some M_0, $\mu_1>0$. Moreover, By Lemma 2.2, $\widetilde{d}_x(0,t)\in L^2(0,\infty)$. Next, we rewrite "$p$-part" and "$\widehat{v}$-part" of (3.38) into operator form of the following:

$$\frac{d}{dt}(p(\cdot,t),p_t(\cdot,t))^\top=A_\varepsilon(p(\cdot,t),p_t(\cdot,t))^\top+\mathbb{B}_1(-\widetilde{d}_x(0,t)) +(c_3+2)\mathbb{B}_2(-r(t))+g(t), \tag{3.40}$$

where A_ε is given by (3.6), $\mathbb{B}_1$ and $\mathbb{B}_2$ are given by (1.4), and $g(t)=(0,-x^2\ddot{r}(t)+2r(t))$.

$$\frac{d}{dt}(\widehat{v}(\cdot,t),\widehat{v}_t(\cdot,t))^\top=A_v(\widehat{v}(\cdot,t),\widehat{v}_t(\cdot,t))^\top+\mathbb{B}_1(f(\widehat{v}(\cdot,t)-p(\cdot,t))) +d(t)-\widetilde{d}_x(0,t))+g(t), \tag{3.41}$$

where A_v is given by (3.20). Similar to the proof of Lemma 3.1, we can prove that system (3.40) admits a unique solution $(p,p_t)\in C(0,\infty;\mathbb{H})$ and

$$\sup_{t\ge 0}\|(p(\cdot,t),p_t(\cdot,t))^\top\|_{\mathbb{H}}<+\infty. \tag{3.42}$$

Moreover, similar to the proof of Proposition 1.1, by the assumption on $f(\cdot)$, we know that system (3.41) admits a unique solution $(\widehat{v},\widehat{v}_t)\in C(0,\infty;\mathbb{H})$. By $w(x,t)=\widehat{v}(x,t)-p(x,t)$, it follows that the "$w$-part" of (3.35) has a unique solution $(w,w_t)\in C(0,\infty;\mathbb{H})$, which, together with Lemma 2.1, implies that the "$\widehat{z}$-part" of (3.35) has a unique solution $(\widehat{z},\widehat{z}_t)\in C(0,\infty;\mathbb{H})$ and satisfies

$$\sup_{t\ge 0}\|(\widehat{z}(\cdot,t),\widehat{z}_t(\cdot,t))^\top\|_{\mathbb{H}}<+\infty. \tag{3.43}$$

Since $v(x,t)=\widehat{v}(x,t)+x^2r(t)$, $z(x,t)=\widehat{z}(x,t)+\varepsilon(x,t)+v(x,t)$, and $\widehat{d}(x,t)=\widetilde{d}(x,t)+\widehat{z}(x,t)$, we know that system (3.35) is well-posed. Obviously, (ii) and (iii) follow from the second equation

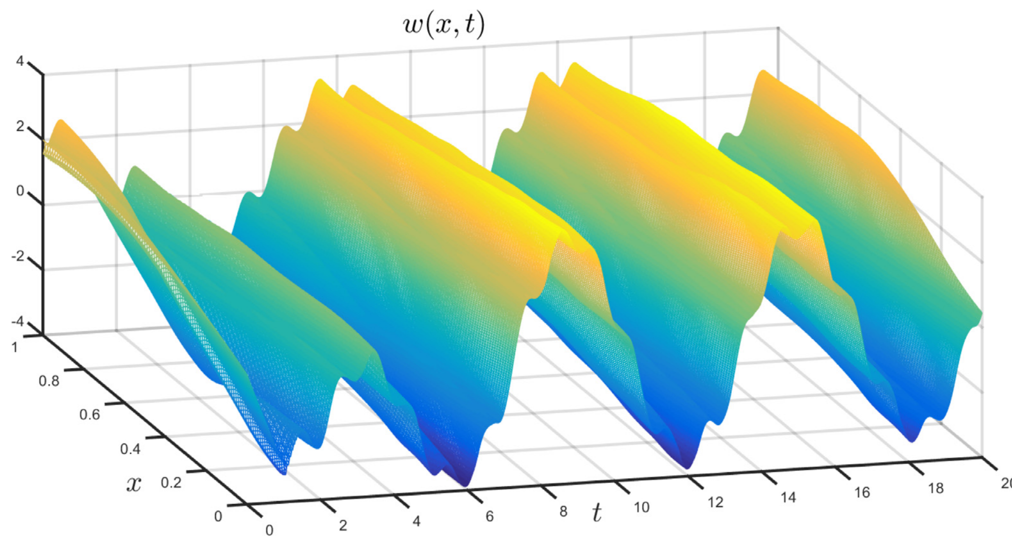

Fig. 1. The state $w(x,t)$.

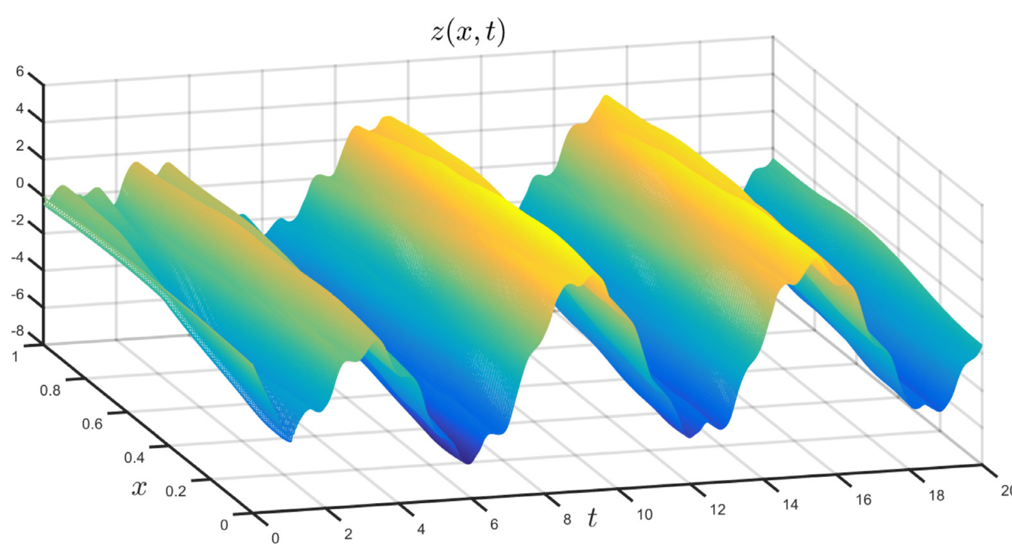

Fig. 2. The state $z(x, t)$.

of (3.39). Since $r \in W^{2,\infty}(0,\infty)$, by (3.39), (3.42), and (3.43), we can conclude (i) and from Theorem 3.1, (iv) is concluded as well. □

Remark 3.2. In Theorem 3.2, we have supposed $c_2 = 1$. When $c_2 > 0(\neq 1)$, by virtue of Lemmas 1.1 and 2.2, it is easy to verify that the expressions in (ii) and (iii) are asymptotically convergent to zero. By Theorem 3.2, the reference model (3.1) can be regarded as a state observer of (1.1). Alteratively, based on estimation of the total disturbance, we can also design the following state observer of (1.1):

$$\begin{cases} \widehat{w}_{tt}(x,t) = \widehat{w}_{xx}(x,t), \quad x \in (0,1), & t > 0, \\ \widehat{w}_x(0,t) = q\widehat{w}_t(0,t) + c_1[\widehat{w}(0,t) - w(0,t)] & t \geq 0, \\ \qquad - \widehat{d}_x(0,t) + c_1\widehat{d}(0,t) + q\widehat{d}_t(0,t), & \\ \widehat{w}_x(1,t) = u(t), & t \geq 0, \\ \widehat{w}(x,0) = \widehat{w}_0(x), \quad \widehat{w}_t(x,0) = \widehat{w}_1(x), & x \in [0,1]. \end{cases} \tag{3.44}$$

Indeed, let $\widetilde{w}(x,t) = \widehat{w}(x,t) - w(x,t)$. A direct computation shows that $\widetilde{w}(x,t)$ satisfies

$$\begin{cases} \widetilde{w}_{tt}(x,t) = \widetilde{w}_{xx}(x,t), \quad x \in (0,1), & t > 0, \\ \widetilde{w}_x(0,t) = q\widetilde{w}_t(0,t) + c_1\widetilde{w}(0,t) - \widetilde{d}_x(0,t), & t \geq 0, \\ \widetilde{w}_x(1,t) = 0, & t \geq 0, \\ \widetilde{w}(x,0) = \widehat{w}_0(x) - w_0(x), \ \ \widetilde{w}_t(x,0) = \widehat{w}_1(x) - w_1(0), & x \in [0,1]. \end{cases} \tag{3.45}$$

Similarly to the proof in Theorem 3.1, and by Lemmas 6.1 and 1.1, we can obtain

$$\int_0^1 [\widetilde{w}_x^2(x,t) + \widetilde{w}_t^2(x,t)]dx + \widetilde{w}^2(0,t) \to 0 \text{ as } t \to \infty.$$

Thus, (3.44) is a state observer of (1.1).

4. Numerical simulation

In this section, we present some numerical simulations for system (3.35) for illustration. For numerical computations, we

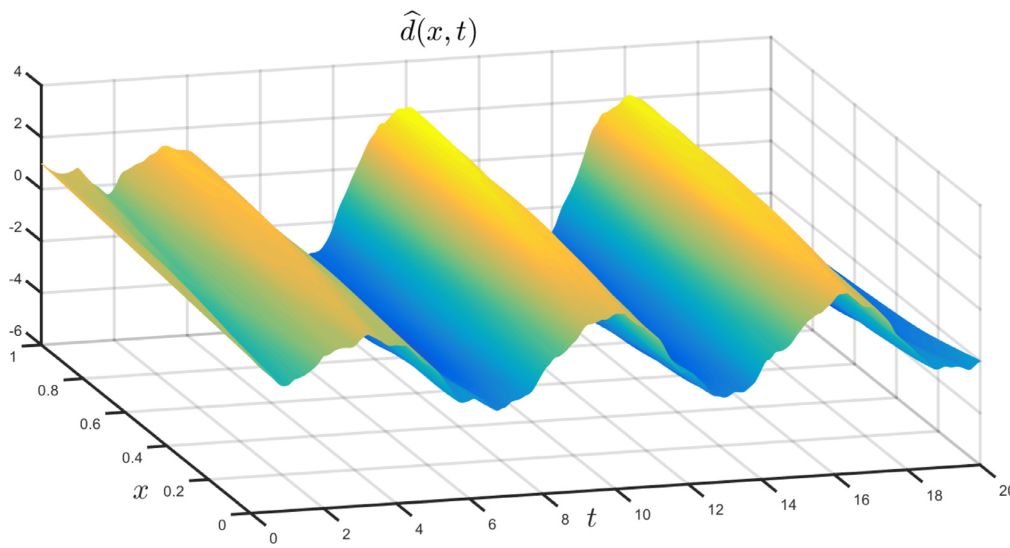

Fig. 3. The state $\widehat{d}(x, t)$.

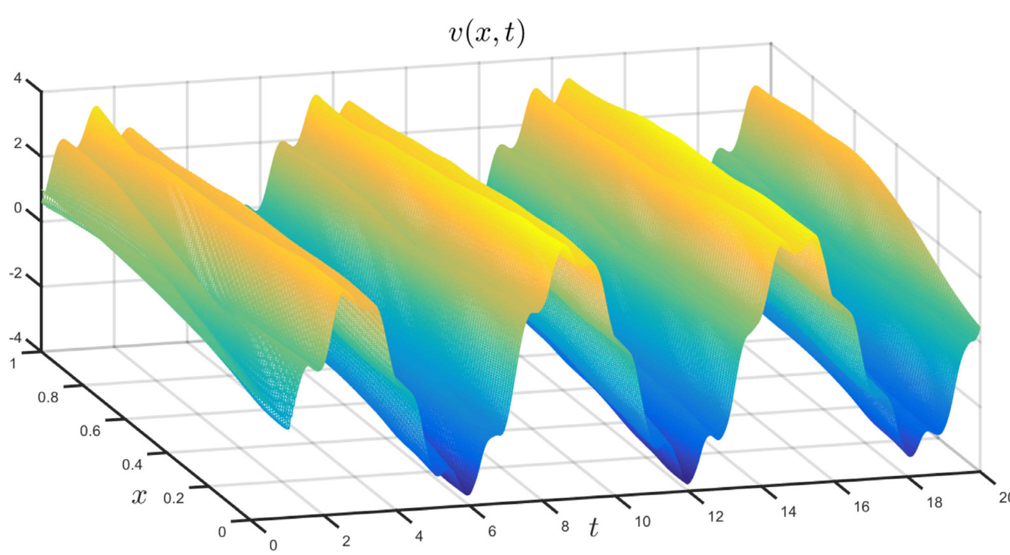

Fig. 4. The state $v(x, t)$.

choose reference signal $r(t) = 2\sin(t) - 0.45\cos(2\pi t) + 1$, the internal nonlinear uncertainty $f(w(\cdot, t)) = \sin(w^2(0, t))$ and the external disturbance $d(t) = 2\sin(t) + 0.7\cos(2\pi t) - 1$. The other parameters are taken as $q = 1.1,\ c_1 = 1,\ c_2 = 1.1,\ c_3 = 1$. The initial values are taken as:

$$\begin{aligned} & w(x,0) = 2x - x^2, \quad w_t(x,0) = -2x + x^2, \quad z(x,0) = x - x^2, \\ & z_t(x,0) = -x + x^2, \\ & \widehat{d}(x,0) = x, \quad \widehat{d}_t(x,0) = -x, \quad v(x,0) = 2x - x^2 - 0.45, \\ & v_t(x,0) = 2x - x^2. \end{aligned} \tag{4.1}$$

It is clear that the above initial value satisfies the compatible condition (3.36). The backward Euler method in time and the Chebvshev spectral method for space variable are used to discretize system (3.35). Here, we take the grid size $N = 20$ for x and the time step $dt = 5 \times 0.001$. The solution of system is plotted in Figs. 1–4. Fig. 5 shows that the reference model (3.1) can be regarded as a state observer of (1.1). Fig. 6 shows that the total disturbance $F(w, t) = \sin(w^2(0, t)) + 2\sin(t) + 0.7\cos(2\pi t) - 1$ and its estimate $-\widehat{d}_x(0, t) + c_1\widehat{d}(0, t) + q\widehat{d}_t(0, t)$. It is seen that the disturbance is estimated effectively. The convergence is very fast and smoothly. Fig. 7 shows that $w(1, t)$ tracks asymptotically the reference signal $r(t)$. Fig. 8 displays the feedback control in time.

5. Concluding remarks

In this paper, we present a new infinite-dimensional disturbance estimator to estimate unknown nonlinear internal uncertainty and external disturbance for a one-dimensional wave equation. An servomechanism is designed by the measured output and the reference signal where the estimation mechanism of unknown total disturbance is presented. Five control objectives are achieved: (a) The performance output tracks exponentially the reference signal; (b) All the internal-loops are bounded; (c) The system state is recovered from input and output; d) The unknown total disturbance can be estimated in the sense that the estimation error belongs to $L^2(0, \infty)$; (e) When the disturbance and reference signal belong to $L^2(0, \infty)$ and $H^2(0, \infty)$, respectively, the closed-loop is asymptotically stable. The last point shows particularly that when the disturbance and reference are disconnected to the system, the closed-loop is asymptotically stable, that is, the system is internally asymptotically stable. This paper is a generalization of a recent work [24] where only boundary external disturbance was considered. Here we use a different estimator compared with [24] to deal with the nonlinearity for which the treatment for output tracking is different to stabilization like a recent paper [8].

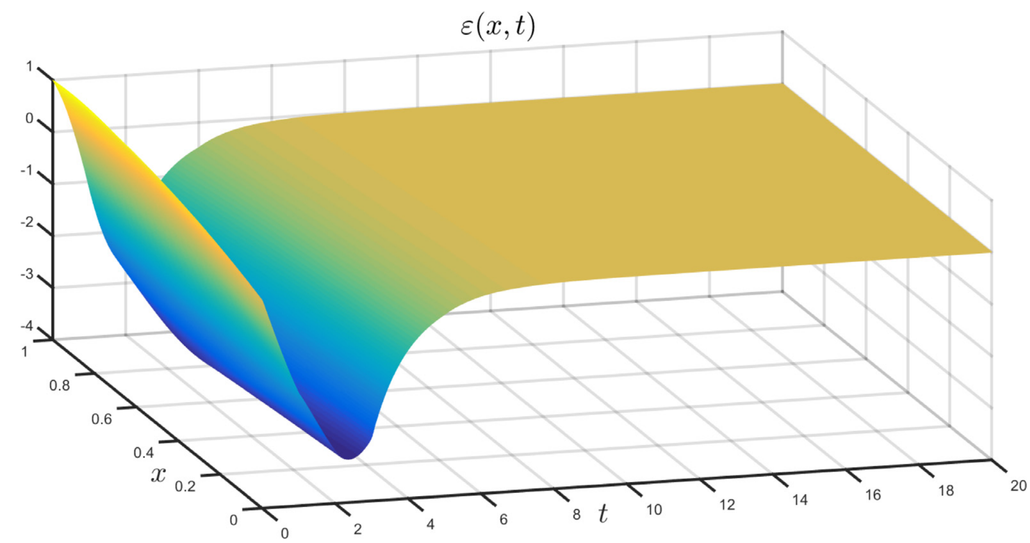

Fig. 5. The error $\varepsilon(x,t) = w(x,t) - v(x,t)$.

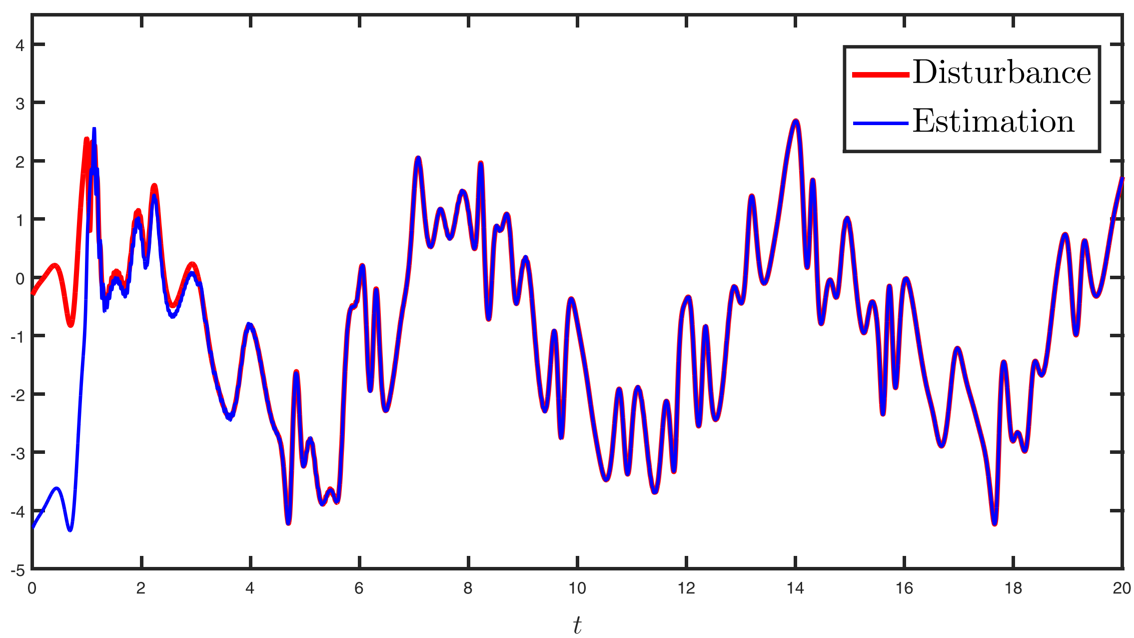

Fig. 6. The total disturbance and its estimation.

Appendix

In this appendix, we present several lemmas concerning the operators being able to generate C_0-groups on the state space. These results are crucial to establishment of the main results of the present paper.

Let $\mathbb{H} := H^1(0,1) \times L^2(0,1)$ with the inner product given by (1.3). Define the operator $\mathbf{A}_1 : D(\mathbf{A}_1)(\subset \mathbb{H}) \to \mathbb{H}$ as follows:

$$\begin{cases} \mathbf{A_1}(\phi,\psi)^\top = (\psi,\phi'')^\top, \quad \forall\ (\phi,\psi)^\top \in \mathbf{D}(\mathbf{A_1}), \\ D(\mathbf{A}_1) = \Big\{(\phi,\psi)^\top \in \mathbb{H} \cap (H^2(0,1)\times H^1(0,1)) \\ \qquad : \phi'(0) = c\psi(0) + \phi(0), \quad \phi'(1) = 0\Big\}. \end{cases} \tag{6.1}$$

Lemma 6.1. *For any $c \in \mathbb{R}$ and $c \neq \pm 1$, $\mathbf{A}_1$ generates a C_0-group on $\mathbb{H}$. Moreover, if $c > 0$ and $c \neq 1$, then $\mathbf{A}_1$ generates an exponentially stable C_0-semigroup on $\mathbb{H}$.*

Proof. We first show that $\mathbf{A}_1$ generates a C_0-group on $\mathbb{H}$. A direct computation shows that

$$\mathbf{A}_1^{-1}(\phi,\psi)^\top = \Big(c\phi(0) + (1+x)\int_0^1 \psi(x)dx - \int_0^x (x-\xi)\psi(\xi)d\xi, -\phi(x)\Big)^\top, \tag{6.2}$$

By the Sobolev embedding theorem, $\mathbf{A}_1^{-1}$ is compact on $\mathbb{H}$, and thus $\sigma(\mathbf{A}_1)$ consists of eigenvalues of $\mathbf{A}_1$ only. It is easily seen that $\lambda \in \sigma(\mathbf{A}_1)$ if and only if there exists $\phi(x) \neq 0$ satisfying

$$\begin{cases} \phi''(x) = \lambda^2\phi(x), \\ \phi'(0) = \lambda c\phi(0) + \phi(0), \quad \phi'(1) = 0, \end{cases} \tag{6.3}$$

and the associated eigenfunction is $(\lambda^{-1}\phi(x), \phi(x))$. Solving (6.3) gives

$$\phi(x) = \cosh(\lambda(x-1)) \text{ with } \lambda \text{ satisfying } e^{2\lambda} = \frac{1-c-\frac{1}{\lambda}}{1+c+\frac{1}{\lambda}}. \tag{6.4}$$

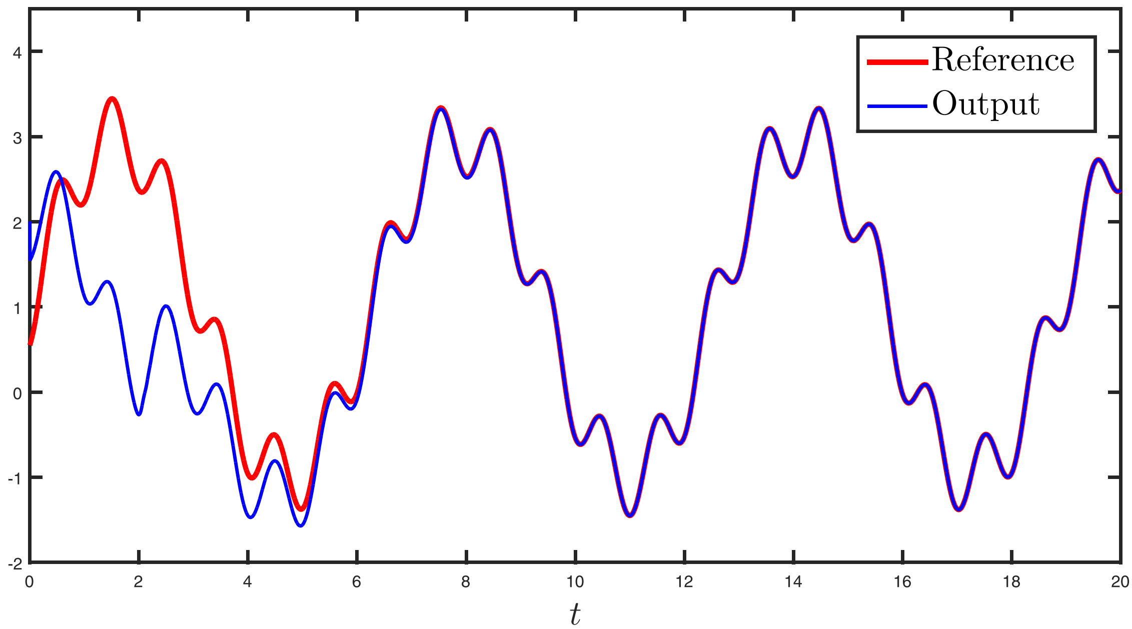

Fig. 7. The reference signal $r(t)$ and the output $w(1, t)$.

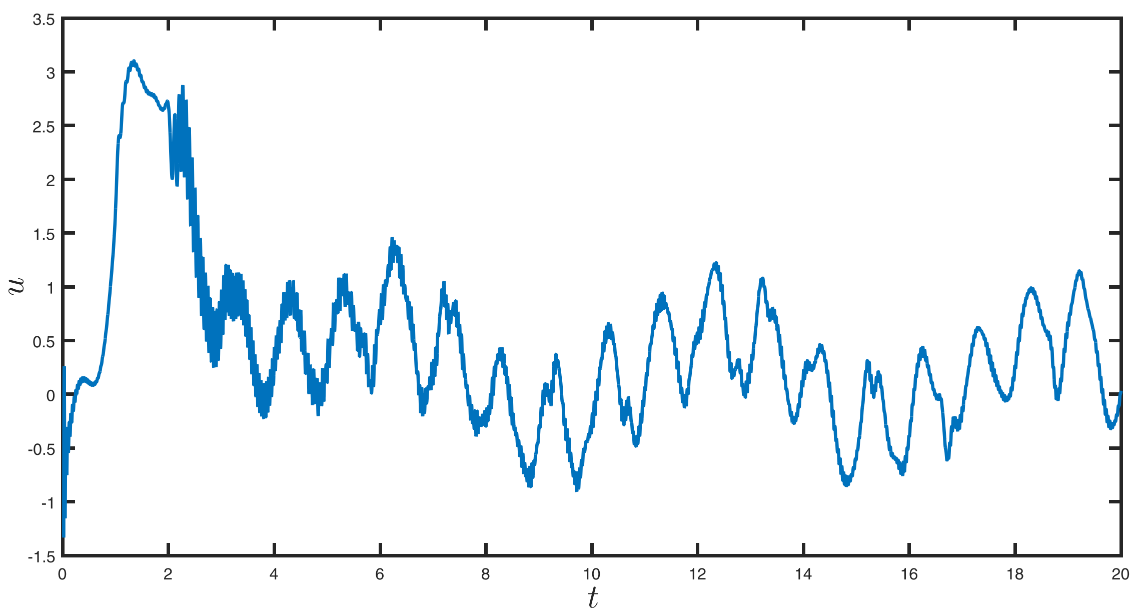

Fig. 8. The control law $u(t)$.

Furthermore, the eigen-pairs are found to have the following asymptotic expression:

$$(\mu_n^{-1}\phi_n, \phi_n) = (\mu_n^{-1}\cosh(\mu_n(x-1)), \quad \cosh(\mu_n(x-1))) + \mathcal{O}(n^{-1}), \quad \lambda_n = \mu_n + \mathcal{O}(n^{-1}),$$

where μ_n is given by

$$\mu_n = \frac{1}{2}\ln\left|\frac{1-c}{1+c}\right| + j\pi\begin{cases} n, & |c|<1, \\ n+\frac{1}{2}, & |c|>1 \end{cases}$$

where $j^2 = -1$. By the method in [23, Section 4], one can show that $\{(\mu_n^{-1}\phi_n, \phi_n)^\top : n = 0, 1, 2, \ldots\}$ forms a Riesz basis for $\mathbb{H}$. Thus, $\mathbf{A}_1$ with $D(\mathbf{A}_1)$ generates a C_0-semigroup on $\mathbb{H}$. Noticing that the eigenvalues and corresponding eigenfunctions of $-\mathbf{A}_1$ are just $\{-\lambda_n : n = 0, 1, 2, \ldots\}$ and $(\mu_n^{-1}\phi_n, \phi_n)^\top$. So, $-\mathbf{A}_1$ with $D(-\mathbf{A}_1) = D(\mathbf{A}_1)$ is also a generator of a C_0-semigroup. It follows from [7, Page 79] that $\mathbf{A}_1$ generates a C_0-group on $\mathbb{H}$.

Next, we show that if $c > 0$ satisfying $c \neq 1$, $\mathbf{A}_1$ generates an exponentially stable C_0-semigroup on $\mathbb{H}$. Since the eigenfunctions of $\mathbf{A}_1$ form a Riesz basis for $\mathbb{H}$, the spectrum-determined growth condition holds. In order to show that $e^{\mathbf{A}_1 t}$ is an exponentially stable semigroup, it suffices to prove that $\text{Re}\lambda < 0$ for any $\lambda \in \sigma(\mathbf{A}_1)$. Actually, a simple computation gives, for any $(\phi, \psi)^\top \in \mathbb{H}$, that

$$\text{Re}\langle \mathbf{A}_1(\phi, \psi)^\top, (\phi, \psi)^\top\rangle_{\mathbb{H}} = -c|\psi(0)|^2 \leq 0, \tag{6.5}$$

which implies that for any $\lambda \in \sigma(\mathbf{A}_1)$ must satisfy $\text{Re}\lambda \leq 0$. Since $\mathbf{A}_1^{-1}$ is compact, we only need to show that there is no eigenvalue on the imaginary axis. Let $\lambda = j\tau^2 \in \sigma(\mathbf{A}_1)$ with $\tau \in \mathbb{R}^+$ and the corresponding eigenfunction $(\phi, \psi)^\top \in D(\mathbf{A}_1)$, by (6.5), we have

$$\text{Re}\langle \mathbf{A}_1(\phi, \psi)^\top, (\phi, \psi)^\top\rangle_{\mathbb{H}} = \text{Re}\langle j\tau^2(\phi, \psi)^\top, (\phi, \psi)^\top\rangle_{\mathbb{H}} = -c|\psi(0)|^2 = 0, \tag{6.6}$$

and hence $\psi(0)=0$. Furthermore, $\mathbf{A}_1(\phi,\psi)^\top = j\tau^2(\phi,\psi)^\top$ gives that $\psi = j\tau^2\phi$ with ϕ satisfying

$$\begin{cases}\phi''(x) = -\tau^4\phi(x),\\ \phi(0)=\phi'(0)=0, \quad \phi'(1)=0,\end{cases} \tag{6.7}$$

It is clear that the above equation admits only zero solution. Thus, there is no eigenvalue on the imaginary axis. □

Let $\mathbb{H} := H^1(0,1)\times L^2(0,1)$ with the inner product given by

$$\langle(\phi_1,\psi_1)^\top,(\phi_2,\psi_2)^\top\rangle = \int_0^1[\phi_1'(x)\overline{\phi_2'(x)} + \psi_1(x)\overline{\psi_2(x)}]dx + c_2\phi_1(1)\overline{\phi_2(1)}, \quad \forall\ (\phi_i,\psi_i)^\top\in\mathbb{H}, \quad i=1,2.$$

Define the operator $\mathbf{A}_2 : D(\mathbf{A}_2)(\subset\mathbb{H})\to\mathbb{H}$ as follows:

$$\begin{cases}\mathbf{A_2}(\phi,\psi)^\top = (\psi,\phi'')^\top, \quad \forall(\phi,\psi)^\top\in\mathbf{D(A_2)},\\ D(\mathbf{A}_2) = \Big\{(\phi,\psi)^\top\in\mathbb{H}\cap(H^2(0,1)\times H^1(0,1))\\ \qquad : \phi'(0)=c_1\psi(0), \quad \phi'(1) = -c_2\phi(1)\Big\},\end{cases} \tag{6.8}$$

Lemma 6.2. *For any $c_1, c_2\in\mathbb{R}$ and $c_1\neq\pm1$, $c_2\neq0$, $\mathbf{A}_2$ generates a C_0-group on $\mathbb{H}$. Moreover, if c_1, $c_2>0$ and $c_1\neq1$, $\mathbf{A}_2$ generates an exponentially stable C_0-semigroup on $\mathbb{H}$.*

Proof. We first show that $\mathbf{A}_2$ generates a C_0-group on $\mathbb{H}$. A direct computation shows that

$$\mathbf{A}_2^{-1}(\phi,\psi)^\top = \Bigg(\frac{1}{c_2}(c_1\phi(0)(c_2+1) + \int_0^1\psi(\xi)d\xi + c_2\int_0^1(1-\xi)\psi(\xi)d\xi\Bigg) - c_1\phi(0)x - \int_0^x(x-\xi)\psi(\xi)d\xi, -\phi(x)\Bigg)^\top, \tag{6.9}$$

By the Sobolev embedding theorem, $\mathbf{A}_2^{-1}$ is compact on $\mathbb{H}$, and thus $\sigma(\mathbf{A}_2)$ consists of eigenvalues of $\mathbf{A}_2$ only. Let λ be any eigenvalue of $\sigma(\mathbf{A}_2)$ and the associated eigenfunction is $(\lambda^{-1}\phi,\phi)$. Similar to the proof in Lemma 6.1, we derive

$$\phi(x) = \cosh(\lambda x) + c_1\sinh(\lambda x), \tag{6.10}$$

and λ satisfies

$$e^{2\lambda} = \frac{(\lambda-c_2)(1-c_1)}{(\lambda+c_2)(1+c_1)}. \tag{6.11}$$

Further, we obtain the eigenfunctions and eigenvalues with the following asymptotic expression:

$$\begin{cases}(\mu_n^{-1}\phi_n,\phi_n) = (\mu_n^{-1}[\cosh(\mu_n x)+c_1\sinh(\mu_n x)],\\ \qquad \cosh(\mu_n x)+c_1\sinh(\mu_n x)) + \mathcal{O}(n^{-1}),\\ \lambda_n = \mu_n + \mathcal{O}(n^{-1}),\end{cases}$$

where μ_n is given by

$$\mu_n = \frac{1}{2}\ln\left|\frac{1-c_1}{1+c_1}\right| + j\pi\begin{cases}n, & |c_1|<1,\\ n+\frac{1}{2}, & |c_1|>1.\end{cases}$$

By the method in [23, Section 4], one can show that $\{(\mu_n^{-1}\phi_n,\phi_n)^\top : n=0,1,2,\ldots\}$ forms a Riesz basis for $\mathbb{H}$. By the asymptotic expression of eigenvalues, $\mathbf{A}_2$ generates a C_0-group on $\mathbb{H}$ as well.

Next, we show that if c_1, $c_2>0$ and $c_1\neq1$, $\mathbf{A}_2$ generates an exponentially stable C_0-semigroup on $\mathbb{H}$. Since the eigenfunctions of $\mathbf{A}_2$ form a Riesz basis for $\mathbb{H}$, the spectrum-determined growth condition holds. In order to show that $e^{\mathbf{A}_2 t}$ is a exponentially stable semigroup, it suffices to prove that $\mathrm{Re}\lambda<0$ for any $\lambda\in\sigma(\mathbf{A}_2)$. Actually, a simple computation gives

$$\mathrm{Re}\langle\mathbf{A}_2(\phi,\psi)^\top,(\phi,\psi)^\top\rangle_{\mathbb{H}} = -c_1|\psi(0)|^2\le0, \tag{6.12}$$

which implies that for any $\lambda\in\sigma(\mathbf{A}_2)$ must satisfy $\mathrm{Re}\lambda\le0$. Since $\mathbf{A}_2^{-1}$ is compact, we only need to show that there is no eigenvalue on the imaginary axis. Let $\lambda = j\tau^2\in\sigma(\mathbf{A}_2)$ with $\tau\in\mathbb{R}^+$ and the corresponding eigenfunction $(\phi,\psi)^\top\in D(\mathbf{A}_2)$. By (6.12),

$$\begin{aligned}\mathrm{Re}\langle\mathbf{A}_2(\phi,\psi)^\top,(\phi,\psi)^\top\rangle_{\mathbb{H}} &= \mathrm{Re}\langle j\tau^2(\phi,\psi)^\top,(\phi,\psi)^\top\rangle_{\mathbb{H}}\\ &= -c_1|\psi(0)|^2 = 0,\end{aligned} \tag{6.13}$$

and hence $\psi(0)=0$. Furthermore, $\mathbf{A}_2(\phi,\psi)^\top = j\tau^2(\phi,\psi)^\top$ gives that $\psi = j\tau^2\phi$ with ϕ satisfying

$$\begin{cases}\phi''(x) = -\tau^4\phi(x),\\ \phi(0)=\phi'(0)=0, \quad \phi'(1) = -c_2\phi(1),\end{cases} \tag{6.14}$$

It is clear that the above equation admits only zero solution. Thus, there is no eigenvalue on the imaginary axis. □

Let $\mathcal{H} := H^1(0,1)\times L^2(0,1)$ with the inter product given by (3.19). Define the operator $\mathbf{A}_3 : D(\mathbf{A}_3)(\subset\mathbb{H})\to\mathcal{H}$ as follows:

$$\begin{cases}\mathbf{A_3}(\phi,\psi)^\top = (\psi,\phi'')^\top, \forall(\phi,\psi)^\top\in\mathbf{D(A_3)},\\ D(\mathbf{A}_3) = \Big\{(\phi,\psi)^\top\in\mathbb{H}\cap(H^2(0,1)\times H^1(0,1))\\ \qquad : \phi'(0)=c\psi(0), \quad \phi(1)=0\Big\},\end{cases} \tag{6.15}$$

Lemma 6.3. *For any $c\in\mathbb{R}$ and $c\neq\pm1$, $\mathbf{A}_3$ generates a C_0-group on $\mathcal{H}$. Moreover, if $c>0$ satisfying $c\neq1$, $\mathbf{A}_2$ generates an exponentially stable C_0-semigroup on $\mathcal{H}$.*

Proof. We first show that $\mathbf{A}_3$ generates a C_0-group on $\mathcal{H}$. A direct computation shows that the eigenvalues λ_n of $\mathbf{A}_3$ is given by

$$\lambda_n = \frac{1}{2}\ln\left|\frac{1-c}{1+c}\right| + j\pi\begin{cases}n+\frac{1}{2}, & |c|<1,\\ n, & |c|>1,\end{cases} \quad n=0,\pm1,\pm2,\ldots,$$

and the corresponding eigenfunctions $(f_n(x),g_n(x))^\top$ of $\mathbf{A}_3$ are given by

$$(f_n,g_n)^\top = \left(\frac{\sinh\lambda_n(x-1)}{\lambda_n}, \sinh\lambda_n(x-1)\right)^\top.$$

It follows from [23, Section 4] that $\{(f_n,g_n)^\top : n=0,1,2,\ldots\}$ forms a Riesz basis for $\mathcal{H}$. Thus, $\mathbf{A}_3$ with $D(\mathbf{A}_3)$ generates a C_0-semigroup on $\mathcal{H}$. By the asymptotic expression of eigenvalues, $\mathbf{A}_3$ generates a C_0-group on $\mathcal{H}$ as well.

Next, we show that if $c>0$, $\mathbf{A}_3$ generates an exponentially stable C_0-semigroup on $\mathcal{H}$. Since the eigenfunctions of $\mathbf{A}_3$ form a Riesz basis for $\mathcal{H}$, the spectrum-determined growth condition holds. Noticing $c>0$ and $c\neq1$, $\mathrm{Re}\lambda<0$ for all $\lambda\in\sigma(\mathbf{A}_3)$, we conclude that $e^{\mathbf{A}_3 t}$ is an exponentially stable C_0-semigroup. □

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