Performance output tracking for one-dimensional wave equation subject to unmatched general disturbance and non-collocated control☆

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ABSTRACT

In this paper, we consider performance output tracking for a boundary controlled one-dimensional wave equation with possibly unknown internal nonlinear uncertainty and external disturbance. We first show that the open-loop system is well-posed and then propose a disturbance estimator. It is shown that the disturbance estimator can estimate successfully the total disturbance that consists of internal uncertainty and external disturbance. An servomechanism based on the estimated total disturbance is then designed. It is shown that the closed-loop system is well-posed. Three control objectives are achieved: (a) the output is tracking the reference signal; (b) all the internal signals are uniformly bounded; (c) the closed-loop system is internally asymptotically stable if both the reference signal and the disturbance vanish or belong to the space $H^2(0, \infty)$ and $L^2(0, \infty)$, respectively. The unmatched performance output tracking control is first time applied to a system described by the partial differential equation for complete general disturbance rejection and reference tracking purpose. Another key feature of this paper is that we do not use the high-gain to estimate total disturbance for unmatched system. The numerical experiments are carried out to illustrate effectiveness of the proposed control law.

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1. Introduction

In this paper, we consider performance output tracking for a one-dimensional wave equation with Neumann boundary control and unknown internal nonlinear uncertainty and external disturbance. The system is governed by the following partial differential equation:

\[
\begin{align*}
W_{tt}(x, t) &= W_{xx}(x, t), & x \in (0, 1), & t > 0, \\
W_{t}(0, t) &= qw_{t}(0, t) + f(w(t)), & t \geq 0, \\
w_{t}(1, t) &= u(t), & t \geq 0, \\
w(x, 0) &= w_{0}(x), & w_{t}(x, 0) = w_{1}(x), & x \in (0, 1), \quad (1.1) \\
y_{m}(t) &= \{w(0, t), w_{t}(1, t)\}, & t \geq 0, \\
y_{o}(t) &= w(1, t), & t \geq 0,
\end{align*}
\]

where we denote by $w'(x, t)$ or $w_{t}(x, t)$ the derivative of $w(x, t)$ with respect to $x$ and by $w^{'}(0, t)$ or $w_{t}(0, t)$ the derivative of $w'(x, t)$ with respect to $t$. The $u(t)$ is the control input, $y_{m}(t)$ the measured output, $y_{o}(t)$ the performance output signal to be regulated, $f : H^{2}(0, 1) \rightarrow \mathbb{R}$ an unknown possibly nonlinear mapping that represents internal uncertainty. Examples include like $f(w) = \sin(w^{2}(0, t))$, $f(w) = \sin(\int_{0}^{t} w(x, t)dx)$. These nonlocal boundary condition arises mainly when the data on the boundary cannot be measured directly [1]. The $d(t)$ is the unknown external disturbance which is only supposed to satisfy $d \in L^\infty(0, \infty)$, and $q > 0, q \neq 1$ is a constant.

For a given reference signal $r(t)$, we are expected to design an output feedback control for uncertain system (1.1) to reject the external disturbance so that

\[
e(t) = y_{o}(t) - r(t) \rightarrow 0 \text{ as } t \rightarrow \infty. \quad (1.2)
\]

System (1.1) is a typical non-collocated Neumann control problem: Control is on the one end and the disturbance is on the other end, which was first investigated in [12] where no internal uncertainty is concerned and the external disturbance is of harmonic disturbance only, i.e.,

\[
d(t) = \sum_{j=1}^{m} (\theta_{j} \sin(\alpha_{j} t) + \theta_{j} \cos(\alpha_{j} t)).
\]

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Except the general external disturbance, another point that is different from [12] is that for asymptotic stability of the closed-loop system, we only assume that the disturbance and reference signal belong to $L^2(0,\infty)$ and $H^2(0,\infty)$, respectively, which does not necessarily vanish at infinity required in [12]. This paper is a non-trivial generalization of the results from [12]. The boundary condition in (1.1) implies that system (1.1) suffers from the external disturbance and the internal uncertainty and system (1.1) is exponentially stable when there are no disturbance and control input involved.

The above problem is a special kind of output regulation problem where the disturbance is not limited to special class. It is well known that the output regulation is one of the fundamental issues in control theory. There are many works dedicated to the output regulation for finite-dimensional systems since from [3,4,9], and the related results have been generalized to the infinite-dimensional systems, see, for instance, [2,5,6,14,17-20], among many others. These results are based on the internal model principle where reference signal and disturbance are generated by exosystem. For finite harmonic signals, this requires the frequencies to be known. A few surveys are contributed to estimate the frequencies [16] where the order of parameter update law is huge as the number of frequencies increase. A first attempt on an infinite-dimensional system is [13] where a general periodic signal that has infinitely many frequencies is considered. However, it does not cover general disturbance considered in this paper. A recent interesting work is [15] where output tracking problem is considered for a general $2 \times 2$ system of first order linear hyperbolic PDEs but no uncertainty and disturbance are taken into consideration.

To the best of our knowledge, the general disturbance signals are not considered in output tracking problems for PDEs. In this paper, we solve the performance output tracking problem with general disturbance by designing a new disturbance estimator that can estimate the total disturbance yet does not use high-gain in estimation. For external disturbance estimation/cancellation only, we refer to [10,31] via active disturbance rejection control. We consider system (1.1) in the state Hilbert space $\mathbb{H} = H^1(0,1) \times L^2(0,1)$ with the inner product given by

$$\langle (\phi_1, \psi_1) \rangle, (\phi_2, \psi_2) \rangle = \int_0^1 \phi_1(x)\overline{\psi_2(x)} + \psi_1(x)\overline{\phi_2(x)} \, dx$$

(1.3)

Define the operators $A : D(A) \subset \mathbb{H} \rightarrow \mathbb{H}$ and $B_1, B_2 : \mathbb{R} \rightarrow \mathbb{H}$ as

$$\langle A(\phi, \psi) \rangle = \langle \psi, \phi \rangle, \quad \forall \phi, \psi \in \mathbb{H},$$

and

$$D(A) = \{ (\phi, \psi) \in H^2(0,1) \times H^1(0,1), \phi^{(0)}(0) = 0, \phi^{(1)}(1) = 0 \}$$

(1.4)

$$D(B_1) = D(B_2) = \mathbb{R},$$

where $\delta_0$ is the Dirac distribution which satisfies $\delta_0(\phi) = \phi(0)$ for all $\phi \in H^2(0,1)$. It is readily found that

$$\langle A(\phi, \psi), \phi' \rangle = -\langle \psi, \phi' \rangle, \quad \forall \phi, \psi \in D(A^*),$$

$$D(A^*) = \{ (\phi, \psi) \in H^2(0,1) \times H^1(0,1), \phi^{(0)}(0) = 0, \phi^{(1)}(1) = 0 \}$$

(1.5)

The system (1.1) can be rewritten as

$$\begin{align*}
\frac{d}{dt} \begin{pmatrix} w(t) \\ \dot{w}(t) \end{pmatrix} &= A \begin{pmatrix} w(t) \\ \dot{w}(t) \end{pmatrix} + B_1(f(w(t)) - w(0,t) + d(t)) + B_2 u(t).
\end{align*}$$

Before stating Proposition 1.1, we introduce some terminology. Let $X$ and $U$ be Hilbert spaces. Suppose that $A$ generates a $C_0$-semigroup $e^{At}$ on $X$. Let $X_{-1}$ be the completion of $X$ with respect to the norm $\|x\|_{-1} = \| (\beta I - A)^{-1} x \|$, where $\beta$ is some element in the resolvent set $\rho(A)$. An operator $B \in L(L(U,X_{-1}))$ is an admissible control operator to $e^{At}$ if for some (hence for any) $\tau > 0$ and for every $u \in L^2([0,\infty); U), \Phi_T u = \int_0^T e^{A(t-\tau)} B_u(s) \, ds \in X$ (the integral is computed in $X_{-1}$). Then, this integral gives the strong solution of $z(t) = Az(t) + B_u(t)$ in the space $X$, corresponding to $z(0) = 0$, evaluated at the time $\tau$. $B$ is called bounded if $B \in L(L(U,X))$, and unbounded otherwise. If $B$ is admissible and $\alpha > 0$, then there exists $M_\alpha > 0$ such that

$$\left\| (sI - A)^{-1} B \right\|_{L(X_{-1})} \leq \frac{M_\alpha}{\sqrt{\Re s - \alpha}}$$

for $s > \alpha$.

For more admissibility and its properties, we refer the reader to [21, Chapter 4].

Proposition 1.1. The operator $A$ defined by (1.4) generates a $C_0$-group $e^{At}$ on $\mathbb{H}$ and $B_1$ and $B_2$ are admissible to $e^{At}$. Suppose that $f : H^1(0,1) \rightarrow \mathbb{R}$ is continuous and satisfies local Lipschitz condition in $H^1(0,1)$. Then, for any $(w_0, w_1) \in \mathbb{H}$, $u \in L^2_{loc}(0,\infty)$, and $d \in L^2_{loc}(0,\infty)$, there exists a unique local solution to (1.1) such that $(w(t), \dot{w}(t)) \in C(0,T; \mathbb{H})$ for some $T > 0$ and for $t \in [0,T)$,

$$w(t) = e^{At} w_0 + \int_0^t e^{A(t-s)} B_1 f(w(s)) - w(0,t) + d(s) \, ds + \int_0^t e^{A(t-s)} B_2 u(s) \, ds.$$

(1.7)

Moreover, if $f : H^1(0,1) \rightarrow \mathbb{R}$ satisfies the uniform Lipschitz condition

$$|f(w_1) - f(w_2)| \leq L \|w_1 - w_2\|_{H^1(0,1)}, \quad \forall w_1, w_2 \in H^1(0,1),$$

for some $L > 0$ or if $f : H^1(0,1) \rightarrow \mathbb{R}$ is bounded. Then, there exists a unique global solution $(w, \dot{w}) \in C(0,\infty; \mathbb{H})$ to (1.1) satisfying (1.7) with $T = +\infty$.

Proof. By Lemma 6.1 in Appendix, $A$ generates a $C_0$-group $e^{At}$ on $\mathbb{H}$. We now show that $B_1$ and $B_2$ are admissible to $e^{At}$. By [21, Theorem 4.4.3], it suffices to show that $B_1^* A_1^{-1}$ and $B_2^* A_2^{-1}$ are bounded from $\mathbb{H}$ to $\mathbb{C}$ and $B_2^* A_2^{-1}$ for every $T > 0$, there exists $M_T > 0$ depending on $T$ only such that the system of the following:

$$\begin{align*}
\dot{w}_n(x, t) &= w_n(x, t), \quad x \in (0,1), \quad t > 0, \\
\dot{w}_n(0, t) &= -\omega_n(x, t) + w_n(0, t), \quad t > 0, \\
\dot{w}_n(1, t) &= 0, \quad t > 0, \\
\dot{w}(x, 0) &= w_0(x), \quad w_0(x) = w_0(x), \quad x \in (0,1), \\
y_w &= (w_0(w_0, t), w_1(1, t), t \geq 0, \\
\int_0^T [\|w_n(t, 0)\|^2 + (\dot{w}_n(1, t))^2] \, dt \leq M_T \|w_0, \dot{w}_n(0, t)\|_{\mathbb{H}}^2.
\end{align*}$$

(1.8)

By Lemma 6.1 in Appendix, it is easy to see that (1.8) admits a unique solution $(w^n(t), \dot{w}(t), \dot{w}(t)) \in C(0,\infty; \mathbb{H})$ and there exist two constants $M_T > 0$ and $\omega_n \in \mathbb{R}$ such that for all $T \geq 0$,

$$\int_0^T [\|w_n(t, 0)\|^2 + (\dot{w}(t))^2] \, dx + (\dot{w}(0, t))^2 \leq M_T e^{\omega_n T} \|w_0, \dot{w}(0, t)\|_{\mathbb{H}}^2.$$

(1.9)

Let

$$\rho_u(t) = \int_0^1 (2x - 1) w_n(x, t) w_t(x, t) \, dx.$$

(1.10)
Then
$$|\varrho(t)| \leq \frac{1}{\Lambda_1} \int_0^t \left[ (w^*_2(x,t))^2 + (w^*_1(x,t))^2 \right] dx.$$  
(1.12)
Finding the derivative of $\varrho(t)$ along the solution of (1.8), we obtain
$$\dot{\varrho}(t) = (w^*_2(0,t))^2 + (w^*_1(1,t))^2 + (w^*_2(0,t))^2 + (w^*_2(1,t))^2$$
$$- \int_0^t \left[ (w^*_2(x,t))^2 + (w^*_1(x,t))^2 \right] dx,$$
(1.13)
which, together with (1.10) and (1.12), gives
$$\int_0^T \left[ (w^*_2(0,t))^2 + (w^*_1(1,t))^2 \right] dt \leq \varrho(t) - \rho_0(0)$$
$$+ \int_0^T \int_0^1 \left[ (w^*_2(x,t))^2 + (w^*_1(x,t))^2 \right] dx$$
$$\leq \left( \frac{1}{\Lambda_1} \right) \left( M_{e^x} \| w_0, w_1 \|^2_{L^2(\mathbb{R}, \mathbb{R})} \right).$$
(1.14)
A direct computation shows that
$$\left( A^{-1} \phi, \psi \right) = \left( \int_0^1 \int_0^1 \phi \psi(x) dx \right)$$
$$- \int_0^1 \left( \int_0^1 (x - \xi) \phi(\xi) \psi(x) dx \right)$$
$$\leq \left( B^*_1 A^{-1} \phi, \psi \right) = \left( \int_0^1 \phi \right),$$
$$\left( B^*_1 A^{-1} \phi, \psi \right) = \left( \int_0^1 \phi(\xi) \right).$$
(1.15)
By the Sobolev embedding theorem, $H^1(0, 1) \subset C(0, 1)$. By $\phi \in H^1(0, 1)$ and (1.15), we know that $\| A^{-1} \phi \|$ is bounded from $\mathbb{R}$ to $\mathbb{R}$, which, together with (1.14), implies that $\mathbb{B}_1$ and $\mathbb{B}_2$ are admissible to $e^A t$. Therefore, for any fixed $T > 0$, and for any given $u, v \in L^2(0, \infty)$, we have
$$\int_0^t e^{e^{-A} t} B_1 d(s) ds \in C(0, T, \mathbb{R}),$$
$$\int_0^t e^{e^{-A} t} B_2 u(s) ds \in C(0, T, \mathbb{R}).$$
(1.16)
Denote by $F(w(t), v(t)) = \{ f(w(t), v(t)) \}$ for any initial value $(w_0, w_1)^T \in \mathbb{R}$, satisfies the local Lipschitz condition, so does $F$. For any initial value $(w_0, w_1)^T \in \mathbb{R}$, let $(\eta_1(t), \eta_2(t))^T = e^{A t} (w_0(t), w_1(t))^T$. For any given $\sigma > \max_{0 \leq t \leq 1} \| \eta_1(t) \|^2_{L^2(0, 1)} > 0$ and $t \in [0, 1]$, define a set $A_1$ given by
$$A_1 = \{ z : \mathbb{C} \times \mathbb{R} \times \mathbb{R} \}, |z - \eta_1(t)| \leq \sigma \}.$$
(1.17)
The admissibility of $\mathbb{B}_1$ implies that for all $t > 0$,
$$\left\| e^{A(t-\sigma)} B_1 \xi \right\|_{L^2(\mathbb{R}, \mathbb{R})} \leq C_\sigma \| \xi \|_{L^2(\mathbb{R}, \mathbb{R})} \| \xi \|_{L^2(\mathbb{R}, \mathbb{R})},$$
(1.18)
for some constant $C_\sigma$ which is independent of $\xi$. By [22, Proposition 2.3], we know that $\xi(t)$ is nondecreasing with respect to $t$. Let $\sigma < 1$. Then $C_\sigma \leq C_1$.

Let $\varphi(t) = (\varphi_1(t), \varphi_2(t), \varphi_3(t))^T \in C(0, T; \mathbb{R})$ such that
$$\varphi(t) = e^{A(t-\sigma)} B_1 d(s) ds$$
$$= e^{A(t-\sigma)} B_2 u(s) ds$$
and
$$\left\| e^{A(t-\sigma)} B_1 \xi \right\|_{L^2(\mathbb{R}, \mathbb{R})} \leq C_\sigma \| \xi \|_{L^2(\mathbb{R}, \mathbb{R})} \| \xi \|_{L^2(\mathbb{R}, \mathbb{R})}.$$
which, by using Gronwall’s inequality, yields
\[
\left\| \int_0^t e^{\Lambda(t-s)} Bu(s) \, ds \right\|_X \leq \left( \int_0^t e^{\Lambda(t-s)} \|u(s)\|_X \right)_{L^1} \leq L_1 \left( \|u\|_{L^2(0,\infty;U)} \right),
\]
that is, \((w, \dot{w})^T\) is uniformly bounded on \(H\) over \([0, T]\). Thus, if \(T < \infty\), similar to the proof of the existence of local solution, we can prove that \((1.1)\) has a unique solution on \([0, T + \sigma_0]\) for some \(\sigma_0 > 0\). This is a contradiction. When \(f : H^1(0, 1) \to \mathbb{R}\) is bounded, similar contradiction also happens. Therefore, \((1.1)\) admits a unique global solution. □

**Remark 1.1.** By Proposition 1.1, we can assume that \(f : H^1(0, 1) \to \mathbb{R}\) is continuous, and system \((1.1)\) admits a unique solution \((w, \dot{w})^T \in C(0, \infty; H)\).

The next Lemma 1.1 is well-known and is not difficult to prove by using the results of [21] and [22]. For the reader’s convenience, we give a simple proof.

**Lemma 1.1.** Let \(A\) be the generator of an exponentially stable \(C_0\)-semigroup \(e^{tA}\) on the Hilbert space \(X\). Assume that \(B \in L(U, X)\) is an admissible control operator for \(e^{tA}\). Then, the initial value problem
\[
\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0,
\]
admits a unique solution \(x \in C(0, \infty; X)\), which tends to zeros as \(t \to \infty\) if \(u \in L^2(0, \infty; U)\) or \(\lim_{t \to +\infty} \|u(t)\|_U = 0\), and is bounded if \(u \in L^\infty(0, \infty; U)\).

**Proof.** Since \(B \in L(U, X)\) is an admissible control operator, by [21, Proposition 4.2.5], the solution can be represented as
\[
x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}Bu(s) \, ds \in X.
\]
Since \(A\) generates an exponentially stable \(C_0\)-semigroup \(e^{tA}\), there exist two constants \(M, \mu > 0\) such that \(\|e^{tA}\| \leq Me^{-\mu t}\). Suppose that \(u \in L^\infty(0, \infty; U)\). By [22, Remark 4.7], the admissibility of \(B\) implies that there exists a constant \(L_1 > 0\) independent of \(u(s)\) such that
\[
\|\int_0^t e^{(t-s)A}Bu(s) \, ds\|_X \leq L_1 \|u\|_{L^\infty(0, \infty; U)},
\]
which, together with (1.23) and the exponential stability of \(e^{tA}\), implies that the solution \(x(t)\) is bounded. Next, suppose that \(u \in L^2(0, \infty; U)\). For any given \(\sigma > 0\), there exists \(t_0 > 0\) such that
\[
\|u\|_{L^2(0, \infty; U)} \leq \sigma.
\]
It follows from the admissibility of \(B\) and [22, Remark 2.6] that
\[
\left\| \int_0^t e^{(t-s)A}Bu(s) \, ds \right\|_X \leq L_2 \|u\|_{L^2(0, \infty; U)} \leq L_2 \sigma,
\]
where \(L_2\) is a constant that is independent of \(u(s)\), and
\[
(u \oplus v)(t) = \begin{cases} u(t), & 0 \leq t \leq \tau, \\ v(t), & t > \tau. \end{cases}
\]
Using the exponential stability of \(e^{tA}\) again, we have
\[
\left\| \int_0^t e^{(t-s)A}Bu(s) \, ds \right\|_X \leq \left\| \int_0^t e^{(t-s)A}Bu(s) \, ds \right\|_X \leq M e^{-\mu(t-t_0)} \left\| \int_0^{t_0} e^{(t-s)A}Bu(s) \, ds \right\|_X.
\]
Rewriting (1.23) as
\[
x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}Bu(s) \, ds + \int_0^t e^{(t-s)A}Bu(s) \, ds,
\]
it follows from (1.26) and (1.27) that
\[
\|x(t)\|_X \leq M e^{-\mu(t-t_0)} \|x_0\| + M e^{-\mu(t-t_0)} \int_0^{t_0} e^{(t-s)A}Bu(s) \, ds + L_2 \sigma.
\]
This shows that \(\lim_{t \to \infty} \|x(t)\|_X \leq L_2 \sigma\). By the arbitrariness of \(\sigma\), \(x(t) \to 0\) as \(t \to \infty\). Finally, suppose that \(\lim_{t \to \infty} \|u(t)\|_U = 0\). For any given \(\sigma > 0\), there exists \(t_0 > 0\) such that
\[
\|u\|_{L^2(0, \infty; U)} \leq \sigma.
\]
It follows from the admissibility of \(B\) and [22, Remark 4.7] that
\[
\left\| \int_0^t e^{(t-s)A}Bu(s) \, ds \right\|_X \leq L_1 \|u\|_{L^2(0, \infty; U)} \leq L_1 \sigma.
\]
The rest of the proof is similar to (1.27)–(1.29). This ends the proof of the lemma. □

**Remark 1.2.** Consider the control problem: \(\dot{x}(t) = Ax(t) + Bu(t) + \tilde{d}(t), \quad x(0) = x_0\), where \(\tilde{d}(t)\) is the disturbance. By Lemma 1.1, if \(d \in L^2(0, \infty; U)\), it does not affect the asymptotic stability of the system. When \(d \in L^\infty(0, \infty; U)\), if we can find estimate \(\tilde{d}(t)\) of \(d\) so that \((\tilde{d} - d) \in L^2(0, \infty; U)\), we do not need to construct disturbance estimator likewise [10,11] such that \(\|\tilde{d}(t) - d(t)\|_U \to 0\) as \(t \to \infty\) which is usually stronger than \((\tilde{d} - d) \in L^\infty(0, \infty; U)\).

We proceed as follows. In Section 2, we design a total disturbance estimator which estimates the total disturbance that consists of internal uncertainty and the external disturbance. Section 3 is devoted to design of servomechanism and output tracking of the closed-loop system. Some numerical simulations are presented in Section 4 for illustration.

2. Estimator design

In this section, we design a disturbance estimator in terms of input and output \((u(t), y_m(t))\) for system (1.1):

\[
\begin{align*}
\dot{z}_1(x, t) &= \tilde{z}_1(x, t), & x \in (0, 1), & t > 0, \\
\dot{z}_2(0, t) &= q_1 \dot{z}_2(0, t) + c_1 \left[ z(0, t) - w(0, t) \right], & t \geq 0, \\
\dot{z}_2(1, t) &= u(t), & t \geq 0, \\
\dot{\tilde{d}}_1(x, t) &= \tilde{d}_1(x, t), & x \in (0, 1), & t > 0, \\
\dot{\tilde{d}}_1(0, t) &= q_1 \dot{\tilde{d}}_1(0, t) + q_2 \dot{z}_2(0, t) - f(w(s, t)) - d(t), & t \geq 0, \\
\dot{\tilde{d}}_1(1, t) &= 0, & t \geq 0, \\
\dot{z}_1(x, 0) &= \tilde{z}_1(x, 0) = \tilde{z}_1(x), & x \in [0, 1], \\
\dot{\tilde{d}}_1(x, 0) &= \tilde{d}_1(x, 0) = \tilde{d}_1(x), & x \in [0, 1],
\end{align*}
\]

(2.1)

where \(c_1, c_2 > 0\) are the design parameters. We will show that this estimator can estimate the total disturbance \(f(w(s, t)) + d(t)\). It is seen that system (2.1) is completely determined by the measured output \(y_m(t)\) of system (1.1) and the control input \(u(t)\).

Let \(\hat{z}(x, t) = z(x, t) - w(x, t)\). Then \(\hat{z}(x, t)\) is governed by

\[
\begin{align*}
\dot{\hat{z}}_1(x, t) &= \hat{z}_1(x, t), & x \in (0, 1), & t > 0, \\
\dot{\hat{z}}_2(0, t) &= c_1 \hat{z}_2(0, t) + q_1 \hat{z}_2(0, t) - f(w(s, t)) - d(t), & t \geq 0, \\
\dot{\hat{z}}_2(1, t) &= 0, & t \geq 0, \\
\hat{z}(0, 0) &= \tilde{z}_1(x, 0) = \tilde{z}_1(x), & x \in [0, 1], \\
\hat{z}(1, t) &= \tilde{z}_1(x, t) = \tilde{z}_1(x), & x \in [0, 1].
\end{align*}
\]

(2.2)
We consider system (2.2) in the state space $\mathbb{H} = H^1(0, 1) \times L^2(0, 1)$ with the inner product given by

$$
\langle (\phi_1, \psi_1), (\phi_2, \psi_2) \rangle^\top = \int_0^1 \left[ \phi_1'(x) \overline{\phi_2'(x)} + \psi_1(x) \overline{\psi_2(x)} \right] dx
+ c_1 \phi_0(0) \overline{\phi_0(0)}.
$$

$\forall (\phi_1, \psi_1) \in \mathbb{H}, \quad i = 1, 2.$

System (2.2) can be written as an evolutionary equation in $\mathbb{H}$:

$$
d t \left[ \begin{array}{c} \tilde{z}(t) \\ \tilde{d}_i(t) \end{array} \right] = A_2 \left[ \begin{array}{c} \tilde{z}(t) \\ \tilde{d}_i(t) \end{array} \right] + B_1 (\tilde{w}(\cdot, t) - d(t)),
$$

(2.3)

where the operator $B_1$ is given by (1.4) and $A_2$ given by

$$
A_2(\phi, \psi)^\top = (\psi, \phi)^\top, \quad \forall (\phi, \psi) \in D(A_2),
$$

$$
D(A_2) = \left\{ (\phi, \psi) \in H^2(0, 1) \times H^1(0, 1) : \phi(0) = \phi(1) = 0 \right\}.
$$

Lemma 2.1. Suppose that $d \in L^\infty(0, \infty)$, $f : H^1(0, 1) \rightarrow \mathbb{R}$ is continuous and that (1.1) admits a unique solution $(w, \tilde{w}) \in C(0, \infty; \mathbb{H})$ which is bounded. For any initial value $(\tilde{z}_0, \tilde{d}_i) \in H^1(0, 1) \times L^2(0, 1)$, there exists a unique solution $(\tilde{z}, \tilde{d}_i) \in C(0, \infty; H^1(0, 1) \times L^2(0, 1))$ to (2.2) such that

$$
\sup_{t \geq 0} \| (\tilde{z}(t), \tilde{d}_i(t)) \|_{H^1(0, 1) \times L^2(0, 1)} < +\infty.
$$

Proof. Let $(\tilde{d}_i(x,t) - \tilde{z}(x,t)) = \tilde{d}_i(x,t) - z(x,t) + w(x,t)$. We can see that for $d(t,x)$ satisfies

$$
\begin{align*}
\tilde{d}_i(t,x) &= \tilde{d}_i(x,t), & x \in (0,1), & t > 0, \\
\tilde{d}_i(0,t) &= 0, & t \geq 0, \\
\tilde{d}_i(t,1) &= -c_1 \tilde{d}_i(t,1), & t \geq 0.
\end{align*}
$$

(2.6)

We consider (2.6) in the energy space $H_0^1(0, 1) \times L^2(0, 1)$, where $H_0^1(0, 1) = \{ \phi \in H^1(0, 1) : \phi(0) = 0 \}$. It is well known that for any initial value $(\tilde{d}_0, \tilde{d}_i) \in H_0^1(0, 1) \times L^2(0, 1)$, system (2.6) admits a unique solution and there exist two constants $M_d, \mu_d > 0$ such that

$$
\int_0^1 \left| \tilde{d}_i(x,t) \right|^2 + \left| \tilde{d}_i(t,x) \right|^2 dx \leq M_d \left( \| \tilde{d}_0(t) \|_{H^1(0, 1) \times L^2(0, 1)}^2 e^{-\mu_d t} \right).
$$

(2.7)

Lemma 2.2. For any initial value $(\tilde{d}_0, \tilde{d}_i) \in H_0^1(0, 1) \times L^2(0, 1)$, the solution of (2.6) satisfies

$$
\int_0^\infty \tilde{d}_i^2(0,t)dt < +\infty.
$$

(2.8)

Proof. Let

$$
\tilde{q}(t) = \int_0^1 (x - 1) \tilde{d}_i(x,t) \tilde{d}_i(x,t) dx.
$$

Then, the derivative of $\tilde{q}(t)$ along the solution of (2.6) yields

$$
\dot{\tilde{q}}(t) = \frac{1}{2} \int_0^1 \left| \tilde{d}_i(x,t) \right|^2 + \left| \tilde{d}_i(t,x) \right|^2 dx
\leq \frac{1}{2} M_d \left( \| \tilde{d}_0(t) \|_{H^1(0, 1) \times L^2(0, 1)}^2 e^{-\mu_d t} \right).
$$

(2.10)
We consider system (3.4) in the state space $\mathcal{H} = H^1(0, 1) \times L^2(0, 1)$ with the inner product given by
\[
\langle (\phi_1, \psi_1)^\top, (\phi_2, \psi_2)^\top \rangle = \int_0^1 \left[ \phi_1'(x)\overline{\phi_2'(x)} + \psi_1(x)\overline{\psi_2(x)} \right] dx + c_3\phi_1(1)\overline{\phi_2(1)}, \quad \forall (\phi_1, \psi_1)^\top \in \mathbb{H}, \quad i = 1, 2.
\]

**Theorem 3.1.** Suppose that $d_{\delta_1}(0, \cdot) \in L^2(0, \infty)$. For any initial value $(t(\cdot), 0, 0, (\cdot, 0)) \in \mathbb{H}$, there exists a unique solution to (3.4) such that $(e_1, e_1^\top) \in C(0, \infty; \mathbb{H})$ satisfying
\[
\lim_{t \to \infty} \left( \int_0^1 (e_2^\top(0, t)) \right)^2 dx + e_2^\top(1, t) = 0.
\]

**Proof.** Define an operator $A_e : D(A_e) \subset \mathbb{H} \to \mathbb{H}$ by
\[
A_e(\phi, \psi)^\top = (\psi, \phi')^\top, \quad \forall (\phi, \psi)^\top \in D(A_e),
\]
where $D(A_e)$ is given by (1.4). It is readily found that
\[
A_e^*(\phi, \psi)^\top = (\psi', \phi)^\top, \quad \forall (\phi, \psi)^\top \in D(A_e^*),
\]
\[
D(A_e^*) = \left\{ (\phi, \psi)^\top \in \mathbb{H} \cap (H^2(0, 1) \times H^1(0, 1)) : \phi(0) = -q\psi(0), \quad \phi'(1) = -c_3\phi(1) \right\}.
\]

By Lemma 6.2 in Appendix, $A_e$ generates an exponentially stable $C_0$-semigroup $e^{A_e t}$ in $\mathbb{H}$. Now we show that $\mathbb{B}_1$ is admissible to $e^{A_e t}$. Once again, this is equivalent to showing that a) $\mathbb{B}_1A_e^{-1}$ is bounded from $\mathbb{H}$ to $C$, and b) for every $t > 0$, there exists $M_t > 0$ depending on $T$-only such that the system of the following:
\[
\begin{align*}
\dot{e}_1^e(0, t) &= e_1^e(0, t), \quad x \in [0, 1], \quad t > 0, \\
\dot{e}_1^e(0, t) &= -q\psi(0), \quad t \geq 0, \\
\dot{e}_1^e(1, t) &= -c_3\phi(1), \quad t \geq 0, \\
\psi(x) &= \psi_0^e(x), \quad \phi'(x) = e_1^e(1, t), \quad x \in [0, 1], \\
\end{align*}
\]
satisfies
\[
\int_0^T (e_1^e(0, t))^2 dt \leq M_t \| e_0^e \|_{\mathbb{H}}^2.
\]

A direct computation shows that
\[
A_e^{-1}(\phi, \psi)^\top = \left( \frac{1}{c_3} - q\psi(0)) \right) \psi(t) + \int_0^1 \psi(x) dx + c_3 \int_0^1 \psi'(x) dx + c_3 \int_0^1 \psi'(x) dx - \phi(x)^\top, \quad \forall (\phi, \psi)^\top \in \mathbb{H}.
\]

Thus, $\mathbb{B}_1A_e^{-1}$ is bounded from $\mathbb{H}$ to $C$. By Lemma 6.2 in Appendix, it is easy to see that (3.9) admits a unique solution $(e_1, e_1^\top) \in C(0, \infty; \mathbb{H})$ and there exist two constants $M > 0$ and $\omega \in \mathbb{R}$ such that for all $t \geq 0$,
\[
E_1^e(t) := \int_0^1 (e_1^e(x, t))^2 dx + c_3(e_1^e(1, t))^2 \leq M e^{\omega t} \| e_0^e \|_{\mathbb{H}}^2
\]
\[
E_1^e(t) = 2q(e_1^e(0, t))^2.
\]

Integrating from 0 to $T$ with respect to $t$ and by (3.12), we obtain
\[
2q \int_0^T (e_1^e(0, t))^2 dt = E_1^e(T) - E_1^e(0) \leq M, (e^{\omega t} + 1) \| e_0^e \|_{\mathbb{H}}^2.
\]

This, together with (3.11), implies that $B_1$ is admissible to $e^{A_e t}$. Therefore, (3.7) admits a unique solution. Since $d_{\delta_1}(0, \cdot) \in L^2(0, \infty)$, it follows from Lemma 1.1 that system (3.7) admits a unique solution that tends to zeros as $t$ goes to infinity, i.e., (3.5) holds.

**Remark 3.1.** By Remark 2.1, if we take $C_2 = 1$ in (2.1), then system (3.4) is exponentially stable. In this case, we can see that the performance output tracking is exponentially convergent.

Now we consider the reference model (3.1) in the state space $\mathcal{H} = H^1(0, 1) \times L^2(0, 1)$ with the inner product given by
\[
\langle (\phi_1, \psi_1)^\top, (\phi_2, \psi_2)^\top \rangle = \int_0^1 \left[ \phi_1'(x)\overline{\phi_2'(x)} + \psi_1(x)\overline{\psi_2(x)} \right] dx + \phi(1)\overline{\phi_2'(1)}, \quad \forall (\phi, \psi)^\top \in \mathbb{H}, \quad i = 1, 2.
\]

**Lemma 3.1.** Suppose that $d \in L^2(0, \infty)$, $z \in W^2(0, \infty)$, $f : H^1(0, 1) \to \mathbb{R}$ is continuous and bounded, and system (1.1) admits a unique solution $(v, u) \in C(0, \infty; \mathbb{H})$. For any initial value $(v_0, u_0) \in \mathbb{H}$, there exists a unique solution to (3.1) such that $(v, u) \in C(0, \infty; \mathbb{H})$. Moreover, there exists a constant $M > 0$ such that
\[
\sup_{t \geq 0} \left( \int_0^1 (v^2(0, t) + \xi^2(0, t)) dx + \xi^2(1, t) \right) \leq M.
\]

**Suppose additionally that** $f(w) + d \in L^2(0, \infty)$ and $r \in H^2(0, \infty)$.

Then,
\[
\lim_{t \to \infty} \left( \int_0^1 (v^2(0, t) + \xi^2(0, t)) dx + \xi^2(1, t) \right) = 0.
\]

**Proof.** Introducing the transformation \(\hat{v}(x, t) = v(x, t) - x^2 r(t)\), we can transform system (3.1) into an equivalent system:
\[
\begin{align*}
\hat{v}_t(x, t) &= \hat{v}_x(x, t) - x^2 r(t) + 2r(t), \quad x \in (0, 1), \quad t > 0, \\
\hat{v}_x(0, t) &= q\hat{v}_x(0, t) - \hat{a}_e(0, t) + c_3d(0, t) + q\hat{d}(0, t), \quad t \geq 0, \\
\hat{v}(1, t) &= 0, \quad t \geq 0, \\
\hat{v}(0, t) &= v(t) - x^2 r(t), \quad \hat{v}_x(0, t) = v_1(x) - x^2 r(t), \quad x \in [0, 1].
\end{align*}
\]

Since $r \in W^2(0, \infty)$, it suffices to prove that there exists a constant $M > 0$ such that
\[
\sup_{t \geq 0} \left( \int_0^1 (\hat{v}_x(0, t) + \xi(0, t)) dx \right) \leq M.
\]

We consider system (3.17) in the state space $\mathcal{H} = H^1_2(0, 1) \times L^2(0, 1)$, where $H^1_2(0, 1) = \{ f \in H^1(0, 1) : \phi(1) = 0 \}$, with the inner product given by
\[
\langle (\phi_1, \psi_1)^\top, (\phi_2, \psi_2)^\top \rangle = \int_0^1 \left[ \phi_1'(x)\overline{\phi_2'(x)} + \psi_1(x)\overline{\psi_2(x)} \right] dx, \quad \forall (\phi, \psi)^\top \in \mathbb{H}, \quad i = 1, 2.
\]

Define the operator $A_e : D(A_e) \subset \mathbb{H} \to \mathbb{H}$ by
\[
A_e(\phi, \psi)^\top = (\psi', \phi)^\top, \quad \forall (\phi, \psi)^\top \in D(A_e),
\]
\[
D(A_e) = \{ (\phi, \psi)^\top \in H^2_2(0, 1) \times H^1_2(0, 1) : \phi(1) = 0 \}.
\]
We can write (3.17) into an operator form of the following:
\[
\frac{d}{dt} (\tilde{y}(t), \tilde{y}_t(t))^T = A_t(\tilde{y}(t), \tilde{y}_t(t))^T + I_0(0, -\chi t \hat{f}(t) + 2r(t) + B_1 g(t),
\]
where \(T\) is an identity operator and \(g(t) = -\tilde{d}_t(0, t) + 2\sigma d_t(0, t) + q_d(0, t)\). It is readily found that
\[
\begin{align*}
A_t(\phi, \psi)^T &= (-\phi, -\phi^T), \\
D(A_t) &= H^2(0, 1) 	imes H^1(0, 1) \\
\phi(0) &= -\psi(0), \\
\psi(0) &= 0.
\end{align*}
\]
By Lemma 63 in Appendix, \(A_t\) generates an exponentially stable \(C_0\)-semigroup \(e^{A_t t}\) on \(\mathbb{H}\). Since \(T\) is a bounded operator, \(T\) is admissible to \(e^{A_t t}\). Now we show that \(B_1\) is admissible to \(e^{A_t t}\). This is equivalent to showing that \(B_1 A_t^{-1} e^{A_t t} B_1^{-1}\) is bounded from \(\mathbb{H}\) to \(C\), and by \(B_1 A_t^{-1} e^{A_t t} B_1^{-1}\) for any \(T > 0\), there exists \(M_t > 0\) depending on \(T\) such that the system of the following:
\[
\begin{align*}
t &> 0, \\
\tilde{y}_t(0, t) &= q_v^T(0, t), \\
\tilde{y}_t(t) &= 0, \\
\tilde{y}_t(t, 0) &= v_t(0, x), \\
\tilde{y}_t(t, x) &= v_t(0, x), \\
\tilde{y}_t(t, x) &= v_t(0, x), \\
\tilde{y}_t(t, x) &= v_t(0, x), \\
\tilde{y}_t(t, x) &= v_t(0, x),
\end{align*}
\]
satisfies
\[
\int_0^T t \tilde{y}_t^2(t, 0) \leq M_t \| v_t(0, t) \|_{H^2}.
\]
A direct computation shows that
\[
\begin{align*}
A_t^{-1}(\phi, \psi)^T &= \left( q\phi(0)(x - 1) + \int_0^x (1 - \xi) \psi(\xi) d\xi \
- \int_0^x (x - \xi) \psi(\xi) d\xi, -\phi(x) \right)^T, \\
B_1 A_t^{-1}(\phi, \psi)^T &= -\phi(0).
\end{align*}
\]
Thus, \(B_1 A_t^{-1} e^{A_t t} B_1^{-1}\) is bounded from \(\mathbb{H}\) to \(C\). By Lemma 63 in Appendix, it is easy to see that (3.23) admits a unique solution \((\tilde{u}_t, \tilde{v}_t)^T \in C(0, \infty; H)\) and there exist two constants \(M > 0\) and \(\omega \in \mathbb{R}\) such that for all \(t \geq 0\),
\[
E_t(t) := \int_0^T (\tilde{u}_t^2 + \tilde{v}_t^2) dx \leq M e^{\omega T}(\tilde{u}_t(0) + 1) \| v_t(0, t) \|_{H^2}^2.
\]
On the other hand, differentiating \(E_t(t)\) with respect to \(t\) along the solution to (3.23), we obtain
\[
\dot{E}_t(t) = 2q(\tilde{v}_t^2(0, t))^2.
\]
Integrating from 0 to \(T\) with respect to \(t\) and by (3.26), we have
\[
\int_0^T (\tilde{v}_t^2(0, t))^2 ds = M_1 e^{\omega T} (\tilde{u}_t(0) + 1) \| v_t(0, t) \|_{H^2}^2.
\]
This, together with (3.25), implies that \(B_1\) is admissible to \(e^{A_t t}\). Therefore, the solution of (3.17) can be written as
\[
(\tilde{y}(t), \tilde{y}_t(t))^T = e^{A_t t}(\tilde{y}(0), \tilde{y}_t(0))^T + \int_0^t e^{A_t(t-s)} I_0(0, -\chi z f(s) + 2r(s)) \tilde{y}^T ds + \int_0^t e^{A_t(t-s)} B_1 g(s) ds.
\]
Since \(r \in W^{\infty}(0, \infty), \) we have \((-\chi z f + 2r) \in L^\infty(0, \infty)\) and since \(I_0\) is admissible control operator to \(e^{A_t t}\), by Lemma 1.1,
\[
\sup_{t \geq 0} \left\| \int_0^t e^{A_t(t-s)} I_0(0, -\chi z f + 2r(s)) \tilde{y}^T ds \right\| < +\infty.
\]
By assumption \(d \in L^\infty(0, \infty)\) and \(f(w)\) being bounded on \([0, \infty)\), it follows from the admissibility of \(B_1\) and Lemma 1.1 that
\[
\sup_{t \geq 0} \left\| \int_0^t e^{A_t(t-s)} B_1 f(w(s)) + d(s) ds \right\| < +\infty.
\]
From Lemma 2.2, \(\tilde{d}_t(0, \cdot) \in L^2(0, \infty),\) and by Lemma 1.1,
\[
\lim_{t \to \infty} \left\| \int_0^t e^{A_t(t-s)} B_1 (\tilde{d}_t(0, \cdot)) ds \right\| = 0.
\]
On the other hand, by Remark 2.2, \(g(t) = f(w(-t)) + d(t) - \tilde{d}_t(0, t).\) This, together with (3.31) and (3.32), yields
\[
\sup_{t \geq 0} \left\| \int_0^t e^{A_t(t-s)} B_1 g(s) ds \right\| < +\infty.
\]
It then follows from (3.29), (3.30), and (3.33) that \((\tilde{y}(t), \tilde{y}_t(t))^T\) is bounded on \(\mathbb{H}\), that is, (3.15) holds.
Suppose that \((f(w) + d) \in L^2(0, \infty)\) and \(r \in H^2(0, \infty).\) It is observed that \((-\chi z f + 2r) \in L^2(0, \infty),\) By Lemma 1.1,
\[
\lim_{t \to \infty} \left( \int_0^T |\tilde{u}_t^2(0, x) + \tilde{v}_t^2(0, x)| dx + \tilde{v}_t^2(1, t) \right) = 0.
\]
Now, we show that (3.16) can be reduced to (3.34). To do this, we claim that \(r \in H^2(0, \infty)\) will lead to \(\lim_{t \to \infty} |r(t)| = 0.\) Actually, since \(r \in L^2(0, \infty), \) \(r \in L^2(0, \infty),\) for any \(t_1, t_2 \geq 0,\)
\[
|\tilde{r}(t_1) - \tilde{r}(t_2)| \leq \int_{t_1}^{t_2} |\tilde{r}(t)| dt \\
\leq 2 \int_{t_1}^{t_2} t \tilde{r}^2(t) dt \\
\leq 2 \int_{t_1}^{t_2} \tilde{r}^2(t) dt \\
\to 0 \text{ as } t_1, t_2 \to \infty.
\]
So for any given \(\varepsilon > 0,\) there exists a \(T > 0\) such that for all \(t_1, t_2 > T,\)
\[
|\tilde{r}(t_1) - \tilde{r}(t_2)| \leq \varepsilon.
\]
This shows that \(\lim_{t \to \infty} \tilde{r}^2(t)\) exists. But since \(r \in L^2(0, \infty),\) this limit must be zero. Hence, \(\lim_{t \to \infty} |r(t)| = 0 \text{ and so } \lim_{t \to \infty} |r(t)| = 0.\) In an analogous way, we can obtain \(\lim_{t \to \infty} |r(t)| = 0.\) Since \(n(x, t) = \tilde{v}(t, x) + x^2 r(t)\),
\[
\lim_{t \to \infty} \left( \int_0^T |\tilde{u}_t^2(0, x) + \tilde{v}_t^2(0, x)| dx + \tilde{v}_t^2(1, t) \right) \\
\leq 2 \lim_{t \to \infty} \left( \int_0^T |\tilde{u}_t^2(0, x) + \tilde{v}_t^2(0, x)| dx + \tilde{v}_t^2(1, t) \right) \\
+ 2 \lim_{t \to \infty} \left( \int_0^T |4x^2 r^2(t) + x^4 r^2(t)| dx + \tilde{v}_t^2(1, t) \right) = 0.
\]
i.e. (3.16) holds. This ends the proof of the lemma. \(\square\)
Now we turn to the closed-loop which is composed of (11), (2.1), and (3.1) as follows:

\[
\begin{align*}
    \dot{w}_1(t) &= w_{ax}(t), \quad x \in (0, 1), \\
    \dot{w}_2(t) &= qw_1(t) + f(w(x,t)) + d(t), \\
    \dot{w}_3(t) &= z_0(t) - z_{ax}(t), \quad x \in (0, 1), \\
    \dot{z}_0(t) &= z_0(t), \quad x \in (0, 1), \\
    \dot{z}_1(t) &= z_1(t) - C_1(w(t) - r(t)), \\
    \dot{d}(t) &= d(t) - z(t) - w(t), \\
    \dot{\delta}_d(t) &= \delta_d(t) = -C_1d(t), \\
    \dot{\delta}_v(t) &= \delta_v(t) = -C_1v(t), \\
    \dot{v}(t) &= \dot{r}(t), \quad t > 0.
\end{align*}
\]

We consider system (3.35) in the state space \( \mathcal{X} = (H^1(0, 1) \times L^2(0, 1))^4 \).

**Theorem 3.2.** Let \( c_1, c_2 > 0 \) and \( c_2 = 1 \). Suppose that \( d \in L^\infty(0, \infty), \quad r \in W^{2, \infty}(0, \infty), \quad f : H^1(0, 1) \to \mathbb{R} \) is continuous, bounded, and satisfies the local Lipschitz condition in \( H^1(0, 1) \). Then, for any initial value \( (w_0, w_1, z_0, \delta_d, \delta_v, v, \delta_v) \in \mathcal{X} \), the closed-loop system solution has the following properties:

(i) \[
\sup_{t \geq 0} \left( \int_0^t \left[ w_{ax}(x,t) + w_{ax}(x,t) + z_{ax}(x,t) + z_{ax}(x,t) + d_{ax}(x,t) \right] \right) < +\infty;
\]

(ii) There exist two constants \( M, \mu > 0 \) such that

\[
\int_0^t \left[ \left( \dot{v}(x,t) - w_0(x,t) \right)^2 + \left( \dot{v}(x,t) - w_0(x,t) \right)^2 \right] dx + \int_0^t \left[ \left( \dot{v}(x,t) + v(x,t) \right)^2 + \left( \dot{v}(x,t) + v(x,t) \right)^2 \right] dx + w^2(0,t) + z^2(0,t) = +\infty;
\]

(iii) There exist two constants \( M, \mu > 0 \) such that

\[
|e(t)| = |w(1,t) - r(t)| \leq Me^{-\mu t}, \quad \forall t \geq 0;
\]

(iv) When \( f = 0 \), \( d \in L^2(0, \infty) \), especially when \( d(t) = 0, \quad r(t) = 0 \),

\[
\lim_{t \to \infty} \left( \int_0^t \left[ w_{ax}(x,t) + w_{ax}(x,t) + z_{ax}(x,t) + z_{ax}(x,t) + d_{ax}(x,t) \right] \right) = 0.
\]

**Proof.** Let \( \tilde{z}(x,t) = z(x,t) - w(x,t), \quad \tilde{d}(x,t) = d(x,t) - z(x,t) - w(x,t), \quad \varepsilon(t) = \varepsilon(x,t) = v(x,t), \quad \tilde{v}(x,t) = \tilde{v}(x,t) - w(x,t). \) It is easy to verify that \( (\tilde{z}(x,t), \tilde{d}(x,t), \varepsilon(x,t)) \) satisfies

\[
\begin{align*}
    \dot{\tilde{z}}_d(t) &= \tilde{z}_{ax}(x,t), \quad x \in (0, 1), \\
    \dot{\tilde{z}}_v(t) &= c_1\tilde{z}_0(t) + q\tilde{z}_0(t) - f(w(x,t)) - d(t), \\
    \dot{\varepsilon}(t) &= 0, \\
    \dot{\delta}_d(t) &= -c_d\delta_d(t), \\
    \dot{\delta}_v(t) &= c_3\delta_v(t), \quad t > 0.
\end{align*}
\]

Note that \( c_2 = 1 \). By the compatibility boundary condition (3.36), (2.7), Lemma 2.2, Theorem 3.1 and Remark 3.1, the \( (\tilde{d}, \varepsilon) \)-part of (3.37) admits a unique solution such that

\[
\int_0^t \left[ \left( \dot{\tilde{d}}_a(x,t) \right)^2 + \left( \dot{\tilde{d}}_a(x,t) \right)^2 + \varepsilon_2^2(x,t) + \varepsilon_2^2(x,t) \right] dx + e^2(t, 1, t) \leq M_0e^{-\mu t}, \quad \forall t \geq 0.
\]

with some \( M_0, \mu > 0 \). Moreover, By Lemma 2.2, \( \tilde{d}_a(0,t) \in L^2(0, \infty) \). Next, we rewrite \( p \)-part and \( \varepsilon \)-part of (3.38) into operator form of the following:

\[
\begin{align*}
    \frac{d}{dt} \left( p(t, \cdot) + p(t, \cdot) \right) &= A_p \left( p(t, \cdot) + p(t, \cdot) \right) + B_p \left( -\tilde{d}_a(0,t) \right) + g(t), \\
    \tilde{v}(t) &= \tilde{v}(t), \quad \varepsilon(t) = \varepsilon(x,t), \quad x \in (0, 1), \\
    \frac{d}{dt} \left( \tilde{v}(t), \tilde{v}(t) \right) &= A_p \left( \tilde{v}(t), \tilde{v}(t) \right) + B_p \left( f(\varepsilon(t) - p(\cdot, t)) \right) + d(t) - \tilde{d}_a(0,t) + g(t), \\
    \frac{d}{dt} \left( \tilde{v}(t), \tilde{v}(t) \right) &= A_p \left( \tilde{v}(t), \tilde{v}(t) \right) + B_p \left( f(\varepsilon(t) - p(\cdot, t)) \right) + d(t) - \tilde{d}_a(0,t) + g(t),
\end{align*}
\]

where \( A_p \) is given by (3.20). Similar to the proof of Lemma 3.1, we can prove that system (3.40) admits a unique solution \( (p, p) \in C(0, \infty; \mathbb{H}) \) and

\[
\sup_{t \geq 0} \left\| (p(t), p(t)) \right\|_{\mathbb{H}} < +\infty.
\]

Moreover, similar to the proof of Proposition 11, by the assumption on \( f(\cdot) \), we know that system (3.41) admits a unique solution \( (\tilde{v}, \tilde{v}) \in C(0, \infty; \mathbb{H}) \). By \( w(x,t) = \tilde{v}(x,t) - p(x,t) \), it follows that the \( \varepsilon \)-part of (3.35) has a unique solution \( w(x,w) \in C(0, \infty; \mathbb{H}) \) which, together with Lemma 2.1, implies that the \( \varepsilon \)-part of (3.35) has a unique solution \( (\tilde{z}, \tilde{z}) \in C(0, \infty; \mathbb{H}) \) and satisfies

\[
\sup_{t \geq 0} \left\| (\tilde{z}(t), \tilde{z}(t)) \right\|_{\mathbb{H}} < +\infty.
\]

Since 

\[
\begin{align*}
    \lim_{t \to \infty} \left( \int_0^t \left[ w_{ax}(x,t) + w_{ax}(x,t) + z_{ax}(x,t) + z_{ax}(x,t) + d_{ax}(x,t) \right] \right) = 0.
\end{align*}
\]
of (3.39). Since \( r \in W^2 \cap (0, \infty) \), by (3.39), (3.42), and (3.43), we can conclude (i) and from Theorem 3.1, (iv) is concluded as well. □

**Remark 3.2.** In Theorem 3.2, we have supposed \( c_2 = 1 \). When \( c_2 > 0 \) (\( \neq 1 \)), by virtue of Lemmas 1.1 and 2.2, it is easy to verify that the expressions in (ii) and (iii) are asymptotically convergent to zero. By Theorem 3.2, the reference model (3.1) can be regarded as a state observer of (1.1). Alternatively, based on estimation of the total disturbance, we can also design the following state observer of (1.1):

\[
\begin{align*}
\dot{\hat{w}}_t(x,t) &= \hat{w}_{xx}(x,t), \quad x \in (0, 1), \quad t > 0, \\
\hat{w}_x(0, t) &= q \hat{w}_t(0, t) + c_1 [\hat{w}(0, t) - w(0, t)] \quad t \geq 0, \\
-\hat{d}_x(0, t) + c_1 \hat{d}(0, t) &+ q \hat{d}_t(0, t), \quad t \geq 0, \\
\hat{w}_x(1, t) &= u(t), \\
\hat{w}(x, 0) &= \hat{w}_0(x), \\
\hat{w}_t(x, 0) &= \hat{w}_1(x), \\
&x \in [0, 1].
\end{align*}
\]

(3.44)

Indeed, let \( \hat{w}(x, t) = \hat{w}(x, t) - w(x, t) \). A direct computation shows that \( \hat{w}(x, t) \) satisfies

\[
\begin{align*}
\dot{\hat{w}}_t(x, t) &= \hat{w}_{xx}(x, t), \quad x \in (0, 1), \quad t > 0, \\
\hat{w}_x(0, t) &= q \hat{w}_t(0, t) + c_1 \hat{w}(0, t) - \hat{d}_x(0, t), \quad t \geq 0, \\
\hat{w}_x(1, t) &= 0, \\
\hat{w}(x, 0) &= \hat{w}_0(x) - w_0(x), \quad \hat{w}_t(x, 0) = \hat{w}_1(x) - w_1(x), \quad x \in [0, 1].
\end{align*}
\]

(3.45)

Similarly to the proof in Theorem 3.1, and by Lemmas 6.1 and 1.1, we can obtain

\[
\int_0^1 [\hat{w}^2_t(x, t) + \hat{w}^2(x, t)] dx + \hat{w}^2(0, t) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty.
\]

Thus, (3.44) is a state observer of (1.1).

4. Numerical simulation

In this section, we present some numerical simulations for system (3.35) for illustration. For numerical computations, we
choose reference signal \( r(t) = 2\sin(t) - 0.45\cos(2\pi t) + 1 \), the internal nonlinear uncertainty \( f(w(t)) = \sin(w^2(0, t)) \) and the external disturbance \( d(t) = 2\sin(t) + 0.7\cos(2\pi t) - 1 \). The other parameters are taken as \( q = 1.1, c_1 = 1, c_2 = 1.1, c_3 = 1 \). The initial values are taken as:

\[
\begin{align*}
 w(x, 0) &= 2x - x^2, \\
 w_1(x, 0) &= -2x + x^2, \\
 z(x, 0) &= x - x^2, \\
 z_1(x, 0) &= -x + x^2, \\
 d(x, 0) &= x, \\
 \hat{d}(x, 0) &= -x, \\
 v(x, 0) &= 2x - x^2 - 0.45, \\
 v_1(x, 0) &= 2x - x^2.
\end{align*}
\]  

(4.1)

It is clear that the above initial value satisfies the compatible condition (3.36). The backward Euler method in time and the Chebyshev spectral method for space variable are used to discretize system (3.35). Here, we take the grid size \( N = 20 \) for \( x \) and the time step \( dt = 5 \times 0.001 \). The solution of system is plotted in Figs. 1–4. Fig. 5 shows that the reference model (3.1) can be regarded as a state observer of (1.1). Fig. 6 shows that the total disturbance \( F(w(t)) = \sin(w^2(0, t)) + 2\sin(t) + 0.7\cos(2\pi t) - 1 \) and its estimate \( -d(0, t) + c_1\hat{d}(0, t) + q\hat{d}(0, t) \). It is seen that the disturbance is estimated effectively. The convergence is very fast and smoothly. Fig. 7 shows that \( w(1, t) \) tracks asymptotically the reference signal \( r(t) \). Fig. 8 displays the feedback control in time.

5. Concluding remarks

In this paper, we present a new infinite-dimensional disturbance estimator to estimate unknown nonlinear internal uncertainty and external disturbance for a one-dimensional wave equation. An servomechanism is designed by the measured output and the reference signal where the estimation mechanism of unknown total disturbance is presented. Five control objectives are achieved: (a) The performance output tracks exponentially the reference signal; (b) All the internal-loops are bounded; (c) The system state is recovered from input and output; (d) The unknown total disturbance can be estimated in the sense that the estimation error belongs to \( L^2(0, \infty) \); (e) When the disturbance and reference signal belong to \( L^2(0, \infty) \) and \( H^p(0, \infty) \), respectively, the closed-loop is asymptotically stable. The last point shows particularly that when the disturbance and reference are disconnected to the system, the closed-loop is asymptotically stable, that is, the system is internally asymptotically stable. This paper is a generalization of a recent work [24] where only boundary external disturbance was considered. Here we use a different estimator compared with [24] to deal with the nonlinearity for which the treatment for output tracking is different to stabilization like a recent paper [8].

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Appendix

In this appendix, we present several lemmas concerning the operators being able to generate $\mathcal{C}_0$-groups on the state space. These results are crucial to establishment of the main results of this paper.

Let $\mathcal{H} := H^1(0, 1) \times L^2(0, 1)$ with the inner product given by (1.3). Define the operator $A_1 : D(A_1) \subset \mathcal{H} \to \mathcal{H}$ as follows:

$$\begin{align*}
A_1(\phi, \psi)^T &= (\psi, \phi'')^T, \quad \forall (\phi, \psi)^T \in D(A_1), \\
D(A_1) &= \left\{ (\phi, \psi)^T \in \mathcal{H} \cap (H^2(0, 1) \times H^1(0, 1)) : \phi'(0) = c\psi(0) + \phi(0), \quad \phi'(1) = 0 \right\}.
\end{align*}
$$

Lemma 6.1. For any $c \in \mathbb{R}$ and $c \neq \pm 1$, $A_1$ generates a $\mathcal{C}_0$-group on $\mathcal{H}$. Moreover, if $c > 0$ and $c \neq 1$, then $A_1$ generates an exponentially stable $\mathcal{C}_0$-semigroup on $\mathcal{H}$.

Proof. We first show that $A_1$ generates a $\mathcal{C}_0$-group on $\mathcal{H}$. A direct computation shows that

$$A_1^{-1}(\phi, \psi)^T = \left( c\phi(0) + (1 + x) \int_0^1 \psi(x)dx - \int_0^x (x - \xi)\psi'(\xi)d\xi, -\phi(x) \right)^T.
$$

By the Sobolev embedding theorem, $A_1^{-1}$ is compact on $\mathcal{H}$, and thus $\sigma(A_1)$ consists of eigenvalues of $A_1$ only. It is easily seen that $\lambda \in \sigma(A_1)$ if and only if there exists $\phi(x) \neq 0$ satisfying

$$\begin{align*}
\phi''(x) &= \lambda^2 \phi(x), \\
\phi'(0) &= \lambda c\phi(0) + \phi(0), \quad \phi'(1) = 0,
\end{align*}
$$

and the associated eigenfunction is $(\lambda^{-1}\phi(x), \phi(x))$. Solving (6.3) gives

$$\phi(x) = \cosh(\lambda(x - 1)) \quad \text{with} \quad \lambda \text{ satisfying } e^{2\lambda} = \frac{1 - c}{1 + c + \frac{1}{c}}.
$$

Fig. 5. The error $\varepsilon(x, t) = w(x, t) - v(x, t)$.

Fig. 6. The total disturbance and its estimation.
Furthermore, the eigen-pairs are found to have the following asymptotic expression:

\[
\mu_n^{-1} \phi_n, \phi_n = (\mu_n^{-1} \cos(\mu_n(x-1)), \cosh(\mu_n(x-1))) + O(n^{-1}), \quad \lambda_n = \mu_n + O(n^{-1}),
\]

where \( \mu_n \) is given by

\[
\mu_n = \frac{1}{2} \ln \left| \frac{1-c}{1+c} \right| + j \pi \begin{cases} n, & |c| < 1, \\ n + \frac{1}{2}, & |c| > 1 \end{cases}
\]

where \( j^2 = -1 \). By the method in [23, Section 4], one can show that \( \{(\mu_n^{-1}\phi_n, \phi_n) : n = 0, 1, 2, \ldots \} \) forms a Riesz basis for \( \mathbb{H} \). Thus, \( A_1 \) with \( D(A_1) \) generates a \( C_0 \)-semigroup on \( \mathbb{H} \). Noticing that the eigenvalues and corresponding eigenfunctions of \( -A_1 \) are just \( \{-\lambda_n : n = 0, 1, 2, \ldots \} \) and \( (\mu_n^{-1}\phi_n, \phi_n) \). So, \(-A_1\) with \( D(-A_1) = D(A_1) \) is also a generator of a \( C_0 \)-semigroup. It follows from [7, Page 79] that \( A_1 \) generates a \( C_0 \)-group on \( \mathbb{H} \).

Next, we show that if \( c > 0 \) satisfying \( c \neq 1 \). \( A_1 \) generates an exponentially stable \( C_0 \)-semigroup on \( \mathbb{H} \). Since the eigenfunctions of \( A_1 \) form a Riesz basis for \( \mathbb{H} \), the spectrum-determined growth condition holds. In order to show that \( c A_1 \) is an exponentially stable semigroup, it suffices to prove that \( \text{Re} \lambda < 0 \) for any \( \lambda \in \sigma(A_1) \). Actually, a simple computation gives, for any \( (\phi, \psi)^\top \in \mathbb{H} \), that

\[
\text{Re}(A_1(\phi, \psi)^\top, (\phi, \psi)^\top) = -c|\psi|^2 \leq 0,
\]

which implies that for any \( \lambda \in \sigma(A_1) \) must satisfy \( \text{Re} \lambda < 0 \). Since \( A_1^{-1} \) is compact, we only need to show that there is no eigenvalue on the imaginary axis. Let \( \lambda = j \tau \in \sigma(A_1) \) with \( \tau \in \mathbb{R}^+ \) and the corresponding eigenfunction \( (\phi, \psi)^\top \in D(A_1) \), by (6.5), we have

\[
\text{Re}(A_1(\phi, \psi)^\top, (\phi, \psi)^\top) = -c|\psi|^2 \leq 0,
\]

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and hence $\psi(0) = 0$. Furthermore, $A_1(\phi, \psi)^\top = j\tau^2(\phi, \psi)^\top$ gives that $\psi = j\tau^2 \phi$ with $\phi$ satisfying
\begin{equation}
\begin{aligned}
\phi''(x) &= -\tau^4 \phi(x), \\
\phi(0) &= \phi'(0) = 0, \quad \phi'(1) = 1.
\end{aligned}
\end{equation}
(6.7)

It is clear that the above equation admits only zero solution. Thus, there is no eigenvalue on the imaginary axis. □

Let $\mathfrak{H} := H^1(0,1) \times L^2(0,1)$ with the inner product given by
\begin{equation}
\langle (\phi_1, \psi_1)^\top, (\phi_2, \psi_2)^\top \rangle = \int_0^1 \left[ \phi_1'(x) \bar{\phi}_2(x) + \phi_1(x) \bar{\psi}_2(x) \right] dx.
\end{equation}

Define the operator $A_2 : D(A_2)(\subset \mathbb{H}) \to \mathfrak{H}$ as follows:
\begin{equation}
\begin{aligned}
A_2(\phi, \psi)^\top &= (\psi, \phi)^\top, \\
D(A_2) &= \{ (\phi, \psi)^\top \in \mathbb{H} \cap (H^2(0,1) \times H^1(0,1)) : \phi(0) = c_1 \psi(0), \quad \phi'(1) = -c_2 \phi(1) \}.
\end{aligned}
\end{equation}
(6.8)

Lemma 6.2. For any $c_1, c_2 \in \mathbb{R}$ and $c_1 \neq \pm 1$, $c_2 \neq 0$, $A_2$ generates a $C_0$-group on $\mathbb{H}$. Moreover, if $c_1, c_2 > 0$ and $c_1 \neq 1$, $A_2$ generates an exponentially stable $C_0$-semigroup on $\mathbb{H}$.

Proof. We first show that $A_2$ generates a $C_0$-group on $\mathbb{H}$. A direct computation shows that
\begin{equation}
A_2^{-1}(\phi, \psi)^\top = \left( \frac{1}{c_1} c_1 \phi(0) (c_2 + 1) + \int_0^1 \psi(\xi) d\xi + c_2 \int_0^1 (1 - \xi) \psi(\xi) d\xi \right) x - c_1 \phi(0) x,
\end{equation}
and $\lambda$ satisfies
\begin{equation}
\psi(x) = \cosh(\lambda x) + c_1 \sinh(\lambda x),
\end{equation}
(6.10)

and $\lambda$ satisfies
\begin{equation}
\lambda^2 = \frac{(\lambda - c_2)(1 - c_1)}{(\lambda + c_2)(1 + c_1)}.
\end{equation}
(6.11)

Further, we obtain the eigenfunctions and eigenvalues with the following asymptotic expression:
\begin{equation}
\begin{aligned}
\left( \mu_n^{-1} \phi_n, \phi_n \right) &= \left( \mu_n^{-1} \left[ \cosh(\mu_n x) + c_1 \sinh(\mu_n x) \right], \cosh(\mu_n x) + c_1 \sinh(\mu_n x) \right), \\
\left| \mu_n \right| &= n + O(n^{-1}),
\end{aligned}
\end{equation}
where $\mu_n$ is given by
\begin{equation}
\mu_n = \frac{1}{2} \ln \left[ \frac{1 - c_1}{1 + c_1} + j \tau \left( \frac{1}{n + \frac{1}{2}}, \left| \frac{c_1}{n} \right| < 1 \right. \right],
\end{equation}
(6.12)

which implies that for any $\lambda \in \sigma(A_2)$ must satisfy $\text{Re}\lambda < 0$. Since $A_2^{-1}$ is compact, we only need to show that there is no eigenvalue on the imaginary axis. Let $\lambda = j\tau^2 \in \sigma(A_2)$ with $\tau \in \mathbb{R}^+$ and the corresponding eigenfunction $(\phi, \psi)^\top \in D(A_2)$. By (6.12),
\begin{equation}
\text{Re}(A_2(\phi, \psi)^\top, (\phi, \psi)^\top)_\mathfrak{H} = \text{Re}(j\tau^2(\phi, \psi)^\top, (\phi, \psi)^\top)_\mathfrak{H} = -c_1 |\psi(0)|^2 < 0,
\end{equation}
(6.13)

and hence $\psi(0) = 0$. Furthermore, $A_2(\phi, \psi)^\top = j\tau^2(\phi, \psi)^\top$ gives that $\psi = j\tau^2 \phi$ with $\phi$ satisfying
\begin{equation}
\phi''(x) = -\tau^4 \phi(x), \\
\phi(0) = \phi'(0) = 0, \quad \phi'(1) = -c_2 \phi(1).
\end{equation}
(6.14)

It is clear that the above equation admits only zero solution. Thus, there is no eigenvalue on the imaginary axis. □

Let $\mathfrak{H} := H^1(0,1) \times L^2(0,1)$ with the inter product given by
\begin{equation}
\langle (\phi, \psi)^\top, (\phi, \psi)^\top \rangle = \int_0^1 \left[ \phi_1'(x) \bar{\phi}_2(x) + \phi_1(x) \bar{\psi}_2(x) \right] dx.
\end{equation}

Define the operator $A_3 : D(A_3)(\subset \mathbb{H}) \to \mathfrak{H}$ as follows:
\begin{equation}
\begin{aligned}
A_3(\phi, \psi)^\top &= (\psi, \phi)^\top, \\
D(A_3) &= \{ (\phi, \psi)^\top \in \mathbb{H} \cap (H^2(0,1) \times H^1(0,1)) : \phi(0) = c_1 \psi(0), \quad \phi'(1) = 0 \}.
\end{aligned}
\end{equation}
(6.15)

Lemma 6.3. For any $c \in \mathbb{R}$ and $c \neq \pm 1$, $A_3$ generates a $C_0$-group on $\mathbb{H}$. Moreover, if $c > 0$ satisfying $c \neq 1$, $A_3$ generates an exponentially stable $C_0$-semigroup on $\mathbb{H}$.

Proof. We first show that $A_3$ generates a $C_0$-group on $\mathbb{H}$. A direct computation shows that the eigenvalues $\lambda_n$ of $A_3$ is given by
\begin{equation}
\lambda_n = \frac{1}{2} \ln \left[ \frac{1 - c}{1 + c} + j \tau \left( \frac{1}{n + \frac{1}{2}}, \left| \frac{c}{n} \right| < 1 \right. \right],
\end{equation}
and the corresponding eigenfunctions $(f_n(x), g_n(x))^\top$ of $A_3$ are given by
\begin{equation}
(f_n, g_n)^\top = \frac{\sinh \lambda_n (x - 1)}{\lambda_n}, \quad \sinh \lambda_n (x - 1)^\top.
\end{equation}

It follows from [23, Section 4] that $(f_n, g_n)^\top : n = 0, 1, 2, \ldots$ forms a Riesz basis for $\mathfrak{H}$. Thus, $A_3$ with $D(A_3)$ generates a $C_0$-semigroup on $\mathfrak{H}$. By the asymptotic expression of eigenvalues, $A_3$ generates a $C_0$-group on $\mathbb{H}$ as well.

Next, we show that if $c > 0$, $A_3$ generates an exponentially stable $C_0$-semigroup on $\mathbb{H}$. Since the eigenfunctions of $A_3$ form a Riesz basis for $\mathfrak{H}$, the spectrum-determined growth condition holds. Noticing $c > 0$ and $c \neq 1$, Re$\lambda < 0$ for all $\lambda \in \sigma(A_3)$, we conclude that $\mathcal{A}^{\text{vis}}$ is an exponentially stable $C_0$-semigroup. □

References

[8] H. Feng, B.Z. Guo, New unknown input observer and output feedback stabiliza-
[10] B.Z. Guo, F.F. Jin, Output feedback stabilization for one-dimensional wave equa-
Please cite this article as: H.-C. Zhou, B.-Z. Guo, Performance output tracking for one-dimensional wave equation subject to unmatched general disturbance and non-collocated control, European Journal of Control (2017), https://doi.org/10.1016/j.ejcon.2017.10.006