Performance Output Tracking for Multidimensional Heat Equation Subject to Unmatched Disturbance and Noncollocated Control

Hua-Cheng Zhou, *Member, IEEE*, Bao-Zhu Guo, *Senior Member, IEEE*, and Shu-Huang Xiang

***Abstract*—This paper investigates performance output tracking for a boundary controlled multidimensional heat equation. It is assumed that the so-called total disturbance (which is composed of internal possibly nonlinear uncertainty and external disturbance) the equation is subject to is on one part of the boundary, and that the control is applied on the rest of the boundary. Using only partial boundary measurement, we first propose an extended state observer to estimate both system state and the total disturbance. This allows us to design a servomechanism and then an output feedback controller.Under the condition that both the reference signal and the disturbance vanish or belong to spaces $H^1(0,\infty;L^1(\Gamma_1))$ and $L^2(0,\infty;L^2(\Gamma_0))$, respectively. We show that the over-all control strategy achieves three objectives on the system performance: first, exponentially output tracking for arbitrary given reference signal; second, uniformly boundedness of all internal signals; and, third, the internal asymptotic stability of the closed-loop system. In addition, the control strategy turns out to be robust to the measurement noise. We provide numerical experiments to illustrate the effectiveness of the proposed control strategy.**

***Index Terms*—Active disturbance rejection control (ADRC), disturbance rejection, heat equation, output tracking.**

Manuscript received April 2, 2018; revised September 7, 2018, January 28, 2019, and June 12, 2019; accepted June 21, 2019. Date of publication July 1, 2019; date of current version April 23, 2020. This work was supported in part by the National Natural Science Foundation of China under Grant 61803386 and Grant 61873260, in part by the Israel Science Foundation under Grant 800/14, and in part by the Project of Department of Education of Guangdong Province under Grant 2017KZDXM087. Recommended by Associate Editor Prof. Y. Le Gorrec. *(Corresponding author: Bao-Zhu Guo.)*

H.-C. Zhou is with the School of Mathematics and Statistics, Central South University, Changsha 410075, China (e-mail: hczhou@amss.ac.cn).

B.-Z. Guo is with the Department of Mathematics and Physics, North China Electric Power University, Beijing 102206, China, and with the Key Laboratory of System and Control, Academy of Mathematics and Systems Science, Academia Sinica, Beijing 100190, China, and also with the School of Mathematical Sciences, Shanxi University, Taiyuan 030006, China (e-mail: bzguo@iss.ac.cn).

S.-H. Xiang is with the School of Mathematics and Statistics, Central South University, Changsha, 410075, China (e-mail: xiangsh@mail.csu.edu.cn).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAC.2019.2926132

I. Introduction

OUTPUT tracking is one of the fundamental issues in control theory. In many situations, output tracking is the only major concern for a control system. For this purpose, it is also required that all loops are uniformly bounded and the system is internally asymptotically stable. The problem of output tracking has been studied systematically for lumped parameter systems under the title of output regulation with modeled disturbance since from [2], [3], and [6]. Part of the results (notably the internal model principle) have been generalized to the infinite-dimensional systems, see, for instance, [1], [4], [5], [16], [21], [25]–[27], among many others. In these output regulation results, the reference signal and disturbance are limited to outputs generated by exosystems. And in the case of finite number of harmonic signals, it is also required that the frequencies are known or determined *a priori*. This requirement seems quite natural in the sense that for nonminimal phase systems, arbitrary reference tracking is not possible unless noncausal feedback is used. Several surveys dedicates to the frequencies estimation [19] where the order of parameter update law is set to be the same as the number of frequencies. A first attempt on handling infinite-dimensional signal is [15] where general periodic signals that has infinitely many frequencies were considered. A recent interesting work is [22] where the output tracking problem was considered for a general 2×2 system of first-order linear hyperbolic partial differential equations (PDEs), though no uncertainty or disturbance were taken into consideration. To the best of our knowledge, very few work considered general disturbance rejection in the context of output tracking for PDEs. In [32], performance output tracking for a one-dimensional (1-D) wave equation with a general boundary disturbance was studied, which has been generalized to include both internal uncertainty and external disturbance in our recent work [34]. In both [32] and [34], a new control method called active disturbance rejection control (ADRC) has been used in achieving output tracking. In the ADRC, the disturbance is first estimated by an extended state observer (ESO) and is then compensated in the feedback loop. A remarkable characteristic of [32] and [34] is that the control and disturbance can be unmatched. This is very different from the stabilization using ADRC, where the control and disturbance need to be matched [8], [9]. Very recently, a noncollocated output tracking problem was investigated in [12]–[14] by the adaptive control method, which has also been used in an early effort in [11]. Note that all these works are limited to 1-D PDEs only.

In this paper, we solve the performance output tracking problem with general disturbance and reference for an uncertain multidimensional heat equation by the ADRC approach. This is a first fruitful effort on output tracking for multidimensional PDE with arbitrary given reference signal and general uncertainty including internal uncertainty and external disturbance. In the same spirit of [7] on stabilization of uncertain PDE via ADRC, here, we do not use high gain for disturbance estimation and the output feedback control is shown to be robust to measurement noise. This paper is motivated from a recent paper [17], which considered a similar problem for 1-D heat equation with general external disturbance only (without internal uncertainty). We confine ourselves to the case where the control and the performance output are on the same part of the boundary (another collocated case).

The system we consider in this paper is described by a multidimensional heat equation with Neumann boundary control and unknown noncollocated internal nonlinear uncertainty and external disturbance

$$\begin{cases} w_t(x,t) = \Delta w(x,t), \ x \in \Omega, \ t > 0 \\ \frac{\partial w(x,t)}{\partial \nu}|_{\Gamma_0} = f(w(\cdot,t)) + d(x,t), \ t \geq 0 \\ \frac{\partial w(x,t)}{\partial \nu}|_{\Gamma_1} = u(x,t), \ t \geq 0 \\ w(x,0) = w_0(x) \\ y_{\rm m}(x,t) = w(x,t)|_{\Gamma_0}, \ t \geq 0 \\ y_{\rm o}(x,t) = w(x,t)|_{\Gamma_1}, \ t \geq 0 \end{cases} \quad (1)$$

where we denote by $w'(x,t)$ or $w_x(x,t)$ the derivative of $w(x,t)$ with respect to x and $\dot{w}(x,t)$ or $w_t(x,t)$ the derivative of $w(x,t)$ with respect to t, and $w_0(x)$ the initial state; $\Omega \subset \mathbb{R}^n (n \geq 2)$ is an open bounded domain with a smooth C^2-boundary $\Gamma = \overline{\Gamma_0} \cup \overline{\Gamma_1}$ with Γ_0 and Γ_1 subsets of Γ satisfying $\text{int}(\Gamma_0) \neq \emptyset$, $\text{int}(\Gamma_1) \neq \emptyset$, $\Gamma_0 \cap \Gamma_1 = \emptyset$; ν is the unit normal vector of Γ pointing the exterior of Ω. $u(x,t)$ is the control input, $y_{\rm m}(x,t)$ is the measured output, $y_{\rm o}(x,t)$ is the performance output signal to be regulated, $f : L^2(\Omega) \to L^2(\Gamma_0)$ is a possibly unknown nonlinear mapping, which represents internal uncertainty.

A typical example is $f(w) = \gamma(x) \int_\Omega w^2(x)dx$ where $\gamma \in L^2(\Gamma_0)$ and $w \in L^2(\Omega)$. By the state for this example, we have $f(w(\cdot,t)) = \gamma(x) \int_\Omega w^2(x,t)dx$, which depends on the value of the state in the whole spatial domain, explaining why we call it the "internal uncertainty." $d(x,t)$ is the unknown external disturbance, which comes from outside of the system and is supposed to satisfy $d \in L^\infty(0,\infty; L^2(\Gamma_0))$. Note that the left boundary side $\frac{\partial w(x,t)}{\partial \nu}|_{\Gamma_0}$ represents physically the heat flux on the boundary Γ_0 and the $d(x,t)$ may represents physically the ambient temperature that affects the heat flux from the boundary Γ_0. For the sake of simplicity, we denote

$$F(w,t) := f(w(\cdot,t)) + d(x,t) \quad (2)$$

as the "total disturbance." System (1) will be discussed in the usual state space $L^2(\Omega)$ and the control space $U = L^2(\Gamma_1)$.

Let $W^{1,\infty}(0,\infty; L^2(\Gamma_1)) = \{\phi : \phi \in L^\infty(0,\infty; L^2(\Gamma_1)), \phi_t \in L^\infty(0,\infty; L^2(\Gamma_1))\}$. Our problem can be stated as follows: For a given reference signal

$$r \in W^{1,\infty}(0,\infty; L^2(\Gamma_1))$$

design an output feedback control for uncertain system (1) to reject the external disturbance and achieve output tracking

$$\|e(\cdot,t)\|_{L^2(\Gamma_1)} = \|y_{\rm o}(\cdot,t) - r(\cdot,t)\|_{L^2(\Gamma_1)} \to 0 \text{ as } t \to \infty. \quad (3)$$

We proceed as follows. In Section II, we design for system (1) an ESO, which serves as an unknown input observer. We show that this ESO gives an asymptotical approximation of the total disturbance. This spirit of ADRC can be found in many other papers [8]–[10], [33] on stabilization of PDEs via ADRC approach. In Section III, we design a servo system in terms of measured output and state of ESO, which is used to make the state of the original system track the state of the servo system. This results in the boundedness of all-loops while achieving output tracking (a big challenge in output tracking for PDEs). An output feedback control is then designed after compensation of the disturbance in the performance output. Section IV is devoted to well posedness and convergence of the closed-loop system. All mathematical proofs are arranged in Section V. Some numerical simulations are presented in Section VI to demonstrate the effectiveness of the proposed control. Some concluding remarks are given in Section VII. The well posedness of the open-loop system is presented in the Appendix.

II. Extended State Observer

In this section, we design an ESO that can estimate not only the state $w(x,t)$ of the controlled system (1) but also the total disturbance $F(w,t)$. This ESO can, thus, serve as a natural unknown input observer for system (1). The ESO is designed as follows:

$$\begin{cases} \widehat{w}_t(x,t) = \Delta \widehat{w}(x,t), \ x \in \Omega, \ t > 0 \\ \widehat{w}(x,t)|_{\Gamma_0} = y_{\rm m}(x,t), \ t \geq 0 \\ \frac{\partial \widehat{w}(x,t)}{\partial \nu}|_{\Gamma_1} = u(x,t), \ t \geq 0 \\ \widehat{w}(x,0) = \widehat{w}_0(x) \end{cases} \quad (4)$$

where $\widehat{w}_0 \in L^2(\Omega)$ is an arbitrary given initial value of (4). It is seen that system (4) is completely determined by the measured output $y_{\rm m}(x,t)$ and the control input $u(x,t)$ of system (1). The well posedness of (4) is presented in the Appendix.

Let $\widetilde{w}(x,t) = \widehat{w}(x,t) - w(x,t)$ be the error, which is governed by

$$\begin{cases} \widetilde{w}_t(x,t) = \Delta \widetilde{w}(x,t), \ x \in \Omega, \ t > 0 \\ \widetilde{w}(x,t)|_{\Gamma_0} = 0, \ t \geq 0 \\ \frac{\partial \widetilde{w}(x,t)}{\partial \nu}|_{\Gamma_1} = 0, \ t \geq 0. \end{cases} \quad (5)$$

We consider (5) in the state space $L^2(\Omega)$ as well.

Lemma II.1: For any initial value $\widetilde{w}(\cdot,0) \in L^2(\Omega)$, system (5) admits a unique solution $\widetilde{w} \in C(0,\infty; L^2(\Omega))$ such that $\|\widetilde{w}(\cdot,t)\|_{L^2(\Omega)} \leq e^{-t} \|\widetilde{w}(\cdot,0)\|_{L^2(\Omega)}$. Moreover, for any given positive integer m, there exist constants $M_m, \mu > 0$ such that

$$\left\|\frac{\partial \widetilde{w}(\cdot,t)}{\partial \nu}\right\|_{L^2(\Gamma_0)} \le \left(\frac{M_m}{t^m} + M_m e^{-\mu t}\right)\|\widetilde{w}(\cdot,0)\|_{L^2(\Omega)} \quad \forall\, t>0 \tag{6}$$

and

$$\|\widetilde{w}(\cdot,t)\|_{L^2(\Gamma_1)} \le \left(\frac{M_m}{t^m} + M_m e^{-\mu t}\right)\|\widetilde{w}(\cdot,0)\|_{L^2(\Omega)} \quad \forall\, t>0. \tag{7}$$

In addition, for any fixed $T>0$, there exist two constants $M_0, \mu_0 > 0$ depending on T such that

$$\left\|\frac{\partial \widetilde{w}(\cdot,t)}{\partial \nu}\right\|_{L^2(\Gamma_0)} \le M_0 e^{-\mu_0 t}\|\widetilde{w}(\cdot,0)\|_{L^2(\Omega)} \quad \forall\, t \ge T \tag{8}$$

and

$$\|\widetilde{w}(\cdot,t)\|_{L^2(\Gamma_1)} \le M_0 e^{-\mu_0 t}\|\widetilde{w}(\cdot,0)\|_{L^2(\Omega)} \quad \forall\, t \ge T. \tag{9}$$

Remark 1: In general, for a function $\widetilde{w} \in L^2(\Omega)$, the Sobolev trace theorem does not imply $\frac{\partial \widetilde{w}(\cdot)}{\partial \nu} \in L^2(\Gamma_0)$ and $\widetilde{w}(\cdot) \in L^2(\Gamma_0)$, and thus, $\frac{\partial \widetilde{w}(\cdot)}{\partial \nu}$ and $\widetilde{w}(\cdot)$ do not make sense in $L^2(\Gamma_0)$. However, when $\widetilde{w}(\cdot,t) \in L^2(\Omega)$ is the (weak) solution of (5), it has some smoothness for $t>0$, which leads to $\frac{\partial \widetilde{w}(\cdot,t)}{\partial \nu} \in L^2(\Gamma_0)$ and $\widetilde{w}(\cdot,t) \in L^2(\Gamma_0)$ for $t>0$ except $t=0$ (hidden regularity). Moreover, the norms $L^2(\Gamma_0)$ and $L^2(\Gamma_1)$ in (6)–(9) can be replaced by the stronger norms $L^\infty(\Gamma_0)$ and $L^\infty(\Gamma_1)$, respectively. Actually, take $m>n$. By (27), the Sobolev embedding theorem, and the trace theorem, we have $\|\widetilde{w}(\cdot,t)\|_{L^\infty(\Gamma)}, \|\nabla \widetilde{w}(\cdot,t)\|_{L^\infty(\Gamma)} \le C_4\|\widetilde{w}(\cdot,t)\|_{H^{2m-1/2}(\Gamma)} \le C_4C_5\|\widetilde{w}(\cdot,t)\|_{H^{2m}(\Omega)}$ for some $C_4, C_5 > 0$. Since $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\frac{\partial \widetilde{w}(\cdot,t)}{\partial \nu} = \nabla \widetilde{w}\cdot\nu$, by (27), we have furthermore that

$$\left\|\frac{\partial \widetilde{w}(\cdot,t)}{\partial \nu}\right\|_{L^\infty(\Gamma_0)} \le \left(\frac{M_m}{t^m} + M_m e^{-\mu t}\right)\|\widetilde{w}(\cdot,0)\|_{L^2(\Omega)} \quad \forall\, t>0$$

and

$$\|\widetilde{w}(\cdot,t)\|_{L^\infty(\Gamma_1)} \le \left(\frac{M_m}{t^m} + M_m e^{-\mu t}\right)\|\widetilde{w}(\cdot,0)\|_{L^2(\Omega)} \quad \forall\, t>0.$$

From Lemma II.1, system (4) can be regarded as an unknown input observer of system (1), and from the fact

$$\frac{\partial \widetilde{w}(\cdot,t)}{\partial \nu} = \frac{\partial \widehat{w}(\cdot,t)}{\partial \nu} - \frac{\partial w(\cdot,t)}{\partial \nu} = \frac{\partial \widehat{w}(\cdot,t)}{\partial \nu} - F(w,t)$$

(6) and (8), we have, for sufficient large t, that

$$\frac{\partial \widehat{w}(x,t)}{\partial \nu}|_{\Gamma_0} \approx F(w,t).$$

Therefore, $\frac{\partial \widehat{w}(x,t)}{\partial \nu}|_{\Gamma_0}$ is an asymptotical estimation of the total disturbance $F(w,t)$. In other words, system (4) is an ESO for system (1).

III. Servomechanism

In this section, we design the following servomechanism for system (1) in terms of the reference signal $r(x,t)$ and the boundary values of ESO (4)

$$\begin{cases} v_t(x,t) = \Delta v(x,t),\ x\in\Omega,\ t>0 \\ \frac{\partial v(x,t)}{\partial \nu}|_{\Gamma_0} = -c_0(v(x,t) - \widehat{w}(x,t))|_{\Gamma_0} + \frac{\partial \widehat{w}(x,t)}{\partial \nu}|_{\Gamma_0} \\ v(x,t)|_{\Gamma_1} = r(x,t),\ t\ge 0 \end{cases} \tag{10}$$

where $c_0 > 0$ is a tuning design parameter. It is seen that system (10) is completely determined by the measured output of system (1), the output of ESO (4), and the reference signal $r(x,t)$ only. The term $\frac{\partial \widehat{w}(x,t)}{\partial \nu}|_{\Gamma_0}$ in (10) is used as a compensation to the total disturbance $F(w,t)$ in the original system (1). Design of system (10) is motivated by the facts: 1) it enables us to find an output feedback control law that allows its perfect tracking by ESO (4); and 2) once ESO (4) converges asymptotically (or exponentially) to servo system (10), then, by

$$\begin{aligned} e(x,t) &= y_o(x,t) - r(x,t) = w(x,t)|_{\Gamma_1} - v(x,t)|_{\Gamma_1} \\ &= [w(x,t)|_{\Gamma_1} - \widehat{w}(x,t)|_{\Gamma_1}] + [\widehat{w}(x,t)|_{\Gamma_1} - v(x,t)|_{\Gamma_1}] \end{aligned}$$

we can expect that $\|e(\cdot,t)\|_{L^2(\Gamma_1)}$ converges to zero as $t\to\infty$ because both $[w(x,t)|_{\Gamma_1} - \widehat{w}(x,t)|_{\Gamma_1}]$ and $[\widehat{w}(x,t)|_{\Gamma_1} - v(x,t)|_{\Gamma_1}]$ are expected to converge to zero on $L^2(\Gamma_1)$. Furthermore, we can show that system (10) is uniformly bounded for all time $t\ge 0$, which guarantees in turn the uniform boundedness of ESO (4) and, hence, (1). The last point is crucial in output regulation for PDEs.

Let $\varepsilon(x,t) = v(x,t) - \widehat{w}(x,t)$ be the error between the state $\widehat{w}(x,t)$ of ESO (4) and the state of servo system (10). Then, $\varepsilon(x,t)$ is governed by

$$\begin{cases} \varepsilon_t(x,t) = \Delta\varepsilon(x,t),\ x\in\Omega,\ t>0 \\ \frac{\partial \varepsilon(x,t)}{\partial \nu}|_{\Gamma_0} = -c_0\varepsilon(x,t)|_{\Gamma_0},\ t\ge 0 \\ \frac{\partial \varepsilon(x,t)}{\partial \nu}|_{\Gamma_1} = \frac{\partial v(x,t)}{\partial \nu}|_{\Gamma_1} - u(x,t),\ t\ge 0. \end{cases} \tag{11}$$

We propose the following output feedback control law

$$u(x,t) = \frac{\partial v(x,t)}{\partial \nu}|_{\Gamma_1} \tag{12}$$

under which, the resulting closed loop of (11) becomes

$$\begin{cases} \varepsilon_t(x,t) = \Delta\varepsilon(x,t),\ x\in\Omega,\ t>0 \\ \frac{\partial \varepsilon(x,t)}{\partial \nu}|_{\Gamma_0} = -c_0\varepsilon(x,t)|_{\Gamma_0},\ t\ge 0 \\ \frac{\partial \varepsilon(x,t)}{\partial \nu}|_{\Gamma_1} = 0,\ t\ge 0. \end{cases} \tag{13}$$

Lemma III.1: For any initial value $\varepsilon(\cdot,0)\in L^2(\Omega)$, system (13) admits a unique solution $\varepsilon \in C(0,\infty; L^2(\Omega))$ and there exist two constants $M,\mu>0$ such that $\|\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le Me^{-\mu t}\|\varepsilon(\cdot,0)\|_{L^2(\Omega)}$ and $\varepsilon \in L^2(0,\infty; L^2(\Gamma_0))$. Moreover, for any given positive integer m, there exists a constant $L_m > 0$ such that

$$\|\varepsilon(\cdot,t)\|_{L^2(\Gamma_1)} \le \left(\frac{L_m}{t^m} + L_m e^{-\mu t}\right)\|\varepsilon(\cdot,0)\|_{L^2(\Omega)} \quad \forall\, t>0. \tag{14}$$

In addition, for any fixed $T>0$, there exist two constants $\tilde{M}, \tilde{\mu} > 0$ depending on T such that

$$\|\varepsilon(\cdot,t)\|_{L^2(\Gamma_1)} \le \tilde{M}e^{-\tilde{\mu}t}\|\varepsilon(\cdot,0)\|_{L^2(\Omega)} \quad \forall\, t\ge T. \tag{15}$$

In the rest of this section, we show that the solution of system (10) is uniformly bounded for all time $t \geq 0$. To this end, we separate system (10) into two subsystems p and q, which are described, respectively, by

$$\begin{cases} p_t(x,t) = \Delta p(x,t),\ x \in \Omega,\ t > 0 \\ \frac{\partial p(x,t)}{\partial \nu}|_{\Gamma_0} = -c_0 \varepsilon(x,t)|_{\Gamma_0} + \frac{\partial \widetilde{w}(x,t)}{\partial \nu}|_{\Gamma_0} \\ \qquad + F(p+q-\varepsilon-\widetilde{w},t),\ t \geq 0 \\ p(x,t)|_{\Gamma_1} = 0,\ t \geq 0 \end{cases} \tag{16}$$

and

$$\begin{cases} q_t(x,t) = \Delta q(x,t),\ x \in \Omega,\ t > 0 \\ \frac{\partial q(x,t)}{\partial \nu}|_{\Gamma_0} = 0,\ t \geq 0 \\ q(x,t)|_{\Gamma_1} = r(x,t),\ t \geq 0. \end{cases} \tag{17}$$

Clearly, the relation among the solution of (10) and the solutions of (16) and (17) is $v(x,t) = p(x,t) + q(x,t)$ and system (17) is independent of system (16).

Lemma III.2: Suppose that $r \in W^{1,\infty}(0,\infty; L^2(\Gamma_1))$. Then, for any initial value $q(\cdot,0)) \in L^2(\Omega)$, system (17) admits a unique solution $q \in C(0,\infty; L^2(\Omega))$, which is uniformly bounded for all $t \geq 0$, i.e, $\sup_{t\geq 0} \|q(\cdot,t)\|_{L^2(\Omega)} < \infty$. Moreover, $\lim_{t\to\infty} \|q(\cdot,t)\|_{L^2(\Omega)} = 0$ whenever $r \in H^1(0,\infty; L^2(\Gamma_1))$.

Now, we claim that system (16) is uniformly bounded for all time $t \geq 0$. To this end, we consider the coupled systems, which are composed of (16) and (5) as follows:

$$\begin{cases} \widetilde{w}_t(x,t) = \Delta \widetilde{w}(x,t),\ x \in \Omega,\ t > 0 \\ \widetilde{w}(x,t)|_{\Gamma_0} = 0,\ \frac{\partial \widetilde{w}(x,t)}{\partial \nu}|_{\Gamma_1} = 0,\ t \geq 0 \\ p_t(x,t) = \Delta p(x,t),\ x \in \Omega,\ t > 0 \\ \frac{\partial p(x,t)}{\partial \nu}|_{\Gamma_0} = -c_0 \varepsilon(x,t)|_{\Gamma_0} + \frac{\partial \widetilde{w}(x,t)}{\partial \nu}|_{\Gamma_0} \\ \qquad + F(p+q-\varepsilon-\widetilde{w},t),\ t \geq 0 \\ p(x,t)|_{\Gamma_1} = 0,\ t \geq 0. \end{cases} \tag{18}$$

Let us consider system (18) in the energy Hilbert state space $[L^2(\Omega)]^2$ with the usual inner product. System (18) can be rewritten as an evolution equation in $[L^2(\Omega)]^2$

$$\frac{d}{dt}(\widetilde{w}(\cdot,t), p(\cdot,t)) = \mathbb{A}(\widetilde{w}(\cdot,t), p(\cdot,t)) + \mathbb{B}[F(p+q-\varepsilon-\widetilde{w},t) - c_0\varepsilon(x,t)] \tag{19}$$

where $\mathbb{B} = (0, \delta|_{\Gamma_0})$ with $\delta|_{\Gamma_0}$ being the Dirac function and the operator $\mathbb{A}$ is given by

$$\begin{cases} \mathbb{A}(\phi,\psi) = (\Delta\phi, \Delta\psi) \quad \forall\ (\phi,\psi) \in D(\mathbb{A}) \\ D(\mathbb{A}) = \Big\{ (\phi,\psi) \in H^2(\Omega):\ \phi|_{\Gamma_0} = 0,\ \frac{\partial \phi}{\partial \nu}|_{\Gamma_1} = 0 \\ \qquad \frac{\partial \psi}{\partial \nu}|_{\Gamma_0} = \frac{\partial \phi}{\partial \nu}|_{\Gamma_0},\ \psi|_{\Gamma_1} = 0 \Big\}. \end{cases} \tag{20}$$

It is readily found that $\mathbb{B}^*(\phi,\psi) = \psi|_{\Gamma_0}$ for all $(\phi,\psi) \in [H^1(\Omega)]^2$ and

$$\begin{cases} \mathbb{A}^*(\phi,\psi) = (\Delta\phi, \Delta\psi) \quad \forall\ (\phi,\psi) \in D(\mathbb{A}^*) \\ D(\mathbb{A}^*) = \Big\{ (\phi,\psi) \in H^2(\Omega):\ \phi|_{\Gamma_0} = -\psi|_{\Gamma_0} \\ \qquad \frac{\partial \phi}{\partial \nu}|_{\Gamma_1} = 0,\ \frac{\partial \psi}{\partial \nu}|_{\Gamma_0} = 0,\ \psi|_{\Gamma_1} = 0 \Big\}. \end{cases} \tag{21}$$

Lemma III.3: The operator $\mathbb{A}$ given by (20) generates an exponentially stable C_0-semigroup $e^{\mathbb{A}t}$. Moreover, $\mathbb{B}$ is admissible for $e^{\mathbb{A}t}$.

Now we give the existence and boundedness of the solution to (16).

Lemma III.4: Suppose that $r \in W^{1,\infty}(0,\infty; L^2(\Gamma_1))$, $d \in L^\infty(0,+\infty; L^2(\Gamma_0))$, $f: L^2(\Omega) \to L^2(\Gamma_0)$ is continuous, bounded, and satisfies the local Lipschitz condition in $L^2(\Omega)$. Let $\varepsilon(x,t)$ be the solution of (13). Then, for any initial value $q(\cdot,0) \in L^2(\Omega)$, system (16) admits a unique solution $p(\cdot,t) \in C(0,\infty; L^2(\Omega))$, which is uniformly bounded for all $t \geq 0$, i.e, $\sup_{t\geq 0} \|p(\cdot,t)\|_{L^2(\Omega)} < \infty$. Moreover, $\lim_{t\to\infty} \|p(\cdot,t)\|_{L^2(\Omega)} = 0$ whenever $f \equiv 0$ and $d \in L^2(0,\infty; L^2(\Gamma_0))$.

Next we state the well posedness and boundedness of servo system (10).

Lemma III.5: Suppose that $r \in W^{1,\infty}(0,\infty; L^2(\Gamma_1))$, $d \in L^\infty(0,+\infty; L^2(\Gamma_0))$, $f: L^2(\Omega) \to L^2(\Gamma_0)$ is continuous, bounded, and satisfies the local Lipschitz condition in $L^2(\Omega)$, and $\frac{\partial \widehat{w}(x,t)}{\partial \nu}|_{\Gamma_0}$ is determined by (4). Then, for any initial value $v(\cdot,0) \in L^2(\Omega)$, system (10) admits a unique solution $v \in C(0,\infty; L^2(\Omega))$, which is uniformly bounded for all $t \geq 0$, i.e., $\sup_{t\geq 0} \|v(\cdot,t)\|_{L^2(\Omega)} < +\infty$. Moreover, $\lim_{t\to\infty} \|v(\cdot,t)\|_{L^2(\Omega)} = 0$ whenever $f \equiv 0$, $d \in L^2(0,\infty; L^2(\Gamma_0))$, and $r \in H^1(0,\infty; L^2(\Gamma_0))$.

IV. Closed-Loop System

In this section, we establish the well posedness and performance output tracking of the closed-loop system of (1). Under the control law (12), the closed-loop system is composed of (1), (4), and (10) as follows:

$$\begin{cases} w_t(x,t) = \Delta w(x,t),\ x \in \Omega,\ t > 0 \\ \frac{\partial w(x,t)}{\partial \nu}|_{\Gamma_0} = f(w(\cdot,t)) + d(x,t),\ t \geq 0 \\ \frac{\partial w(x,t)}{\partial \nu}|_{\Gamma_1} = \frac{\partial v(x,t)}{\partial \nu}|_{\Gamma_1},\ t \geq 0 \\ \widehat{w}_t(x,t) = \Delta \widehat{w}(x,t),\ x \in \Omega,\ t > 0 \\ \widehat{w}(x,t)|_{\Gamma_0} = y_{\rm m}(x,t),\ t \geq 0 \\ \frac{\partial \widehat{w}(x,t)}{\partial \nu}|_{\Gamma_1} = \frac{\partial v(x,t)}{\partial \nu}|_{\Gamma_1},\ t \geq 0 \\ v_t(x,t) = \Delta v(x,t),\ x \in \Omega,\ t > 0 \\ \frac{\partial v(x,t)}{\partial \nu}|_{\Gamma_0} = -c_0(v(x,t) - \widehat{w}(x,t))|_{\Gamma_0} \\ \qquad + \frac{\partial \widehat{w}(x,t)}{\partial \nu}|_{\Gamma_0} \\ v(x,t)|_{\Gamma_1} = r(x,t),\ t \geq 0. \end{cases} \tag{22}$$

We consider system (22) in the state space $[L^2(\Omega)]^3$ with the usual inner product.

Theorem IV.1: Let $c_0 > 0$. Suppose that $d \in L^\infty(0,+\infty; L^2(\Gamma_0))$, $r \in W^{1,\infty}(0,\infty; L^2(\Gamma_1))$, and $f : L^2(\Omega) \to L^2(\Gamma_0)$ is continuous, bounded, and satisfies the local Lipschitz condition in $L^2(\Omega)$. Then, for any initial value $(w(\cdot,0), \widehat{w}(\cdot,0), v(\cdot,0)) \in [L^2(\Omega)]^3$, the closed-loop system (22) admits a unique solution $(w,\widehat{w},v) \in C(0,\infty;[L^2(\Omega)]^3)$, which has the following properties:

i)
$$\sup_{t\geq 0}\left(\int_\Omega [w^2(x,t)+\widehat{w}^2(x,t)+v^2(x,t)]dx\right) < +\infty.$$

ii) There exist two constants M_L, which depends on the initial value $(w(\cdot,0),\widehat{w}(\cdot,0))$ only and $\mu_L > 0$, which is independent of initial value, such that
$$\int_\Omega [\widehat{w}(x,t)-w(x,t)]^2 dx \leq M_L e^{-\mu_L t} \quad \forall\, t \geq 0.$$

iii) For any given positive integer m, there exist two constants $M_p, \mu_p > 0$ such that
$$\|e(\cdot,t)\|_{L^2(\Gamma_1)} = \|y_o(\cdot,t) - r(\cdot,t)\|_{L^2(\Gamma_1)} \leq \frac{M_p}{t^m} + M_p e^{-\mu_p t} \quad \forall\, t > 0$$
and for any fixed $T > 0$, there exist two constants M' which depends on initial value $(w(\cdot,0),\widehat{w}(\cdot,0),v(\cdot,0))$ only, and $\mu' > 0$ which is independent of initial value, such that
$$\|e(\cdot,t)\|_{L^2(\Gamma_1)} = \|y_o(\cdot,t) - r(\cdot,t)\|_{L^2(\Gamma_1)} \leq M' e^{-\mu' t} \quad \forall\, t \geq T$$
where M_p and M' depend on the initial value $(w(\cdot,0), \widehat{w}(\cdot,0), v(\cdot,0))$.

iv) When $f \equiv 0$, $d \in L^2(0,\infty; L^2(\Gamma_0))$, $r \in H^1(0,\infty; L^2(\Gamma_1))$, system (22) is internally asymptotically stable:
$$\lim_{t\to\infty}\left(\int_\Omega [w^2(x,t)+\widehat{w}^2(x,t)+v^2(x,t)]dx\right) = 0.$$

v) When $f \equiv 0$, $d \equiv 0$, and $r \equiv 0$, system (22) is internally exponentially stable, i.e., there exists two constants M'', which depends on initial value only and $\mu'' > 0$, which is independent of initial value, such that
$$\int_\Omega [w^2(x,t)+\widehat{w}^2(x,t)+v^2(x,t)]dx \leq M'' e^{-\mu'' t}.$$

Remark 2: From Remark 1, it is seen that the norm $L^2(\Gamma_1)$ in (iii) of Theorem IV.1 can be replaced by the stronger norm $L^\infty(\Gamma_1)$.

Remark 3: From Lemmas II.1 and III.1, and the fact $\widetilde{w}(x,t) = \widehat{w}(x,t) - w(x,t)$, $\varepsilon(x,t) = v(x,t) - \widehat{w}(x,t)$, in the closed-loop system (22), both the $\widehat{w}$-part and the v-part are regarded as the state observer of (1). However, their roles are different in that the $\widehat{w}$-part is used to estimate the total disturbance whereas the v-part is used to be a servo system, which is essentially a duplicate of the original system.

We point out that in Theorem IV.1, the boundary (surface) temperature measurement $y_{\rm m}(x,t) = w(x,t)|_{\Gamma_0}$ is assumed to be error-free, which is a (part) boundary spatially distributed noise-free measurement. Note that the surface temperature measurement which in today's industrial environment encompasses a wide variety of needs and applications. To meet this wide array of needs, the process controls industry has developed a large number of sensors and devices to handle this demand. Actually, there are a wide variety of temperature measurement probes in use today, which include thermometers (such as thermometers), temperature probe (such as thermocouples), and noncontact temperature sensor (such as optical devices). For instance, the surface temperature of a concrete wall can be measured by inserting half of a thermocouple into the wall, which is probably the most-often-used and least-understood of the temperature measuring device.

The temperature measurement error is often unavoidable. When the measurement is corrupted by noise, we write $y_{\rm m}(x,t) = w(x,t)|_{\Gamma_0}$ by $y_{\rm m}(x,t) = w(x,t)|_{\Gamma_0} + \sigma(x,t)$ with the noise $\sigma \in W^{2,\infty}(0,\infty; L^2(\Gamma_0)))$. Here, we denote $W^{2,\infty}(0,\infty; L^2(\Gamma_0)) := \{\phi : \phi \in L^\infty(0,\infty; L^2(\Gamma_0)),\ \phi_t \in L^\infty(0,\infty; L^2(\Gamma_0)), \phi_{tt} \in L^\infty(0,\infty; L^2(\Gamma_0))\}$ with the norm given by $\|\phi\|_{W^{2,\infty}(0,\infty;L^2(\Gamma_0))} = \|\phi\|_{L^\infty(0,\infty;L^2(\Gamma_0))} + \|\phi_t\|_{L^\infty(0,\infty;L^2(\Gamma_0))} + \|\phi_{tt}\|_{L^\infty(0,\infty;L^2(\Gamma_0))}$. Then, it turns out that our control is still working with small tracking error as long as $\sigma(x,t)$ is small.

Theorem IV.2: Let $c_0 > 0$. Suppose that $d \in L^\infty(0,+\infty; L^2(\Gamma_0))$, $r \in W^{1,\infty}(0,\infty; L^2(\Gamma_1))$, and $f : L^2(\Omega) \to L^2(\Gamma_0)$ is continuous, bounded, and satisfies the local Lipschitz condition in $L^2(\Omega)$. Suppose that $y_{\rm m}(x,t) = w(x,t)|_{\Gamma_0} + \sigma(x,t)$ with the noise $\sigma \in W^{2,\infty}(0,\infty; L^2(\Gamma_0))$. Then, for any initial value $(w(\cdot,0),\widehat{w}(\cdot,0),v(\cdot,0)) \in [L^2(\Omega)]^3$, the closed-loop system (22) admits a unique solution $(w,\widehat{w},v) \in C(0,\infty;[L^2(\Omega)]^3)$. Moreover, the output tracking is robust with respect to σ in the sense that for any fixed $T > 0$, there exist two constants $M_1, M_2 > 0$, which depend only on initial value and T, and $\mu > 0$, which is independent of initial value, such that
$$\|e(\cdot,t)\|_{L^2(\Gamma_1)} = \|y_o(\cdot,t) - r(\cdot,t)\|_{L^2(\Gamma_1)} \leq M_1 e^{-\mu t} + M_2\|\sigma\|_{W^{2,\infty}(0,\infty;L^2(\Gamma_0))} \quad \forall\, t \geq T. \tag{23}$$

V. Proof of Main Results

Proof of Lemma II.1 Let $A = \Delta$ be the usual Laplacian with $D(A) = \left\{\phi \in H^2(\Omega) : \phi|_{\Gamma_0} = 0,\ \frac{\partial\phi}{\partial\nu}|_{\Gamma_1} = 0\right\}$. It is easy to verify that $-A$ is a strongly elliptic operator. It follows from [24, Th. 2.7, Ch. 7] that A generates an analytic semigroup $S(t)$, which implies that system (5) admits a unique solution $\widetilde{w}(\cdot,t) = S(t)\widetilde{w}(\cdot,0) \in C(0,\infty; L^2(\Omega))$.

Next, we claim that $S(t)$ is exponentially stable, for which it suffices to prove that the solution of (5) is exponentially convergent to zero (we consider the real solution only without loss the generality). Indeed, let
$$V_0(t) = \frac{1}{2}\int_\Omega \widetilde{w}^2(x,t)dx.$$

Differentiating $V(t)$ along the solution of (5), and using Green's formula and Poincare's inequality yield

$$\dot{V}_0(t) = \int_\Omega \widetilde{w}(x,t)\Delta\widetilde{w}(x,t)dx = -\int_\Omega |\nabla\widetilde{w}(x,t)|^2dx$$
$$\leq -\int_\Omega \widetilde{w}^2(x,t)dx = -2V_0(t)$$

which gives the exponential stability of $S(t)$, i.e.,

$$\|\widetilde{w}(x,t)\|_{L^2(\Omega)} = \|S(t)\widetilde{w}(\cdot,0)\|_{L^2(\Omega)} \leq e^{-t}\|\widetilde{w}(\cdot,0)\|_{L^2(\Omega)}. \tag{24}$$

Since $S(t)$ is an analytic semigroup, by [24, Corollary 4.4 and Th. 5.2, Ch. 2], for any positive integer m, $S(t)w(\cdot,0)\in D(A^m)$ for all $t>0$, and there exists a constant $C_1>0$ such that

$$\|AS(t)\| \leq \frac{C_1}{t} \text{ for all } t>0. \tag{25}$$

Since $A^mS(t)\widetilde{w}(\cdot,0) = (AS(t/m))^m\widetilde{w}(\cdot,0)$, by (25), it follows that

$$\|\Delta^m\widetilde{w}(x,t)\|_{L^2(\Omega)} = \|A^m\widetilde{w}(x,t)\|_{L^2(\Omega)}$$
$$= \|A^mS(t)\widetilde{w}(\cdot,0)\|_{L^2(\Omega)} = \|(AS(t/m))^m\widetilde{w}(\cdot,0)\|_{L^2(\Omega)}$$
$$\leq \|AS(t/m)\|^m\|\widetilde{w}(\cdot,0)\|_{L^2(\Omega)}$$
$$\leq \frac{C_1^m m^m}{t^m}\|\widetilde{w}(\cdot,0)\|_{L^2(\Omega)}. \tag{26}$$

Since $\widetilde{w}(\cdot,t)\in L^2(\Omega)$, it follows from (24), (26), and the Sobolev embedding theorem that $\widetilde{w}(\cdot,t)\in H^{2m}(\Omega)$ and there exist a constant $C_2>0$ such that

$$\|\widetilde{w}(\cdot,t)\|_{H^{2m}(\Omega)}$$
$$\leq C_2\left[\|\Delta^m\widetilde{w}(x,t)\|_{L^2(\Omega)} + \|\widetilde{w}(x,t)\|_{L^2(\Omega)}\right]$$
$$\leq \left(\frac{C_2C_1^m m^m}{t^m} + C_2Me^{-\mu t}\right)\|\widetilde{w}(\cdot,0)\|_{L^2(\Omega)} \tag{27}$$

with $\mu=1$. The Sobolev trace theorem implies that

$$\left\|\frac{\partial\widetilde{w}(\cdot,t)}{\partial\nu}\right\|_{L^2(\Gamma_0)} \leq C_3\|\widetilde{w}(\cdot,t)\|_{H^{2m}(\Omega)}$$
$$\|\widetilde{w}(\cdot,t)\|_{L^2(\Gamma_0)} \leq C_3\|\widetilde{w}(\cdot,t)\|_{H^{2m}(\Omega)} \tag{28}$$

for some constant $C_3>0$. Therefore, (6) and (7) follow from (27) and (28). It remains to show (8) and (9). By (24)

$$\|\Delta^m\widetilde{w}(x,t)\|_{L^2(\Omega)} = \|A^m\widetilde{w}(x,t)\|_{L^2(\Omega)}$$
$$= \|A^mS(t)\widetilde{w}(\cdot,0)\|_{L^2(\Omega)} = \|S(t-T)A^mS(T)\widetilde{w}(\cdot,0)\|_{L^2(\Omega)}$$
$$\leq Me^{-\mu(t-T)}\|A^mS(T)\widetilde{w}(\cdot,0)\|_{L^2(\Omega)}$$

which, together with (24), (28), and the Sobolev embedding theorem, gives (8) and (9). ■

Proof of Lemma III.1: Let $A_0=\Delta$ be the usual Laplacian with $D(A_0)=\left\{\phi\in H^2(\Omega): \frac{\partial\phi}{\partial\nu}|_{\Gamma_0} = -c_0\phi|_{\Gamma_0},\ \frac{\partial\phi}{\partial\nu}|_{\Gamma_1}=0\right\}$. It is easy to verify that $-A_0$ is a strongly elliptic operator. It follows from [24, Th. 2.7, Ch. 7] that A_0 generates an analytic semigroup $S_0(t)$, which implies that system (5) admits a unique solution $\varepsilon(\cdot,t)=S_0(t)\varepsilon(\cdot,0)\in C(0,\infty;L^2(\Omega))$.

Next, we claim that $S_0(t)$ is exponentially stable for which it suffices to prove that the solution of (13) is exponentially convergent to zero. Indeed, let (again we only consider the real solution without loss of generality)

$$V_1(t) = \frac{1}{2}\int_\Omega \varepsilon^2(x,t)dx.$$

Differentiating $V(t)$ along the solution of (5) and using Green's formula yield

$$\dot{V}_1(t) = \int_\Omega \varepsilon(x,t)\Delta\varepsilon(x,t)dx$$
$$= -c_0\int_{\Gamma_0}|\varepsilon(x,t)|^2dx - \int_\Omega|\nabla\varepsilon(x,t)|^2dx. \tag{29}$$

Since

$$\|\phi\|_0^2 = c_0\int_{\Gamma_0}|\phi(x)|^2dx + \int_\Omega|\nabla\phi(x)|^2dx$$

and

$$\|\phi\|^2 = \int_\Omega|\phi(x)|^2dx + \int_\Omega|\nabla\phi(x)|^2dx$$

are two equivalent norms on $H^1(\Omega)$, there exists a constant $C_*>0$ such that

$$C_*\int_\Omega[|\varepsilon(x,t)|^2 + |\nabla\varepsilon(x,t)|^2]dx$$
$$\leq c_0\int_{\Gamma_0}|\varepsilon(x,t)|^2dx + \int_\Omega|\nabla\varepsilon(x,t)|^2dx. \tag{30}$$

It follows from (29) and (30) that

$$\dot{V}_1(t) \leq -C_*\int_\Omega[|\varepsilon(x,t)|^2 + |\nabla\varepsilon(x,t)|^2]dx$$
$$\leq -C_*\int_\Omega \varepsilon^2(x,t)dx = -2C_*V_1(t)$$

which gives the exponential stability of $S_0(t)$, i.e.,

$$\|\varepsilon(x,t)\|_{L^2(\Omega)} = \|S_0(t)\varepsilon(\cdot,0)\| \leq Me^{-\mu t}\|\varepsilon(\cdot,0)\|_{L^2(\Omega)} \tag{31}$$

with $M=1,\mu=C_*$. From (29) and (31), we obtain

$$c_0\int_0^\infty\int_{\Gamma_0}\varepsilon^2(x,t)dx \leq V_1(0)-V_1(t)\leq V_1(0)$$

which gives $\varepsilon(\cdot,t)\in L^2(0,\infty;L^2(\Gamma_0))$. The rest of the proof is exactly the same as the proof of Lemma II.1. We thus omit the details. ■

Proof of Lemma III.2: In order to prove this lemma, we first introduce the Dirichlet map $\Upsilon\in\mathscr{L}(H^s(\Gamma_1),H^{1/2+s}(\Omega))$ ([20, pp. 188–189]), i.e., $\Upsilon(r)=z$ if and only if

$$\begin{cases}\Delta z = 0 \text{ in } \Omega\\ \frac{\partial z}{\partial\nu}\big|_{\Gamma_0}=0,\ z|_{\Gamma_1}=r.\end{cases} \tag{32}$$

Since $r\in W^{1,\infty}(0,\infty;L^2(\Gamma_1))$, it is obvious that $z\in W^{1,\infty}(0,\infty;H^{1/2}(\Omega))$ and

$$\|z(\cdot,t)\|_{H^{1/2}(\Omega)} \leq C_q\|r(\cdot,t)\|_{L^2(\Gamma_1)}$$
$$\|z_t(\cdot,t)\|_{H^{1/2}(\Omega)} \leq C_q\|r_t(\cdot,t)\|_{L^2(\Gamma_1)} \tag{33}$$

for some constant $C_q > 0$. By the Sobolev embedding theorem, there exists a constant $C_p > 0$ such that

$$\|z_t(\cdot,t)\|_{L^2(\Omega)} = \|z_t(\cdot,t)\|_{H^0(\Omega)} \le C_p\|z_t(\cdot,t)\|_{H^{1/2}(\Omega)} \le C_q C_p \|r_t(\cdot,t)\|_{L^2(\Gamma_1)}. \tag{34}$$

Using the Dirichlet map (32) and letting $\widehat{q}(x,t) = q(x,t) - z(x,t)$, we can verify from (17) that $\widehat{q}(x,t)$ is governed by

$$\begin{cases} \widehat{q}_t(x,t) = \Delta\widehat{q}(x,t) - z_t(x,t),\ x\in\Omega,\ t>0 \\ \frac{\partial \widehat{q}(x,t)}{\partial\nu}|_{\Gamma_0} = 0,\ t\ge 0 \\ \widehat{q}(x,t)|_{\Gamma_1} = 0,\ t\ge 0. \end{cases} \tag{35}$$

System (35) is then written as

$$\frac{d}{dt}\widehat{q}(\cdot,t) = A_1\widehat{q}(\cdot,t) + B_1 z_t(\cdot,t) \tag{36}$$

where $B_1 = -I$ and the operator A_1 is given by

$$\begin{cases} A_1\phi = \Delta\phi \quad \forall\ \phi\in D(A_1) \\ D(A_1) = \left\{\phi\in H^2(\Omega)|\ \frac{\partial\phi}{\partial\nu}|_{\Gamma_0} = 0,\ \phi|_{\Gamma_1} = 0\right\}. \end{cases} \tag{37}$$

It is well known that A_1 generates an exponentially stable C_0-semigroup $e^{A_1 t}$, which, together with (34), $r\in W^{1,\infty}(0,\infty; L^2(\Gamma_1))$, and [34, Lemma 1.1] implies that system (36) admits a unique solution that is uniformly bounded for all $t\ge 0$. In particular when $r\in H^1(0,\infty;L^2(\Gamma_1))$, by [34, Lemma 1.1], $\lim_{t\to\infty}\|q(\cdot,t)\|_{L^2(\Omega)} = 0$. ■

Proof of Lemma III.3: We first show the first assertion. For this purpose, we define a new inner product in $[L^2(\Omega)]^2$

$$\begin{aligned}&\langle(\phi_1,\psi_1)^\top,(\phi_2,\psi_2)^\top\rangle_* \\ &= \int_\Omega \Big[\phi_1(x)\overline{\phi_2(x)} + \psi_1(x)\overline{\psi_2(x)} \\ &\qquad - \frac{1}{2}\mathrm{Re}(\phi_1(x)\overline{\psi_2(x)} + \phi_2(x)\overline{\psi_1(x)})\Big]dx\end{aligned}$$

for $(\phi_i,\psi_i)^\top\in[L^2(\Omega)]^2,\ i=1,2$. The induced norm is given by

$$\|(\phi,\psi)\|_*^2 = \int_\Omega\left[|\phi(x)|^2 + |\psi(x)|^2 - \mathrm{Re}(\phi(x)\overline{\psi(x)})\right]dx \tag{38}$$

for all $(\phi,\psi)\in[L^2(\Omega)]^2$. Since $|\mathrm{Re}(\phi(x)\overline{\psi(x)})|\le|\phi(x)|^2/2 + |\psi(x)|^2/2$, it follows that

$$\begin{aligned}&\frac{1}{2}\|(\phi,\psi)\|^2 \\ &\le \int_\Omega\left[|\phi(x)|^2 + |\psi(x)|^2 - \mathrm{Re}(\phi(x)\overline{\psi(x)})\right]dx \\ &\le \frac{3}{2}\|(\phi,\psi)\|^2\end{aligned} \tag{39}$$

which implies that (38) is well-defined and is equivalent to the original norm. For any $(\phi,\psi)\in D(\mathbb{A})$, by Green's formula, Poincare's inequality, and (39), we have

$$\begin{aligned}&\mathrm{Re}\langle\mathbb{A}(\phi,\psi),(\phi,\psi)\rangle_* \\ &= \mathrm{Re}\bigg(\int_\Omega \Delta\phi(x)\overline{\phi(x)} + \Delta\psi(x)\overline{\psi(x)} - \Delta\phi(x)\overline{\psi(x)}]dx\bigg) \\ &= -\int_\Omega|\nabla\phi(x)|^2dx - \int_\Omega|\nabla\psi(x)|^2dx \\ &\quad - \mathrm{Re}\int_\Omega\nabla\phi(x)\cdot\overline{\nabla\psi}dx \\ &\le -\frac{1}{2}\int_\Omega|\nabla\phi(x)|^2dx - \frac{1}{2}\int_\Omega|\nabla\psi(x)|^2dx \\ &\le -\frac{1}{2}\int_\Omega|\phi(x)|^2dx - \frac{1}{2}\int_\Omega|\psi(x)|^2dx \\ &\le -\|(\phi,\psi)\|_*^2/3.\end{aligned} \tag{40}$$

This shows that $\mathbb{A} + I/3$ is dissipative in $[L^2(\Omega)]^2$. Now we show that $\mathbb{A}^{-1}\in\mathscr{L}([L^2(\Omega)]^2)$. Solve the equation

$$\mathbb{A}(\phi,\psi) = (\Delta\phi,\Delta\psi) = (\widehat{\phi},\widehat{\psi})\in[L^2(\Omega)]^2$$

to obtain

$$\begin{cases}\Delta\phi(x) = \widehat{\phi}(x) \\ \phi|_{\Gamma_0} = 0,\ \frac{\partial\phi}{\partial\nu}|_{\Gamma_1} = 0\end{cases} \tag{41}$$

and

$$\begin{cases}\Delta\psi(x) = \widehat{\psi}(x) \\ \frac{\partial\psi}{\partial\nu}|_{\Gamma_0} = \frac{\partial\phi}{\partial\nu}|_{\Gamma_0},\ \ \psi|_{\Gamma_1} = 0.\end{cases} \tag{42}$$

By the elliptic partial differential equation theory, we know that (41) admits a unique solution $\phi\in H^2(\Omega)$ and there exists a constant $C_L > 0$ such that

$$\|\phi\|_{H^2(\Omega)} \le C_L\|\widehat{\phi}\|_{L^2(\Omega)}.$$

By the trace theorem, $\frac{\partial\phi}{\partial\nu}\in H^{1/2}(\Gamma_0)$ and there exists a constant $C_M > 0$ such that

$$\left\|\frac{\partial\phi}{\partial\nu}\right\|_{H^{1/2}(\Gamma_0)} \le C_M\|\phi\|_{H^2(\Omega)}.$$

By the elliptic partial differential equation theory again, equation (42) admits a unique solution $\phi\in H^2(\Omega)$ and there exists a constant $C_N > 0$ such that

$$\begin{aligned}\|\psi\|_{H^2(\Omega)} &\le C_N\left[\|\widehat{\psi}\|_{L^2(\Omega)} + \left\|\frac{\partial\phi}{\partial\nu}\right\|_{H^{1/2}(\Gamma_0)}\right] \\ &\le C_N\left[\|\widehat{\psi}\|_{L^2(\Omega)} + C_1C_2\|\widehat{\phi}\|_{L^2(\Omega)}\right].\end{aligned}$$

Hence $\mathbb{A}^{-1}(\widehat{\phi},\widehat{\psi}) = (\phi,\psi)$. It follows from the Lumer–Phillips theorem [24, Th. 1.4.3] that $\mathbb{A} + I/3$ generates a C_0-semigroup of contractions $e^{(\mathbb{A}+I/3)t}$ on $[L^2(\Omega)]^2$, which implies that $e^{\mathbb{A}t}$ is an exponentially stable C_0-semigroup on $[L^2(\Omega)]^2$.

Next, we show that $\mathbb{B}$ is admissible for $e^{\mathbb{A}t}$ (see [31]). By [29, Th. 4.4.3], it suffices to show that $\mathbb{B}^*$ is admissible observation operator for the adjoint semigroup $e^{\mathbb{A}^*t}$. This amounts to showing that a) $\mathbb{B}^*\mathbb{A}^{*-1}$ is bounded from $[L^2(\Omega)]^2$ to $L^2(\Gamma_0)$, and

b) for every $T_* > 0$, there exists $M_{T_*} > 0$ depending on T_* only such that the system of the following:

$$\begin{cases} \widetilde{w}_t^*(x,t) = \Delta \widetilde{w}^*(x,t),\ x \in \Omega,\ t > 0 \\ \widetilde{w}^*(x,t)|_{\Gamma_0} = -p^*(x,t)|_{\Gamma_0},\ \frac{\partial \widetilde{w}^*(x,t)}{\partial \nu}|_{\Gamma_1} = 0,\ t \geq 0 \\ p_t^*(x,t) = \Delta p^*(x,t),\ x \in \Omega,\ t > 0 \\ \frac{\partial p^*(x,t)}{\partial \nu}|_{\Gamma_0} = 0,\ p^*(x,t)|_{\Gamma_1} = 0,\ t \geq 0 \\ \widetilde{w}^*(x,0) = \widetilde{w}_0^*(x),\ p^*(x,0) = p_0^*(x),\ x \in \Omega \\ y_{\rm m}^* = p^*(x,t)|_{\Gamma_0},\ t \geq 0 \end{cases} \tag{43}$$

satisfies

$$\int_0^{T_*} \int_{\Gamma_0} (p^*(x,t))^2 dx dt \leq M_{T_*} \|(w_0^*, p_0^*)\|^2_{[L^2(\Omega)]^2}. \tag{44}$$

According to the first assertion, $\mathbb{A}$ generates a C_0-semigroup on $[L^2(\Omega)]^2$. Thus, the same is true for $\mathbb{A}^*$. As a result, system (43) admits a unique solution $(\widetilde{w}^*, p^*) \in C(0,\infty; [L^2(\Omega)]^2)$ and there exist two constants $\hat{M}, \hat{\mu} > 0$ such that

$$\int_\Omega [(\widetilde{w}^*(x,t))^2 + (p^*(x,t))^2] dx \leq \hat{M} e^{\hat{\mu} t} \int_\Omega [(\widetilde{w}^*(x,0))^2 + (p^*(x,0))^2] dx. \tag{45}$$

Define

$$\rho(t) = \int_\Omega G(x) \widetilde{w}^*(x,t) p^*(x,t) dx \tag{46}$$

where $G(x)$ is the solution of the following PDE

$$\begin{cases} \Delta G(x) = 0 \\ G|_{\Gamma_0} = 1,\ \ G|_{\Gamma_1} = 0. \end{cases} \tag{47}$$

Since $1 \in H^{n+3/2}(\Gamma_0)$ and $0 \in H^{n+3/2}(\Gamma_1)$, it follows from the elliptic partial differential equation theory that (47) has a unique solution $G \in H^{n+2}(\Omega)$. By the Sobolev embedding theorem, $G \in C^1(\Omega)$ and there exists a constant $C_4 > 0$ such that

$$\max\{\|G\|_{L^\infty(\Omega)}, \|\nabla G\|_{L^\infty(\Omega)}\} \leq C_4. \tag{48}$$

Thus, (46) is well-defined. Since $p^*(x,t)|_{\Gamma_1} = 0$, by Poincare's inequality, there exists a constant $C_5 > 0$ such that

$$\int_\Omega |p^*(x,t)|^2 \leq C_5 \int_\Omega |\nabla p^*(x,t)|^2 dx. \tag{49}$$

From the Sobolev embedding theorem and the trace theorem, there exist constants $C_6, C_7 > 0$ such that

$$\int_{\Gamma_0} (p^*(x,t))^2 dx \leq C_6 \|p^*(\cdot,t)\|^2_{H^{1/2}(\Gamma_0)} \leq C_6 C_7 \|p^*(\cdot,t)\|^2_{H^1(\Omega)} = C_6 C_7 \int_\Omega |\nabla p^*(x,t)|^2 dx. \tag{50}$$

Since

$$\|\phi\|_0^2 = \int_{\Gamma_0} |\phi(x)|^2 dx + \int_\Omega |\nabla \phi(x)|^2 dx$$

and

$$\|\phi\|^2 = \int_\Omega |\phi(x)|^2 dx + \int_\Omega |\nabla \phi(x)|^2 dx$$

are two equivalent norms on $H^1(\Omega)$, there exists a constant $C_8 > 0$ such that

$$\int_\Omega [|\widetilde{w}^*(x,t)|^2 + |\nabla \widetilde{w}^*(x,t)|^2] dx \leq C_8 \int_{\Gamma_0} |\widetilde{w}^*(x,t)|^2 dx + C_8 \int_\Omega |\nabla \widetilde{w}^*(x,t)|^2 dx. \tag{51}$$

By (48)–(51), using the fact that $\widetilde{w}^*(x,t)|_{\Gamma_0} = p^*(x,t)|_{\Gamma_0}$, we have

$$\begin{aligned} &\int_\Omega \left[\frac{1}{4\delta} |\nabla(G(x)p^*(x,t)|^2 + \delta |\nabla(G(x)\widetilde{w}^*(x,t))|^2\right] dx \\ &= \int_\Omega \left[\frac{1}{4\delta} |p^*(x,t)\nabla G(x) + G(x)\nabla p^*(x,t)|^2 \right. \\ &\quad \left. + \delta |\widetilde{w}^*(x,t)\nabla G(x) + G(x)\nabla \widetilde{w}^*(x,t)|^2\right] dx \\ &\leq \frac{C_4^2}{2\delta} \int_\Omega [|p^*(x,t)|^2 + |\nabla p^*(x,t)|^2] dx \\ &\quad + 2C_4^2 \delta \int_\Omega [|\widetilde{w}^*(x,t)|^2 + |\nabla \widetilde{w}^*(x,t)|^2] dx \\ &\leq \frac{C_4^2(C_5+1)}{2\delta} \int_\Omega |\nabla p^*(x,t)|^2 dx + 2C_4^2 C_8 \delta \\ &\quad \times \int_{\Gamma_0} |p^*(x,t)|^2 dx + 2C_4^2 C_8 \delta \int_\Omega |\nabla \widetilde{w}^*(x,t)|^2 dx \\ &\leq \left(\frac{C_4^2(C_5+1)}{2\delta} + 2C_4^2 C_6 C_7 C_8 \delta\right) \int_\Omega |\nabla p^*(x,t)|^2 dx \\ &\quad + 2C_4^2 C_8 \delta \int_\Omega |\nabla \widetilde{w}^*(x,t)|^2 dx. \end{aligned} \tag{52}$$

By the boundary conditions of (43) and (47), differentiating $\rho(t)$ along the solution of (43) and using the Green's formula yield

$$\begin{aligned} \dot{\rho}(t) &= \int_\Omega \Delta \widetilde{w}^*(x,t) G(x) p^*(x,t) dx \\ &\quad + \int_\Omega \Delta p^*(x,t) G(x) \widetilde{w}^*(x,t) dx \\ &= -\int_\Omega \nabla \widetilde{w}^*(x,t) \cdot \nabla(G(x)p^*(x,t)) dx \\ &\quad - \int_\Omega \nabla p^*(x,t) \cdot \nabla(G(x)\widetilde{w}^*(x,t)) dx \\ &\quad + \int_{\Gamma_0 \cup \Gamma_1} \frac{\partial \widetilde{w}^*(x,t)}{\partial \nu} G(x) p^*(x,t) dx \\ &\quad + \int_{\Gamma_0 \cup \Gamma_1} \frac{\partial p^*(x,t)}{\partial \nu} G(x) \widetilde{w}^*(x,t) dx \\ &= -\int_\Omega \nabla \widetilde{w}^*(x,t) \cdot \nabla(G(x)p^*(x,t)) dx \\ &\quad - \int_\Omega \nabla p^*(x,t) \cdot \nabla(G(x)\widetilde{w}^*(x,t)) dx \\ &\quad + \int_{\Gamma_0} \frac{\partial \widetilde{w}^*(x,t)}{\partial \nu} p^*(x,t) dx \end{aligned}$$

$$
\begin{aligned}
&\leq \int_\Omega \left[\delta|\nabla\widetilde{w}^*(x,t)|^2 + \frac{1}{4\delta}|\nabla p^*(x,t)|^2\right]dx \\
&+ \frac{1}{4\delta}\int_\Omega |\nabla(G(x)p^*(x,t)|^2 dx \\
&+ \delta\int_\Omega |\nabla(G(x)\widetilde{w}^*(x,t))|^2 dx \\
&+ \int_{\Gamma_0} \frac{\partial\widetilde{w}^*(x,t)}{\partial\nu}p^*(x,t)dx. \qquad (53)
\end{aligned}
$$

In the last step of (53), the inequality $ab \leq \delta a^2 + \frac{1}{4\delta}b^2$ was used and δ was chosen so that

$$0 < \delta < \frac{1}{2(2C_4^2C_8+1)}. \qquad (54)$$

Denote

$$E(t) =: \frac{1}{2}\int_\Omega [(\widetilde{w}^*(x,t))^2 + C_9(p^*(x,t))^2]dx \qquad (55)$$

where the constant C_9 is chosen so that

$$C_9 > \frac{C_4^2(C_5+1)}{2\delta} + 2C_4^2C_6C_7C_8\delta + \frac{1}{4\delta} + 1. \qquad (56)$$

Differentiating $\rho(t)$ along the solution of (43) gives

$$
\begin{aligned}
&\dot{E}(t) \\
&= \int_\Omega [\Delta\widetilde{w}^*(x,t)\widetilde{w}^*(x,t) + C_9\Delta p^*(x,t)p^*(x,t)]dx \\
&= -\int_\Omega [|\nabla\widetilde{w}^*(x,t)|^2 + C_9|\nabla p^*(x,t)|^2]dx \\
&+ \int_{\Gamma_0} \frac{\partial\widetilde{w}^*(x,t)}{\partial\nu}\widetilde{w}^*(x,t)dx \\
&+ C_9\int_{\Gamma_0} \frac{\partial p^*(x,t)}{\partial\nu}p^*(x,t)dx \\
&+ \int_{\Gamma_1} \frac{\partial\widetilde{w}^*(x,t)}{\partial\nu}\widetilde{w}^*(x,t)dx \\
&+ C_9\int_{\Gamma_1} \frac{\partial p^*(x,t)}{\partial\nu}p^*(x,t)dx \\
&= -\int_\Omega [|\nabla\widetilde{w}^*(x,t)|^2 + C_9|\nabla p^*(x,t)|^2]dx \\
&+ \int_{\Gamma_0} \frac{\partial\widetilde{w}^*(x,t)}{\partial\nu}\widetilde{w}^*(x,t)dx. \qquad (57)
\end{aligned}
$$

Let $V_2(t) = \rho(t) + E(t)$, where $\rho(t)$ and $E(t)$ are given by (46) and (55), respectively. Since $C_9 > 1$, by Cauchy's inequality and (48), it follows that

$$|V_2(t)| \leq \left(\frac{C_4}{2} + C_9\right)\int_\Omega [(\widetilde{w}^*(x,t))^2 + (p^*(x,t))^2]dx. \quad (58)$$

Differentiating $V_2(t)$ along the solution of (43) and by (53), (54), (56), and (57), we obtain

$$
\begin{aligned}
&\dot{V}_2(t) \\
&\leq \int_\Omega \left[\delta|\nabla\widetilde{w}^*(x,t)|^2 + \frac{1}{4\delta}|\nabla p^*(x,t)|^2\right]dx \\
&+ \left(\frac{C_4^2(C_5+1)}{2\delta} + 2C_4^2C_6C_7C_8\delta\right)\int_\Omega |\nabla p^*(x,t)|^2 dx \\
&+ \int_{\Gamma_0} \frac{\partial\widetilde{w}^*(x,t)}{\partial\nu}p^*(x,t)dx + \int_{\Gamma_0} \frac{\partial\widetilde{w}^*(x,t)}{\partial\nu}\widetilde{w}^*(x,t)dx \\
&- \int_\Omega [|\nabla\widetilde{w}^*(x,t)|^2 + C_9|\nabla p^*(x,t)|^2]dx \\
&+ 2C_4^2C_8\delta\int_\Omega |\nabla\widetilde{w}^*(x,t)|^2 dx \\
&\leq -\left(1 - (2C_4^2C_8+1)\delta\right)\int_\Omega |\nabla\widetilde{w}^*(x,t)|^2 dx \\
&- \left(C_9 - \left(\frac{C_4^2(C_5+1)}{2\delta} + 2C_4^2C_6C_7C_8\delta + \frac{1}{4\delta}\right)\right) \\
&\times \int_\Omega |\nabla p^*(x,t)|^2 dx \\
&\leq -\frac{1}{2}\int_\Omega |\nabla\widetilde{w}^*(x,t)|^2 dx - \int_\Omega |\nabla p^*(x,t)|^2 dx \qquad (59)
\end{aligned}
$$

which, together with (58) and (45), gives

$$
\begin{aligned}
&\int_0^t \int_\Omega |\nabla p^*(x,t)|^2 dxdt \leq V_2(0) - V_2(t) \\
&\leq \left(\frac{C_4}{2} + C_9\right)\left(\|(\widetilde{w}^*(x,t), p^*(x,t))\|^2_{[L^2(\Omega)]^2}\right. \\
&\left. + \|(\widetilde{w}^*(x,0), p^*(x,0))\|^2_{[L^2(\Omega)]^2}\right) \\
&\leq \left(\frac{C_4}{2} + C_9\right)\hat{M}(1 + e^{\hat{\mu}t})\|(\widetilde{w}^*(x,0), p^*(x,0))\|^2_{[L^2(\Omega)]^2}. \qquad (60)
\end{aligned}
$$

Combining with (50) and (60), we obtain

$$
\begin{aligned}
&\int_0^t \int_{\Gamma_0} (p^*(x,t))^2 dx \leq C_6C_7\left(\frac{C_4}{2} + C_9\right) \\
&\times \hat{M}(1 + e^{\hat{\mu}t})\|(\widetilde{w}^*(x,0), p^*(x,0))\|^2_{[L^2(\Omega)]^2}. \qquad (61)
\end{aligned}
$$

A direct computation shows that

$$\mathbb{A}^{*-1}(\widehat{\phi}, \widehat{\psi}) = (\phi, \psi), \quad \mathbb{B}^*\mathbb{A}^{*-1}(\widehat{\phi}, \widehat{\psi}) = \psi|_{\Gamma_0} \qquad (62)$$

where (ϕ, ψ) satisfies the following PDEs:

$$\begin{cases} \Delta\phi(x) = \widehat{\phi}(x) \\ \phi|_{\Gamma_0} = -\psi|_{\Gamma_0}, \ \frac{\partial\phi}{\partial\nu}|_{\Gamma_1} = 0 \end{cases} \qquad (63)$$

and

$$\begin{cases} \Delta\psi(x) = \widehat{\psi}(x) \\ \frac{\partial\psi}{\partial\nu}|_{\Gamma_0} = 0, \ \psi|_{\Gamma_1} = 0. \end{cases} \qquad (64)$$

By the elliptic partial differential equation theory, we know that (64) admits a unique solution $\psi \in H^2(\Omega)$ and that there exists a constant $C_{10} > 0$ such that $\|\psi\|_{H^2(\Omega)} \le C_{10}\|\widehat{\psi}\|_{L^2(\Omega)}$. By the trace theorem, $\psi \in H^{3/2}(\Gamma_0)$ and there exists a constant $C_{11} > 0$ such that $\|\psi\|_{H^{3/2}(\Gamma_0)} \le C_{11}\|\psi\|_{H^2(\Omega)} \le C_{11}C_{12}\|\widehat{\psi}\|_{L^2(\Omega)}$. By the elliptic partial differential equation theory again, we have that (63) admits a unique solution $\phi \in H^2(\Omega)$ and that there exists a constant $C_{12} > 0$ such that

$$\begin{aligned}\|\phi\|_{H^2(\Omega)} &\le C_{12}\left[\|\widehat{\phi}\|_{H^{3/2}(\Gamma_0)} + \|\widehat{\phi}\|_{L^2(\Omega)}\right] \\ &\le C_{12}\left[\|C_{11}C_{12}\|\widehat{\psi}\|_{L^2(\Omega)} + \|\widehat{\phi}\|_{L^2(\Omega)}\right].\end{aligned}$$

By the Sobolev embedding theorem and the trace theorem, $H^2(\Omega) \hookrightarrow L^2(\Gamma_0)$. Since $\psi \in H^2(\Omega)$ and (62), we know that $\mathbb{B}^*\mathbb{A}^{*-1}$ is bounded from $[L^2(\Omega)]^2$ to $L^2(\Gamma_0)$, which, together with (61), implies that $\mathbb{B}$ is admissible for $e^{\mathbb{A}t}$. ■

Proof of Lemma III.4: By Lemma III.3, $e^{\mathbb{A}t}$ is exponentially stable and $\mathbb{B}$ is admissible for $e^{\mathbb{A}t}$. Note that $F = f(p + q - \varepsilon - \widetilde{w}, t) + d(x,t)$ satisfies the local Lipschitz condition with respect to $(\widetilde{w}, p)$. By the same argument used in [34, Proposition 1.1], we know that (19) has a global solution $(\widetilde{w}, p) \in C(0,\infty; [L^2(\Omega)]^2)$. Thus, $F = f(p + q - \varepsilon - \widetilde{w}, t) + d(x,t)$ makes sense for all $t \ge 0$. Since $f(\cdot)$ is bounded and $d \in L^\infty(0, +\infty; L^2(\Gamma_0))$, $F \in L^\infty(0, +\infty; L^2(\Gamma_0))$. Rewrite (19) as

$$\frac{d}{dt}(\widetilde{w}(\cdot,t), p(\cdot,t)) = \mathbb{A}(\widetilde{w}(\cdot,t), p(\cdot,t)) + \mathbb{B}F(p + q - \varepsilon - \widetilde{w}, t) + \mathbb{B}(-c_0\varepsilon(x,t)). \tag{65}$$

Since $\varepsilon(\cdot,t) \in L^2(0,\infty; L^2(\Gamma_0))$ by virtue of Lemma III.1, it follows from [34, Lemma 1.1] or [35, Lemma 2.1] that the solution of (65) is uniformly bounded for all $t \ge 0$. Next, suppose $f \equiv 0$ and $d \in L^2(0,\infty; L^2(\Gamma_0))$. Then, $F \in L^2(0,\infty; L^2(\Gamma_0))$ and by [34, Lemma 1.1] again, we know that $\lim_{t\to\infty}\|(\widetilde{w}(\cdot,t), p(\cdot,t))\|_{[L^2(\Omega)]^2} = 0$. ■

Proof of Lemma III.5: Since system (10) can be written as the sum of p-system (16) and q-system (17), the result follows immediately from Lemmas III.2 and III.4. ■

Proof of Theorem IV.1: Let $\widetilde{w}(x,t) = \widehat{w}(x,t) - w(x,t)$, $\varepsilon(x,t) = v(x,t) - \widehat{w}(x,t)$. Then, it is readily shown that the closed-loop system (22) is equivalent to the one following:

$$\begin{cases} \widetilde{w}_t(x,t) = \Delta\widetilde{w}(x,t),\ x \in \Omega,\ t > 0 \\ \widetilde{w}(x,t)|_{\Gamma_0} = \frac{\partial \widetilde{w}(x,t)}{\partial \nu}|_{\Gamma_1} = 0,\ t \ge 0 \\ \varepsilon_t(x,t) = \Delta\varepsilon(x,t),\ x \in \Omega,\ t > 0 \\ \frac{\partial \varepsilon(x,t)}{\partial \nu}|_{\Gamma_0} = -c_0\varepsilon(x,t)|_{\Gamma_0}, \frac{\partial \varepsilon(x,t)}{\partial \nu}|_{\Gamma_1} = 0,\ t \ge 0 \\ v_t(x,t) = \Delta v(x,t),\ x \in \Omega,\ t > 0 \\ \frac{\partial v(x,t)}{\partial \nu}|_{\Gamma_0} = -c_0(v(x,t) - \widehat{w}(x,t))|_{\Gamma_0} \\ \qquad + \frac{\partial \widehat{w}(x,t)}{\partial \nu}|_{\Gamma_0} \\ v(x,t)|_{\Gamma_1} = r(x,t),\ t \ge 0. \end{cases} \tag{66}$$

By Lemmas II.1 and III.1, the "$(\widetilde{w}, \varepsilon)$-part" of (66) has a unique solution and for any fixed $T > 0$ and integer $m > 0$, there exist six constants $M_1, M_2, M_3, \mu_1, \mu_2, \mu_3 > 0$ such that

$$\int_\Omega [\widetilde{w}^2(x,t) + \varepsilon^2(x,t)]dx \le M_1 e^{-\mu_1 t} \quad \forall\, t \ge 0 \tag{67}$$

$$\int_{\Gamma_1} [\widetilde{w}^2(x,t) + \varepsilon^2(x,t)]dx \le \frac{M_2}{t^m} + M_2 e^{-\mu_2 t} \quad \forall\, t > 0 \tag{68}$$

and

$$\int_{\Gamma_1} [\widetilde{w}^2(x,t) + \varepsilon^2(x,t)]dx \le M_3 e^{-\mu_3 t} \quad \forall\, t \ge T. \tag{69}$$

It is seen from Lemmas II.1 and III.1 that M_1, M_2, and M_3 depend on $(\widetilde{w}(\cdot,0), \varepsilon(\cdot,0))$ yet $\mu_i, i = 1,2,3$, do not. By Lemma III.5, the "v-part" of (66) has a unique solution and there exists a constant $M_4 > 0$ depending on $(v(\cdot,0), \widehat{w}(\cdot,0))$ such that

$$\int_\Omega v^2(x,t)dx \le M_4 \quad \forall\, t \ge 0. \tag{70}$$

Note that

$$\begin{pmatrix} w(x,t) \\ \widehat{w}(x,t) \\ v(x,t) \end{pmatrix} = \begin{pmatrix} -I & -I & I \\ 0 & -I & I \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} \widetilde{w}(x,t) \\ \varepsilon(x,t) \\ v(x,t) \end{pmatrix}. \tag{71}$$

The $(w(x,t), \widehat{w}(x,t), v(x,t))$ is well-defined for all $t \ge 0$. Thus, (22) admits a unique solution. The claim (i) follows from (67) and (71). The claim (ii) follows from (67). Since $e(x,t) = y_o(x,t) - r(x,t) = -\widetilde{w}(x,t)|_{\Gamma_1} - \varepsilon(x,t)|_{\Gamma_1}$, the claim (iii) follows from (68) and (69). By the last assertion of Lemma III.5, (67), (68), and (71), we can see that (iv) holds as well.

Now suppose $f \equiv 0$, $d \equiv 0$, and $r \equiv 0$. Since A_1 given by (37) and $\mathbb{A}$ given by (20) generate two exponentially stable C_0-semigroups $e^{A_1 t}$ and $e^{\mathbb{A}t}$, respectively, which are proved in Lemmas III.2 and III.3, we see that when $r \equiv 0$, the "v-part" of (66) is exponentially stable, which, together with (67) and (71), implies that (v) holds. ■

Proof of Theorem IV.2: Let $\widetilde{w}(x,t) = \widehat{w}(x,t) - w(x,t)$ be the error, which is governed by

$$\begin{cases} \widetilde{w}_t(x,t) = \Delta\widetilde{w}(x,t),\ x \in \Omega,\ t > 0 \\ \widetilde{w}(x,t)|_{\Gamma_0} = \sigma(x,t),\ t \ge 0 \\ \frac{\partial \widetilde{w}(x,t)}{\partial \nu}|_{\Gamma_1} = 0,\ t \ge 0. \end{cases} \tag{72}$$

Then, $\varepsilon(x,t) = v(x,t) - \widehat{w}(x,t)$ satisfies (13). Clearly, the closed-loop system (22) is equivalent to a coupled system composed of (10), (13), and (72). Since the proof of well posedness of the closed-loop system is similar to that of Theorem IV.1, we only prove the robustness of the output tracking. For this purpose, we introduce a Dirichlet map $\Upsilon_1 \in \mathscr{L}(H^s(\Gamma_0), H^{1/2+s}(\Omega))$ ([20, pp. 188–189]): $\Upsilon_1(\sigma) = z$ if and only if

$$\begin{cases} \Delta z = 0 \text{ in } \Omega \\ z|_{\Gamma_0} = \sigma,\ \frac{\partial z}{\partial \nu}\Big|_{\Gamma_1} = 0. \end{cases} \tag{73}$$

Since $\sigma \in W^{2,\infty}(0,\infty;L^2(\Gamma_0))$, it is obvious that $z \in W^{2,\infty}(0,\infty;H^{1/2}(\Omega))$ and

$$\begin{aligned}\|z(\cdot,t)\|_{H^{1/2}(\Omega)} &\le C_1\|\sigma(\cdot,t)\|_{L^2(\Gamma_0)}\\ \|z_t(\cdot,t)\|_{H^{1/2}(\Omega)} &\le C_1\|\sigma_t(\cdot,t)\|_{L^2(\Gamma_0)}\\ \|z_{tt}(\cdot,t)\|_{H^{1/2}(\Omega)} &\le C_1\|\sigma_{tt}(\cdot,t)\|_{L^2(\Gamma_0)}\end{aligned} \tag{74}$$

for some constant $C_1 > 0$. By the Sobolev embedding theorem, there exists a constant $C_2 > 0$ such that

$$\begin{aligned}\|z_t(\cdot,t)\|_{L^2(\Omega)} &= \|z_t(\cdot,t)\|_{H^0(\Omega)} \le C_2\|z_t(\cdot,t)\|_{H^{1/2}(\Omega)}\\ &\le C_1C_2\|\sigma_t(\cdot,t)\|_{L^2(\Gamma_0)}\\ &\le C_1C_2\|\sigma\|_{W^{2,\infty}(0,\infty;L^2(\Gamma_0))}\end{aligned} \tag{75}$$

and

$$\begin{aligned}&\|z_t(\cdot,t_1)-z_t(\cdot,t_2)\|_{L^2(\Omega)} = |t_1-t_2|\|z_{tt}(\cdot,\xi)\|_{H^0(\Omega)}\\ &\le C_2|t_1-t_2|\|z_{tt}(\cdot,\xi)\|_{H^{1/2}(\Omega)}\\ &\le C_1C_2|t_1-t_2|\|\sigma_{tt}(\cdot,\xi)\|_{L^2(\Gamma_0)}\\ &\le C_1C_2|t_1-t_2|\|\sigma\|_{W^{2,\infty}(0,\infty;L^2(\Gamma_0))}.\end{aligned} \tag{76}$$

Using the Dirichlet map (73) and setting $W(x,t) = \widetilde{w}(x,t) - z(x,t)$, we can verify from (72) that $W(x,t)$ is governed by

$$\begin{cases} W_t(x,t) = \Delta W(x,t) - z_t(x,t),\ x\in\Omega,\ t>0\\ W(x,t)|_{\Gamma_0} = 0,\ t\ge 0\\ \frac{\partial W(x,t)}{\partial \nu}|_{\Gamma_1} = 0,\ t\ge 0.\end{cases} \tag{77}$$

System (77) is then written as

$$\tfrac{d}{dt}W(\cdot,t) = AW(\cdot,t) - z_t(\cdot,t) \tag{78}$$

where $A=\Delta$ is the usual Laplacian with $D(A) = \{\phi \in H^2(\Omega) : \phi|_{\Gamma_0} = 0,\ \frac{\partial\phi}{\partial\nu}|_{\Gamma_1} = 0\}$. From the proof of Lemma II.1, A generates an analytic semigroup $S(t)$ and there exist $M_0,\mu_0 > 0$ such that $\|S(t)\| \le M_0e^{-\mu_0 t}$. By [24, Th. 3.2, p. 111] and its proof, it follows from (75) and (76) that for any $W(\cdot,0)\in L^2(\Omega)$, the mild solution of (78) given by

$$\begin{aligned}W(\cdot,t) &= S(t)W(\cdot,0) - \int_0^t S(t-s)z_s(\cdot,s)ds\\ &=: S(t)W(\cdot,0) + W_1(t)\end{aligned} \tag{79}$$

is the classical solution, that is, $W_1(t)\in D(A)$ and $AW_1(t)$ is continuous in t over $(0,+\infty)$. From (75), we obtain

$$\begin{aligned}\|W_1(t)\|_{L^2(\Omega)} &\le \int_0^t M_0e^{-\mu_0(t-s)}\|z_s(\cdot,s)\|_{L^2(\Omega)}ds\\ &\le \frac{M_0C_1C_2}{\mu_0}\|\sigma\|_{W^{2,\infty}(0,\infty;L^2(\Gamma_0))}\end{aligned} \tag{80}$$

and

$$\begin{aligned}\|AW_1(t)\|_{L^2(\Omega)} &\le \left\|\int_0^t AS(t-s)z_s(\cdot,s)ds\right\|_{L^2(\Omega)}\\ &= \left\|\int_0^t S(t-T-s)AS(T)z_s(\cdot,s)ds\right\|_{L^2(\Omega)}\\ &\le \int_0^t M_0e^{-\mu_0(t-T-s)}\|AS(T)\|\|z_s(\cdot,s)\|_{L^2(\Omega)}ds\\ &\le \frac{M_0e^{\mu_0T}\|AS(T)\|C_1C_2}{\mu_0}\|\sigma\|_{W^{2,\infty}(0,\infty;L^2(\Gamma_0))}.\end{aligned} \tag{81}$$

Since for any $t\ge T$

$$\begin{aligned}&\|\Delta(S(t)W(\cdot,0))\|_{L^2(\Omega)} = \|AS(t)W(\cdot,0)\|_{L^2(\Omega)}\\ &= \|S(t-T)AS(T)W(\cdot,0)\|_{L^2(\Omega)}\\ &\le M_0e^{-\mu_0(t-T)}\|AS(T)\|\|W(\cdot,0)\|_{L^2(\Omega)}\end{aligned}$$

it follows from (79), (80), (81) and the Sobolev embedding theorem that $W(\cdot,t)\in H^2(\Omega)$ and that there exist a constant $C_3>0$ such that

$$\begin{aligned}&\|W(\cdot,t)\|_{H^2(\Omega)}\\ &\le C_3\left[\|\Delta W(x,t)\|_{L^2(\Omega)} + \|W(x,t)\|_{L^2(\Omega)}\right]\\ &\le C_4e^{-\mu_0t}\|W(\cdot,0)\|_{L^2(\Omega)} + C_5\|\sigma\|_{W^{2,\infty}(0,\infty;L^2(\Gamma_0))}\end{aligned} \tag{82}$$

with $C_4 = C_3M_0 + C_3M_0e^{\mu_0T}\|AS(T)\|$, $C_5 = (M_0C_1C_2C_3 + M_0e^{\mu_0T}\|AS(T)\|C_1C_2C_3)/\mu_0$. Furthermore, by the Sobolev trace theorem, it follows from (82) that

$$\begin{aligned}&\|W(\cdot,t)\|_{L^2(\Gamma_1)} \le C_6\|W(\cdot,t)\|_{H^2(\Omega)}\\ &\le C_4C_6e^{-\mu_0t}\|W(\cdot,0)\|_{L^2(\Omega)} + C_5C_6\|\sigma\|_{W^{2,\infty}(0,\infty;L^2(\Gamma_0))}\end{aligned} \tag{83}$$

for some $C_6>0$. Since $z\in W^{2,\infty}(0,\infty;H^{1/2}(\Omega))$, by the Sobolev trace theorem and (74), we obtain

$$\begin{aligned}\|z(\cdot,t)\|_{L^2(\Gamma_1)} &\le C_7\|z(\cdot,t)\|_{H^{1/2}(\Omega)}\\ &\le C_1C_7\|\sigma\|_{W^{2,\infty}(0,\infty;L^2(\Gamma_0))}\end{aligned} \tag{84}$$

for some constant $C_7>0$. Since

$$\begin{aligned}e(x,t) &= y_o(x,t) - r(x,t) = -\widetilde{w}(x,t)|_{\Gamma_1} - \varepsilon(x,t)|_{\Gamma_1}\\ &= -W(x,t)|_{\Gamma_1} - z(x,t)|_{\Gamma_1} - \varepsilon(x,t)|_{\Gamma_1}\end{aligned}$$

the (23) then follows from (83), (84), and (15). ■

From the proof of Theorem IV.2, we see that the assumption that the measurement noise $\sigma(x,t)$ is in $W^{2,\infty}(0,\infty;L^2(\Gamma_0))$ is used to guarantee (76) so that the mild solution of (78) is the classical solution (see [24, Th. 3.2, p. 111]).

VI. Numerical Simulation

In this section, we present numerical simulations for the closed-loop system (22) to illustrate the effectiveness of the proposed feedback control. For numerical computations, we take $\Omega = \{(x_1,x_2)\in\mathbb{R}^2 |\ 1 < x_1^2+x_2^2 < 4\}$, $\Gamma = \partial\Omega$, $\Gamma_0 = \{(x_1,x_2)\in\mathbb{R}^2|\ x_1^2+x_2^2 = 1\}$, $\Gamma_1 = \Gamma\setminus\Gamma_0$. The parameter is taken as $c_0 = 1$. The external disturbance, the internal uncertainty, and the reference signal are taken, as $d(x_1, x_2, t) = \sin(x_1 t)$, $f(w) = 0.1\sin(\int_\Omega w(x,t)dx)$, and $r(x_1, x_2, t) = \sin(x_2 t)$. The initial values are taken as

$$\begin{cases} w(x_1,x_2,0) = \frac{4x_1^3}{(x_1^2+x_2^2)^{1.5}} - \frac{3x_1}{\sqrt{x_1^2+x_2^2}} \\ \widehat{w}(x_1,x_2,0) = \frac{3x_2}{\sqrt{x_1^2+x_2^2}} - \frac{4x_2^3}{(x_1^2+x_2^2)^{3/2}} \\ v(x_1,x_2,0) = \frac{4x_1^3-4x_2^3}{(x_1^2+x_2^2)^{3/2}} + \frac{-3x_1+3x_2}{\sqrt{x_1^2+x_2^2}} \end{cases} \tag{85}$$

where $(x_1, x_2) \in \mathbb{R}^2$ satisfies $1 \le x_1^2 + x_2^2 \le 4$. Since the spatial domain consists of a 2-D annulus, it is easier to solve (22) in the polar coordinate (γ, θ). The results can then be converted back to the original coordinate for some figures if necessary. Under the polar coordinate, system (22) can be written as

$$\begin{cases} \frac{\partial w(\gamma,\theta,t)}{\partial t} = \frac{\partial^2 w(\gamma,\theta,t)}{\partial \gamma^2} + \frac{1}{\gamma}\frac{\partial w(\gamma,\theta,t)}{\partial \gamma} \\ \qquad + \frac{1}{\gamma^2}\frac{\partial^2 w(\gamma,\theta,t)}{\partial \theta^2},\ 1 < \gamma < 2,\ 0 < \theta < 2\pi \\ \frac{\partial w(1,\theta,t)}{\partial \gamma} = f(w(\cdot,t)) + d(\cos(\theta),\sin(\theta),t) \\ \frac{\partial w(2,\theta,t)}{\partial \gamma} = \frac{\partial v(2,\theta,t)}{\partial \gamma},\ 0 \le \theta \le 2\pi,\ t \ge 0 \\ \frac{\partial \widehat{w}(\gamma,\theta,t)}{\partial t} = \frac{\partial^2 \widehat{w}(\gamma,\theta,t)}{\partial \gamma^2} + \frac{1}{\gamma}\frac{\partial \widehat{w}(\gamma,\theta,t)}{\partial \gamma} \\ \qquad + \frac{1}{\gamma^2}\frac{\partial^2 \widehat{w}(\gamma,\theta,t)}{\partial \theta^2},\ 1 < \gamma < 2,\ 0 < \theta < 2\pi \\ \widehat{w}(1,\theta,t) = w(1,\theta,t),\ 0 \le \theta \le 2\pi,\ t \ge 0 \\ \frac{\partial \widehat{w}(2,\theta,t)}{\partial \gamma} = \frac{\partial v(2,\theta,t)}{\partial \gamma},\ 0 \le \theta \le 2\pi,\ t \ge 0 \\ \frac{\partial v(\gamma,\theta,t)}{\partial t} = \frac{\partial^2 v(\gamma,\theta,t)}{\partial \gamma^2} + \frac{1}{\gamma}\frac{\partial v(\gamma,\theta,t)}{\partial \gamma} \\ \qquad + \frac{1}{\gamma^2}\frac{\partial^2 v(\gamma,\theta,t)}{\partial \theta^2},\ 1 < \gamma < 2,\ 0 < \theta < 2\pi \\ \frac{\partial v(1,\theta,t)}{\partial \gamma} = -c_0(v(1,\theta,t) - \widehat{w}(1,\theta,t)) + \frac{\partial \widehat{w}(1,\theta,t)}{\partial \gamma} \\ v(2,\theta,t) = r(2\cos(\theta), 2\sin(\theta), t),\ 0 \le \theta \le 2\pi,\ t \ge 0 \end{cases} \tag{86}$$

where we still use w, $\widehat{w}$, and v to denote the states under the polar coordinate for notation simplicity (the exact coordinate should be clear from the context). The corresponding initial value (85) is transformed into

$$\begin{cases} w(\gamma,\theta,0) = \cos(3\theta), 1 \le \gamma \le 2, 0 \le \theta \le 2\pi \\ \widehat{w}(\gamma,\theta,0) = \sin(3\theta), 1 \le \gamma \le 2, 0 \le \theta \le 2\pi \\ v(\gamma,\theta,0) = \cos(3\theta) + \sin(3\theta), 1 \le \gamma \le 2, 0 \le \theta \le 2\pi. \end{cases}$$

The backward Euler method in time and the Chebyshev spectral method for polar variables are used to discretize system (86). Here, we take the grid sizes $\gamma_N = 30$ for γ and $\theta_N = 50$ for θ, and $dt = 5 \times 10^{-4}$ for the time step. The numerical algorithm is programmed in MATLAB [28] and the numerical results are plotted in Figs. 1–5.

Fig. 1 plots the tracking errors for the output signal to be regulated and the reference for $\theta \in [0, 2\pi]$. The convergence of tracking error is clearly observed from the particular direction $\theta = \pi/4$ in the polar coordinate in Fig. 2. The numerical results for $w(x,t)$, $\widehat{w}(x,t)$, and $v(x,t)$ are shown in Figs. 3–5 for direction $\theta = \pi/4$. It is seen that states $w(x,t)$, $\widehat{w}(x,t)$, and $v(x,t)$ are all bounded.

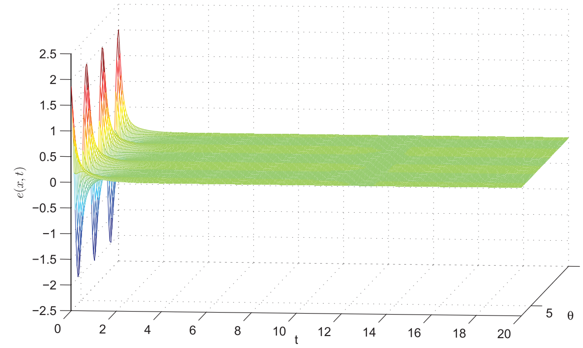

Fig. 1. Tracking error $e(x,t) = y_o(x,t) - r(x,t)$ (for interpretation of the references to color of the figure's legend in this section, we refer to the PDF version of this paper).

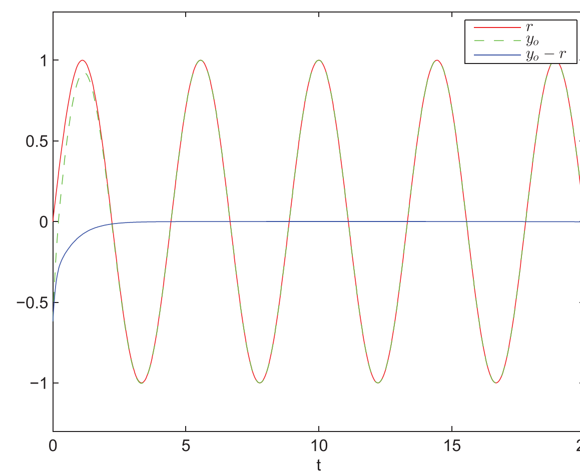

Fig. 2. Tracking error $e(x,t) = y_o(x,t) - r(x,t)$ in the radial direction of $\theta = \frac{1}{4}\pi$ (for interpretation of the references to color of the figure's legend in this section, we refer to the PDF version of this paper).

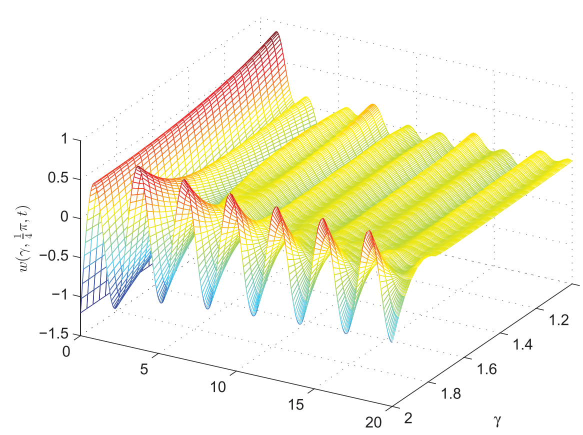

Fig. 3. Evolution of $w(\gamma, \frac{1}{4}\pi, t)$ under the polar coordinate (for interpretation of the references to color of the figure's legend in this section, we refer to the PDF version of this paper).

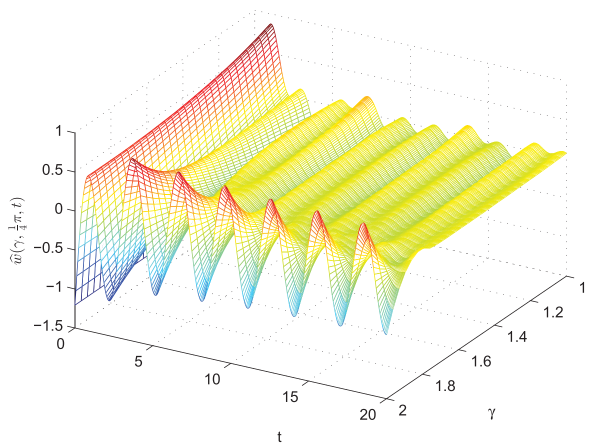

Fig. 4. Evolution of $\widehat{w}(\gamma, \frac{1}{4}\pi, t)$ under the polar coordinate (for interpretation of the references to color of the figure's legend in this section, we refer to the PDF version of this paper).

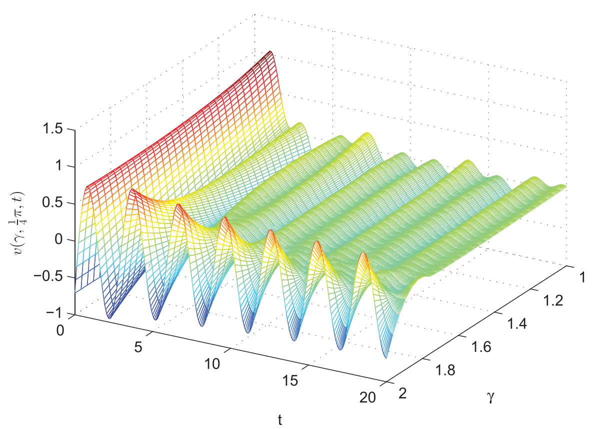

Fig. 5. Evolution of $v(\gamma, \frac{1}{4}\pi, t)$ under the polar coordinate (for interpretation of the references to color of the figure's legend in this section, we refer to the PDF version of this paper).

VII. Concluding Remarks

In this paper, we present for the first time the ADRC for performance output tracking to a boundary controlled multidimensional PDE. A new infinite-dimensional ESO is proposed to asymptotically estimates both unknown nonlinear internal uncertainty and external disturbance for a multidimensional heat equation. The speciality of this problem lies in the following three points:

1) the uncertainty and control are not matched;
2) the boundary observation is almost minimal in the sense that the signal makes system approximately observable only;
3) the performance output is not used in the control design.

A servomechanism is designed by virtue of the measured output and the reference signal after compensation of the total disturbance from its estimation obtained from the ESO. The following three major control objectives are achieved.

1) The performance output exponentially tracks the reference signal.
2) All internal-loops are bounded.
3) When the disturbance and reference signal belong to $L^2(0,\infty;L^2(\Gamma_0))$ and $H^1(0,\infty;L^2(\Gamma_1))$, respectively, the closed-loop is asymptotically stable.

In particular, The last point states that in the absence of disturbance and reference, the closed loop is exponentially stable, that is, the overall system is internally asymptotically stable. In addition, the feedback control is shown to be robust to the measurement noise.

An related important problem in the ADRC control of PDEs arises when the control and the performance output are not on the same part of the boundary (noncollocated). This is a difficult problem because the control must go through the whole spatial domain to exert its force from one part to another part of the boundary. A possible approach could be the backstepping design, which has been applied systematically to 1-D problems in [18] and to the domain of $\mathbb{R}^n$ balls in [30]. Moreover it would be more intriguing to develop ADRC configuration for general framework of state-space representation of uncertain infinite-dimensional systems to cover more 1-D and multidimensional PDEs.

Appendix
Well Posedness of Observer (4)

Theorem A.1: Suppose that the input and measured output of system (1) satisfy the following smoothness conditions:

$$u \in H^1_{\rm loc}(0,\infty;L^2(\Gamma_1)), \quad y_{\rm m} \in H^1_{\rm loc}(0,\infty;L^2(\Gamma_0)).$$

Then, for any $\widehat{w}(\cdot,0)\in L^2(\Omega)$, there exists a unique solution $\widehat{w}\in C(0,\infty;L^2(\Omega))$ to (4). Furthermore, if $u\in W^{1,\infty}(0,\infty;L^2(\Gamma_1))$, $y_{\rm m}\in W^{1,\infty}(0,\infty;L^2(\Gamma_0))$, then

$$\sup_{t\geq 0}\|\widehat{w}(\cdot,t)\|_{L^2(\Omega)}<\infty.$$

Proof: Let $\widehat{w}(x,t)$ denote the solution of (4) and write $\widehat{w}(x,t)=Y(x,t)+Z(x,t)$, where $Y(x,t)$ and $Z(x,t)$ are described, respectively, by

$$\begin{cases} Y_t(x,t)=\Delta Y(x,t),\ x\in\Omega,\ t>0\\ Y(x,t)|_{\Gamma_0}=y_{\rm m}(x,t)|_{\Gamma_0},\ t\geq 0\\ \frac{\partial Y(x,t)}{\partial\nu}|_{\Gamma_1}=0,\ t\geq 0\end{cases} \tag{87}$$

and

$$\begin{cases} Z_t(x,t)=\Delta Z(x,t),\ x\in\Omega,\ t>0\\ Z(x,t)|_{\Gamma_0}=0,\ t\geq 0\\ \frac{\partial Z(x,t)}{\partial\nu}|_{\Gamma_1}=u(x,t),\ t\geq 0.\end{cases} \tag{88}$$

First, by Lemma III.2 and exchanging Γ_0 and Γ_1, we can obtain the well posedness of system (87): For any $Y(\cdot,0)\in L^2(\Omega)$, system (87) admits a unique solution $Y\in C(0,\infty;L^2(\Omega))$ whence $y_{\rm m}\in H^1_{\rm loc}(0,\infty;L^2(\Gamma_0))$, which is uniformly bounded for all $t\geq 0$, i.e, $\sup_{t\geq 0}\|Y(\cdot,t)\|_{L^2(\Omega)}<\infty$ if $y_{\rm m}\in W^{1,\infty}(0,\infty;L^2(\Gamma_0))$.

Next we discuss the well posedness of system (88). To this purpose, we introduce a Neumann map $\Upsilon_0\in\mathscr{L}(H^s(\Gamma_1),$

$H^{3/2+s}(\Omega)$) (see[23, p. 668]): $\Upsilon_0(r) = z$ if and only if

$$\begin{cases} \Delta z = 0 \text{ in } \Omega \\ z|_{\Gamma_0} = 0, \ \frac{\partial z}{\partial \nu}\big|_{\Gamma_1} = u. \end{cases}$$

Since $u \in W^{1,\infty}(0,\infty; L^2(\Gamma_1))$, it is obvious that $z \in W^{1,\infty}(0,\infty; H^{3/2}(\Omega))$ and

$$\begin{cases} \|z(\cdot,t)\|_{H^{3/2}(\Omega)} \le C_1 \|u(\cdot,t)\|_{L^2(\Gamma_1)} \\ \|z_t(\cdot,t)\|_{H^{3/2}(\Omega)} \le C_1 \|u_t(\cdot,t)\|_{L^2(\Gamma_1)} \end{cases}$$

for some constant $C_1 > 0$. By the Sobolev embedding theorem, there exists a constant $C_2 > 0$ such that

$$\|z_t(\cdot,t)\|_{L^2(\Omega)} = \|z_t(\cdot,t)\|_{H^0(\Omega)} \le C_2 \|z_t(\cdot,t)\|_{H^{3/2}(\Omega)} \le C_1 C_2 \|u_t(\cdot,t)\|_{L^2(\Gamma_1)}.$$

Using the Neumann map $\Upsilon_0(r) = z$ and setting $\widehat{Z}(x,t) = Z(x,t) - z(x,t)$, we can verify that $\widehat{Z}(x,t)$ is governed by

$$\begin{cases} \widehat{Z}_t(x,t) = \Delta \widehat{Z}(x,t) - z_t(x,t), \ x \in \Omega, \ t > 0 \\ \widehat{Z}(x,t)|_{\Gamma_0} = 0, \ t \ge 0 \\ \frac{\partial \widehat{Z}(x,t)}{\partial \nu}|_{\Gamma_1} = 0, \ t \ge 0 \end{cases}$$

which can be written as

$$\frac{d}{dt}\widehat{Z}(\cdot,t) = A\widehat{Z}(\cdot,t) - z_t(\cdot,t)$$

where $A = \Delta$ is the usual Laplacian with $D(A) = \big\{\phi \in H^2(\Omega) : \phi|_{\Gamma_0} = 0, \ \frac{\partial \phi}{\partial \nu}|_{\Gamma_1} = 0\big\}$. The rest proof is exactly the same as the proof of Lemma III.2, from which we can conclude that for $u \in H^1_{\text{loc}}(0,\infty; L^2(\Gamma_1))$ and $Z(\cdot,0) \in L^2(\Omega)$, the system (87) admits a unique solution $Z \in C(0,\infty; L^2(\Omega))$, which is uniformly bounded for all $t \ge 0$ if $u \in W^{1,\infty}(0,\infty; L^2(\Gamma_1))$, i.e, $\sup_{t\ge 0} \|Z(\cdot,t)\|_{L^2(\Omega)} < \infty$. ■

In Theorem A.1, the output smoothness condition $y_{\text{m}} \in H^1_{\text{loc}}(0,\infty; L^2(\Gamma_0))$ for open-loop system (1) can be further guaranteed by some conditions of system (1), which is a pure mathematical problem, less relevant to control problem discussed in this paper. We only address this problem for linear case of $f \equiv 0$. The nonlinear case is much complicated.

Theorem A.2: Suppose that $u \in L^2_{\text{loc}}(0,\infty; L^2(\Gamma_1))$, $d \in L^2_{\text{loc}}(0,\infty; L^2(\Gamma_0))$, $f : L^2(\Omega) \to L^2(\Gamma_0)$ is continuous, bounded, and satisfies the local Lipschitz condition in $L^2(\Omega)$. Then, for any $w_0 \in L^2(\Omega)$, the open-loop system (1) admits a unique local solution $w \in C(0,\tau; L^2(\Omega))$ for some $\tau > 0$. Moreover, if $f(\cdot)$ satisfies the global Lipschitz condition, then (1) admits a global solution $w \in C(0,\infty; L^2(\Omega))$ and $y_{\text{m}} \in L^2_{\text{loc}}(0,\infty; L^2(\Gamma_0))$. For the linear case of $f \equiv 0$, if $u \in H^1_{\text{loc}}(0,\infty; L^2(\Gamma_1))$ and $d \in H^1_{\text{loc}}(0,\infty; L^2(\Gamma_0))$, then $y_{\text{m}} \in H^1_{\text{loc}}(0,\infty; L^2(\Gamma_0))$.

Proof: We first write system (1) as

$$\frac{d}{dt}w(\cdot,t) = \mathscr{A} w(\cdot,t) + \mathscr{B}_1[f(w) + d(\cdot,t)] + \mathscr{B}_2 u(\cdot,t) \tag{89}$$

where $\mathscr{A} = \Delta$ is the Laplacian with domain $D(\mathscr{A}) = \{\phi \in H^2(\Omega) : \frac{\partial \phi}{\partial \nu}|_{\partial\Omega} = 0\}$, and $\mathscr{B}_1 = \delta|_{\Gamma_0}$, $\mathscr{B}_2 = \delta|_{\Gamma_1}$ are two Dirac functions. It is well known that $\mathscr{A} = \mathscr{A}^*$ and $\mathscr{A}$ generates a C_0-semigroup $e^{\mathscr{A}t} = e^{\mathscr{A}^*t}$. Now we show that $\mathscr{B}_1$ and $\mathscr{B}_2$ are admissible for $e^{\mathscr{A}t}$. By [29, Th. 4.4.3], it suffices to show that $\mathscr{B}_1^*$ and $\mathscr{B}_2^*$ are admissible observation operators for the adjoint semigroup $e^{\mathscr{A}^*t}$. This amounts to showing that a) $\mathscr{B}_1^*(\mathscr{A}^* - I)^{-1} : L^2(\Omega) \to L^2(\Gamma_1)$ and $\mathscr{B}_2^*(\mathscr{A}^* - I)^{-1} : L^2(\Omega) \to L^2(\Gamma_0)$ are bounded, respectively; and b) for every $T^* > 0$, there exists $M_{T^*} > 0$ depending on T^* only such that the system of the following:

$$\begin{cases} w_t^*(x,t) = \Delta w^*(x,t), \ x \in \Omega, \ t > 0 \\ \frac{\partial w^*(x,t)}{\partial \nu}|_{\partial\Omega} = 0, \ t \ge 0 \\ w^*(x,0) = w_0^*(x), \ x \in \Omega \\ y_{\text{m}}^* = w^*(x,t)|_{\partial\Omega}, \ t \ge 0 \end{cases} \tag{90}$$

satisfies

$$\int_0^{T^*} \int_{\partial\Omega} (w^*(x,t))^2 dx dt \le M_{T^*} \|w_0^*\|^2_{L^2(\Omega)}.$$

A simple computation shows that $(\mathscr{A}^* - I)^{-1}\widehat{\phi} = \phi$, and

$$\mathscr{B}_1^*(\mathscr{A}^* - I)^{-1}\widehat{\phi} = \phi|_{\Gamma_0}, \ \ \mathscr{B}_2^*(\mathscr{A}^* - I)^{-1}\widehat{\phi} = \phi|_{\Gamma_1} \tag{91}$$

where ϕ satisfies the following PDEs

$$\Delta\phi(x) - \phi(x) = \widehat{\phi}(x), \ x \in \Omega, \ \ \frac{\partial \phi}{\partial \nu}\bigg|_{\partial\Omega} = 0. \tag{92}$$

From the elliptic partial differential equation theory, it is seen that (92) admits a unique solution $\phi \in H^2(\Omega)$ and there exists a constant $\widehat{C}_1 > 0$ such that $\|\phi\|_{H^2(\Omega)} \le \widehat{C}_1 \|\widehat{\phi}\|_{L^2(\Omega)}$. By the trace theorem, $\phi \in H^{1/2}(\Gamma_0)$, $\phi \in H^{1/2}(\Gamma_1)$, and there exists a constant $\widehat{C}_2 > 0$ such that $\max\{\|\phi\|_{H^{1/2}(\Gamma_0)}, \|\phi\|_{H^{1/2}(\Gamma_1)}\} \le \widehat{C}_2 \|\phi\|_{H^2(\Omega)} \le \widehat{C}_1 \widehat{C}_2 \|\widehat{\phi}\|_{L^2(\Omega)}$. This implies that $\mathscr{B}_1^*(\mathscr{A}^* - I)^{-1}$ and $\mathscr{B}_2^*(\mathscr{A}^* - I)^{-1}$ are bounded.

Since $\mathscr{A}^*$ generates a C_0-semigroup on $L^2(\Omega)$, system (90) admits a unique solution $w^* \in C(0,\infty; L^2(\Omega))$ and there exist two constants $\hat{M}, \hat{\mu} > 0$ such that

$$\int_\Omega (w^*(x,t))^2 dx \le \hat{M} e^{\hat{\mu}t} \int_\Omega (w^*(x,0))^2 dx. \tag{93}$$

Let $E(t) = \int_\Omega (w^*(x,t))^2 dx$. Differentiating $E(t)$ along the solution of (90) gives $\dot{E}(t) = -\int_\Omega |\nabla w^*(x,t)|^2 dx$, which yields

$$\int_0^{T^*} \int_\Omega |\nabla w^*(x,t)|^2 dx dt \le E(0). \tag{94}$$

Since by the Sobolev embedding theorem and the trace theorem

$$\int_{\partial\Omega} \psi^2(x) dx \le C_1 \|\psi\|^2_{H^{1/2}(\partial\Omega)} \le C_1 C_2 \|\psi\|^2_{H^1(\Omega)} = C_1 C_2 [\|\psi\|^2_{L^2(\Omega)} + \||\nabla\psi|\|^2_{L^2(\Omega)}] \tag{95}$$

for some $C_1, C_2 > 0$, it follows from (93) and (94) that

$$\int_0^{T^*}\int_{\partial\Omega}(w^*(x,t))^2dxdt$$
$$\leq C_1C_2\left[\int_0^{T^*}\int_\Omega (w^*(x,t))^2dxdt + \int_0^{T^*}\int_\Omega |\nabla w^*(x,t)|^2dxdt\right]$$
$$\leq C_1C_2[\hat{M}T^*e^{\hat{\mu}T^*} + T^*]\|w_0^*\|^2_{L^2(\Omega)}.$$

Hence, $\mathscr{B}_1$ and $\mathscr{B}_2$ are admissible for $e^{\mathscr{A}t}$. Therefore, similar to the proof of [33, Proposition 1.2], one can get the local solution and global solution of system (89).

Next, we claim $y_{\rm m} \in L^2_{\rm loc}(0,\infty;L^2(\Gamma_0))$. Since $w \in C(0,\infty;L^2(\Omega))$, for any fixed $T > 0$, $\|w(\cdot,t)\|_{L^2(\Omega)} \leq M_T$ for some $M_T > 0$. Since $f(\cdot)$ is bounded, $\|f(w)\|_{L^2(\Gamma_0)} \leq M_0$ for some $M_0 > 0$. Let $V(t) = \frac{1}{2}\int_\Omega w^2(x,t)dx$. Finding $\dot{V}(t)$ along the solution of (1) gives

$$\dot{V}(t) = -\int_\Omega [|\nabla w(x,t)|^2dx + \int_{\Gamma_0} w(x,t)(f(w)+d(x,t))dx$$
$$+ \int_{\Gamma_1} w(x,t)u(x,t)dx$$
$$\leq -\int_\Omega [|\nabla w(x,t)|^2dx + \kappa\int_{\partial\Omega} w^2(x,t)dx$$
$$+ \frac{1}{2\kappa}\int_{\Gamma_0}[M_0^2 + d^2(x,t)]dx + \frac{1}{4\kappa}\int_{\Gamma_1} u^2(x,t)dx$$

where $\kappa > 0$ is chosen so that $\kappa C_1C_2 = 1/2$. This produces

$$\int_0^T\int_\Omega [|\nabla w(x,t)|^2dx \leq \kappa\int_0^T\int_{\partial\Omega} w^2(x,t)dx$$
$$+ V(0) + \frac{T}{2\kappa}M_0^2 + \frac{1}{2\kappa}\|d\|^2_{L^2(0,T;L^2(\Gamma_0))}$$
$$+ \frac{1}{4\kappa}\|u\|^2_{L^2(0,T;L^2(\Gamma_1))}$$

which, together with (95) and $\|w(\cdot,t)\|_{L^2(\Omega)} \leq M_T$, yields

$$\int_0^T\int_{\partial\Omega} w^2(x,t)dxdt$$
$$\leq C_1C_2\left[\int_0^T\int_\Omega (w(x,t))^2dxdt + \int_0^T\int_\Omega |\nabla w(x,t)|^2dxdt\right]$$
$$\leq C_1C_2M_T^2 + \frac{1}{2}\int_0^T\int_{\partial\Omega} w^2(x,t)dx + C_1C_2V(0)$$
$$+ \frac{C_1C_2T}{2\kappa}M_0^2 + \frac{C_1C_2}{2\kappa}\|d\|^2_{L^2(0,T;L^2(\Gamma_0))}$$
$$+ \frac{C_1C_2}{4\kappa}\|u\|^2_{L^2(0,T;L^2(\Gamma_1))}.$$

This implies $w|_{\partial\Omega} \in L^2_{\rm loc}(0,\infty;L^2(\partial\Omega))$ and, hence, $y_{\rm m} \in L^2_{\rm loc}(0,\infty;L^2(\Gamma_0))$.

Finally, we claim the last assertion. When $f \equiv 0$, $u \in H^1_{\rm loc}(0,\infty;L^2(\Gamma_1))$, and $d \in H^1_{\rm loc}(0,\infty;L^2(\Gamma_0))$, we can see that $\dot{w}(x,t)$ is governed by

$$\begin{cases} \dot{w}_t(x,t) = \Delta\dot{w}(x,t),\ x\in\Omega,\ t>0 \\ \frac{\partial\dot{w}(x,t)}{\partial\nu}|_{\Gamma_0} = \dot{d}(x,t),\ t\geq 0 \\ \frac{\partial\dot{w}(x,t)}{\partial\nu}|_{\Gamma_1} = \dot{u}(x,t),\ t\geq 0 \\ \dot{y}_{\rm m} = \dot{w}(x,t)|_{\Gamma_0} \end{cases} \tag{96}$$

where $\dot{u} \in L^2_{\rm loc}(0,\infty;L^2(\Gamma_1))$ and $\dot{d} \in L^2_{\rm loc}(0,\infty;L^2(\Gamma_0))$. However, for system (96), we have shown that $\dot{y}_{\rm m} \in L^2_{\rm loc}(0,\infty;L^2(\Gamma_0))$. This gives $y_{\rm m} \in H^1_{\rm loc}(0,\infty;L^2(\Gamma_0))$. ■

Acknowledgment

The authors would like to thank Dr. Z.-Z. Zhang from Donghua University of China for discussion on the simulation and Prof. R. Yu from Eastern Mediterranean University, North Cyprus for polishing the writing. One anonymous reviewer indicated the well posedness of observer (4), which seems difficult because the open-loop plant (1) is a nonlinear PDE. Nevertheless, under some smoothness conditions of (1), we present a partial result on well posedness of observer (4) and open-loop system (1) in the Appendix. The constructive suggestions from reviewers are fully acknowledged.

References

[1] C. I. Byrnes, I. G. Laukó, D. S. Gilliam, and V. I. Shubov, "Output regulation for linear distributed parameter systems," *IEEE Trans. Automat. Control*, vol. 45, no. 12, pp. 2236–2252, Dec. 2000.
[2] E. J. Davison, "The robust control of a servomechanism problem for linear time-invariant multivariable systems," *IEEE Trans. Automat. Control*, vol. AC-21, no. 1, pp. 25–34, Feb. 1976.
[3] C. A. Desoer and C. A. Lin, "Tracking and disturbance rejection of MIMO nonlinear systems with PI controller," *IEEE Trans. Automat. Control*, vol. AC-30, no. 9, pp. 861–867, Sep. 1985.
[4] J. Deutscher, "A backstepping approach to the output regulation of boundary controlled parabolic PDEs," *Automatica*, vol. 57, pp. 56–64, 2015.
[5] J. Deutscher, "Output regulation for linear distributed-parameter systems using finite-dimensional dual observers," *Automatica*, vol. 47, pp. 2468–2473, 2011.
[6] B. A. Francis and W. M. Wonham, "The internal model principle of control theory," *Automatica*, vol. 12, pp. 457–465, 1976.
[7] H. Feng and B. Z. Guo, "A new active disturbance rejection control to output feedback stabilization for a one-dimensional anti-stable wave equation with disturbance," *IEEE Trans. Automat. Control*, vol. 62, no. 8, pp. 3774–3787, Aug. 2017.
[8] B. Z. Guo and F. F. Jin, "Output feedback stabilization for one-dimensional wave equation subject to boundary disturbance," *IEEE Trans. Automat. Control*, vol. 60, no. 3, pp. 824–830, Mar. 2015.
[9] B. Z. Guo and H. C. Zhou, "The active disturbance rejection control to stabilization for multi-dimensional wave equation with boundary control matched disturbance," *IEEE Trans. Automat. Control*, vol. 60, no. 1, pp. 143–157, Jan. 2015.
[10] B. Z. Guo and H. C. Zhou, "Active disturbance rejection control for rejecting boundary disturbance from multidimensional Kirchhoff plate via boundary control," *SIAM J. Control Optim.*, vol. 52, pp. 2800–2830, 2014.
[11] W. Guo and B. Z. Guo, "Performance output tracking for a wave equation subject to unmatched general boundary harmonic disturbance," *Automatica*, vol. 68, pp. 194–202, 2016.
[12] W. Guo, Z. C. Shao, and M. Krstic, "Adaptive rejection of harmonic disturbance anticollocated with control in 1D wave equation," *Automatica*, vol. 79, pp. 17–26, 2017.
[13] W. Guo, H. C. Zhou, and M. Krstic, "Adaptive error feedback output regulation for 1D wave equation," *Int. J. Robust Nonlinear Control*, vol. 28, pp. 4309–4329, 2018.

[14] W. Guo and H. C. Zhou, "Adaptive error feedback regulation problem for an Euler–Bernolli beam equation with unmatched general unmatched boundary harmonic disturbance," *SIAM J. Control Optim.*, vol. 57, no. 3, pp. 1890–1928, 2019.

[15] E. Immonen and S. Pohjolainen, "Output regulation of periodic signals for DPS: An infinite-dimensional signal generator," *IEEE Trans. Automat. Control*, vol. 50, no. 11, pp. 1799–1804, Nov. 2005.

[16] E. Immonen and S. Pohjolainen, "Feedback and feedforward output regulation of bounded uniformly continuous signals for infinite-dimensional systems," *SIAM J. Control Optim.*, vol. 45, pp. 1714–1735, 2006.

[17] F. F. Jin and B. Z. Guo, "Performance boundary output tracking for one-dimensional heat equation suffered boundary unmatched disturbance," *Automatica*, vol. 96, pp. 1–10, 2018.

[18] M. Krstic and A. Smyshlyaev, *Boundary Control of PDEs: A Course on Backstepping Designs*. Philadelphia, PA, USA: SIAM, 2008.

[19] L. Liu, Z. Chen, and J. Huang, "Parameter convergence and minimal internal model with an adaptive output regulation problem," *Automatica*, vol. 45, pp. 1306–1311, 2009.

[20] J. L. Lions and E. Magens, *Non-Homegeneous Boundary-Valued Problems and Applications*. Berlin, Germany: Springer-Verlag, 1972.

[21] H. Logemann and A. Ilchmann, "An adaptive servomechanism for a class of infinite-dimensional systems," *SIAM J. Control Optim.*, vol. 32, pp. 917–936, 1994.

[22] P. O. Lamare and N. Bekiaris-Liberis, "Control of 2×2 linear hyperbolic systems: Backstepping-based trajectory generation and PI-based tracking," *Systems Control Lett.*, vol. 86, pp. 24–33, 2015.

[23] I. Lasiecka and R. Triggiani, *Control Theory for Partial Differential Equations: Continuous and Approximation Theories*, vol. II. Cambridge, U.K.: Cambridge Univ. Press, 2000.

[24] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*. New York, NY, USA: Springer-Verlag, 1983.

[25] A. Pisano, Y. Orlov, and E. Usai, "Tracking control of the uncertain heat and wave equation via power-fractional and sliding-mode techniques," *SIAM J. Control Optim.*, vol. 49, pp. 363–382, 2011.

[26] L. Paunonen and S. Pohjolainen, "The internal model principle for systems with unbounded control and observation," *SIAM J. Control Optim.*, vol. 52, pp. 3967–4000, 2014.

[27] R. Rebarber and G. Weiss, "Internal model based tracking and disturbance rejection for stable well-posed systems," *Automatica*, vol. 39, pp. 1555–1569, 2003.

[28] L. N. Trefethen, *Spectral Methods in MATLAB*. Philadelphia, PA, USA: SIAM, 2000.

[29] M. Tucsnak and G. Weiss, *Observation and Control for Operator Semigroups*. Basel, Switzerland: Birkhäuser, 2009.

[30] R. Vazquez and M. Krstic, "Explicit output-feedback boundary control of reaction-diffusion PDEs on arbitrary-dimensional balls," *ESAIM Control Optim. Calculus Variations*, vol. 22, pp. 1078–1096, 2016.

[31] G. Weiss, "Admissibility of unbounded control operators," *SIAM J. Control Optim.*, vol. 27, pp. 527–545, 1989.

[32] X. Zhang, H. Feng, and S. G. Chai, "Performance output exponential tracking for a wave equation with a general boundary disturbance," *Syst. Control Lett.*, vol. 98, pp. 79–85, 2016.

[33] H. C. Zhou and B. Z. Guo, "Unknown input observer design and output feedback stabilization for multi-dimensional wave equation with boundary control matched uncertainty," *J. Differ. Equ.*, vol. 263, pp. 2213–2246, 2017.

[34] H. C. Zhou and B. Z. Guo, "Performance output tracking for one-dimensional wave equation subject to unmatched general disturbance and non-collocated control," *Eur. J. Control*, vol. 39, pp. 39–52, 2018.

[35] H. C. Zhou and G. Weiss, "Output feedback exponential stabilization for one-dimensional unstable wave equations with boundary control matched disturbance," *SIAM J. Control Optim.*, vol. 56, pp. 4098–4129, 2018.

Hua-Cheng Zhou (M'18) received the Ph.D. degree in applied mathematics from the Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing, China, in 2015.

From 2015 to 2018, he was a Postdoctoral Fellow with the School of Electrical Engineering, Tel Aviv University, Tel Aviv, Israel. He is currently with the School of Mathematics and Statistics, the Central South University, Changsha, China. His research interests focus on distributed parameter systems control.

Bao-Zhu Guo (SM'09) received the Ph.D. degree in applied mathematics from the Chinese University of Hong Kong, Hong Kong, in 1991.

From 1985 to 1987, he was a Research Assistant with the Beijing Institute of Information and Control, Beijing, China. During the period 1993–2000, he was with the Beijing Institute of Technology, Beijing, China, first as an Associate Professor (1993–1998) and subsequently a Professor (1998–2000). Since 2000, he has been with the Academy of Mathematics and Systems Science, the Chinese Academy of Sciences, where he is a Research Professor in Mathematical System Theory. From 2004 to 2019, he was a Chair Professor with the School of Computer Science and Applied Mathematics, University of the Witwatersrand, Johannesburg, South Africa. He is the author or coauthor of the books: *Stability and Stabilization of Infinite Dimensional Systems with Applications* (Springer-Verlag, 1999), *Active Disturbance Rejection Control for Nonlinear Systems: An Introduction* (Wiley, 2016), and *Control of Wave and Beam PDEsThe Riesz Basis Approach*" (Springer-Verlag, 2019). His research interests include theory of infinite-dimensional systems and active disturbance rejection control.

Dr. Guo received the One Hundred Talent Program from the Chinese Academy of Sciences (1999), and the National Science Fund for Distinguished Young Scholars (2003).

Shu-Huang Xiang received the Ph.D. degree in computational mathematics from the Xi'an Jiaotong University, Xi'an, China, in 1997.

From 1997 to 1999, he was a Postdoctoral Fellow with the Nankai University. During the period 1992–1997, he was with Xi'an Jiaotong University. Since 1999, he has been with the School of Mathematics and Statistics, Central South University, Changsha, China, where he is currently a Full Professor in Computational Mathematics. His research interests include the robustness, high performance computing methods, and high performance numerical integration algorithm for high oscillation differential equations.