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BOUNDARY FEEDBACK STABILIZATION FOR AN UNSTABLE TIME FRACTIONAL REACTION DIFFUSION EQUATION*

HUA-CHENG ZHOU† AND BAO-ZHU GUO‡

Abstract. In this paper, we consider boundary feedback stabilization for unstable time fractional reaction diffusion equations. New state feedback controls with actuation on one end are designed by the backstepping method for both Dirichlet and Neumann boundary controls. By the Riesz basis approach and the fractional Lyapunov method, we prove the existence and uniqueness and the Mittag–Leffler stability for the closed-loop systems. For both cases, the observers and the observer-based output feedback are designed to stabilize the systems.

Key words. time fractional reaction diffusion equation, boundary control, output feedback, stabilization

AMS subject classifications. 35R11, 35K57, 37L15, 37B25, 47B06, 93D15, 93B51, 93B52

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1. Introduction.

Output feedback stabilization is one of the fundamental issues in control theory. The key idea in output feedback design is that the control and output should be as few in number as possible. For stabilization purpose, the control should make the system stabilizable, and the output should make the system detectable. However, stabilizability and detectability are very difficult to be checked for systems described by partial differential equations (PDEs). They are replaced usually by controllability and observability in PDEs. There are extensive studies on output feedback stabilization for PDEs, yet few results are available even on state feedback stabilization for fractional differential equations. In [22, 29], output feedback controls are designed for finite dimensional fractional order systems by linear matrix inequality and the direct Lyapunov approach. For controllability and observability aspects, there are some results for fractional PDEs. In [7], approximate controllability for fractional diffusion equations with Dirichlet boundary control was considered. In [17, 21], approximate controllability for abstract fractional equations was discussed, which can be applied to fractional diffusion equations but is not applicable to boundary control because the control operator there was supposed to be bounded while the boundary control leads usually unbounded control operator. A first attempt on boundary stabilization for time fractional diffusion-wave equations was investigated in [19], where mainly numerical simulations were presented to illustrate the effectiveness of boundary control and no rigorous mathematical proof was presented. The backstepping method was first applied to control of fractional ordinary differential equations in [6]. In this respect, a recent development can be found in [5]. In [25], stabilization for a one-dimensional wave equation via boundary fractional derivative control was dis-

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cussed. Recently, stabilization for a fractional order linear system subject to input saturation has been discussed in [24]. However, to the best of our knowledge, the backstepping method has not been applied for fractional PDEs. In this paper, we adopt the backstepping approach to achieve output feedback stabilization for a class of unstable time fractional reaction diffusion equations.

We begin with the Mittag–Leffer function $E_\alpha(z)$ with $\alpha > 0$, which is defined by the following series representation, valid in the whole complex plane:

$$E_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)}, \ \alpha > 0, \ z \in \mathbb{C}, \tag{1.1}$$

where $\Gamma(\cdot)$ is the gamma function. The two-parameter Mittag–Leffler function $E_{\alpha,\beta}(z)$, $\alpha, \beta > 0$ is defined by the following series representation:

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \ \alpha > 0, \ z \in \mathbb{C}. \tag{1.2}$$

The most interesting properties of the Mittag–Leffer function are associated with its asymptotic property as $z \to \infty$ in various sectors of the complex plane. These properties can be summarized as follows (see, e.g., [16, p. 41]). For $0 < \alpha \leq 1$,

$$E_\alpha(z) \sim \frac{1}{\alpha} \exp(z^{1/\alpha}) - \sum_{j=1}^{\infty} \frac{z^{-j}}{\Gamma(1 - \alpha j)}, \ |z| \to \infty, \ |\mathrm{Arg} z| < \frac{\alpha\pi}{2} \tag{1.3}$$

$$E_\alpha(z) \sim - \sum_{j=1}^{\infty} \frac{z^{-j}}{\Gamma(1 - \alpha j)}, |z| \to \infty, \ \frac{\alpha\pi}{2} < |\mathrm{Arg} z| \leq \pi. \tag{1.4}$$

The following Lemma 1.1 is brought from [27, Chapter 1, Theorem 1.6].

LEMMA 1.1. *For $\alpha \in (0, 2)$, arbitrary real number β, and $\frac{\pi}{2}\alpha < \eta < \min\{\pi, \pi\alpha\}$, there exists an $M > 0$ such that*

$$|E_{\alpha,\beta}(z)| \leq \frac{M}{1 + |z|}, \ \ \eta \leq |\arg(z)| \leq \pi, |z| \geq 0. \tag{1.5}$$

In this paper, we consider stabilization for the following time fractional reaction diffusion equation (TFRDE) with Dirichlet boundary control and Neumann boundary control, respectively:

$$\begin{cases} {}_0^C D_t^\alpha w(x,t) = \varepsilon w_{xx}(x,t) + \lambda(x) w(x,t) + g(x) w(0,t) \\ \qquad\qquad + \displaystyle\int_0^x f(x,y) w(y,t) \mathrm{d}y, \ x \in (0,1), \ t \geq 0, \\ w_x(0,t) = -q w(0,t), \ t \geq 0, \\ w(1,t) = u(t), \ t \geq 0, \\ w(x,0) = w_0(x), \ 0 \leq x \leq 1, \\ y_o(t) = w(0,t), \ t \geq 0, \end{cases} \tag{1.6}$$

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and

$$
(1.7)\qquad \begin{cases} {}_0^C D_t^\alpha w(x,t) = \varepsilon w_{xx}(x,t) + \lambda(x) w(x,t) + g(x) w(0,t) \\ \qquad\qquad + \displaystyle\int_0^x f(x,y) w(y,t)\mathrm{d}y, \quad x \in (0,1),\ t \geq 0, \\ w_x(0,t) = -q w(0,t),\ t \geq 0, \\ w_x(1,t) = u(t),\ t \geq 0, \\ w(x,0) = w_0(x),\ 0 \leq x \leq 1, \\ y_o(t) = w(1,t),\ t \geq 0, \end{cases}
$$

where $q \geq 0$, $\alpha \in (0,1]$ is the order of the fractional derivative, $u(t)$ is the input (control), $y_o(t)$ is the output (measurement), $w(x,t)$ is the state, $\varepsilon > 0$ is the diffusion coefficient, and $g, \lambda \in C[0,1]$ and $f \in C(\mathcal{F})$, where $\mathcal{F} := \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq x \leq 1\}$. The ${}_0^C D_t^\alpha w(x,t)$ is the Caputo derivative, which is also called Caputo–Dzhrbashyan derivative, and is a regularized fractional derivative of $w(x,t)$ with respect to time variable t, that is,

$$
{}_0^C D_t^\alpha w(x,t) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{\partial}{\partial t} \int_0^t (t-s)^{-\alpha} w(x,s)\mathrm{d}s - t^{-\alpha} w(x,0) \right].
$$

The time fractional reaction diffusion equation is perhaps one of the most important fractional order linear PDEs for description of the "memory" occurring in physics such as plasma turbulence [4], where Caputo–Dzhrbashyan derivative accounts for the trapping effect of the turbulent eddies. Fractional diffusion can also arise in finance [26] and hydrology [3] and in the context of levy flights [32]. In a physical model presented in [32], the fractional diffusion corresponds to a diverging jump length variance in the random walk, and a fractional time derivative arises when the characteristic waiting time diverges. It is well known that

$$
\lim_{\alpha \to 1^-} {}_0^C D_t^\alpha w(x,t) = \frac{\partial w(x,t)}{\partial t}.
$$

In other words, when $\alpha = 1$, the systems (1.6) and (1.7) are reduced to the classic reaction diffusion equations. For more about fractional calculus and fractional PDEs, we refer to the monographs [16, 20, 27] and the references therein. For notational simplicity, we drop the domains of time t and spatial variable x for associated equations in the rest of the paper.

The objective of this paper is to design output feedback controls to stabilize systems (1.6) and (1.7), respectively. First of all, we present examples to show that systems (1.6) and (1.7) can be unstable without control.

Example 1.2. Let $g(x) = 0$ on $[0,1]$, $f(x,y) = 0$ on $\mathcal{F}$, $\lambda(x) = \frac{\pi^2 \varepsilon}{2}$, and $q = 0$. Let the initial value be $w_0(x) = \sin(\frac{\pi}{2}(x-1))$. Then, system (1.6) without control admits a solution $w(x,t) = E_\alpha(\frac{\pi^2\varepsilon}{4} t^\alpha) \sin(\frac{\pi}{2}(x-1))$ satisfying $\|w(\cdot,t)\|_{L^2(0,1)} \to \infty$ as $t \to \infty$.

Example 1.3. Let $g(x) = 0$ on $[0,1]$, $f(x,y) = 0$ on $\mathcal{F}$, $\lambda(x) = \frac{5\pi^2\varepsilon}{4}$, and $q = 0$. Let the initial value be $w_0(x) = \cos(\pi x)$. Then, system (1.7) without control admits a solution $w(x,t) = E_\alpha(\frac{\pi^2\varepsilon}{4} t^\alpha) \cos(\pi x)$ satisfying also $\|w(\cdot,t)\|_{L^2(0,1)} \to \infty$ as $t \to \infty$.

In general, for large positive $q, \lambda(x), g(x)$, or $f(x,y)$, the systems (1.6) and (1.7) are unstable. This is because the operator A given by

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$$\begin{cases} (A\phi)(x) = \varepsilon\phi''(x) + \lambda(x)\phi(x) + g(x)\phi(0) + \int_0^x f(x,y)\phi(y)\mathrm{d}y, \\ D(A) = \{\phi \in H^2(0,1) | \phi'(0) = -q\phi(0), \phi(1) = 0\}, \text{ or} \\ D(A) = \{\phi \in H^2(0,1) | \phi'(0) = -q\phi(0), \phi'(1) = 0\} \end{cases}$$

has at least one positive eigenvalue whenever $q, \lambda(x), g(x)$, or $f(x,y)$ are sufficiently large.

DEFINITION 1.4. *(Mittag–Leffler stability). The solution of* (1.6) *or* (1.7) *is said to be Mittag–Leffler stable if*

$$\|w(\cdot,t)\|_{L^2(0,1)} \leq \{m(\|w(\cdot,0)\|_{L^2(0,1)})E_\alpha(-\lambda t^\alpha)\}^b,$$

where $\alpha \in (0,1)$, $\lambda > 0$, $b > 0$, $m(0) = 0$, $m(s) \geq 0$, *and* $m(s)$ *is locally Lipschitz on* $s \in \mathbb{R}$ *with Lipschitz constant* m_0.

The Mittag–Leffler stability implies the asymptotic stability. This is because by (1.4) and Lemma 1.1,

$$E_\alpha(-\lambda t^\alpha) \leq \frac{M}{1+\lambda t^\alpha} \text{ for all } t \geq 0 \text{ and } E_\alpha(-\lambda t^\alpha) = \mathcal{O}\left(\frac{1}{\lambda t^\alpha}\right) \text{ as } t \to \infty. \tag{1.8}$$

Thus, the Mittag–Leffler stability is actually polynomial stability when $\alpha \in (0,1)$. From (1.8), we can see that the parameter λ can be used to regulate the convergence speed.

Consider the following Cauchy problem in a Banach space H:

$$\begin{cases} {}_0^C D_t^\alpha X(t) = A_0 X(t), \\ X(0) = x, \end{cases} \tag{1.9}$$

where A_0 is a closed linear operator in H.

DEFINITION 1.5. *A function* $X \in C(\mathbb{R}^+; H)$ *is called a strong solution to* (1.9) *if* $X \in C(\mathbb{R}^+; D(A_0))$, $\int_0^t (X(s) - X(0))/(t-s)^\alpha ds \in C^1(\mathbb{R}^+; H)$, *and* (1.9) *holds on* $\mathbb{R}^+$. *The problem* (1.9) *is called well-posed if for any* $x \in D(A_0)$ *there exists a unique strong solution* $X(t,x)$ *of* (1.9), *and* $x_n \to 0$ *as* $n \to \infty$ *implies* $X(t,x_n) \to 0$ *as* $n \to \infty$ *in* H, *uniformly on compact intervals.*

The following lemma which can be found in [2] plays an important role in establishing the well-posedness of fractional PDEs.

LEMMA 1.6. *Suppose that* $\alpha \in (0,1)$. *Let* A_0 *be a closed linear operator densely defined in a Banach space* H. *If* A_0 *generates a* C_0*-semigroup on* H, *then Cauchy problem* (1.9) *admits a unique strong solution* $X \in C(0,\infty; H)$.

The study of the stabilization of fractional time derivative PDEs has just begun to catch researchers' attention. This paper provides one of the early results. Precisely, the main contributions of this paper are (1) to introduce a backstepping method for fractional reaction and diffusion equations which has potential applications to other equations and (2) to achieve the Mittag–Leffler stability for the closed-loop systems for both the Dirichlet control and the Neumann control problems by utilizing the measured outputs only. We proceed as follows. In section 2, we consider stabilization for an unstable time fractional reaction diffusion equation with the Dirichlet boundary control. Section 3 is about the stabilization via the Neumann boundary control. Some concluding remarks are presented in section 4.

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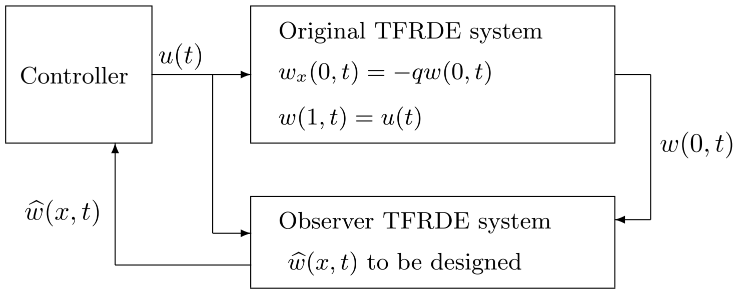

FIG. 1. *Block diagram of output feedback for time fractional reaction diffusion equation with Dirichlet boundary control* (1.6).

2. Backstepping with Dirichlet boundary control. In this section, we apply the backstepping approach to design an output feedback stabilizing for system (1.6) as depicted in Figure 1 in the sense of Mittag–Leffler stability.

2.1. Target system. We introduce a target system,

$$\begin{cases} {}_0^C D_t^\alpha z(x,t) = \varepsilon z_{xx}(x,t) - cz(x,t), \\ z_x(0,t) = z(1,t) = 0, \\ z(x,0) = z_0(x), \end{cases} \tag{2.1}$$

where the parameter c is used to regulate the convergence speed, which is seen from Lemma 2.1.

LEMMA 2.1. *For any initial value $z_0 \in L^2(0,1)$, system* (2.1) *admits a unique solution $z \in C(0,\infty; L^2(0,1))$. Moreover, the solution is Mittag–Leffler stable in $L^2(0,1)$:*

$$\|z(\cdot,t)\|^2_{L^2(0,1)} \le E^2_\alpha\left(-\left[c + \varepsilon\frac{\pi^2}{4}\right]t^\alpha\right)\|z_0\|^2_{L^2(0,1)}. \tag{2.2}$$

Proof. Define the operator $A_D : D(A_D)(\subset L^2(0,1)) \to L^2(0,1)$ as follows:

$$\begin{cases} [A_D f](x) = \varepsilon f''(x) - cf(x), \\ D(A_D) = \{f \in H^2(0,1) | f'(0) = 0, f(1) = 0\}. \end{cases} \tag{2.3}$$

It is well known that A_D is a generator of C_0-semigroup. By Lemma 1.6, we know that (2.1) has a unique solution $z \in C(0,\infty; L^2(0,1))$. Moreover, a simple computation shows that A_D is self-adjoint in $L^2(0,1)$ with the eigenvalues $\{\mu_j\}$ and the corresponding eigenfunctions $\{e_j(x)\}$ given by

$$\mu_j = -c - \varepsilon\Big(j + \frac{1}{2}\Big)^2\pi^2,\ e_j(x) = \sqrt{2}\sin\left(\left(j + \frac{1}{2}\right)\pi(x-1)\right),\ j = 0,1,2,\ldots. \tag{2.4}$$

Moreover, $\{e_j(x)\}$ forms an orthnormal basis for $L^2(0,1)$. Therefore, the solution of (2.1) can be represented as

$$z(x,t) = \sum_{j\ge 0}\varphi_j(t)e_j(x) \text{ with } z_0(x) = \sum_{j\ge 0} a_j e_j(x),$$

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where $\{a_j\} \in l^2$ and $\varphi_j \in C(0,\infty;\mathbb{R})$ satisfies the following linear fractional differential equation:

$$ {}_0^C D_t^\alpha \varphi_j(t) = \mu_j \varphi_j(t),\ \varphi_j(0) = a_j. $$

By [16, Chapter 4, Theorem 4.3], $\varphi_j(t) = a_j E_\alpha(\mu_j t^\alpha)$. Thus,

$$ z(x,t) = \sum_{j\ge 0} a_j E_\alpha(\mu_j t^\alpha) e_j(x). \tag{2.5} $$

Moreover, since $\{e_j(x)\}$ is an orthonormal basis for $L^2(0,1)$, it follows from (2.5) that

$$ \|z(\cdot,t)\|^2_{L^2(0,1)} = \sum_{j\ge 0} a_j^2 E_\alpha^2(\mu_j t^\alpha) \le E_\alpha^2(\mu_1 t^\alpha) \sum_{j\ge 0} a_j^2 = E_\alpha^2\left(-\left[c+\varepsilon\frac{\pi^2}{4}\right]t^\alpha\right)\|z_0\|^2_{L^2(0,1)}. $$

This completes the proof of the lemma. ∎

Remark 2.2. For stability of system (2.1), we can also apply alternatively fractional version of the Lyapunov method without solving equation (2.1). Actually, by following inequality [1, Lemma 1],

$$ {}_0^C D_t^\alpha z^2(t) \le 2z(t) {}_0^C D_t^\alpha z(t) \text{ for all } z \in C(0,\infty) $$

and Wirtinger's inequality [15, p. 182]

$$ \int_0^1 f^2(x)\mathrm{d}x \le \frac{4}{\pi^2}\int_0^1 (f'(x))^2 \mathrm{d}x \text{ for } f \in H^1(0,1) \text{ with } f'(0) = f(1) = 0, $$

we have

$$ \begin{aligned} {}_0^C D_t^\alpha \int_0^1 z^2(x,t)\mathrm{d}x &\le 2\int_0^1 z(x,t) {}_0^C D_t^\alpha z(x,t)\mathrm{d}x \\ &= 2\int_0^1 z(x,t)[\varepsilon z_{xx}(x,t) - cz(x,t)]\mathrm{d}x \\ &= -2\varepsilon\int_0^1 z_x^2(x,t)\mathrm{d}x - 2c\int_0^1 z^2(x,t)\mathrm{d}x \le -2\left(c+\varepsilon\frac{\pi^2}{4}\right)\int_0^1 z^2(x,t)\mathrm{d}x. \end{aligned} $$

It follows from the fractional Lyapunov method [23, Theorem 5] that the solution of (2.1) is Mittag–Leffler stable with

$$ \|z(\cdot,t)\|^2_{L^2(0,1)} \le E_\alpha\left(-2\left[c+\varepsilon\frac{\pi^2}{4}\right]t^\alpha\right)\|z_0\|^2_{L^2(0,1)}. \tag{2.6} $$

By (1.8), for sufficiently large t, we have

$$ E_\alpha^2\left(-\left[c+\varepsilon\frac{\pi^2}{4}\right]t^\alpha\right) < E_\alpha\left(-2\left[c+\varepsilon\frac{\pi^2}{4}\right]t^\alpha\right). $$

This shows that, comparing to (2.6), the estimation (2.2) is better and cannot be improved since for $z_0(x) = e_1(x)$,

$$ \|z(\cdot,t)\|^2_{L^2(0,1)} = E_\alpha^2\left(-\left[c+\varepsilon\frac{\pi^2}{4}\right]t^\alpha\right)\|z_0\|^2_{L^2(0,1)}. $$

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Remark 2.3. When $\alpha = 1$, the target system (2.1) becomes a classic heat equation, which is exponentially stable. This is because by (1.1) and (2.2),

$$\|z(\cdot,t)\|^2_{L^2(0,1)} \le E_1^2\left(-\left[c+\varepsilon\frac{\pi^2}{4}\right]t^1\right)\|z_0\|^2_{L^2(0,1)} = e^{-2\left[c+\varepsilon\frac{\pi^2}{4}\right]t}\|z_0\|^2_{L^2(0,1)}.$$

However, when $\alpha \in (0,1)$, the system (2.1) is never exponentially stable but only Mittag–Leffler stable. Furthermore, by Lemma 1.1,

$$\|z(\cdot,t)\|^2_{L^2(0,1)} \le E_\alpha^2\left(-\left[c+\varepsilon\frac{\pi^2}{4}\right]t^\alpha\right)\|z_0\|^2_{L^2(0,1)} \le \frac{1}{\left(1+\left[c+\varepsilon\frac{\pi^2}{4}\right]t^\alpha\right)^2}\|z_0\|^2_{L^2(0,1)}.$$

This is exactly the polynomial stability, and the parameter c is used to regulate the speed of convergence.

Remark 2.4. Generally speaking, we cannot expect exponential stability for a fractional PDE. Actually, even for fractional ordinary differential equations, since there is a memory effect in the equation due to the tail of time fractional derivative, there is no exponential stability. A typically example can be constructed as

$$ {}_0^C D_t^\alpha x(t) = -\lambda x(t), x(0) = x_0 \tag{2.7}$$

with $\lambda > 0$. The solution of (2.7) is explicitly found to be $x(t) = x_0 E_\alpha(-\lambda t^\alpha)$, which, by (1.8), is asymptotically stable but not exponentially stable when $\alpha \in (0,1)$.

2.2. Backstepping transform via state feedback. To find a state feedback control law for system (1.6), we introduce a transformation $w \to z$ [31],

$$z(x,t) = w(x,t) - \int_0^x k(x,y)w(y,t)\mathrm{d}y, \tag{2.8}$$

to transform system (1.6) into the target system (2.1), for which the stability is clearly presented in Lemma 2.1. When the transformation is invertible, stability for original system (1.6) can be obtained from the target system (2.1).

Taking Caputo's fractional derivative for (2.8) and using the first equation of (1.6), through performing the integration by parts, we obtain

$$\begin{aligned}
{}_0^C D_t^\alpha z(x,t) &= {}_0^C D_t^\alpha w(x,t) - \int_0^x k(x,y)\,{}_0^C D_t^\alpha w(y,t)\mathrm{d}y \\
&= {}_0^C D_t^\alpha w(x,t) - \int_0^x k(x,y)\bigg(\varepsilon w_{yy}(y,t) + \lambda(y)w(y,t) + g(y)w(0,t) \\
&\qquad\qquad + \int_0^y f(y,\xi)w(x,\xi)\mathrm{d}\xi\bigg)\mathrm{d}y \\
&= {}_0^C D_t^\alpha w(x,t) - \varepsilon\big(k(x,x)w_x(x,t) - k(x,0)w_x(0,t) \\
&\quad - [k_y(x,x)w(x,t) - k_y(x,0)w(0,t)]\big) - \int_0^x (k_{yy}(x,y) + \lambda(y)k(x,y))w(y,t)\mathrm{d}y \\
&\quad - w(0,t)\int_0^x k(x,y)g(y)\mathrm{d}y - \int_0^x w(y,t)\left(\int_y^x k(x,\xi)f(\xi,y)\mathrm{d}\xi\right)\mathrm{d}y
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
z_{xx}(x,t) &= w_{xx}(x,t) - \frac{d}{dx}(k(x,x))w(x,t) - k(x,x)w_x(x,t) \\
&\quad - k_x(x,x)w(x,t) - \int_0^x k_{xx}(x,y)w(y,t)\mathrm{d}y.
\end{aligned} \tag{2.10}$$

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Substituting (2.9) and (2.10) into (2.1), it follows that the kernel function $k(x,y)$ should satisfy the following PDE:

$$\begin{cases} \varepsilon k_{xx}(x,y)-\varepsilon k_{yy}(x,y)=(\lambda(y)+c)k(x,y)-f(x,y)+\displaystyle\int_y^x k(x,\xi)f(\xi,y)\mathrm{d}\xi, \\ \varepsilon k_y(x,0)+\varepsilon q k(x,0)=g(x)-\displaystyle\int_0^x k(x,y)g(y)\mathrm{d}y, \\ k(x,x)=-q-\dfrac{1}{2\varepsilon}\displaystyle\int_0^x(\lambda(y)+c)\mathrm{d}y. \end{cases} \tag{2.11}$$

By [30, Theorem 2.1], the PDE (2.11) has a unique solution $k \in C^2(\bar{\mathcal{F}})$.

To find the inverse of transform (2.8), suppose

$$w(x,t)=z(x,t)+\int_0^x l(x,y)z(y,t)\mathrm{d}y. \tag{2.12}$$

Similarly, taking Caputo's fractional derivative for (2.12) and using the first equation of (2.1) through performing the integration by parts, we have

$$\begin{aligned} {}_0^C D_t^\alpha w(x,t) &= {}_0^C D_t^\alpha z(x,t)+\int_0^x l(x,y){}_0^C D_t^\alpha z(y,t)\mathrm{d}y \\ &= {}_0^C D_t^\alpha z(x,t)+\int_0^x l(x,y)\big(\varepsilon z_{xx}(y,t)-cz(y,t)\big)\mathrm{d}y \\ &= \varepsilon\big(l(x,x)z_x(x,t)-l(x,0)z_x(0,t)-[l_y(x,x)z(x,t)-l_y(x,0)z(0,t)]\big) \\ &\quad+\int_0^x (l_{yy}(x,y)-c)z(y,t)\mathrm{d}y+\varepsilon z_{xx}(x,t)-cz(x,t) \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} w_{xx}(x,t) &= z_{xx}(x,t)+\frac{d}{dx}(l(x,x))z(x,t)+l(x,x)z_x(x,t) \\ &\quad+l_x(x,x)z(x,t)+\int_0^x l_{xx}(x,y)z(y,t)\mathrm{d}y. \end{aligned} \tag{2.14}$$

Substituting (2.12), (2.13), and (2.14) into (1.6), it follows that the kernel function $l(x,y)$ satisfies the following PDE:

$$\begin{cases} \varepsilon l_{xx}(x,y)-\varepsilon l_{yy}(x,y)=-(\lambda(x)+c)l(x,y)-f(x,y)-\displaystyle\int_y^x l(\xi,y)f(x,\xi)\mathrm{d}\xi, \\ \varepsilon l_y(x,0)+\varepsilon q l(x,0)=g(x), \\ l(x,x)=-q-\dfrac{1}{2\varepsilon}\displaystyle\int_0^x(\lambda(y)+c)\mathrm{d}y. \end{cases} \tag{2.15}$$

Once again, by [30, Theorem 2.2], the PDE (2.15) has a unique solution $l \in C^2(\bar{\mathcal{F}})$.

Now, we design a state feedback for system (1.6) as follows:

$$u(t)=\int_0^1 k(1,y)w(y,t)\mathrm{d}y. \tag{2.16}$$

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Under this state feedback, the closed loop of system (1.6) is

(2.17)
$$\begin{cases} {}_0^C D_t^\alpha w(x,t) = \varepsilon w_{xx}(x,t) + \lambda(x)w(x,t) + g(x)w(0,t) + \int_0^x f(x,y)w(x,y)\mathrm{d}y, \\ w_x(0,t) = -qw(0,t), \\ w(1,t) = \int_0^1 k(1,y)w(y,t)\mathrm{d}y, \\ w(x,0) = w_0(x), \end{cases}$$

which is shown to be equivalent to the target system (2.1). We thus have Proposition 2.5.

PROPOSITION 2.5. *For any initial value $w_0 \in L^2(0,1)$, the closed-loop system (2.17) admits a unique solution $w \in C(0,\infty; L^2(0,1))$ given by*

$$w(x,t) = \sum_{j\geq 0} a_j E_\alpha \left(- \left[c + \varepsilon \left(j + \frac{1}{2} \right)^2 \pi^2 \right] t^\alpha \right) \phi_j(x),$$

where

$$a_j = \sqrt{2} \int_0^1 \left(w_0(x) - \int_0^x k(x,y) w_0(y)\mathrm{d}y \right) \sin\left(\left(j + \frac{1}{2} \right) \pi(x-1) \right) \mathrm{d}x, j \geq 0,$$
$$\phi_j(x) = \sqrt{2} \sin\left(\left(j+\frac{1}{2} \right) \pi(x-1) \right) + \sqrt{2} \int_0^x l(x,y) \sin\left(\left(j+\frac{1}{2} \right) \pi(y-1) \right) \mathrm{d}y, j \geq 0,$$

and hence is Mittag–Leffler stable in $L^2(0,1)$.

2.3. Observer design. To design an output feedback control, we need to recover the state $w(x,t)$ of system (1.6), which is used in the feedback control (2.16), through an observer.

We design the following observer for system (1.6):

$$\begin{cases} {}_0^C D_t^\alpha \widehat{w}(x,t) = \varepsilon \widehat{w}_{xx}(x,t) + \lambda(x)\widehat{w}(x,t) + p_1(x)(\widehat{w}(0,t) - y_o(t)) \\ \qquad\qquad + g(x)y_o(t) + \int_0^x f(x,y)\widehat{w}(x,y)\mathrm{d}y, \\ \widehat{w}_x(0,t) = -qy_o(t) + p_0(\widehat{w}(0,t) - y_o(t)), \\ \widehat{w}(1,t) = u(t), \\ \widehat{w}(x,0) = \widehat{w}_0(x). \end{cases} \tag{2.18}$$

This observer is designed similarly as that in [31] for a parabolic system.

Let $\widetilde{w}(x,t) = \widehat{w}(x,t) - w(x,t)$ be the observer error. Then $\widetilde{w}(x,t)$ is governed by the following fractional PDE:

(2.19)
$$\begin{cases} {}_0^C D_t^\alpha \widetilde{w}(x,t) = \varepsilon \widetilde{w}_{xx}(x,t) + \lambda(x)\widetilde{w}(x,t) + p_1(x)\widetilde{w}(0,t) + \int_0^x f(x,y)\widetilde{w}(x,y)\mathrm{d}y, \\ \widetilde{w}_x(0,t) = p_0\widetilde{w}(0,t), \ \widetilde{w}(1,t) = 0, \\ \widetilde{w}(x,0) = \widehat{w}_0(x) - w_0(x). \end{cases}$$

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It is noticed that the observer gains $p_1(x)$ and p_0 should be designed to stabilize system (2.19).

We investigate the stability of (2.19) by similar integral transformation for the state feedback. The difference is that here we introduce $\widetilde{z} \to \widetilde{w}$ by

$$\widetilde{w}(x,t) = \widetilde{z}(x,t) - \int_0^x p(x,y)\widetilde{z}(y,t)\mathrm{d}y, \tag{2.20}$$

which is expected to transform (2.19) into the following Mittag–Leffler stable system for $\widetilde{c} > -\varepsilon\frac{\pi^2}{4}$:

$$\begin{cases} {}_0^C D_t^\alpha \widetilde{z}(x,t) = \varepsilon \widetilde{z}_{xx}(x,t) - \widetilde{c}\widetilde{z}(x,t), \\ \widetilde{z}_x(0,t) = \widetilde{z}(1,t) = 0, \\ \widetilde{z}(x,0) = \widetilde{z}_0(x). \end{cases} \tag{2.21}$$

The parameter $\widetilde{c}$ is set to regulate the observer convergence speed, which is seen from Lemma 2.1 and is generally different from the analogous coefficient c in control design if one expects convergence of the closed-loop system via output feedback to be as good as via state feedback, referred to by (2.31) later.

Substituting (2.20) into (2.19), we obtain the following fractional PDE for $\widetilde{z}(x,t)$:

$$\begin{aligned} {}_0^C D_t^\alpha \widetilde{z}(x,t) &= \int_0^x p(x,y) {}_0^C D_t^\alpha \widetilde{z}(y,t)\mathrm{d}y + \varepsilon\left(\widetilde{z}(x,t) - \int_0^x p(x,y)\widetilde{z}(y,t)\mathrm{d}y\right)_{xx} \\ &\quad + \lambda(x)\left(\widetilde{z}(x,t) - \int_0^x p(x,y)\widetilde{z}(y,t)\mathrm{d}y\right) + p_1(x)\widetilde{z}(0,t) \\ &\quad + \int_0^x f(x,y)\left(\widetilde{z}(y,t) - \int_0^y p(y,\xi)\widetilde{z}(\xi,t)\mathrm{d}\xi\right)\mathrm{d}y \\ &= \varepsilon\widetilde{z}_{xx}(x,t) + \varepsilon\int_0^x p(x,y)\widetilde{z}_{yy}(y,t)\mathrm{d}y - \varepsilon\left(\int_0^x p(x,y)\widetilde{z}(y,t)\mathrm{d}y\right)_{xx} \\ &\quad - \int_0^x (\widetilde{c} + \lambda(x))p(x,y)\widetilde{z}(y,t)\mathrm{d}y \\ &\quad + \int_0^x \widetilde{z}(y,t)\left(f(x,y) - \int_y^x f(x,\xi)p(\xi,y)\mathrm{d}\xi\right)\mathrm{d}y + \lambda(x)\widetilde{z}(x,t) + p_1(x)\widetilde{z}(0,t) \\ &= \varepsilon\widetilde{z}_{xx}(x,t) + \int_0^x [\varepsilon p_{yy}(x,y) - \varepsilon p_{xx}(x,y) - \widetilde{c}p(x,y) - \lambda(x)p(x,y)]\widetilde{z}(y,t)\mathrm{d}y \\ &\quad + \int_0^x \widetilde{z}(y,t)\left(f(x,y) - \int_y^x f(x,\xi)p(\xi,y)d\xi\right)\mathrm{d}y + (\varepsilon p_y(x,0) + p_1(x))\widetilde{z}(0,t) \\ &\quad + \left(\lambda(x) - 2\varepsilon\frac{d}{dx}p(x,x)\right)\widetilde{z}(x,t) - \varepsilon\widetilde{z}_x(0,t)p(x,0) \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} \widetilde{z}_x(0,t) &= \widetilde{w}_x(0,t) + p(0,0)\widetilde{z}(0,t) = \widetilde{w}_x(0,t) + p(0,0)\widetilde{w}(0,t) \\ &= (p_0 + p(0,0))\widetilde{w}(0,t), \end{aligned}$$

$$\widetilde{z}(1,t) = \int_0^1 p(1,y)\widetilde{z}(y,t)\mathrm{d}y. \tag{2.23}$$

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Comparing the fractional PDE (2.22) and boundary conditions (2.23) with (2.21), it follows that $p(x,y)$ needs to satisfy the following PDE:

$$\begin{cases} \varepsilon p_{yy}(x,y) - \varepsilon p_{xx}(x,y) = (\widetilde{c} + \lambda(x))p(x,y) - f(x,y) + \displaystyle\int_y^x f(x,\xi)p(\xi,y)\mathrm{d}\xi, \\ 2\varepsilon \dfrac{d}{dx} p(x,x) = \lambda(x) + \widetilde{c}, \\ p(1,y) = 0, \end{cases} \tag{2.24}$$

and the observer gains should be chosen as

$$p_1(x) = -\varepsilon p_y(x,0), \ p_0 = -p(0,0). \tag{2.25}$$

Once again we seek the inverse of transformation (2.20) by setting

$$\widetilde{z}(x,t) = \widetilde{w}(x,t) + \int_0^x r(x,y)\widetilde{w}(y,t)\mathrm{d}y. \tag{2.26}$$

Notice that the form of system (2.1) and (2.21) and the form of system (1.6) and (2.19) are similar to the system (2.1) converting into system (1.6) by replacing $g(x)$, q, and c in (2.15) with $p_1(x)$, $-p_0$, and $\widetilde{c}$, respectively. We then have that $r(x,y)$ satisfies the following PDE:

$$\begin{cases} \varepsilon r_{xx}(x,y) - \varepsilon r_{yy}(x,y) = -(\lambda(x) + \widetilde{c})r(x,y) - f(x,y) - \displaystyle\int_y^x r(\xi,y)f(x,\xi)\mathrm{d}\xi, \\ \varepsilon r_y(x,0) - \varepsilon p_0 r(x,0) = p_1(x), \\ r(x,x) = p_0 - \dfrac{1}{2\varepsilon}\displaystyle\int_0^x (\lambda(y) + \widetilde{c})\mathrm{d}y, \end{cases} \tag{2.27}$$

where $p_1(x)$ and p_0 are given by (2.25).

It is shown in [30, Theorem 4.1] that (2.24) and (2.27) have a unique solution $p, r \in C^2(\bar{\mathcal{F}})$. With the transforms (2.20) and (2.26), system (2.19) is equivalent to system (2.21). Thus, we have immediately the following convergence for observer (2.18).

THEOREM 2.6. *For any control input $u \in L^2_{loc}(0,\infty)$ and initial value $(w_0, \widehat{w}_0) \in L^2(0,1) \times L^2(0,1)$, the observer error system* (2.19) *admits a unique solution $\widetilde{w} \in C(0,\infty; L^2(0,1))$ given by*

$$\widetilde{w}(x,t) = \sum_{j\geq 0} a_j E_\alpha \left(- \left[\widetilde{c} + \varepsilon \left(j + \frac{1}{2} \right)^2 \pi^2 \right] t^\alpha \right) \phi_j(x), \tag{2.28}$$

where

$$a_j = \sqrt{2}\int_0^1 \left((\widehat{w}_0(x) - w_0(x)) + \int_0^x r(x,y)(\widehat{w}_0(y) - w_0(y))\mathrm{d}y \right) \sin\left(\left(j + \frac{1}{2} \right) \pi (x-1) \right) \mathrm{d}x,$$

$$\phi_j(x) = \sqrt{2}\sin\left(\left(j + \frac{1}{2} \right) \pi(x-1) \right) - \sqrt{2}\int_0^x p(x,y)\sin\left(\left(j + \frac{1}{2} \right) \pi(y-1) \right) \mathrm{d}y, j \geq 0,$$

and hence is Mittag–Leffler stable in $L^2(0,1)$.

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2.4. Observer-based output feedback. Since we obtain an approximated state $\widehat{w}(x,t)$ from the output by observer (2.18), it follows from the state feedback (2.16) that an observer-based feedback should be designed naturally as

$$u(t) = \int_0^1 k(1,y)\widehat{w}(y,t)\mathrm{d}y. \tag{2.29}$$

Under feedback (2.29), we have the closed-loop system of (1.6):

(2.30)
$$\begin{cases} {}_0^C D_t^\alpha w(x,t) = \varepsilon w_{xx}(x,t) + \lambda(x)w(x,t) + g(x)w(0,t) + \int_0^x f(x,y)w(x,y)\mathrm{d}y, \\ w_x(0,t) = -qw(0,t), \\ w(1,t) = \int_0^1 k(1,y)\widehat{w}(y,t)\mathrm{d}y, \\ {}_0^C D_t^\alpha \widehat{w}(x,t) = \varepsilon \widehat{w}_{xx}(x,t) + \lambda(x)\widehat{w}(x,t) + p_1(x)(\widehat{w}(0,t) - y_o(t)) \\ \qquad\qquad +g(x)y_o(t) + \int_0^x f(x,y)\widehat{w}(x,y)\mathrm{d}y, \\ \widehat{w}_x(0,t) = -qy_o(t) + p_0(\widehat{w}(0,t) - y_o(t)), \\ \widehat{w}(1,t) = \int_0^1 k(1,y)\widehat{w}(y,t)\mathrm{d}y, \\ w(x,0) = w_0(x), \;\; \widehat{w}(x,0) = \widehat{w}_0(x). \end{cases}$$

THEOREM 2.7. *For any initial value* $(w_0, \widehat{w}_0) \in \mathcal{H} := L^2(0,1) \times L^2(0,1)$, *the closed-loop system* (2.30) *admits a unique solution* $(w, \widehat{w}) \in C(0,\infty; L^2(0,1)\times L^2(0,1))$. *Moreover, there exists a constant* $C > 0$ *such that*

(2.31)
$$\|(w(\cdot,t), \widehat{w}(\cdot,t))\|_{\mathcal{H}} \le CE_\alpha\left(-\min\left\{\left[c+\varepsilon\frac{\pi^2}{4}\right], \left[\widetilde{c}+\varepsilon\frac{\pi^2}{4}\right]\right\} t^\alpha\right) \|(w_0, \widehat{w}_0)\|_{\mathcal{H}}.$$

Proof. Since $\widetilde{w}(x,t) = \widehat{w}(x,t) - w(x,t)$ is an observer error, it is obvious that system (2.30) is equivalent to the following system:

(2.32)
$$\begin{cases} {}_0^C D_t^\alpha w(x,t) = \varepsilon w_{xx}(x,t) + \lambda(x)w(x,t) + g(x)w(0,t) + \int_0^x f(x,y)w(x,y)\mathrm{d}y, \\ w_x(0,t) = -qw(0,t), \\ w(1,t) = \int_0^1 k(1,y)(w(y,t) + \widetilde{w}(y,t))\mathrm{d}y, \\ {}_0^C D_t^\alpha \widetilde{w}(x,t) = \varepsilon \widetilde{w}_{xx}(x,t) + \lambda(x)\widetilde{w}(x,t) + p_1(x)\widetilde{w}(0,t) + \int_0^x f(x,y)\widetilde{w}(x,y)\mathrm{d}y, \\ \widetilde{w}_x(0,t) = p_0\widetilde{w}(0,t), \quad \widetilde{w}(1,t) = 0, \\ w(x,0) = w_0(x), \;\; \widetilde{w}(x,0) = \widehat{w}_0(x) - w_0(x). \end{cases}$$

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Under transforms (2.8) and (2.26), (2.32) is equivalent to the following system:

$$\begin{cases} {}_0^C D_t^\alpha z(x,t) = \varepsilon z_{xx}(x,t) - cz(x,t), \\ z_x(0,t) = 0, \\ z(1,t) = \displaystyle\int_0^1 k(1,y)\left(\widetilde{z}(y,t) - \int_0^x p(y,\xi)\widetilde{z}(\xi,t)\mathrm{d}\xi\right)\mathrm{d}y, \\ {}_0^C D_t^\alpha \widetilde{z}(x,t) = \varepsilon \widetilde{z}_{xx}(x,t) - \widetilde{c}\widetilde{z}(x,t), \\ \widetilde{z}_x(0,t) = \widetilde{z}(1,t) = 0, \\ z(x,0) = z_0(x), \quad \widetilde{z}(x,0) = \widetilde{z}_0(x). \end{cases} \tag{2.33}$$

Define the operator $\mathbb{A}_D : D(\mathbb{A}_D)(\mathcal{H}) \to \mathcal{H}$ as follows:

$$\begin{cases} [\mathbb{A}_D(f,g)](x) = (\varepsilon f''(x) - cf(x), \varepsilon g''(x) - \widetilde{c}g(x)), \\ D(\mathbb{A}_D) = \Big\{(f,g) \in H^2(0,1) \times H^2(0,1) : f'(0) = 0, g'(0) = 0, g(1) = 0, \\ \qquad\qquad f(1) = \displaystyle\int_0^1 k(1,y)\Big(g(y) - \int_0^y p(y,\xi)g(\xi)\mathrm{d}\xi\Big)\mathrm{d}y\Big\}. \end{cases} \tag{2.34}$$

We compute the eigenvalues and the corresponding eigenfunctions of $\mathbb{A}_D$. Solve $\mathbb{A}_D(f,g) = \mu(f,g)$, where $\mu \in \sigma(\mathbb{A}_D)$ and $(f,g) \in D(\mathbb{A}_D)$, to obtain

$$\begin{cases} \varepsilon f''(x) - cf(x) = \mu f(x), \\ \varepsilon g''(x) - \widetilde{c}g(x) = \mu g(x), \\ f'(0) = 0, \ g'(0) = 0, \ g(1) = 0, \\ f(1) = \displaystyle\int_0^1 k(1,y)\Big(g(y) - \int_0^y p(y,\xi)g(\xi)\mathrm{d}\xi\Big)\mathrm{d}y. \end{cases} \tag{2.35}$$

There are two cases.

Case I: $g(x) \equiv 0$. In this case, (2.35) becomes

$$\begin{cases} \varepsilon f''(x) - cf(x) = \mu f(x), \\ f'(0) = 0, \quad f(1) = 0, \end{cases} \tag{2.36}$$

which has nontrivial solutions $(\mu_{1n}, f_{1n}(x))$:

$$\mu_{1n} = -c - \varepsilon\left(n\pi + \frac{\pi}{2}\right)^2, \quad f_{1n}(x) = \cos\left(n\pi + \frac{\pi}{2}\right)x, \ n = 0,1,2,\ldots. \tag{2.37}$$

Hence, $(\mu_{1n}, F_{1n}(x)) = (\mu_{1n}, (f_{1n}, 0))$ is an eigenpair of $\mathbb{A}_D$.

Case II: $g(x) \neq 0$. In this case,

$$\begin{cases} \varepsilon g''(x) - \widetilde{c}g(x) = \mu g(x), \\ g'(0) = 0, \quad g(1) = 0, \end{cases} \tag{2.38}$$

which has nontrivial solutions $(\mu_{2n}, g_{2n}(x))$:

$$\mu_{2n} = -\widetilde{c} - \varepsilon\left(n\pi + \frac{\pi}{2}\right)^2, \ g_{2n}(x) = \cos\left(n\pi + \frac{\pi}{2}\right)x, \ n = 0,1,2,\ldots. \tag{2.39}$$

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Substituting $(\mu_{2n}, g_{2n}(x))$ into the equation

$$\begin{cases} \varepsilon f''(x) - cf(x) = \mu_{2n} f(x), \\ f'(0) = 0, \quad f(1) = \displaystyle\int_0^1 k(1,y)\left(g_{2n}(y) - \int_0^y p(y,\xi) g_{2n}(\xi)\mathrm{d}\xi\right)\mathrm{d}y, \end{cases} \tag{2.40}$$

we have the solution $f_{2n}(x)$ given by

$$\begin{aligned} f_{2n}(x) = &\frac{\displaystyle\int_0^1 k(1,y)\left(\cos\left(n\pi + \frac{\pi}{2}\right)y - \int_0^y p(y,\xi)\cos\left(n\pi + \frac{\pi}{2}\right)\xi\mathrm{d}\xi\right)\mathrm{d}y}{\cos\left(\sqrt{(n\pi + \frac{\pi}{2})^2 + \frac{\widetilde{c}-c}{\varepsilon}}\right)} \\ &\times \cos\left(\sqrt{\left(n\pi + \frac{\pi}{2}\right)^2 + \frac{\widetilde{c}-c}{\varepsilon}}\right)x. \end{aligned} \tag{2.41}$$

We thus have another eigenpair: $(\mu_{2n}, F_{2n}(x)) = (\mu_{2n}, (f_{2n}(x), g_{2n}(x)))$ of $\mathbb{A}_D$.

Now we show that $\{F_{1n}(x), F_{2n}(x)\}$ forms a Riesz basis for $L^2(0,1) \times L^2(0,1)$. First, since $\{\cos(n+\frac{\pi}{2})x, n = 0,1,2,\ldots\}$ forms a Riesz (orthogonal) basis for $L^2(0,1)$, $\{F^*_{1n}(x) = (\cos(n+\frac{\pi}{2})x, 0), n = 0,1,2,\ldots\} \cup \{F_{2n}(x) = (0, \cos(n+\frac{\pi}{2})x), n = 0,1,2,\ldots\}$ forms a Riesz basis for $L^2(0,1) \times L^2(0,1)$. Moreover,

$$\begin{aligned} &\sum_{n=0}^{\infty} \|F_{1n}(x) - F^*_{1n}(x)\|^2_{\mathcal{H}} + \sum_{n=0}^{\infty} \|F_{2n}(x) - F^*_{2n}(x)\|^2_{\mathcal{H}} \\ &= \sum_{n=0}^{\infty} \left|\int_0^1 k(1,y)\left(\cos\left(n\pi + \frac{\pi}{2}\right)y - \int_0^y p(y,\xi)\cos\left(n\pi + \frac{\pi}{2}\right)\xi\mathrm{d}\xi\right)\mathrm{d}y\right|^2 \\ &\qquad \left\|\frac{\cos\sqrt{(n\pi + \frac{\pi}{2})^2 + \frac{\widetilde{c}-c}{\varepsilon}}x}{\cos\sqrt{\left(n\pi + \frac{\pi}{2}\right)^2 + \frac{\widetilde{c}-c}{\varepsilon}}}\right\|^2_{L^2(0,1)} \\ &\le \sum_{n=0}^{\infty} \frac{2\left|\displaystyle\int_0^1 k(1,y)\left(\cos\left(n\pi + \frac{\pi}{2}\right)y - \int_0^y p(y,\xi)\cos\left(n\pi + \frac{\pi}{2}\right)\xi\mathrm{d}\xi\right)\mathrm{d}y\right|^2}{1 + \cos\left(2\sqrt{(n\pi + \frac{\pi}{2})^2 + \frac{\widetilde{c}-c}{\varepsilon}}\right)} \\ &= \sum_{n=0}^{\infty} \frac{2\left|\displaystyle\int_0^1 k(1,y)\cos\left(n\pi+\frac{\pi}{2}\right)y\mathrm{d}y - \int_0^1\left(\int_y^1 k(1,\xi)p(\xi,y)\mathrm{d}\xi\right)\cos\left(n\pi+\frac{\pi}{2}\right)y\mathrm{d}y\right|^2}{1 + \cos\left(2\sqrt{(n\pi + \frac{\pi}{2})^2 + \frac{\widetilde{c}-c}{\varepsilon}}\right)}. \end{aligned}$$

Since

$$\begin{aligned} &\lim_{n\to\infty} \varepsilon\left(2\sqrt{\left(n\pi + \frac{\pi}{2}\right)^2 + \frac{\widetilde{c}-c}{\varepsilon}} + (2n\pi + \pi)\right)\frac{1 + \cos\left(2\sqrt{(n\pi + \frac{\pi}{2})^2 + \frac{\widetilde{c}-c}{\varepsilon}}\right)}{4(\widetilde{c} - c)} \\ &\qquad = \lim_{n\to\infty} \frac{1 + \cos\left(2\sqrt{\left(n\pi + \frac{\pi}{2}\right)^2 + \frac{\widetilde{c}-c}{\varepsilon}}\right)}{2\sqrt{\left(n\pi + \frac{\pi}{2}\right)^2 + \frac{\widetilde{c}-c}{\varepsilon}} - (2n\pi + \pi)} = 2, \end{aligned}$$

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there exists a positive integer $N > \frac{\widetilde{c}-c}{\varepsilon}$ such that for all $n \geq N+1$,

$$1+\cos(2\sqrt{\left(n\pi+\frac{\pi}{2}\right)^2+\frac{\widetilde{c}-c}{\varepsilon}}) \geq \frac{4(\widetilde{c}-c)}{\varepsilon\left(2\sqrt{\left(n\pi+\frac{\pi}{2}\right)^2+\frac{\widetilde{c}-c}{\varepsilon}}+(2n\pi+\pi)\right)}.$$

It follows that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\|F_{1n}(x)-F_{1n}^*(x)\|_{\mathcal{H}}^2+\sum_{n=0}^{\infty}\|F_{2n}(x)-F_{2n}^*(x)\|_{\mathcal{H}}^2\\
&\leq \sum_{n=0}^{\infty}\frac{4\left|\int_0^1 k(1,y)\cos\left(n\pi+\frac{\pi}{2}\right)y\mathrm{d}y\right|^2+\left|\int_0^1\left(\int_y^1 k(1,\xi)p(\xi,y)\mathrm{d}\xi\right)\cos\left(n\pi+\frac{\pi}{2}\right)y\mathrm{d}y\right|^2}{1+\cos\left(2\sqrt{\left(n\pi+\frac{\pi}{2}\right)^2+\frac{\widetilde{c}-c}{\varepsilon}}\right)}\\
&\leq \sum_{n=0}^{N}\frac{4\left|\int_0^1 k(1,y)\cos\left(n\pi+\frac{\pi}{2}\right)y\mathrm{d}y\right|^2+\left|\int_0^1\left(\int_y^1 k(1,\xi)p(\xi,y)\mathrm{d}\xi\right)\cos\left(n\pi+\frac{\pi}{2}\right)y\mathrm{d}y\right|^2}{1+\cos\left(2\sqrt{\left(n\pi+\frac{\pi}{2}\right)^2+\frac{\widetilde{c}-c}{\varepsilon}}\right)}\\
&\quad+\varepsilon\sum_{n=N+1}^{\infty}5\left(n\pi+\frac{\pi}{2}\right)^2\left|\int_0^1 k(1,y)\cos\left(n\pi+\frac{\pi}{2}\right)y\mathrm{d}y\right|^2\\
&\quad+\varepsilon\sum_{n=N+1}^{\infty}5\left(n\pi+\frac{\pi}{2}\right)^2\left|\int_0^1\left(\int_y^1 k(1,\xi)p(\xi,y)\mathrm{d}\xi\right)\cos\left(n\pi+\frac{\pi}{2}\right)y\mathrm{d}y\right|^2.
\end{aligned}$$

Since $k,p\in C^2(\overline{\mathcal{F}})$, $k(1,\cdot)\in C^1[0,1]$ and $\int_y^1 k(1,\xi)p(\xi,y)d\xi\in C^1[0,1]$, by [18, p. 25, Theorem],

$$\left\{\left(n\pi+\frac{\pi}{2}\right)\int_0^1 k(1,y)\cos\left(n\pi+\frac{\pi}{2}\right)y\mathrm{d}y\right\}_{n=0}^{\infty}\in l^2$$

and

$$\left\{\left(n\pi+\frac{\pi}{2}\right)\int_0^1\left(\int_y^1 k(1,\xi)p(\xi,y)d\xi\right)\cos\left(n\pi+\frac{\pi}{2}\right)y\mathrm{d}y\right\}_{n=0}^{\infty}\in l^2.$$

By the classical Bari theorem, $\{F_{1n}(x),F_{2n}(x),n=0,1,2,\ldots\}$ forms a Riesz basis for $L^2(0,1)\times L^2(0,1)$. So $\mathbb{A}_D$ generates a C_0-semigroup in $L^2(0,1)\times L^2(0,1)$. Next, the solution of (2.33) can be expressed as

$$(z(x,t),\widetilde{z}(x,t))=\sum_{n\geq 0}\varphi_{1n}(t)F_{1n}(x)+\varphi_{2n}(t)F_{2n}(x) \tag{2.42}$$

and the initial value

$$(z_0(x),\widetilde{z}_0(x))=\sum_{n\geq 0}a_{1n}F_{1n}(x)+a_{2n}F_{2n}(x), \tag{2.43}$$

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where $\{a_{1n}\}, \{a_{2n}\} \in l^2$, and $\varphi_{1n}, \varphi_{2n} \in C(0,\infty;\mathbb{R})$ satisfy the following linear fractional differential equations

$$ {}_0^C D_t^\alpha \varphi_{1n}(t) = \mu_{1n}\varphi_{1n}(t), \quad {}_0^C D_t^\alpha \varphi_{2n}(t) = \mu_{2n}\varphi_{2n}(t),\ n=0,1,2,\ldots, \tag{2.44}$$

with the initial values

$$\varphi_{1n}(0) = a_{1n},\ \ \varphi_{2n}(0) = a_{2n},\ n=0,1,2,\ldots. \tag{2.45}$$

By [16, Theorem 4.3], the solutions of (2.44) with the initial values (2.45) are found to be $\varphi_{1n}(t) = a_{1n}E_\alpha(\mu_{1n}t^\alpha)$ and $\varphi_{2n}(t) = a_{2n}E_\alpha(\mu_{2n}t^\alpha)$. Thus, the solution of (2.33) is finally represented by

$$(z(x,t),\widetilde{z}(x,t)) = \sum_{n\ge 0} a_{1n}E_\alpha(\mu_{1n}t^\alpha)F_{1n}(x) + a_{2n}E_\alpha(\mu_{2n}t^\alpha)F_{2n}(x). \tag{2.46}$$

Since $\{F_{1n}(x), F_{2n}(x), n=0,1,2,\ldots\}$ forms a Riesz basis for $L^2(0,1)\times L^2(0,1)$, there exists constants $C_1, C_2>0$ such that for all $\xi_{1n},\xi_{2n}\in\mathbb{R}$,

$$C_1\sum_{n\ge 0}(\xi_{1n}^2+\xi_{2n}^2) \le \|\xi_{1n}F_{1n}(x)+\xi_{2n}F_{2n}(x)\|_{\mathcal{H}}^2 \le C_2\sum_{n\ge 0}(\xi_{1n}^2+\xi_{2n}^2). \tag{2.47}$$

It then follows from (2.46) and (2.47) that

$$\begin{aligned}\|\,(z(\cdot,t),\widetilde{z}(\cdot,t))\,\|_{\mathcal{H}}^2 &\le C_2\sum_{n\ge 0}\left(a_{1n}^2E_\alpha^2(\mu_{1n}t^\alpha)+a_{2n}^2E_\alpha^2(\mu_{2n}t^\alpha)\right)\\ &\le C_2E_\alpha^2\left(\max\{\mu_{10},\mu_{20}\}\,t^\alpha\right)\sum_{n\ge 0}\left(a_{1n}^2+a_{2n}^2\right)\\ &\le \frac{C_2}{C_1}E_\alpha^2\left(\max\{\mu_{10},\mu_{20}\}t^\alpha\right)^2\|(z_0,\widetilde{z}_0\|_{\mathcal{H}}^2.\end{aligned} \tag{2.48}$$

Since

$$\begin{aligned}\begin{pmatrix} w\\ \widehat{w}\end{pmatrix} &= \begin{pmatrix} I & 0\\ I & I\end{pmatrix}\begin{pmatrix} w\\ \widetilde{w}\end{pmatrix} = \begin{pmatrix} I & 0\\ I & I\end{pmatrix}\begin{pmatrix} I+\mathbb{P}_1 & 0\\ 0 & I+\mathbb{P}_2\end{pmatrix}\begin{pmatrix} z\\ \widetilde{z}\end{pmatrix}\\ &= \begin{pmatrix} I+\mathbb{P}_1 & 0\\ I+\mathbb{P}_1 & I+\mathbb{P}_2\end{pmatrix}\begin{pmatrix} z\\ \widetilde{z}\end{pmatrix},\end{aligned} \tag{2.49}$$

where $\mathbb{P}_1$ and $\mathbb{P}_2$ are Volterra transformations, which, in terms of (2.12) and (2.20), are given by

$$\mathbb{P}_1 f(x) = \int_0^x l(x,t)f(x)\mathrm{d}x, \quad \mathbb{P}_2 f(x) = -\int_0^x p(x,t)f(x)\mathrm{d}x,\ \forall\, f\in L^2(0,1),$$

the inequality (2.31) then follows from (2.48) for some constant $C>0$. ∎

Remark 2.8. In the proof of Theorem 2.7, we actually give an explicit expression of solution of the closed-loop system (2.30). Indeed, by (2.46) and (2.49), we have

$$\begin{aligned} w(x,t) &= \sum_{n\ge 0}(a_{1n}E_\alpha(\mu_{1n}t^\alpha)(I+\mathbb{P}_1)f_{1n}(x) + a_{2n}E_\alpha(\mu_{2n}t^\alpha)(I+\mathbb{P}_1)f_{2n}(x)),\\ \widehat{w}(x,t) &= \sum_{n\ge 0}(a_{1n}E_\alpha(\mu_{1n}t^\alpha)(I+\mathbb{P}_1)f_{1n}(x) + a_{2n}E_\alpha(\mu_{2n}t^\alpha)(I+\mathbb{P}_1)f_{2n}(x))\\ &\quad + \sum_{n\ge 0} a_{2n}E_\alpha(\mu_{2n}t^\alpha)(I+\mathbb{P}_2)g_{2n}(x),\end{aligned}$$

where $f_{1n}(x)$, $g_{2n}(x)$, and $f_{2n}(x)$ are given in (2.37), (2.39), and (2.41), respectively.

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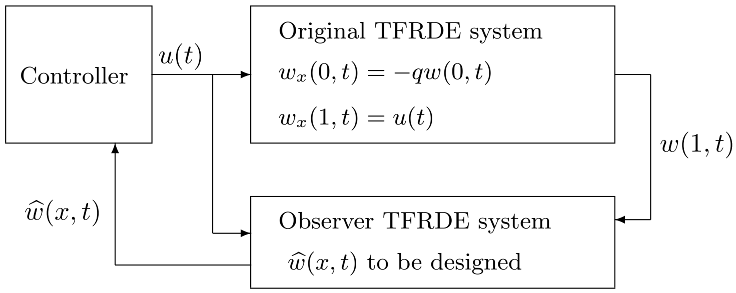

FIG. 2. *Block diagram of output feedback for time fractional reaction diffusion equations with Neumann boundary control* (1.7).

3. Backstepping with Neumann boundary control. In this section, we apply the backstepping approach to design an output feedback stabilizer for system (1.7) as depicted in Figure 2.

3.1. Target system. We introduce a target system,

$$\begin{cases} {}_0^C D_t^\alpha z(x,t) = \varepsilon z_{xx}(x,t) - cz(x,t), \\ z_x(0,t) = z_x(1,t) = 0, \\ z(x,0) = z_0(x), \end{cases} \tag{3.1}$$

where the parameter c is used to regulate the convergence speed, which is seen from Lemma 3.1.

LEMMA 3.1. *For any initial value $z_0 \in L^2(0,1)$, system* (3.1) *admits a unique solution $z(\cdot,t) \in C(0,\infty; L^2(0,1))$. Moreover, the solution is Mittag–Leffler stable in $L^2(0,1)$:*

$$\|z(\cdot,t)\|^2_{L^2(0,1)} \le E^2_\alpha(-ct^\alpha)\,\|z_0\|^2_{L^2(0,1)}. \tag{3.2}$$

Proof. Define the operator $A_N : D(A_N)(\subset L^2(0,1)) \to L^2(0,1)$ as follows:

$$\begin{cases} [A_N f](x) = \varepsilon f''(x) - cf(x), \\ D(A_N) = \{f \in H^2(0,1) | f'(0) = 0, f'(1) = 0\}. \end{cases} \tag{3.3}$$

A simple computation shows that A_N is self-adjoint in $L^2(0,1)$ with the eigenpairs $\{\mu_j, e_j(x)\}$ given by

$$\mu_0 = -c,\ e_0(x) = 1, \text{ and } \mu_j = -c - \varepsilon j^2\pi^2,\ e_j(x) = \sqrt{2}\cos j\pi x, j = 1, 2, \ldots. \tag{3.4}$$

Since $\{e_j(x)\}$ forms an orthnormal basis for $L^2(0,1)$, we can express the solution of (2.1) as

$$z(x,t) = \sum_{j\ge 0} \varphi_j(t) e_j(x) \text{ with } z_0(x) = \sum_{j\ge 0} a_j e_j(x), \tag{3.5}$$

where $\{a_j\} \in l^2$ and $\varphi_j \in C(0,\infty;\mathbb{R})$ satisfies the following linear fractional differential equation:

$${}_0^C D_t^\alpha \varphi_j(t) = \mu_j \varphi_j(t),\ \varphi_j(0) = a_j. \tag{3.6}$$

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By [16, Theorem 4.3], the solution of (3.6) is found to be $\varphi_j(t) = a_j E_\alpha(\mu_j t^\alpha)$. Thus, the solution of (3.1) is finally given by

$$z(x,t) = \sum_{j\geq 0} a_j E_\alpha(\mu_j t^\alpha) e_j(x).$$

Therefore,

$$\|z(\cdot,t)\|^2_{L^2(0,1)} = \sum_{j\geq 0} a_j^2 E_\alpha^2(\mu_j t^\alpha) \leq E_\alpha^2(\mu_1 t^\alpha) \sum_{j\geq 0} a_j^2 = E_\alpha^2(-ct^\alpha)\|z_0\|^2_{L^2(0,1)}.$$

This proves (3.2). ∎

Remark 3.2. Since for $z_0(x) = e_1(x)$,

$$\|z(\cdot,t)\|^2_{L^2(0,1)} = E_\alpha^2(-ct^\alpha)\|z_0\|^2_{L^2(0,1)}.$$

So (3.2) gives the optimal estimation for the solution of (3.1).

By the backstepping transforms (2.8) and (2.12) and the analysis in subsection 2.2 as well, we can design a state feedback control for system (1.7) as follows:

$$u(t) = k(1,1)w(1,t) + \int_0^1 k_x(1,y)w(y,t)\mathrm{d}y. \tag{3.7}$$

Under the feedback control (3.7), the closed-loop of system (3.7) is

$$\begin{cases} {}_0^C D_t^\alpha w(x,t) = \varepsilon w_{xx}(x,t) + \lambda(x)w(x,t) + g(x)w(0,t) + \displaystyle\int_0^x f(x,y)w(x,y)\mathrm{d}y, \\ w_x(0,t) = -qw(0,t), \\ w_x(1,t) = k(1,1)u(1,t) + \displaystyle\int_0^1 k_x(1,y)w(y,t)\mathrm{d}y, \\ w(x,0) = w_0(x). \end{cases} \tag{3.8}$$

Since the transforms (2.8) and (2.12) are invertible, system (1.7) is equivalent to the target system (3.1). Thus, we have Proposition 3.3.

PROPOSITION 3.3. *For any initial value $w_0 \in L^2(0,1)$, the closed-loop system (3.8) admits a unique solution $w \in C(0,\infty; L^2(0,1))$ given by*

$$w(x,t) = \sum_{j\geq 0} a_j E_\alpha\left(-\left[c + \varepsilon j^2\pi^2\right] t^\alpha\right)\phi_j(x), \tag{3.9}$$

where

$$\begin{aligned} a_0 &= \int_0^1 \left(w_0(x) - \int_0^x k(x,y)w_0(y)dy\right)\mathrm{d}x, \\ \phi_0(x) &= 1 + \int_0^x l(x,y)\mathrm{d}y, \\ a_j &= \sqrt{2}\int_0^1 \left(w_0(x) - \int_0^x k(x,y)w_0(y)\mathrm{d}y\right)\cos j\pi x\mathrm{d}x, j\geq 1, \\ \phi_j(x) &= \sqrt{2}\cos j\pi x + \sqrt{2}\int_0^x l(x,y)\cos j\pi y\mathrm{d}y, j\geq 1, \end{aligned}$$

and hence is Mittag–Leffler stable in $L^2(0,1)$.

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3.2. Observer design. In this section, we design an observer for system (1.7) to recover the state $w(x,t)$ by the output $y_o(t) = w(1,t)$. To do this we suppose that $f(x,y) \equiv 0$, $g(x) \equiv 0$. The general case seems complicated to find the corresponding kernel functions in what follows.

The model we consider in this section is described by the fractional PDE of the following:

$$\begin{cases} {}_0^C D_t^\alpha w(x,t) = \varepsilon w_{xx}(x,t) + \lambda(x) w(x,t),\ x \in (0,1),\ t \geq 0, \\ w_x(0,t) = -q w(0,t),\ t \geq 0, \\ w_x(1,t) = u(t),\ t \geq 0, \\ w(x,0) = w_0(x),\ 0 \leq x \leq 1, \\ y_o(t) = w(1,t),\ t \geq 0. \end{cases} \tag{3.10}$$

We design the following observer:

$$\begin{cases} {}_0^C D_t^\alpha \widehat{w}(x,t) = \varepsilon \widehat{w}_{xx}(x,t) + \lambda(x)\widehat{w}(x,t) + p_1(x)(\widehat{w}(1,t) - y_o(t)), \\ \widehat{w}_x(0,t) = -q\widehat{w}(0,t), \\ \widehat{w}_x(1,t) = p_0(\widehat{w}(1,t) - y_o(t)) + u(t), \\ \widehat{w}(x,0) = \widehat{w}_0(x). \end{cases} \tag{3.11}$$

Let

$$\widetilde{w}(x,t) = \widehat{w}(x,t) - w(x,t) \tag{3.12}$$

be the observer error. Then, by (3.10) and (3.11), $\widetilde{w}(x,t)$ satisfies

$$\begin{cases} {}_0^C D_t^\alpha \widetilde{w}(x,t) = \varepsilon \widetilde{w}_{xx}(x,t) + \lambda(x)\widetilde{w}(x,t) + p_1(x)\widetilde{w}(1,t), \\ \widetilde{w}_x(0,t) = -q\widetilde{w}(0,t), \\ \widetilde{w}_x(1,t) = p_0\widetilde{w}(1,t), \\ \widetilde{w}(x,0) = \widehat{w}_0(x) - w_0(x). \end{cases} \tag{3.13}$$

We look for the transformation,

$$\widetilde{w}(x,t) := (1 + \mathbb{P}_3)\widetilde{z}(x,t) = \widetilde{z}(x,t) - \int_x^1 p(x,y)\widetilde{z}(y,t)\mathrm{d}y, \tag{3.14}$$

that transforms (3.13) into the following Mittag–Leffler stable system for $\widetilde{c} > 0$:

$$\begin{cases} {}_0^C D_t^\alpha \widetilde{z}(x,t) = \varepsilon \widetilde{z}_{xx}(x,t) - \widetilde{c}\widetilde{z}(x,t), \\ \widetilde{z}_x(0,t) = \widetilde{z}_x(1,t) = 0, \\ \widetilde{z}(x,0) = \widetilde{z}_0(x). \end{cases} \tag{3.15}$$

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Substituting (3.14) into (3.13) gives the equation satisfied by $\widetilde{z}(x,t)$:

$$\begin{aligned}
{}_0^C D_t^\alpha \widetilde{z}(x,t) &= \int_x^1 p(x,y)\,{}_0^C D_t^\alpha \widetilde{z}(y,t)\mathrm{d}y + \varepsilon\left(\widetilde{z}(x,t) - \int_x^1 p(x,y)\widetilde{z}(y,t)\mathrm{d}y\right)_{xx} \\
&\quad + \lambda(x)\left(\widetilde{z}(x,t) - \int_x^1 p(x,y)\widetilde{z}(y,t)\mathrm{d}y\right) + p_1(x)\widetilde{z}(1,t) \\
&= \varepsilon\widetilde{z}_{xx}(x,t) + \varepsilon\int_x^1 p(x,y)\widetilde{z}_{yy}(y,t)\mathrm{d}y - \varepsilon\left(\int_x^1 p(x,y)\widetilde{z}(y,t)\mathrm{d}y\right)_{xx} \\
&\quad - \int_x^1 (\widetilde{c} + \lambda(x))\, p(x,y)\widetilde{z}(y,t)\mathrm{d}y + \lambda(x)\widetilde{z}(x,t) + p_1(x)\widetilde{z}(1,t) \\
&= \varepsilon\widetilde{z}_{xx}(x,t) + \int_x^1 \left[\varepsilon p_{yy}(x,y) - \varepsilon p_{xx}(x,y) - \widetilde{c}p(x,y) - \lambda(x)p(x,y)\right]\widetilde{z}(y,t)\mathrm{d}y \\
&\quad + \left(\lambda(x) + 2\varepsilon\frac{d}{dx}p(x,x)\right)\widetilde{z}(x,t) + \varepsilon\widetilde{z}_x(1,t)p(x,1) \\
&\quad + (p_1(x) - \varepsilon p_y(x,1))\widetilde{z}(0,t)
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
&\widetilde{z}_x(0,t) = -(p(0,0)+q)\widetilde{z}(0,t) + \int_0^1 (p_x(0,y) + qp(0,y))\widetilde{z}(y,t)\mathrm{d}y, \\
&\widetilde{z}_x(1,t) = \widetilde{w}_x(1,t) - p(1,1)\widetilde{z}(1,t) = \widetilde{w}_x(0,t) - p(1,1)\widetilde{w}(1,t) \\
&\qquad\quad = (p_0 - p(1,1))\widetilde{w}(1,t).
\end{aligned} \tag{3.17}$$

Comparing (3.16) and boundary conditions (3.17) with (3.15), it follows that $p(x,y)$ satisfies the following PDE:

$$\begin{cases}
\varepsilon p_{yy}(x,y) - \varepsilon p_{xx}(x,y) = (\widetilde{c} + \lambda(x))p(x,y), \\
p(x,x) = -q + \dfrac{1}{2\varepsilon}\displaystyle\int_0^x (\lambda(\xi) + \widetilde{c})\mathrm{d}\xi, \\
p_x(0,y) = -qp(0,y).
\end{cases} \tag{3.18}$$

The observer gains should be chosen as

$$p_1(x) = \varepsilon p_y(x,1), \quad p_0 = p(1,1). \tag{3.19}$$

To give existence of the solution of (3.18) and the invertibility of transform (3.14), we introduce new variables:

$$\bar{x} = y, \;\; \bar{y} = x, \;\; \bar{p}(\bar{x},\bar{y}) = p(x,y). \tag{3.20}$$

Then (3.18) becomes

$$\begin{cases}
\varepsilon \bar{p}_{\bar{x}\bar{x}}(\bar{x},\bar{y}) - \varepsilon \bar{p}_{\bar{y}\bar{y}}(\bar{x},\bar{y}) = (\widetilde{c} + \lambda(\bar{y}))\bar{p}(\bar{x},\bar{y}), \\
\bar{p}(\bar{x},\bar{x}) = -q + \dfrac{1}{2\varepsilon}\displaystyle\int_0^{\bar{x}} (\lambda(\xi) + \widetilde{c})\mathrm{d}\xi, \\
\bar{p}_{\bar{y}}(\bar{x},0) = -q\bar{p}(\bar{x},0).
\end{cases} \tag{3.21}$$

It is noticed that (3.21) is exactly the same as (2.11) for $k(x,y)$ with $f(x,y) = 0$, $g(x) = 0$, and c being replaced by $\widetilde{c}$. Thus, (3.18) admits a unique solution $p \in C^2(\bar{\mathcal{F}})$, and the transformation (3.14) is invertible. With this invertible transform (3.14), we have immediately the following convergence for observer (3.11).

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THEOREM 3.4. *For any control input $u \in L^2_{loc}(0,\infty)$ and initial value $(w_0, \widehat{w}_0) \in L^2(0,1)\times L^2(0,1)$, the closed-loop system* (3.13) *admits a unique solution $\widetilde{w} \in C(0,\infty; L^2(0,1))$ given by*

$$\widetilde{w}(x,t) = \sum_{j\ge 0} a_j E_\alpha\left(-\left[\widetilde{c} + \varepsilon j^2\pi^2\right] t^\alpha\right)\phi_j(x), \tag{3.22}$$

where

$$\begin{aligned} a_0 &= \int_0^1 (I+\mathbb{P}_3)^{-1}(\widehat{w}_0(x) - w_0(x))\mathrm{d}x,\ \phi_0(x) = 1 - \int_x^1 p(x,y)\mathrm{d}y,\\ a_j &= \sqrt{2}\int_0^1 (I+\mathbb{P}_3)^{-1}(\widehat{w}_0(x) - w_0(x))\cos j\pi x \mathrm{d}x, j\ge 1,\\ \phi_j(x) &= \sqrt{2}\cos j\pi x - \sqrt{2}\int_x^1 p(x,y)\cos j\pi y\mathrm{d}y, j\ge 1. \end{aligned}$$

Moreover, the solutions is Mittag–Leffler stable:

$$\|\widetilde{w}(\cdot,t)\|^2_{L^2(0,1)} \le CE^2_\alpha\left(-\widetilde{c}t^\alpha\right)\|\widetilde{w}_0\|^2_{L^2(0,1)} \text{ for some } C>0. \tag{3.23}$$

3.3. Observer-based output feedback. In this section, we discuss output feedback stabilization of system (3.10). Since by observer (3.11) we obtain an approximate $\widehat{w}(x,t)$ of the state $w(x,t)$, a natural output feedback control, inspired by state feedback (3.7), should be

$$u(t) = k(1,1)\widehat{w}(1,t) + \int_0^1 k_x(1,y)\widehat{w}(y,t)\mathrm{d}y. \tag{3.24}$$

Under feedback (3.24), we have the following closed loop of (3.10):

$$\begin{cases} {}_0^C D_t^\alpha w(x,t) = \varepsilon w_{xx}(x,t) + \lambda(x)w(x,t),\\ w_x(0,t) = -qw(0,t),\\ w_x(1,t) = k(1,1)\widehat{w}(1,t) + \displaystyle\int_0^1 k_x(1,y)\widehat{w}(y,t)\mathrm{d}y,\\ {}_0^C D_t^\alpha \widehat{w}(x,t) = \varepsilon\widehat{w}_{xx}(x,t) + \lambda(x)\widehat{w}(x,t) + p_1(x)(\widehat{w}(1,t) - y_o(t)),\\ \widehat{w}_x(0,t) = -q\widehat{w}(0,t),\\ \widehat{w}_x(1,t) = p_0(\widehat{w}(1,t) - y_o(t)) + k(1,1)\widehat{w}(1,t) + \displaystyle\int_0^1 k_x(1,y)\widehat{w}(y,t)\mathrm{d}y,\\ w(x,0) = w_0(x),\quad \widehat{w}(x,0) = \widehat{w}_0(x). \end{cases} \tag{3.25}$$

THEOREM 3.5. *For any initial value $(w_0, \widehat{w}_0) \in L^2(0,1)\times L^2(0,1)$, the closed-loop system* (3.25) *admits a unique solution $(w,\widehat{w}) \in C(0,\infty; L^2(0,1)\times L^2(0,1))$. Moreover, there exist two positive constants $C,\mu>0$ such that*

$$\|(w(\cdot,t),\widehat{w}(\cdot,t))\|_{L^2(0,1)\times L^2(0,1)} \le CE_\alpha(-\mu t^\alpha)\|(w_0,\widehat{w}_0)\|_{L^2(0,1)\times L^2(0,1)}. \tag{3.26}$$

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Proof. Using the error variables $\widetilde{w}(x,t)$ defined in (3.12), we can write an equivalent system of (3.25) as follows:

$$
\begin{cases}
{}_0^C D_t^\alpha \widehat{w}(x,t) = \varepsilon \widehat{w}_{xx}(x,t) + \lambda(x)\widehat{w}(x,t) + p_1(x)(\widehat{w}(1,t) - y_o(t)),\\
\widehat{w}_x(0,t) = -q\widehat{w}(0,t),\\
\widehat{w}_x(1,t) = p_0(\widehat{w}(1,t) - y_o(t)) + k(1,1)\widehat{w}(1,t) + \displaystyle\int_0^1 k_x(1,y)\widehat{w}(y,t)\mathrm{d}y,\\
{}_0^C D_t^\alpha \widetilde{w}(x,t) = \varepsilon \widetilde{w}_{xx}(x,t) + \lambda(x)\widetilde{w}(x,t) + p_1(x)\widetilde{w}(1,t),\\
\widetilde{w}_x(0,t) = -q\widetilde{w}(0,t),\\
\widetilde{w}_x(1,t) = p_0\widetilde{w}(1,t),\\
\widehat{w}(x,0) = \widehat{w}_0(x), \quad \widetilde{w}(x,0) = \widehat{w}_0(x) - w_0(x).
\end{cases} \tag{3.27}
$$

Under the transformation (3.14) and

$$
\widehat{z}(x,t) = \widehat{w}(x,t) - \int_0^x k(x,y)\widehat{w}(y,t)\mathrm{d}y, \tag{3.28}
$$

system (3.27) is transformed into the following system:

$$
\begin{cases}
{}_0^C D_t^\alpha \widehat{z}(x,t) = \varepsilon \widehat{z}_{xx}(x,t) - c\widehat{z}(x,t) + G_1(x)\widetilde{z}(1,t),\\
\widehat{z}_x(0,t) = 0,\\
\widehat{z}_x(1,t) = p_0\widetilde{z}(1,t),\\
{}_0^C D_t^\alpha \widetilde{z}(x,t) = \varepsilon \widetilde{z}_{xx}(x,t) - \widetilde{c}\widetilde{z}(x,t),\\
\widetilde{z}_x(0,t) = \widetilde{z}_x(1,t) = 0,\\
\widehat{z}(x,0) = \widehat{z}_0(x), \ \widetilde{z}(x,0) = \widetilde{z}_0(x),
\end{cases} \tag{3.29}
$$

where

$$
G_1(x) = p_1(x) - \int_0^x k(x,y)p_1(y)\mathrm{d}y.
$$

To show well-posedness and Mittag–Leffler stability for system (3.29), let us introduce a new variable, $\bar{z}(x,t) = \widehat{z}(x,t) - p_0x^2(x-1)\widetilde{z}(x,t)$. The purpose of this change of variables is to make boundary conditions be homogeneous. In this way, $\bar{z}(x,t)$ satisfies

$$
\begin{cases}
{}_0^C D_t^\alpha \bar{z}(x,t) = \varepsilon \bar{z}_{xx}(x,t) - c\bar{z}(x,t) + G_1(x)\widetilde{z}(1,t)\\
\qquad\qquad\qquad +G_2(x)\widetilde{z}(x,t) + G_3(x)\widetilde{z}_x(x,t),\\
\bar{z}_x(0,t) = \bar{z}_x(1,t) = 0,\\
\bar{z}(x,0) = \bar{z}_0(x),
\end{cases} \tag{3.30}
$$

where

$$
G_2(x) = \varepsilon p_0(6x-2) - p_0x^2(x-1)(c-\widetilde{c}), \ G_3(x) = \varepsilon p_0(3x^2-2x).
$$

We now prove the well-posedness and the Mittag–Leffler stability for system (3.30) coupled with "$\widetilde{z}$-part" of (3.29). To this end, we introduce an equivalent inner product induced norm in $\mathbb{H} = L^2(0,1) \times L^2(0,1)$:

$$
\|(f,g)\|^2 = \int_0^1 f^2(x)\mathrm{d}x + \kappa\int_0^1 g^2(x)\mathrm{d}x,
$$

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where $\kappa > 0$ is a constant to be determined later in (3.37). Define operators A and B as follows:

$$\begin{cases} [A(f,g)](x) = (-\varepsilon f''(x) - cf(x), -\varepsilon g''(x) - \widetilde{c}g(x)), \\ D(A) = \{(f,g) \in [H^2(0,1)]^2 : f'(0) = f'(1) = g'(0) = g'(1) = 0\}, \\ [B(f,g)](x) = (G_1(x)g(1) + G_2(x)g(x) + G_3(x)g'(x), 0), \\ D(B) = H^1(0,1) \times H^1(0,1). \end{cases} \tag{3.31}$$

We claim that for any given sufficiently small $a > 0$, there exists positive constant $b_a > 0$ such that

$$\|B(f,g)\|_{\mathbb{H}} \le a\|A(f,g)\|_{\mathbb{H}} + b_a\|(f,g)\|_{\mathbb{H}}. \tag{3.32}$$

Actually,

$$\begin{aligned} \|B(f,g)\|_{\mathbb{H}}^2 &= \int_0^1 (G_1(x)g(1) + G_2(x)g(x) + G_3(x)g'(x))^2 \,\mathrm{d}x \\ &\le 3G_{10}^2|g(1)|^2 + 3G_{20}^2 \int_0^1 g^2(x)\mathrm{d}x + 3G_{30}^2 \int_0^1 (g'(x))^2\mathrm{d}x \\ &\le \left(6G_{10}^2 + 3G_{20}^2\right) \int_0^1 g^2(x)dx + \left(6G_{10}^2 + 3G_{30}^2\right) \int_0^1 (g'(x))^2\mathrm{d}x, \end{aligned} \tag{3.33}$$

where $G_{10} = \max_{x\in[0,1]}|G_1(x)|$, $G_{20} = \max_{x\in[0,1]}|G_2(x)|$, and $G_{30} = \max_{x\in[0,1]}|p_0(3x^2 - 2x)|$. Since for any given $\sigma > 0$ there exists $C_\sigma > 0$ such that for all $g \in H^2(0,1)$,

$$\int_0^1 (g'(x))^2\mathrm{d}x \le \sigma \int_0^1 (g''(x))^2\mathrm{d}x + C_\sigma \int_0^1 g^2(x)\mathrm{d}x$$

and

$$\begin{aligned} \|A(f,g)\|_{\mathbb{H}}^2 &= \int_0^1 [\varepsilon f'' + cf]^2\mathrm{d}x + \kappa \int_0^1 [\varepsilon g'' + \widetilde{c}g]^2\mathrm{d}x \\ &\ge \kappa\frac{\varepsilon^2}{2} \int_0^1 (g''(x))^2\mathrm{d}x - \kappa\widetilde{c}^2 \int_0^1 g^2(x)\mathrm{d}x, \end{aligned}$$

we have

$$\begin{aligned} \|B(f,g)\|_{\mathbb{H}}^2 &\le \left(6G_{10}^2 + 3G_{20}^2\right)\sigma \int_0^1 (g''(x))^2 \,\mathrm{d}x + \left(6G_{10}^2 + 3G_{20}^2 + C_\sigma\right) \int_0^1 g^2(x)\mathrm{d}x \\ &\le \frac{\left(12G_{10}^2 + 6G_{20}^2\right)}{\kappa\varepsilon^2}\sigma\|A(f,g)\|_{\mathbb{H}}^2 \\ &\quad + \left(6G_{10}^2 + 3G_{20}^2 + C_\sigma + \frac{(12G_{10}^2 + 6G_{20}^2)}{\varepsilon^2}\sigma\widetilde{c}^2\right) \int_0^1 g^2(x)\mathrm{d}x. \end{aligned}$$

This, together with (3.33), shows that (3.32) holds by arbitrariness of σ. Since A generates an analytic C_0-semigroup on $\mathbb{H}$, it follows from [28, Theorem 2.1] that $A + B$ generates an analytic C_0-semigroup on $\mathbb{H}$ as well. By Lemma 1.6, the system (3.30) coupled with "$\widetilde{z}$-part" of (3.29) is well-posed.

Next, we show that system (3.30) coupled with "$\widetilde{z}$-part" of (3.29) is Mittag–Leffler stable. Let

$$V_1(t) = \frac{1}{2} \int_0^1 \bar{z}^2(x,t)\mathrm{d}x.$$

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Then

$$
\begin{aligned}
{}_0^C D_t^\alpha V_1(t) &= \frac{1}{2}\int_0^1 {}_0^C D_t^\alpha \bar{z}^2(x,t)\mathrm{d}x \le \int_0^1 \bar{z}(x,t){}_0^C D_t^\alpha \bar{z}(x,t)\mathrm{d}x \\
&= \int_0^1 \bar{z}(x,t)\big[\varepsilon\bar{z}_{xx}(x,t) - c\bar{z}(x,t) + G_1(x)\widetilde{z}(1,t) \\
&\qquad\qquad + G_2(x)\widetilde{z}(x,t) + G_3(x)\widetilde{z}_x(x,t)\big]\mathrm{d}x \\
&= -\varepsilon\int_0^1 \bar{z}_x^2(x,t)\mathrm{d}x - c\int_0^1 \bar{z}^2(x,t)\mathrm{d}x \\
&\quad + \int_0^1 \bar{z}(x,t)[G_1(x)\widetilde{z}(1,t) + G_2(x)\widetilde{z}(x,t) + G_3(x)\widetilde{z}_x(x,t)]\mathrm{d}x. \qquad (3.34)
\end{aligned}
$$

Since

$$
\begin{aligned}
\left|\int_0^1 \bar{z}(x,t)G_1(x)\widetilde{z}(1,t)\mathrm{d}x\right| &\le G_{10}\int_0^1 \bar{z}^2(x,t)\mathrm{d}x|\widetilde{z}(1,t)|^2 \\
&\le \frac{\varepsilon}{6}\int_0^1 \bar{z}^2(x,t)\mathrm{d}x + \frac{3G_{10}^2}{2\varepsilon}|\widetilde{z}(1,t)|^2 \\
&\le \frac{\varepsilon}{6}\int_0^1 \bar{z}^2(x,t)\mathrm{d}x + \frac{3G_{10}^2}{\varepsilon}\left(\int_0^1 \widetilde{z}_x^2(x,t)\mathrm{d}x + \int_0^1 \widetilde{z}^2(x,t)\mathrm{d}x\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\int_0^1 \bar{z}(x,t)G_2(x)\widetilde{z}(x,t)\mathrm{d}x\right| &\le \frac{\varepsilon}{6}\int_0^1 \bar{z}^2(x,t)\mathrm{d}x + \frac{3G_{20}^2}{2\varepsilon}\int_0^1 \widetilde{z}^2(x,t)\mathrm{d}x, \\
\left|\int_0^1 \bar{z}(x,t)G_3(x)\widetilde{z}_x(x,t)\mathrm{d}x\right| &\le \frac{\varepsilon}{6}\int_0^1 \bar{z}^2(x,t)\mathrm{d}x + \frac{3G_{30}^2}{2\varepsilon}\int_0^1 \widetilde{z}_x^2(x,t)\mathrm{d}x,
\end{aligned}
$$

we estimate the solution of (3.34) as

$$
\begin{aligned}
{}_0^C D_t^\alpha V_1(t) &\le -\frac{\varepsilon}{2}\int_0^1 \bar{z}_x^2(x,t)\mathrm{d}x - c\int_0^1 \bar{z}^2(x,t)\mathrm{d}x \\
&\quad + \left(\frac{3G_{10}^2}{\varepsilon} + \frac{3G_{30}^2}{2\varepsilon}\right)\int_0^1 \widetilde{z}_x^2(x,t)\mathrm{d}x + \left(\frac{3G_{10}^2}{\varepsilon} + \frac{3G_{20}^2}{2\varepsilon}\right)\int_0^1 \widetilde{z}^2(x,t)\mathrm{d}x. \qquad (3.35)
\end{aligned}
$$

Let

$$V_2(t) = \frac{1}{2}\int_0^1 \widetilde{z}^2(x,t)\mathrm{d}x.$$

Then,

$$
{}_0^C D_t^\alpha V_2(t) \le -\varepsilon\int_0^1 \widetilde{z}_x^2(x,t)\mathrm{d}x - \widetilde{c}\int_0^1 \widetilde{z}^2(x,t)\mathrm{d}x. \qquad (3.36)
$$

Let $V(t) = V_1(t) + \kappa V_2(t)$ be the Lyapunov function, where $\kappa > 0$ is the design parameter. It follows from (3.35) and (3.36) that

$$
\begin{aligned}
{}_0^C D_t^\alpha V(t) &\le -\frac{\varepsilon}{2}\int_0^1 \bar{z}_x^2(x,t)\mathrm{d}x - c\int_0^1 \bar{z}^2(x,t)\mathrm{d}x \\
&\quad - \left(\kappa\widetilde{c} - \frac{3G_{10}^2}{\varepsilon} - \frac{3G_{30}^2}{2\varepsilon}\right)\int_0^1 \widetilde{z}_x^2(x,t)\mathrm{d}x \\
&\quad - \left(\kappa\varepsilon - \frac{3G_{10}^2}{\varepsilon} - \frac{3G_{20}^2}{2\varepsilon}\right)\int_0^1 \widetilde{z}^2(x,t)\mathrm{d}x.
\end{aligned}
$$

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Choose $\kappa > 0$ so that

$$\kappa\widetilde{c} > \frac{3G_{10}^2}{\varepsilon} + \frac{3G_{30}^2}{2\varepsilon}, \quad \kappa\varepsilon > \frac{3G_{10}^2}{\varepsilon} + \frac{3G_{20}^2}{2\varepsilon}. \tag{3.37}$$

Then ${}_0^C D_t^\alpha V(t) \le -C_1 V(t)$, where

$$C_1 = \min\left\{2c, \frac{2}{\kappa}\left(\kappa\varepsilon - \frac{3G_{10}^2}{\varepsilon} - \frac{3G_{20}^2}{2\varepsilon}\right)\right\}.$$

By fractional version of the Lyapunov method [23, Theorem 5], we can obtain that $V(t) \le V(0)E_\alpha(-C_1 t^\alpha)$. Since $\bar{z}(x,t) = \widehat{z}(x,t) - p_0 x^2(x-1)\widetilde{z}(x,t)$ and system (3.29) is equivalent to system (3.25), there exist two constants $C, \mu > 0$ such that (3.26) holds. This completes the proof of the theorem. ∎

Remark 3.6. From the explicit expression of the solution of closed-loop system (2.30), it is clearly seen how the parameters c and $\widetilde{c}$ influence the convergence speed of system energy. However, in Theorem 3.5, we use operator method and Lyapunov method instead of the Riesz basis method to show stability for the closed-loop system (3.25). This is because it seems hard to prove that the operator A defined by (3.31) is a Riesz spectral operator, and hence the explicit expression of the solution is not available.

Remark 3.7. From the analysis, we see that to stabilize an unstable time fractional reaction diffusion system, the control design can be borrowed from those for the classical reaction diffusion equations in [30] and [31]. However, the stability analyses rely on the Riesz basis method for (1.6) and the fractional Lyapunov method for (1.7), which are very different from those of [30] and [31]. The Riesz basis method leads to an optimal decay estimation which was not given in [30] and [31]. The derived asymptotic stability for the closed-loop system is of polynomial type with respect to time and cannot be exponential. It is worth noting that due to the memory effect and complexity of the fractional derivative, the fractional Lyapunov method, which is different from the classical Lyapunov method, was not developed until 2009 in [23], and the applicability of the fractional Lyapunov method was not available until 2014 in [1]. The results here could provide some insights into the qualitative analysis of fractional PDEs. Finally, when the fractional order $\alpha = 1$, our results recover the results of [30] and [31]. In particular, our results provide the optimal decay estimation for the case $\alpha = 1$, which was not available in [30] and [31].

Remark 3.8. As indicated in Remark 3.7, when the parameters in (1.6) and (1.7) are known, the design used in [30] and [31] can be applied. However, the design method of [30] and [31] is not always applicable for the time fractional reaction diffusion equations. This happens when the parameters in (1.6) and (1.7) are uncertain. In this case, the computation of the fractional derivatives for composite functions is complicated, and boundary control for (1.6) and (1.7) with uncertainty is never trivial and should be considered in further study.

To end this paper, we mention the physical feasibility of feedback control presented in the closed-loop systems (2.30) and (3.25). Since in both (1.6) and (1.7) only the boundary temperature $w(0,t)$ or $w(1,t)$ is measured, it is easily physically implementable. The observers (2.18) and (3.11) can be implemented by discretization technique presented in [33].

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4. Concluding remarks. This paper is a first effort to attempt boundary feedback stabilization for fractional PDE systems. Both Dirichlet and Neumann boundary controls are discussed. In the Neumann case, we use collocated control and observation to showcase the different technique with the Dirichlet case. The backstepping transformation is used in designing the state feedback laws. The observers are designed and the observer-based feedback control is obtained based on the stabilizing state feedback. However, the observer for the Neumann control is only considered for a simplified model. The Mittag–Leffler stability is concluded in each case for the closed loop. The idea is potentially promising for treating other fractional PDEs. There are some other interesting problems that are not touched in the field. One of them is stabilization for uncertain fractional PDE systems, which has been discussed for classic PDE systems in [8, 9, 10, 11, 12, 13, 14, 34, 35] and the abundant references therein.

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