Stability of optimal control of heat equation with singular potential

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In this paper, we study stability of an optimal control for a multi-dimensional heat equation with a singular potential term. A family of perturbed optimal control problems with lower powers singular potentials are formulated. It is shown that when the lower powers tend to the critical power two, the optimal controls are convergent to the optimal control of the original system.

1. Introduction

The optimal control theory deals with problems of finding a control law for a given system such that a certain optimality criterion is achieved. This theory has tremendous applications in science and engineering. There are huge works attributed to optimal control for lumped parameter systems. The optimal control theory for systems governed by partial differential equations (PDEs) is one of the main research topics in distributed parameter systems control since beginning of 1960s. In the last several years, some significant progresses have been made on optimal control problems of PDEs, we refer to [1–3] and many references therein. However, the optimal control and optimal cost of a controlled system under a small perturbation have not been fully addressed. In this paper, we attempt stability (sensitivity) analysis for a multi-dimensional heat equation with a singular potential term. This heat equation is a special case of the second-order parabolic partial differential equations, which is described by

\begin{align}
\partial_t y(x, t) - \Delta y(x, t) - V(x) y(x, t) = \chi_\omega u(x, t) \\
\text{in } \Omega \times (0, T], \\
y(x, t) = 0 \text{ on } \partial \Omega \times (0, T], \\
y(x, 0) = y_0(x) \text{ in } \Omega,
\end{align}

where \( T \) is a positive number, \( \Omega \subset \mathbb{R}^d \) \((d \geq 3)\) is a convex and bounded domain, with smooth boundary \( \partial \Omega \) and \( 0 \in \Omega \), \( \omega \) is a nonempty open domain of \( \Omega \), and \( \chi_\omega \) stands for the characteristic function of \( \omega \). The singular term

\[ V(x) = \frac{\lambda}{|x|^2}, \quad \lambda < \lambda_* = \frac{(d-2)^2}{4} \]

represents a potential function. The assumption (1.2) on the constant \( \lambda \) is crucial for the discussions in the present paper. This is because it is proved in [4] that if the initial value \( y_0 \) in the space \( L^2(\Omega) \) is non-negative, then Eq. (1.1) with control \( u = 0 \) admits a unique global weak solution under assumption (1.2), and when \( \lambda > \lambda_* \), even the local solution may not exist. For the existence and many other properties of the solutions to Eq. (1.1), we refer to [5–8], name just a few. In particular, in [8], the well-posedness of Eq. (1.1) without the sign restriction for the solution and control is thoroughly discussed from PDE’s point of view. The stabilization of Eq. (1.1) is investigated in [9].

We point out that the singular potentials occur in many physical phenomena. In non-relativistic quantum mechanics, the harmonic...
oscillator and the Coulomb central potential are typical examples of such kind (see, e.g., [10]). The other applications can be found in the study of near-horizon structure of black holes and dipoles.

In this paper, we are concerned with an optimal control problem of system (1.1) in the state space $L^2(\Omega)$. It is assumed that the admissible controls are taken from the following set:

$$\mathcal{U}_{ad} = \{ u \in L^2(0, T; L^2(\Omega)) \, | \, \| u(\cdot, t) \|_{L^2(\Omega)} \leq 1 \}$$

for almost all $t \in [0, T]$. (1.3)

By classical theory, it can be easily shown that for any $y_0 \in L^2(\Omega)$, $u \in L^2(0, T; L^2(\Omega))$, there exists a unique solution $y \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega))$ for Eq. (1.1) under assumption (1.2) (see also Lemma 2.4 in Section 2) and we denote this solution by $y(\cdot; y_0, u)$ to represent the dependence of the solution with the control $u$ and the initial value $y_0$. Throughout the paper, we use $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$ to denote the usual norm and the inner product in the space $L^2(\Omega)$ respectively without specific explanation.

Let $B(0, 1) = \{ w \in L^2(\Omega) \, | \, \| w \| \leq 1 \}$ be the closed unit ball of $L^2(\Omega)$. The optimal control problem that we are concerned in this paper is a LQ problem with control constraint (1.3), and target state constraint $B(0, 1)$ for system (1.1). The latter reads as follows:

$$\inf \{ J(u, y) \, | \, u \in \mathcal{U}_{ad}, y = y(\cdot; y_0, u) \}$$

is the solution of (1.1) satisfying $y(T; y_0, u) \in B(0, 1)$, where

$$f(u, y) = \frac{1}{2} \int_0^T \int_\Omega [y^2(x, t) + u^2(x, t)] \, dx \, dt.$$ (1.4)

From practical implementable standpoint, the optimal control problem with state and control constraints is significant and natural. On this regard for PDEs, we refer to [11,12]. We also mention the work [13] where the nonlinear boundary control of semi-linear parabolic problems with pointwise state constraints is concerned.

The mathematical model (1.1) is the critical situation where the power of the potential term is two: $V(x) = \lambda/|x|^2$. However, this does not include all reasonable potentials. In reality, this power may vary in $(0, 2)$ which is the most interested case in applications. More precisely, the potential function may take the form of

$$V_\alpha(x) = \frac{\lambda}{|x|^{2\alpha}},$$

where the power parameter $\alpha \in (0, 2)$. We remark that it is venerable physical folklore that potentials of the form $V_\alpha(x)$ product reasonable quantum dynamics as $\alpha \in (0, 2)$. This is explained in details in [14, Section X.2, p. 169] and [15, Section XLI, p. 64], respectively. This gives rise to a family of natural optimal control problems under such potential functions as counterparts of problem (1.3), which is stated as follows:

$$\inf \{ J_\alpha(u, y_\alpha) \, | \, u \in \mathcal{U}_{ad}, y_\alpha = y_\alpha(\cdot; y_0, u_\alpha) \}$$

is the solution to Eq. (1.6) satisfying $y_\alpha(T; y_0, u_\alpha) \in B(0, 1)$, where

$$J_\alpha(u_\alpha, y_\alpha) = \frac{1}{2} \int_0^T \int_\Omega [y_\alpha^2(x, t) + u_\alpha^2(x, t)] \, dx \, dt,$$ (1.5)

subject to control constraint $\mathcal{U}_{ad}$ defined by (1.3) and $y_\alpha = y_\alpha(\cdot; y_0, u_\alpha)$ defined by

$$y_\alpha(x, 0) = 0 \quad \text{on } \partial\Omega \times (0, T),$$

$$y_\alpha(x, t) = y_\alpha(x) \quad \text{in } \Omega.$$ (1.6)

It is seen that system (1.6) is just system (1.1) by simply replacing $V_{\alpha}$ with $V$.

Same as Eq. (1.1), under assumption (1.2), for any $y_0 \in L^2(\Omega)$, $u_\alpha \in L^2(0, T; L^2(\Omega))$, it can be shown that for any $\alpha \in (2 - \varepsilon, 2)$ with sufficiently small $\varepsilon > 0$, there exists a unique (weak) solution $y_\alpha \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega))$ to Eq. (1.6). This is discussed in Lemma 2.4 in detail in the next section.

Clearly, problem $(\mathcal{P}_\alpha)$ can be regarded as a perturbed problem of problem $(\mathcal{P})$ with the perturbed operator:

$$V_{\alpha}(x) = \left( \frac{\lambda}{|x|^2} - \frac{\lambda}{|x|^{2\alpha}} \right) I, \quad \alpha \in (0, 2)$$ (1.7)

where $I$ is the identity operator in $L^2(\Omega)$. By the Hardy–Poincaré inequality, this perturbed operator is not a bounded operator in $L^2(\Omega)$ for any $\alpha \in (2 - \varepsilon, 2)$. This fact measures the degree of major difficulty of the problem and surmounting the obstacle is the main contribution of the present work. Our objective is to relate the optimal controls between the problem $(\mathcal{P})$ and the perturbed problem $(\mathcal{P}_\alpha)$. To this purpose, we must first ensure the existence of feasible pairs for problems $(\mathcal{P})$ and $(\mathcal{P}_\alpha)$, respectively. The following assumption is made for problem $(\mathcal{P}_\alpha)$ throughout the paper.

**Assumption (S).** For given $y_0 \in L^2(\Omega)$, there exists an admissible control $u_\alpha \in \mathcal{U}_{ad}$ such that the corresponding solution $y_\alpha(\cdot; y_0, u_\alpha)$ to Eq. (1.1) reaches the interior of $B(0, 1)$: $y(T; y_0, u_\alpha) \in \text{int}(B(0, 1))$.

The **Assumption (S)** is called the Slater condition (see, e.g., [12]) which guarantees the existence of feasible pairs for problem $(\mathcal{P})$ and $(\mathcal{P}_\alpha)$ for all $\alpha \in (2 - \varepsilon, 2)$ with sufficiently small $\varepsilon > 0$. The latter is explained in the next section. We point out that the **Assumption (S)** is always true for any sufficient large $T > 0$ or for any fixed $T > 0$ with sufficiently small initial norm $\| y_0 \|_{L^2(\Omega)}$, which is explained in Remark 2.2 in next section.

Since both $f(\cdot, \cdot)$ and $J_\alpha(\cdot, \cdot)$ are strictly convex functionals, it is easily shown that under **Assumption (S)**, the optimal control problems $(\mathcal{P})$ and $(\mathcal{P}_\alpha)$ for all $\alpha \in (2 - \varepsilon, 2)$ with sufficiently small $\varepsilon > 0$ admit unique solutions which are denoted by $(u^*_\alpha, y^*_\alpha)$ and $(u^*_\alpha, y^*_\alpha)$, respectively. We refer this conclusion to Theorem 1.1 of [16]. The main result of this paper is the following **Theorem 1.1**.

**Theorem 1.1.** Let $y_0 \in L^2(\Omega)$ and assume that **Condition (S)** stands. Suppose that $(u^*, y^*)$ is the optimal pair for problem $(\mathcal{P})$ and $(u^*_\alpha, y^*_\alpha)$ is the optimal pair for problem $(\mathcal{P}_\alpha)$. Then

$$u^*_\alpha \rightarrow u^* \quad \text{strongly in } L^2(0, T; L^2(\Omega)) \quad \text{as } \alpha \uparrow 2, \tag{1.8}$$

and

$$J_\alpha(u^*_\alpha, y^*_\alpha) \rightarrow J(u^*, y^*) \quad \text{as } \alpha \uparrow 2. \tag{1.9}$$

In addition,

$$y^*_\alpha(T; y_0, u^*_\alpha) \rightarrow y^*(T; y_0, u^*) \quad \text{strongly in } L^2(\Omega) \quad \text{as } \alpha \uparrow 2. \tag{1.10}$$

It is apparently that **Theorem 1.1** can be regarded as a sensitivity or stability for the optimal control pair and optimal cost for problem $(\mathcal{P})$. Mathematically, the major difficulty in proving **Theorem 1.1** lies in the regularity of the solution caused by the singular potential terms. Simply speaking, we cannot expect $H^1(\Omega)$ regularity in spatial variable either for solution of Eq. (1.1) or (1.6) because the perturbed operator $V_{\alpha}$ defined by (1.7) is not a bounded operator in $L^2(\Omega)$ as $\alpha \uparrow 2$. This gives rise to the difficulty in application of the classical perturbation theory of $C_0$-semigroups (see, e.g., [17]). This difficulty is surmounted, however, by setting up new functional space in terms of the Hardy–Poincaré inequality, which enables us to improve the regularity in the space $H^1(\Omega)$.

We proceed as follows. In Section 2, we give some preliminary results. Section 3 is devoted to the proof of **Theorem 1.1**. Some concluding remarks are presented in Section 4.
2. Preliminary results

Once again, we suppose that $\Omega \subset \mathbb{R}^d$ ($d \geq 3$) is an open domain with smooth boundary and $0 \in \Omega$. Let us first recall the well-known Hardy–Poincaré inequality that there exists a positive constant $C(\Omega)$ depending on $\Omega$ only such that
\[
\int_{\Omega} \left| \nabla \psi(x) \right|^2 - \frac{\psi^2(x)}{|x|^2} \, dx \geq C(\Omega) \int_{\Omega} \psi^2(x) \, dx,
\]
\[
\forall \psi \in H_0^1(\Omega),
\]
where $\lambda_*$ is defined in (1.2). The proof for inequality (2.1) can be found in [18,7,8]. Considering $L^2(\Omega)$ as a pivot space, we have the following compact embedding (see, e.g., Section 2.8 of [19]):
\[
H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega) \quad \text{and}
\]
\[
(f, g)_{H^{-1}(\Omega), H_0^1(\Omega)} = (f, g)_{L^2(\Omega)}, \quad \forall f \in L^2(\Omega), g \in H_0^1(\Omega).
\]

Lemma 2.1. For any $v \in H_0^1(\Omega)$, \( \frac{\lambda_2}{|x|^2} v \) \( \in \) \( H^{-1}(\Omega) \) and
\[
\left\| \frac{\lambda_2}{|x|^2} v \right\|_{H^{-1}(\Omega)} \leq \| v \|_{H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega).
\]

Proof. By Eq. (2.1), if $\psi \in H_0^1(\Omega)$, then $\psi(x)/|x| \in L^2(\Omega)$. Hence for supposed $v, \psi(x)/|x| \in L^2(\Omega)$. It then follows from Eqs. (2.1), (2.2), and the Cauchy–Schwarz inequality that
\[
\left\| \frac{\lambda_2}{|x|^2} v \right\|_{H^{-1}(\Omega)} \leq \int_{\Omega} \frac{\lambda_2}{|x|^2} v(x) \psi(x) \, dx \leq \left( \int_{\Omega} \frac{\lambda_2}{|x|^2} v^2(x) \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} \frac{\lambda_2}{|x|^2} \psi^2(x) \, dx \right)^{\frac{1}{2}} \leq \| v \|_{H_0^1(\Omega)} \| \psi \|_{H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega).
\]

This shows that $\lambda_2/|x|^2 v \in H^{-1}(\Omega)$ and inequality (2.3) holds. □

Lemma 2.2. There exists a positive number $\varepsilon > 0$ and $C(\Omega) > 0$ such that for any $\alpha \in [2 - \varepsilon, 2]$,
\[
\int_{\Omega} \left| \nabla v(x) \right|^2 - \frac{\lambda_2 v^2(x)}{|x|^2} \, dx \geq C(\Omega) \int_{\Omega} v^2(x) \, dx,
\]
\[
\forall \alpha < \lambda_*, \ v \in H_0^1(\Omega).
\]

This deduces, same as (2.3), that for any $v \in H_0^1(\Omega)$, \( \frac{\lambda_2}{|x|^2} v \) \( \in \) \( H^{-1}(\Omega) \) for $\lambda < \alpha$, and $\alpha \in [2 - \varepsilon, 2]$ and
\[
\left\| \frac{\lambda_2}{|x|^2} v \right\|_{H^{-1}(\Omega)} \leq \| \nabla v \|_{L^2(\Omega)}, \quad \forall v \in H_0^1(\Omega).
\]

Furthermore, for $\lambda < \alpha$ and $\alpha \in [2 - \varepsilon, 2]$,
\[
\int_{\Omega} \left| \nabla v(x) \right|^2 - \frac{\lambda_2 v^2(x)}{|x|^2} \, dx \geq C(\Omega) \left( 1 - \frac{\lambda}{\lambda_*} \right) \int_{\Omega} |v(x)|^2 \, dx
\]
\[
\quad + \frac{C(\Omega) \lambda}{\lambda_*} \int_{\Omega} v^2(x) \, dx.
\]

Proof. Since $\Omega$ is bounded in $\mathbb{R}^d$, there exists a positive number $D > 1$ so that
\[
|\psi| < D, \quad \forall \psi \in \Omega.
\]

By (2.1),
\[
\int_{\Omega} \left| \nabla v(x) \right|^2 - \frac{\lambda_2 v^2(x)}{|x|^2} \, dx \geq C(\Omega) \int_{\Omega} |v(x)|^2 \, dx
\]
\[
\geq \int_{\Omega} |\nabla v(x)|^2 - \lambda_* v^2(x) \, dx \geq C(\Omega) \int_{\Omega} v^2(x) \, dx.
\]

This is (2.4). Furthermore, by (2.4), it follows that
\[
\int_{\Omega} \left| \nabla v(x) \right|^2 - \frac{\lambda_2 v^2(x)}{|x|^2} \, dx = \left( 1 - \frac{\lambda}{\lambda_0} \right) \int_{\Omega} |\nabla v(x)|^2 \, dx
\]
\[
\quad + \frac{\lambda}{\lambda_0} \int_{\Omega} \left| \nabla v(x) \right|^2 - \lambda_0 v^2(x) \, dx \geq C(\Omega) \left( 1 - \frac{\lambda}{\lambda_*} \right) \int_{\Omega} |\nabla v(x)|^2 \, dx
\]
\[
\quad + \frac{C(\Omega) \lambda}{\lambda_*} \int_{\Omega} v^2(x) \, dx,
\]
\[
\forall v \in H_0^1(\Omega), \lambda < \lambda_*.
\]

This is (2.6). □

In the rest of the paper, the assumption of $\alpha \in [2 - \varepsilon, 2]$ and $\lambda < \lambda^*$ claimed by Lemma 2.2 is acquiesced without repeat mentioning. By (2.6), we can equip $H_0^1(\Omega)$ with the following inner product:
\[
(\langle f, g \rangle)_{H_0^1(\Omega)} = \int_{\Omega} (\nabla f(x) \cdot \nabla g(x) - V_\alpha(f(x)g(x))) \, dx,
\]
\[
(\forall f, g \in H_0^1(\Omega)).
\]

By abuse of notation, we do not distinguish, in what follows, the above $H_0^1(\Omega)$ inner product which is clear from the context although different $\alpha$ and $\lambda$ produce different inner product.

Define the operator $L_\alpha : H_0^1(\Omega) \subset H^{-1}(\Omega)) \rightarrow H^{-1}(\Omega)$ as follows:
\[
(L\alpha f, g)_{H_0^1(\Omega)} = \int_{\Omega} [\nabla f(x) \cdot \nabla g(x) - V_\alpha(f(x)g(x))] \, dx,
\]
\[
(\forall f, g \in H_0^1(\Omega)).
\]

According to the Lax–Milgram theorem, $L_\alpha$ is a canonical isomorphism from $D(L_\alpha) = H_0^1(\Omega)$ onto $H^{-1}(\Omega)$. By (2.9),
\[
\| L\alpha f \|_{H^{-1}(\Omega)} = \| f \|_{H_0^1(\Omega)}, \quad \forall f \in H_0^1(\Omega).
\]

By (2.2), (2.8), and (2.9), we have the formula
\[
(L\alpha f, g)_{L^2(\Omega)} = (Lf, g)_{H_0^1(\Omega)} = (f, g)_{H_0^1(\Omega)},
\]
\[
(\forall f, g \in H_0^1(\Omega), L\alpha f \in L^2(\Omega)).
\]

Remark 2.1. It is easily seen that
\[
(L\alpha f)(x) = -\nabla f(x) - V_\alpha(x)f(x), \quad \forall f \in H^2(\Omega) \cap H_0^1(\Omega).
\]

However, $L\alpha f$ does not necessarily belong to $L^2(\Omega)$ for general $f \in H^2(\Omega) \cap H_0^1(\Omega)$ due to singularity of potential term. Actually, it was proved in [20] that when $d > 4$, the second order Hardy inequality
\[
\int_{\Omega} \frac{\nabla^2(x)}{|x|^2} \, dx \leq \int_{\Omega} (\nabla f(x))^2 \, dx,
\]
holds true, which shows that for $d > 4$, $\alpha = 2$, we can indeed confirm that the operator defined by (2.12) can be considered as an operator in $L^2(\Omega)$ with the domain $D(L_\alpha) = H^2(\Omega) \cap H_0^1(\Omega)$, but the situation for general $d \geq 3$ is not cleared up to present.
Since from (2.9), \( L_a \) is a positive self-adjoint operator in \( H^{-1}(\Omega) \) and by the Sobolev embedding theorem, \( L_a^{-1} : H^1(\Omega) \to H^{-1}(\Omega) \) is compact on \( H^{-1}(\Omega) \), there exists a sequence of subspaces \( Z_n \) such that
\[
L_a f_n = \lambda_n f_n, \quad \forall f_n \in Z_n, \quad \text{dim} Z_n < \infty, \quad n \in \mathbb{N}^*, \quad \lambda_n \uparrow +\infty, \quad \lambda_n \geq 0, \quad Z_n \perp Z_m \quad \text{if} \quad n \neq m, \quad (2.14)
\]
and the vector space \( Z \) generated by \( \cup Z_n \) is dense in \( H^1_0(\Omega) \). Clearly, \( Z_n \) is the eigenspace for the eigenvalue \( \lambda_n \), and \( \text{dim} Z_n \) is equal to the multiplicity of \( \lambda_n \). It is obvious that each \( Z_n \) depends on \( \alpha \) and so does \( f_n \). However, since this dependence does not affect the result in what follows, we just use the same notation \( Z_n \) and \( f_n \) for different \( \alpha \) without confusion. By (2.11) and (2.10), it follows that
\[
\lambda_n \| f_n \|^2_{L^2(\Omega)} = \| f_n \|^2_{H^1(\Omega)} = \lambda_n \| f_n \|^2_{H^{-1}(\Omega)}, \quad \forall \ n \in \mathbb{N}^*. \quad (2.15)
\]
For any \( f \in L^2(\Omega) \) or \( H^1_0(\Omega) \),
\[
f = \sum_{n=1}^{\infty} f_n, \quad f_n \in Z_n, \quad \forall \ n \geq 1, \quad (2.16)
\]
it follows from (2.15) that
\[
\| f \|_{L^2(\Omega)}^2 = \sum_{n=1}^{\infty} \lambda_n \| f_n \|^2_{H^{-1}(\Omega)} = \sum_{n=1}^{\infty} \| f_n \|^2_{L^2(\Omega)}, \quad (2.17)
\]
\[
\| f \|_{H^1_0(\Omega)}^2 = \sum_{n=1}^{\infty} \lambda_n \| f_n \|^2_{L^2(\Omega)}, \quad \forall f \in H^1_0(\Omega). \quad (2.18)
\]

**Lemma 2.3.** There exists \( \theta > 0 \) such that the first eigenvalue of \( L_a \) satisfies
\[
\lambda_1 \geq \theta, \quad \forall \alpha \in [2 - \varepsilon, 2], \quad (2.19)
\]
where \( \varepsilon \) is determined in **Lemma 2.2**.

**Proof.** It follows directly from (2.11), (2.14), and (2.4) that for any \( f_1 \neq 0, f_1 \in Z_1, \)
\[
\lambda_1 = \frac{L_a f_1, f_1}{\| f_1 \|^2_{L^2(\Omega)}} = \frac{\| f_1 \|^2_{L^2(\Omega)}}{\| f_1 \|^2_{L^2(\Omega)}} = \lambda_1, \quad (2.20)
\]
where \( C_{\lambda, \Omega} > 0 \) is a constant depending on \( \lambda, \Omega, \) and \( \alpha \in [2 - \varepsilon, 2] \). Taking \( \theta = C_{\lambda, \Omega}, \) we obtain (2.19). \( \square \)

Let \( S_\alpha(t) \) be the \( C_0 \)-semigroup generated by the operator \( L_a \) on \( H^{-1}(\Omega) \). Its restrictions on \( L^2(\Omega) \) and \( H^1_0(\Omega) \) are also \( C_0 \) semigroups which are defined by (see, e.g., [19, Proposition 2.10.4, p. 62]):
\[
S_\alpha(t) |_{L^2(\Omega)} = L_a^{-1/2} S_\alpha(t) L_a^{1/2}, \quad S_\alpha(t) |_{H^1_0(\Omega)} = L_a^{-1} S_\alpha(t) L_a. \quad (2.20)
\]

We do not distinguish all these semigroups which are clear from the context. By (2.18),
\[
\begin{cases}
S_\alpha(t)f = \sum_{n=1}^{\infty} e^{-\lambda_n t} f_n, \quad \forall f = \sum_{n=1}^{\infty} f_n \in L^2(\Omega), \\
\| S_\alpha(t)f \|^2_{L^2(\Omega)} = \sum_{n=1}^{\infty} e^{-2\lambda_n t} \| f_n \|^2_{L^2(\Omega)}, \\
\| S_\alpha(t)f \|^2_{H^1_0(\Omega)} = \sum_{n=1}^{\infty} e^{-2\lambda_n t} \lambda_n \| f_n \|^2_{L^2(\Omega)}, \\
\forall f = \sum_{n=1}^{\infty} f_n \in H^1_0(\Omega). \quad (2.21)
\end{cases}
\]

It then follows from (2.19), (2.21), and (2.17) that the semigroup \( S_\alpha(t) \) is uniformly exponentially stable in \( L^2(\Omega) \):
\[
\| S_\alpha(t)f \|_{L^2(\Omega)} \leq e^{-\theta t} \| f \|_{L^2(\Omega)}, \quad \forall \alpha \in [2 - \varepsilon, 2], \quad f \in L^2(\Omega), \quad (2.22)
\]
where \( \theta \) is determined in **Lemma 2.3**. With \( S_\alpha(t) \), we write the (weak) solution of (1.1) or (1.6) as
\[
y_\alpha(t; y_0, u) = S_\alpha(t)y_0 + \int_0^t S_\alpha(t-s)\chi_\alpha u_\alpha(\cdot, s)ds. \quad (2.23)
\]

Starting from (2.23), we are now able to prove the following **Lemma 2.4**, which shows in particular the existence of solution of Eq. (1.1) or (1.6), and it also indicates that the solution always belongs to \( L^2(0, T; H^1_0(\Omega)) \).

**Lemma 2.4.** For any \( y_0 \in L^2(\Omega), \) \( u_\alpha \in L^2(0, T; L^2(\Omega)) \), and \( T > 0 \), it always holds:
\[
\begin{align*}
\| y_\alpha(t; y_0, u) \|_{C([0, T]; L^2(\Omega))} & \leq e^{-\theta T} \| y_0 \|_{L^2(\Omega)} + \frac{1}{\sqrt{2\omega}} \| u_\alpha \|_{L^2(0, T; L^2(\Omega))}, \\
\| y_\alpha(t; y_0, u_\alpha) \|_{L^2(0, T; H^1_0(\Omega))} & \leq \| y_0 \|_{L^2(\Omega)} + \| u_\alpha \|_{L^2(0, T; L^2(\Omega))},
\end{align*}
\]
where \( \theta \) is the constant claimed by **Lemma 2.3** and \( \alpha \in [2 - \varepsilon, 2] \) with \( \varepsilon \) determined by **Lemma 2.2**.

**Proof.** The proof of the first inequality of (2.24) is straightforward by virtue of (2.19) and (2.21). We only show the second inequality. Suppose that \( y_\alpha = \sum_{n=1}^{\infty} f_n \in L^2(\Omega) \), it follows from (2.18) and (2.21) that
\[
\| S_\alpha(t)y_0 \|^2_{L^2(0, T; H^1_0(\Omega))} = \int_0^T \| S_\alpha(t)y_0 \|^2_{H^1_0(\Omega)} dt = \sum_{n=1}^{\infty} e^{-2\lambda_n t} \| f_n \|^2_{L^2(\Omega)} dt = \sum_{n=1}^{\infty} \frac{1}{2} \| f_n \|^2_{L^2(\Omega)} \leq \sum_{n=1}^{\infty} \| f_n \|^2_{L^2(\Omega)} = \| y_0 \|^2_{L^2(\Omega)} \quad (2.25)
\]

Next suppose that \( \chi_\alpha u_\alpha(\cdot, s) = \sum_{n=1}^{\infty} f_n(s) \in L^2(\Omega), \)
\[
\int_0^t S_\alpha(t-s)\chi_\alpha u_\alpha(\cdot, s)ds = \sum_{n=1}^{\infty} e^{-2\lambda_n(t-s)}f_n(s) \in L^2(\Omega).
\]

Since from (2.14) \( f_n(s) \in Z_n \) for all \( s \in [0, T] \) and \( \text{dim} Z_n < \infty \), one must have \( \int_0^t f_n(s)ds \in Z_n \) for all \( n \geq 1 \). It then follows from (2.18) and (2.21) that
\[
\begin{align*}
\| S_\alpha(t-s)\chi_\alpha u_\alpha(\cdot, s)ds \|^2_{H^1_0(\Omega)} & = \sum_{n=1}^{\infty} \| e^{-2\lambda_n(t-s)}f_n(s) \|^2_{L^2(\Omega)} \leq \sum_{n=1}^{\infty} \lambda_n \| e^{-2\lambda_n(t-s)}f_n(s) \|^2_{L^2(\Omega)} \leq \sum_{n=1}^{\infty} \| f_n \|^2_{L^2(\Omega)} = \| y_0 \|^2_{L^2(\Omega)} \quad (2.27)
\end{align*}
\]
Therefore,\n
\[
\int_0^T \int_0^1 S_\alpha(t-s)\chi \omega u_\epsilon(\cdot, t) ds \leq \|\chi \omega u_\epsilon(\cdot, t)\|^2_{L^2(\Omega)},
\]

(2.27)

Combining (2.25) and (2.27), we obtain the second inequality of (2.24). \qed

**Remark 2.2.** From the first inequality of (2.24), we obtain immediately that under the conditions of Lemma 2.4, the Slater condition \((S)\) holds true for sufficiently large \(T > 0\) or for fixed \(T > 0\) but with sufficiently small \(\|y_0\|_{L^2(\Omega)}\), both with \(u_\epsilon \equiv 0\).

For any \(T > 0\), define an operator \(A_\alpha : L^2(0, T; L^2(\Omega)) \to L^2(\Omega)\) as follows:

\[
(A_\alpha u)(s) = \int_0^T S_\alpha(T-s)\chi \omega u(\cdot, s) ds.
\]

(2.28)

The following Lemma 2.5 is a direct consequence of (2.26) and the Sobolev embedding theorem.

**Lemma 2.5.** Let \(\varepsilon\) be the constant determined by Lemma 2.2 and \(\alpha \in [2 - \varepsilon, 2]\). Then the operator \(A_\alpha\) defined by (2.28) is a compact operator.

**Proposition 2.1.** For any given \(y_0 \in L^2(\Omega)\), \(u \in L^2(0, T; L^2(\Omega))\),

\[
y_\alpha(t; y_0, u) \to y(t; y_0, u) \quad \text{strongly in } L^2(\Omega) \quad \text{as } \alpha \uparrow 2, \quad \forall t \in [0, T].
\]

**Proof.** Let \(\varepsilon\) be the constant determined by Lemma 2.2 and \(\alpha \in [2 - \varepsilon, 2]\). We write the solution \(y_\alpha(t; y_0, u)\) of Eq. (1.6) as

\[
\begin{aligned}
\partial_t y_\alpha(x, t) - \Delta y_\alpha(x, t) &= \frac{\lambda}{|x|^2} y_\alpha(x, t) \\
&= \left(\frac{\lambda}{|x|^2} - \frac{\lambda}{|x|^2}\right) y_\alpha(x, t) + \chi \omega u(x, t),
\end{aligned}
\]

(2.29)

\[
y_\alpha(x, t) = 0,
\]

\[
y_\alpha(x, 0) = y_0(x).
\]

Therefore,

\[
y_\alpha(t; y_0, u) = S_2(t)y_0 + \int_0^t S_\alpha(t-s)\chi \omega u(\cdot, s) ds \\
+ \int_0^t S_2(t-s) \left(\frac{\lambda}{|x|^2} - \frac{\lambda}{|x|^2}\right) y_\alpha(\cdot, s) ds \\
= y(t; y_0, u) + \int_0^t S_\alpha(t-s) \left(1 - |x|^{2-\alpha}\right) ds \\
\times \frac{\lambda y_\alpha(\cdot, s)}{|x|^2} ds.
\]

(2.30)

Thus

\[
\int_0^t S_\alpha(t-s) \left(1 - |x|^{2-\alpha}\right) ds \in H^1_0(\Omega), \quad \forall t \in [0, T].
\]

The proof is accomplished if we can show that

\[
\int_0^t S_\alpha(t-s) \left(1 - |x|^{2-\alpha}\right) \frac{\lambda y_\alpha(\cdot, s)}{|x|^2} ds \to 0
\]

strongly in \(L^2(\Omega)\) as \(\alpha \uparrow 2, \forall t \in [0, T]\). (2.31)

By Lemma 2.1, \(\frac{\lambda y_\alpha(\cdot, s)}{|x|^2} \in L^2(\Omega)\) and since \(\left(1 - |x|^{2-\alpha}\right) \in L^\infty(\Omega)\),

\[
\left(1 - |x|^{2-\alpha}\right) \frac{\lambda y_\alpha(\cdot, s)}{|x|^2} = \sum_{n=1}^\infty f_n(s) \in L^2(\Omega),
\]

\[
S_\alpha(t-s) \left(1 - |x|^{2-\alpha}\right) \frac{\lambda y_\alpha(\cdot, s)}{|x|^2} = \sum_{n=1}^\infty e^{-\lambda(t-s)} f_n(s) \in L^2(\Omega).
\]

Once again, since from (2.14), \(f_n(s) \in Z_\varepsilon\) for all \(s \in [0, T]\) and \(\dim Z_\varepsilon < \infty\), one must have \(\int_0^t f_n ds \in Z_\varepsilon\) for all \(n \geq 1\). It then follows from (2.17) and (2.21) that

\[
\begin{aligned}
\left\| \int_0^t S_\alpha(t-s) \left(1 - |x|^{2-\alpha}\right) \frac{\lambda y_\alpha(\cdot, s)}{|x|^2} ds \right\|_{L^2(\Omega)}^2
&= \sum_{n=1}^\infty \left\| \int_0^t e^{-\lambda(t-s)} f_n ds \right\|_{L^2(\Omega)}^2 \\
&= \sum_{n=1}^\infty \lambda_{\alpha} \left\| \int_0^t e^{-\lambda(t-s)} f_n ds \right\|_{L^2(\Omega)}^2 \\
&\leq \sum_{n=1}^\infty \lambda_{\alpha} \left\| \int_0^t e^{-2\lambda(t-s)} ds \int_0^t \left| f_n(s) \right|^2_{L^2(\Omega)} ds \right\|_{L^2(\Omega)}^2 \\
&\leq \int_0^t \sum_{n=1}^\infty \left\| f_n(s) \right\|_{L^2(\Omega)}^2 ds \\
&\leq \int_0^t \left(1 - |x|^{2-\alpha}\right) \frac{\lambda y_\alpha(\cdot, s)}{|x|^2} \right\|_{L^2(\Omega)}^2 ds \\
&\leq \left\| \left(1 - |x|^{2-\alpha}\right) \frac{\lambda y_\alpha(\cdot, s)}{|x|^2} \right\|_{L^2(\Omega)}^2 \|y_\alpha\|^2_{L^2(0, T; L^2(\Omega))}.
\end{aligned}
\]

(2.32)

This together with the second inequality in (2.24) gives (2.31) because

\[
\|1 - |x|^{2-\alpha}\|^2_{L^2(\Omega)} \to 0 \quad \text{as } \alpha \uparrow 2
\]

by virtue of the dominated convergence theorem. \qed

The following Corollary 2.1 is a direct consequence of Proposition 2.1, which is actually the Slater property for problem \((P_\varepsilon)\).

**Corollary 2.1.** Suppose that the Slater condition \((S)\) stands. Then there exists a positive number \(0 < \varepsilon_1 < \varepsilon\) such that for any \(\alpha \in (2 - \varepsilon_1, 2)\), there exists an \(u_\alpha^0 \in U_{ad}\) such that \(y_\alpha(T; y_0, u_\alpha^0) \in \text{int}(B(0, 1))\).

3. The proof of main result

**Proof of Theorem 1.1.** The proof will be split into three steps.

**Step 1:** There exists an \(\varepsilon_1 > 0\) such that \(J_\alpha(u_\alpha^0, y_\alpha^0)\) is bounded for all \(\alpha \in (2 - \varepsilon_1, 2)\).

Let \(\varepsilon_1\) be the positive number specified by Corollary 2.1. By (2.24), for \(y_0 \in L^2(\Omega)\), \(u_\alpha^0 \in U_{ad}\), the solution of Eq. (1.6) satisfies

\[
J_\alpha(u_\alpha^0, y_\alpha^0(\cdot; y_0, u_\alpha^0)) = \frac{1}{2} \int_0^T \int_\Omega \frac{\lambda^2}{|x|^2} |y_\alpha^2(t; y_0, u_\alpha^0) + u_\alpha^2(x, t)| dx dt \\
\leq \|y_0\|^2_{L^2(\Omega)} + (T + 1)\|u_\alpha^0\|^2_{L^2(0, T; L^2(\Omega))},
\]

where
which is true for all $\alpha \in (2 - \epsilon_1, 2)$. As a result, both $\{u^{\epsilon}_j\}_{\epsilon \in (2 - \epsilon_1, 2)}$ and $\{y^{\epsilon}_j\}_{\epsilon \in (2 - \epsilon_1, 2)}$ are bounded in $L^2(0, T; L^2(\Omega))$.

**Step 2:** By **Step 1** and the Banach–Alaoglu theorem, there exists a sequence $\alpha_j \in (2 - \epsilon_1, 2)$ such that

$$u^{\epsilon}_j \rightharpoonup \bar{u} \quad \text{weakly in } L^2(0, T; L^2(\Omega)),
$$

$$y^{\epsilon}_j \rightharpoonup \bar{y} \quad \text{weakly in } L^2(0, T; L^2(\Omega)) \quad \text{as } j \to \infty. \tag{3.1}$$

Let $y^{\epsilon}_j(t; y_0, u^{\epsilon}_j)$ be the solution of Eq. (1.6). We claim that for any $t \in [0, T]$, $y^{\epsilon}_j(t; y_0, u^{\epsilon}_j) \to y(t; y_0, \bar{u})$ strongly in $L^2(\Omega)$ as $j \to \infty$. Actually, it follows from (2.30) that

$$y^{\epsilon}_j(t; y_0, u^{\epsilon}_j) = S_2(t)y_0 + \int_0^t S_2(t - s) \left[ \frac{\lambda}{|x|^\alpha} - \frac{\lambda}{|x|^2} \right] y^{\epsilon}_j + \chi_2 u^{\epsilon}_j ds. \tag{3.2}$$

Set

$$y(t; y_0, \bar{u}) = S_2(t)y_0 + \int_0^t S_2(t - s) \chi_2 \bar{u}(\cdot, s) ds.$$

By (3.1) and Lemma 2.5,

$$\int_0^t S_2(t - s) \chi_2 u^{\epsilon}_j (-, t) ds \to \int_0^t S_2(t - s) \chi_2 \bar{u}(-, s) ds
$$

strongly in $L^2(\Omega)$ as $j \to \infty$, $\forall t \in [0, T].$ \tag{3.3}

By (2.31),

$$\left\| \int_0^t S_2(t - s) \left[ \frac{\lambda}{|x|^\alpha} - \frac{\lambda}{|x|^2} \right] y^{\epsilon}_j ds \right\|_{L^2(\Omega)} \to 0
$$

as $j \to \infty$, $\forall t \in [0, T].$ \tag{3.4}.

Combining (3.2)–(3.4), we obtain that

$$y^{\epsilon}_j(t; y_0, u^{\epsilon}_j) \to y(t; y_0, \bar{u}) \quad \text{as } j \to \infty, \tag{3.5}$$

and

$$y^{\epsilon}_j(t; y_0, u^{\epsilon}_j) \to y(t; y_0, \bar{u}) \quad \text{strongly in } L^2(0, T; L^2(\Omega)) \quad \text{as } j \to \infty. \tag{3.6}$$

This together with (3.1) shows that $\bar{y} = y(t; y_0, \bar{u}).$

**Step 3:** By strict convexity of $f$, (3.1), and Proposition 2.1,

$$f(\bar{u}, \bar{y}) \leq \liminf_{j \to \infty} f_0(u^{\epsilon}_j, y^{\epsilon}_j) \leq \lim_{j \to \infty} f_0(u^{\epsilon}_j, y_0, u^{\epsilon}_j) = f(u^*, y^*).$$

Since the optimal control is unique, we finally obtain that

$$u^* = \bar{u}, \quad f(\bar{u}, \bar{y}) = f(u^*, y^*),
$$

$$f_0(u^{\epsilon}_j, y^{\epsilon}_j) \to f(u^*, y^*) \quad \text{as } j \to \infty. \tag{3.7}$$

From (3.7) and (3.5), we can easily deduce (1.9) and (1.10) because for any sequence $\{\alpha_j\} \subset (2 - \epsilon_1, 2)$ there is a subsequence $\{\alpha_j\}$ such that (3.7) and (3.5) hold. In addition, by (3.7) and (3.6), we also have

$$\|u^{\epsilon}_j\|_{L^2(0, T; L^2(\Omega))} \to \|u^*\|_{L^2(0, T; L^2(\Omega))} \quad \text{as } j \to \infty.
$$

This together with (3.1) shows that

$$u^{\epsilon}_j \to u^* \quad \text{strongly in } L^2(0, T; L^2(\Omega)) \quad \text{as } j \to \infty,$n

which leads to (1.8) with the same argument that for any sequence $\{\alpha_j\} \subset (2 - \epsilon_1, 2)$ there is a subsequence $\{\alpha_j\}$ such that (3.7) holds. This completes the proof.

4. Concluding remarks

In this paper, we investigate an optimal control problem (P) for a multi-dimensional heat equation with a singular inverse-square potential. A family of corresponding perturbation problems (P$_{\epsilon}$) is formulated. We show that the optimal controls of the perturbed systems converge to the optimal control of the original system. In addition, the corresponding costs also converge to the cost of the original system. As a result, the optimal control system is stable for both optimal control and optimal cost. The difficulty of the problem is caused by the singularity of the potential functions, which gives rise to the unboundedness of the perturbation operator $(\frac{\lambda}{|x|^\alpha} - \frac{\lambda}{|x|^2}) I$ in the space $L^2(\Omega)$. With the aid of spectral analysis, we finally achieve the convergence. The methods developed in this paper could be applied to solve other stability of control problems of PDEs.

Finally, we mention that there have been many works on stability property for the optimal control problems, see, for instance, [21,22] and the references therein.

References