Uniform Exponential Stability for a Schrödinger Equation and its Semi-Discrete Approximation

Bao-Zhu Guo and Fu Zheng

***Abstract*—In this paper, we investigate the uniform exponential stability of a semi-discrete scheme for the Schrödinger equation under boundary stabilizing feedback control within the natural state space $L^2(0,1)$. This research holds significant value, as a time domain energy multiplier that validates the exponential stability of this continuous Schrödinger system remains elusive, posing a significant mathematical challenge to the uniform exponential stability of its corresponding semi-discretization systems, a long-standing open problem. While the potent frequency domain energy multiplier approach has been successfully applied to prove exponential stability for PDEs since the 1980s, its extension to the uniform exponential stability of semi-discrete schemes for PDEs remains unexplored. The challenge of achieving uniformity lies in the need to simultaneously consider an infinite number of matrices across various state spaces, stemming from the step size parameter. Drawing inspiration from the Huang-Prüss frequency domain criterion for the uniform exponential stability of a family of C_0-semigroups in Hilbert spaces, we tackle this problem for the first time by establishing the uniform boundedness of all resolvents of these matrices on the imaginary axis. The proof closely follows the procedure for the exponential stability of its continuous counterpart, underscoring the merit of this discretization method.**

***Index Terms*—Schrödinger equation, boundary damping, frequency domain multiplier, semi-discretization, uniform exponential stability.**

I. Introduction

Control systems described by partial differential equations (PDEs) are infinite-dimensional. Consequently, controllers such as observer-based feedback control are also infinite-dimensional. Therefore, discretization is essential in almost all implementations of PDE controls [12], [17]. Among various discretization methods, the finite-difference method stands out as the most popular due to its simplicity and appeal to engineers. One of the most commonly employed discretization methods is the semi-discrete scheme, which maintains time continuity while discretizing the spatial variable. This scheme has been extensively studied in the literature. Its primary advantage lies in converting an infinite-dimensional system into infinitely many ordinary differential equation systems, which are well-known to control researchers. However, it has long been recognized that the uniform exponential stability with respect to the spatial discrete step size cannot be maintained in the classical semi-discrete scheme for PDEs, primarily due to the presence of high-frequency spurious components. In addition, several other crucial control properties, such as uniform observability and uniform exact controllability, are not guaranteed in this context. The primary reason for this is that spurious modes are weakly damped during the semi-discretization process. A detailed explanation of this phenomenon can be found in [25]. To address this challenge in wave equations, several solutions have been proposed, including Tichonoff regularization [7], mixed-finite elements [2], [19], high-frequency filtering [10], and non-uniform meshes [3]. Specifically, [6] examines time semi-discrete approximations for a class of exponentially stable infinite-dimensional systems. It also discusses a fully discrete approximation scheme under a CFL-type assumption for the space and time discretization parameters. A similar scheme to the one presented in this paper was proposed in [17]. The paper [12] explores the observability of space semi-discrete beam equations, which share some similarities with the Schrödinger equation. Among these remedies, the numerical viscosity damping introduced in [21], [22], [6] is particularly popular. However, this approach artificially introduces a viscosity term into the classical discrete scheme. The coefficients of this numerical viscosity damping vary depending on the specific PDE [18]. Recently, a novel natural semi-discrete scheme based on an order reduction finite difference method was introduced in [13] and has been applied to various systems [8], [24]. This approach offers several advantages, including preserving uniform exponential stability. As a natural semi-discrete scheme, it enables one to prove uniform exponential stability in a manner analogous to its continuous PDE counterpart.

Constructing a suitable Lyapunov functional for a partial differential equation (PDE) typically relies on a time-domain energy multiplier, which is not always readily available and often involves intricate technical details. In the 1980s, a frequency-domain energy multiplier approach was pioneered for analyzing exponential stability, initially applied to a single PDE ([15]). This method is based on a frequency-domain characterization of exponential stability for C_0-semigroups in Hilbert spaces. Originally developed independently in [9] and [20], the result was later proven in [1], [16] to be valid for uniform exponential stability of families of C_0-semigroups in Hilbert spaces. Furthermore, uniform admissibility and observ-

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ability for finite element space semi-discretizations of abstract Schrödinger systems and second-order infinite-dimensional vibrating systems have also been established [4], [5].

In this paper, we investigate the uniform exponential stability of an order reduction semi-discrete scheme for the Schrödinger equation under boundary control, utilizing the frequency domain multiplier approach. This approach is crucial as finding a suitable time-domain Lyapunov functional for both the continuous PDE and its discrete counterpart is challenging. Consequently, existing methods presented in [14], [13], [8], [24] are not applicable here. In fact, to employ the Lyapunov method, the work in [14] had to consider the Schrödinger system in the higher-order state space $H^1(0,1)$, whereas our state space is the standard $L^2(0,1)$. The stability problem in $L^2(0,1)$ has remained unresolved for a considerable period. Therefore, this paper offers a novel approach to proving the uniform exponential stability of semi-discrete schemes for PDEs. Furthermore, it is noteworthy that the proofs for both the continuous PDE and its discrete counterpart exhibit analogous structures, once again highlighting the merits of the order reduction semi-discretization approach.

We proceed as follows. In the next section, Section II, we prove the exponential stability of the continuous PDE using the frequency domain multiplier method. Although this PDE has been extensively studied in the literature, it serves as a useful starting point in constructing a frequency domain multiplier for its semi-discrete counterpart. In Section III, we design a semi-discretized scheme and derive a family of finite-dimensional systems. In Section IV, we establish the uniform exponential stability by employing the frequency domain multiplier approach. We introduce the concept of shadow elements to aid in understanding the numerical approximating scheme, which plays a pivotal role in the proof of uniform stability. Finally, Section V contains some concluding remarks.

II. Stability of Schrödinger system via frequency domain multiplier

Consider the following closed-loop Schrödinger equation with proportional boundary feedback control:

$$\begin{cases} w_t(x,t) = -iw_{xx}(x,t),\ t>0,\ x\in(0,1), \\ w(0,t)=0,\ t\geq 0, \\ w_x(1,t) = -ikw(1,t),\ k>0,\ t\geq 0, \\ w(x,0)=w^0(x),\ x\in[0,1], \end{cases} \tag{1}$$

where the boundary condition $w_x(1,t)=U(t)$ is actually the control input $U(t)$ set as $U(t)=-ikw(1,t)$. We analyze the system (1) in the natural state space $L^2(0,1)$. Define the system operator of (1) as follows:

$$\begin{cases} Af = -if'',\ \forall f\in D(A)=\{f\in L^2(0,1)| \\ \quad f\in H^2(0,1),\ f(0)=0,\ f'(1)=-ikf(1)\}. \end{cases} \tag{2}$$

Then, (1) can be written as an evolution equation in $L^2(0,1)$:

$$\dot{w}(\cdot,t) = Aw(\cdot,t),\ w(x,0)=w^0(x). \tag{3}$$

It is observed that

$$\mathrm{Re}\langle Af,f\rangle_{L^2(0,1)} = \mathrm{Re}\int_0^1 -ikf''(x)\overline{f(x)}dx = -k|f(1)|^2, \tag{4}$$

which implies that the operator A is dissipative. Furthermore, the operator A is invertible and

$$\begin{aligned} A^{-1}f(x) = &\frac{-kx\int_0^1 \tau f(\tau)d\tau}{1+ik} \\ &-i\int_x^1 (x-\tau)f(\tau)d\tau - i\int_0^1 \tau f(\tau)d\tau, \end{aligned} \tag{5}$$

which is bounded in $L^2(0,1)$. Consequently, due to the Lumer-Phillips theorem ([23, Theorem 3.8.4]), the operator A generates a C_0-semigroup of contractions on $L^2(0,1)$. Since A^{-1} is compact, the spectrum of A comprises only isolated eigenvalues.

Furthermore, we define the system energy for the system (1) as

$$E(t) = \frac{1}{2}\int_0^1 |w(x,t)|^2 dt, \tag{6}$$

which is non-increasing as a consequence of (4):

$$\dot{E}(t) = -k|w(1,t)|^2. \tag{7}$$

We note that a variant of system (1) was examined in [14]. However, (1) represents a rather uncommon system, and as of yet, no time-domain energy multiplier has been identified for it. The first exponential stability result for (1) was established in [11] using the Riesz basis approach. While the Riesz basis method is potent and often yields deeper insights than the multiplier method, for instance, the spectrum-determined growth condition is often a consequence of the Riesz basis approach, this is not typically the case with the multiplier method. Unfortunately, applying the Riesz basis method to achieve uniform exponential stability for the semi-discrete model of (1) presented in this paper is currently highly challenging.

A well-known result from the Hung-Prüss theorem [9], [20] establishes that the C_0-semigroup generated by an operator A is exponentially stable if and only if it satisfies the following two properties:

1) Every imaginary number belongs to the resolvent set of A, that is, $i\mathbb{R}\subset\rho(A)$.
2) The inverse operator of $i\omega - A$ is uniformly bounded for all imaginary numbers, that is,

$$\sup_{\omega\in\mathbb{R}} \|(i\omega - A)^{-1}\| < \infty. \tag{8}$$

The first property is proved in Lemma 2.1 following.

Lemma 2.1: Let A be defined as in (2). Then, $i\mathbb{R}\subset\rho(A)$, where $\rho(A)$ denotes the resolvent set of A.

Proof. If there exist $\beta\in\mathbb{R}$ with $\beta\neq 0$ and a nonzero $f\in D(A)$ such that $i\beta f = Af$, then

$$\begin{cases} i\beta f(x) = -if''(x), \\ f'(1) = -ikf(1),\ f(0)=0. \end{cases} \tag{9}$$

By taking the inner product of both sides of the first equation in (9) with $f(\cdot)$ over $[0,1]$, we arrive at

$$i\beta\|f\|^2 = -k|f(1)|^2 + i\int_0^1 |f'(x)|^2 dx, \tag{10}$$

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which gives $f(1)=0$ and hence $f'(1)=0$. This demonstrates that (9) only has the zero solution, which is a contradiction. ∎

Theorem 2.1: Let A be defined by (2). Then, (8) holds true. Consequently, the C_0-semigroup e^{At} generated by A is exponentially stable in $L^2(0,1)$.

Proof. We prove by assuming the contrary of (8) that there exists a sequence $\omega_n \to \infty$, and $f_n \in D(A)$ with $\|f_n\|=1$ such that

$$\lim_{n\to\infty} \|(i\omega_n - A)f_n\| = 0,$$

i.e.,

$$i\omega_n f_n + i f_n'' \to 0 \text{ in } L^2(0,1). \tag{11}$$

Since

$$\mathrm{Re}\langle (i\omega_n - A)f_n, f_n\rangle_{L^2(0,1)} = k|f_n(1)|^2 \to 0, \tag{12}$$

by the boundary condition $f_n'(1) = -ikf_n(1)$, it has

$$f_n'(1) \to 0. \tag{13}$$

From (11) and $\|f_n\|=1$, it follows that $\frac{f_n''(\cdot)}{\omega_n}$ is bounded in $L^2(0,1)$. Using the inequality $|f_n'(x) - f_n'(1)| \le \|f_n''\|$, combined with (13) and $\omega_n \to \infty$, we can deduce that

$$\frac{f_n'(\cdot)}{\omega_n} \text{ is bounded in } L^2(0,1). \tag{14}$$

Since

$$\mathrm{Re}\left\langle \omega_n f_n + f_n'', \frac{x f_n'}{\omega_n}\right\rangle_{L^2(0,1)} = \frac{|f_n(1)|^2}{2} - \frac{1}{2}\int_0^1 |f_n(x)|^2 dx + \frac{1}{2\omega_n}|f_n'(1)|^2 - \frac{1}{2\omega_n}\int_0^1 |f_n'(x)|^2 dx,$$

and

$$\left\langle \omega_n f_n + f_n'', \frac{x f_n'}{\omega_n}\right\rangle_{L^2(0,1)} \to 0,$$

we have by (12) and (13) that

$$\int_0^1 |f_n(x)|^2 dx + \frac{1}{\omega_n}\int_0^1 |f_n'(x)|^2 dx \to 0, \tag{15}$$

which shows that when $\omega_n > 0$, $\|f_n\|^2 \to 0$, which contradicts to $\|f_n\|=1$. On the other hand, since from (11) and $\omega_n \to \infty$, we have

which indicates that when $\omega_n > 0$, $\|f_n\|^2 \to 0$. However, this contradicts the given condition that $\|f_n\|=1$. On the other hand, considering (11) and the fact that $\omega_n \to \infty$, we have

$$\begin{aligned}
&\int_0^1 \left| f_n(x) + \frac{f_n''(x)}{\omega_n}\right|^2 dx \\
&= \int_0^1 \left[|f_n(x)|^2 + \frac{|f_n''(x)|^2}{\omega_n^2}\right] dx \\
&\quad + \frac{1}{\omega_n}\int_0^1 [f_n(x)\overline{f_n''(x)} + \overline{f_n(x)} f_n''(x)] dx \\
&= \int_0^1 \left[|f_n(x)|^2 + \frac{|f_n''(x)|^2}{\omega_n^2}\right] dx \\
&\quad + \frac{1}{\omega_n}[f_n(x)\overline{f_n'(x)} + \overline{f_n(x)} f_n'(x)]_0^1 \\
&\quad - \frac{2}{\omega_n}\int_0^1 |f_n'(x)|^2 dx \to 0,
\end{aligned} \tag{16}$$

Substituting $f_n'(1) = -ikf_n(1)$ and $f_n(0)=0$ into (16), and utilizing (12) and (13), we obtain

$$\int_0^1 |f_n(x)|^2 dx + \int_0^1 \frac{|f_n''(x)|^2}{\omega_n^2} dx - \frac{2}{\omega_n}\int_0^1 |f_n'(x)|^2 dx \to 0. \tag{17}$$

This indicates that when $\omega_n < 0$, $\|f_n\|^2 \to 0$ which is also a contradiction. ∎

III. Semi-discrete scheme of Schrödinger equation

In this section, we utilize the order reduction method to derive a semi-discrete scheme for (1). To achieve this, we introduce an intermediate variable $v(x,t) = w_x(x,t)$ to reduce the order of the spatial derivative in (1). By doing so, the Schrödinger equation (1) can be rewritten into the following equivalent form:

$$\begin{cases}
w_t(x,t) + i v_x(x,t) = 0, \ t>0, \ x\in(0,1), \\
v(x,t) = w_x(x,t), \ t>0, \ x\in(0,1), \\
w(0,t) = 0, \ t>0, \\
v(1,t) = -ikw(1,t), \ t>0, \\
w(x,0) = w^0(x), x\in[0,1].
\end{cases} \tag{18}$$

We omit certain details of the semi-discretization process as they are analogous to those in [14]. Instead, we focus on the semi-discretized finite difference scheme of (18), which is given as follows:

$$\begin{cases}
w'_{j+\frac12}(t) + i\delta_x v_{j+\frac12}(t) = 0, \ 0\le j\le N, \\
v_{j+\frac12}(t) = \delta_x w_{j+\frac12}(t), \ 0\le j\le N, \\
v_{N+1}(t) = -ikw_{N+1}(t), \ t\ge 0 \\
w_0(t) = 0, \\
w_j(0) = w_j^0, \ 0\le j\le N+1,
\end{cases} \tag{19}$$

where $v_j(t) \approx v(x_j,t)$ and $w_j(t)\approx w(x_j,t)$ are grid functions evaluated at grid points x_j (with $0\le j\le N+1$), $w_{j+\frac12}$ represents the average operator defined as $w_{j+\frac12} := \frac{w_{j+1}+w_j}{2}$, $\delta_x w_{j+\frac12}$ is the difference operator given by $\delta_x w_{j+\frac12} := \frac{w_{j+1}-w_j}{h}$ with $h(N+1)=1$ and N being a positive integer, and w_j^0 is the approximation of the initial value $w^0(x_j)$.

Now, we eliminate $v_j(t)$ from (19). To achieve this, let $W_h(t) = (w_1(t),\cdots,w_{N+1}(t))^\top$ be the unknown variable of (19), and $V_h(t) = (v_0(t),\cdots,v_N(t))^\top$ be the auxiliary variable. We express (19) in vector form as:

$$\begin{cases}
D_h W_h'(t) = -iM_h V_h(t) - kh^{-1} w_{N+1}(t) e_h, \\
D_h^\top V_h(t) = -M_h^\top W_h(t) + i2^{-1} k w_{N+1}(t) e_h, \\
W_h(0) = (w_0^0, w_1^0, \cdots, w_N^0)^\top,
\end{cases} \tag{20}$$

where $e_h = (0,0,\cdots,0,1)^\top \in \mathbb{C}^{N+1}$ and D_h, M_h are given by

$$D_h = \frac12 \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \end{pmatrix}_{(N+1)\times(N+1)},$$

$$M_h = \frac{1}{h}\begin{pmatrix} -1 & 1 & & \\ & -1 & \ddots & \\ & & \ddots & 1 \\ & & & -1 \end{pmatrix}_{(N+1)\times(N+1)}. \tag{21}$$

Obviously, both D_h and M_h are invertible. System (20) System (20) is naturally discussed in the state space $\mathbb{Y}_h = \mathbb{C}^{N+1}$. To establish a relationship between $\mathbb{Y}_h$ in (20) and the step size, we define a new inner product for $\mathbb{Y}_h$:

$$\left\langle Y_h, \widetilde{Y}_h \right\rangle_{\mathbb{Y}_h} = h\left\langle D_h Y_h, D_h \widetilde{Y}_h \right\rangle, \forall Y_h, \widetilde{Y}_h \in \mathbb{Y}_h,$$

where $\langle \cdot,\cdot \rangle$ is the standard inner product of $\mathbb{C}^{N+1}$. For $Y_h = (y_1, \cdots, y_{N+1})^\top \in \mathbb{Y}_h$, we choose the vector $Z_h = (z_0, \cdots, z_N)^\top \in \mathbb{Y}_h$ satisfying $D_h^\top Z_h = -M_h^\top Y_h + 2^{-1} iky_{N+1} e_h$. We refer to Z_h as the shadow element of Y_h as it significantly simplifies the notation in the subsequent proofs.

System (19) or (20) can be expressed as a system of ordinary differential equations in $\mathbb{Y}_h$:

$$\begin{cases} W_h'(t) = \mathscr{A}_h W_h(t), \quad W_h(t) \in \mathbb{Y}_h, \\ W_h(0) = (w_1^0, w_2^0, \cdots, w_{N+1}^0)^\top \in \mathbb{Y}_h, \end{cases} \tag{22}$$

where the matrices $\mathscr{A}_h$ are defined as

$$\begin{aligned} \mathscr{A}_h Y_h = D_h^{-1}\left[iM_h \left(D_h^\top\right)^{-1} \left(M_h^\top Y_h - i2^{-1} k y_{N+1} e_h\right)\right] \\ - kh^{-1} y_{N+1} D_h^{-1} e_h, \ \forall Y_h = (y_1, \cdots, y_{N+1})^\top \in \mathbb{Y}_h. \end{aligned} \tag{23}$$

The classical semi-discrete scheme is similar to (22), where the average operator $D_h = I_{N+1}$, i.e.,

$$\begin{cases} W_h'(t) = \hat{\mathscr{A}}_h W_h(t), \quad W_h(t) \in \mathbb{Y}_h, \\ W_h(0) = (w_1^0, w_2^0, \cdots, w_{N+1}^0)^\top \in \mathbb{Y}_h, \end{cases} \tag{24}$$

in which the $\hat{\mathscr{A}}_h$ is defined by

$$\begin{aligned} \hat{\mathscr{A}}_h Y_h = iM_h \left(D_h^\top\right)^{-1} \left(M_h^\top Y_h - i2^{-1} k y_{N+1} e_h\right) \\ - kh^{-1} y_{N+1} D_h^{-1} e_h. \end{aligned} \tag{25}$$

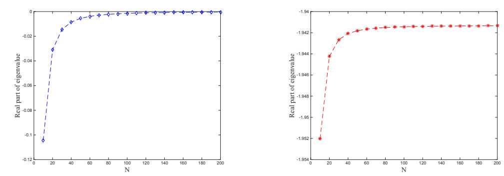
At the end of this section, we highlight the importance of the discrete scheme (22). To illustrate this, we present two figures in Figure 1. The left figure depicts the maximum real parts of the eigenvalues of the classical semi-discrete scheme (22) with the step size h. From this figure, we observe that the real parts of the eigenvalues tend towards zero. The right figure shows the maximum real parts of the eigenvalues of the order reduction semi-discrete scheme (22) using the same step sizes. Here, we see that the real parts of the eigenvalues approach a negative number. For both figures, we set $k = 1$.

IV. Proof of Uniform Exponential Stability

This section focuses on proving the uniform exponential stability of (22). To initiate the proof, we first establish Lemma 4.1, which is partallel to (4).

Lemma 4.1: For the matrix $\mathscr{A}_h$ defined by (23), there holds

$$\mathrm{Re}\,\langle \mathscr{A}_h Y_h, Y_h \rangle_{\mathbb{Y}_h} = -k|y_{N+1}|^2, \ \forall \ Y_h \in \mathbb{Y}_h. \tag{26}$$



(a) classical method (24) (b) order reduction (22)

Fig. 1. Maximal real parts of eigenvalues of the semi-discrete scheme.

Proof. For $Y_h = (y_1, \cdots, y_{N+1})^\top \in \mathbb{Y}_h$, let $Z_h = (z_0 \cdots, z_N)^\top$ be the shadow element of Y_h:

$$\begin{cases} D_h^\top Z_h = -M_h^\top Y_h + i2^{-1} k y_{N+1} e_h, \\ \mathscr{A}_h Y_h = D_h^{-1}[-iM_h Z_h - kh^{-1} y_{N+1} e_h]. \end{cases} \tag{27}$$

Set $y_0 = 0$ and $z_{N+1} = -iky_{N+1}$ and introduce $\widetilde{Y}_h = (\widetilde{y}_1, \cdots, \widetilde{y}_{N+1})^\top \in \mathbb{Y}_h$ such that $\mathscr{A}_h Y_h = \widetilde{Y}_h$ with $\widetilde{y}_0 = 0$. Then, we have $D_h^\top Z_h + 2^{-1} z_{N+1} e_h = -M_h^\top Y_h$, which is equivalent to

$$z_{j+\frac{1}{2}} = \delta_x y_{j+\frac{1}{2}}, \ j = 0, 1, \cdots, N, \tag{28}$$

and $D_h \widetilde{Y}_h = -iM_h Z_h - ih^{-1} z_{N+1} e_h$, which is equivalent to

$$\widetilde{y}_{j+\frac{1}{2}} = -i\delta_x z_{j+\frac{1}{2}}, \ j = 0, 1, \cdots, N. \tag{29}$$

Taking the inner product between $\mathscr{A}_h Y_h$ and Y_h in $\mathbb{Y}_h$ by considering (28) and (29), we obtain

$$\begin{aligned} &\mathrm{Re}\,\langle \mathscr{A}_h Y_h, Y_h \rangle_{\mathbb{Y}_h} = \mathrm{Re}\left\langle \widetilde{Y}_h, Y_h \right\rangle_{\mathbb{Y}_h} \\ &= \frac{h}{2}\left\langle D_h \widetilde{Y}_h, D_h Y_h \right\rangle + \frac{h}{2}\left\langle D_h Y_h, D_h \widetilde{Y}_h \right\rangle \\ &= \frac{h}{2}\sum_{j=0}^{N} \widetilde{y}_{j+\frac{1}{2}} \bar{y}_{j+\frac{1}{2}} + \frac{h}{2}\sum_{j=0}^{N} y_{j+\frac{1}{2}} \bar{\widetilde{y}}_{j+\frac{1}{2}}, \text{ (using (29))} \\ &= -\frac{ih}{2}\sum_{j=0}^{N}\left[\delta_x z_{j+\frac{1}{2}} \bar{y}_{j+\frac{1}{2}} + z_{j+\frac{1}{2}} \delta_x \bar{y}_{j+\frac{1}{2}}\right] \\ &\quad + \frac{ih}{2}\sum_{j=0}^{N}\left[y_{j+\frac{1}{2}} \delta_x \bar{z}_{j+\frac{1}{2}} + \delta_x y_{j+\frac{1}{2}} \bar{z}_{j+\frac{1}{2}}\right]. \text{ (using (28))} \end{aligned} \tag{30}$$

A simple calculation reveals that

$$\begin{aligned} &-\frac{ih}{2}\sum_{j=0}^{N}\left[\delta_x z_{j+\frac{1}{2}} \bar{y}_{j+\frac{1}{2}} + z_{j+\frac{1}{2}} \delta_x \bar{y}_{j+\frac{1}{2}}\right] \\ &\quad + \frac{ih}{2}\sum_{j=0}^{N}\left[y_{j+\frac{1}{2}} \delta_x \bar{z}_{j+\frac{1}{2}} + \delta_x y_{j+\frac{1}{2}} \bar{z}_{j+\frac{1}{2}}\right] \\ &= -\frac{i}{2}\sum_{j=0}^{N}[z_{j+1}\bar{y}_{j+1} - z_j \bar{y}_j] + \frac{i}{2}\sum_{j=0}^{N}[y_{j+1}\bar{z}_{j+1} - y_j \bar{z}_j] \\ &= \frac{i}{2}[z_0 \bar{y}_0 - z_{N+1}\bar{y}_{N+1}] + \frac{i}{2}[y_{N+1}\bar{z}_{N+1} - y_0 \bar{z}_0] \\ &= -k|y_{N+1}|^2. \end{aligned} \tag{31}$$

Using (30) and (31), we can derive (26). ∎

Define the energy of (22) as

$$E_h(t) = \frac{h}{2}\sum_{j=0}^{N}\left|w_{j+\frac{1}{2}}(t)\right|^2 = \frac{1}{2}\langle W_h(t), W_h(t) \rangle_{\mathbb{Y}_h}. \tag{32}$$

which is the discretized version of the continuous energy (6). The following Lemma 4.2 is the discrete counterpart of (7), stemming from (26).

Lemma 4.2: The discrete energy $E_h(t)$ defined by (32) satisfies

$$\dot{E}_h(t) = -k|w_{N+1}(t)|^2. \quad (33)$$

The dissipativity of $\mathscr{A}_h$ ensures that the spectral set $\sigma(\mathscr{A}_h)$ of $\mathscr{A}_h$ lies within the closed left half-plane of the complex plane $\mathbb{C}$. However, we can strengthen this result. Specifically, for any $0<h<1$, the spectral set $\sigma(\mathscr{A}_h)$ of $\mathscr{A}_h$ is contained entirely within the open left half-plane of $\mathbb{C}$. This finding serves as a discrete counterpart of Lemma 2.1.

Lemma 4.3: For every $h\in(0,1)$, $i\mathbb{R}\subset\rho(\mathscr{A}_h)$.

Proof. If there exist $\beta\in\mathbb{R}$ and nonzero $Y_h\in\mathbb{Y}_h$ such that $i\beta Y_h=\mathscr{A}_hY_h$, then it follows from (26) that

$$0=\mathrm{Re}\,\langle i\beta Y_h,\ Y_h\rangle_{\mathbb{Y}_h}=\mathrm{Re}\,\langle \mathscr{A}_hY_h,\ Y_h\rangle_{\mathbb{Y}_h}=-k|y_{N+1}|^2. \quad (34)$$

From $\mathscr{A}_hY_h=i\beta Y_h$, we obtain

$$\begin{cases} \beta y_{j+\frac12}+\delta_x z_{j+\frac12}=0, & 0\le j\le N,\\ z_{j+\frac12}-\delta_x y_{j+\frac12}=0, & 0\le j\le N,\end{cases} \quad (35)$$

where Z_h is the shadow element of Y_h defined in (27), with $y_0:=0$ and $z_{N+1}:=-iky_{N+1}$. From (34), we have $z_{N+1}=y_{N+1}=0$. Substituting $j=N$ into (35) gives $\beta hy_N=2z_N$, $z_N=-\frac{2}{h}y_N$. It follows that $y_N=z_N=0$ whenever βh^2+4 is nonzero. Under the condition $\beta h^2+4\neq0$, $z_{j+1}=y_{j+1}=0$ and (35) imply that $z_j=y_j=0$ for all j. This leads to $Y_h=0$ by induction, which is a contradiction. On the other hand, when $\beta h^2+4=0$, it follows from (35) that

$$\begin{cases} \frac{1}{h}(y_{j+1}+y_j)=\frac12(z_{j+1}-z_j), & j=0,1,\cdots,N,\\ \frac{1}{h}(y_{j+1}-y_j)=\frac12(z_{j+1}+z_j), & j=0,1,\cdots,N.\end{cases} \quad (36)$$

which implies that $y_j=2^{-1}hz_j$ for $j=1,\cdots,N+1$ and $y_j=-2^{-1}hz_j$ for $j=0,\cdots,N$, respectively. This, together with $y_0=0$ and $y_{N+1}=0$, gives $y_j=0$ $(j=1,2,\cdots,N)$ which is also a contradiction. ∎

Lemma 4.4 is brought from [13].

Lemma 4.4: Let $\{u_i\}_i$, $\{v_i\}_i$ and $\{w_i\}_i$ be the sequences of complex numbers. Then,

$$\begin{aligned} &h\sum_{i=0}^{N}\delta_x u_{i+\frac12}v_{i+\frac12}w_{i+\frac12}+\frac{h^3}{4}\sum_{i=0}^{N}\delta_x u_{i+\frac12}\delta_x v_{i+\frac12}\delta_x w_{i+\frac12}\\ &+h\sum_{i=0}^{N}u_{i+\frac12}\delta_x v_{i+\frac12}w_{i+\frac12}+h\sum_{i=0}^{N}u_{i+\frac12}v_{i+\frac12}\delta_x w_{i+\frac12}\\ &=u_{N+1}v_{N+1}w_{N+1}-u_0v_0w_0.\end{aligned}$$

The following uniform stability criterion, which was presented in [15] or [1], will be utilized in the proof of our main result, Theorem 4.2, subsequently.

Theorem 4.1: Let $h^*>0$ and let $\{S_h(t)\}_{h\in(0,h^*)}$ be a family of semigroups of contractions on the Hilbert space H_h, and let $\widetilde{A}_h$ be the corresponding infinitesimal generators. The family $\{S_h(t)\}$ is uniformly exponentially stable if and only if the following two conditions are fulfilled:

- For every $h\in(0,h^*)$, $i\mathbb{R}\subset\rho(\widetilde{A}_h)$;
- $\sup_{h\in(0,h^*),\beta\in\mathbb{R}}\|(i\beta I-\widetilde{A}_h)^{-1}\|<\infty$.

Now, we are in a position to give the main result of this paper.

Theorem 4.2: For the matrices $\mathscr{A}_h$ defined by (23), the associated family of C_0-semigroups $T_h(t)$ generated by $\mathscr{A}_h$ is uniformly exponentially stable. Specifically, there exist two constants $M>0$ and $\omega>0$, both independent of $h\in(0,1)$, such that

$$\|T_h(t)\|\le Me^{-\omega t},\ \forall t\ge0. \quad (37)$$

Proof. By virtue of Lemma 4.1, for every $h\in(0,1)$, $T_h(t)$ is a C_0-semigroup of contractions. Lemma 4.3 has already established that $\mathscr{A}_h$ satisfies the first condition of Theorem 4.1. To demonstrate that the family $\mathscr{A}_h$ also satisfies the second condition of Theorem 4.1, we proceed by contradiction. If the second condition of Theorem 4.1 is false, then there exists a sequence $\{(\beta_n,h_n,Y^n_{h_n})\}_{n\in\mathbb{N}^+}$ with $\beta_n\in\mathbb{R}$, $h_n\in(0,1)$, and $Y^n_{h_n}\in\mathbb{Y}_{h_n}$ such that

$$\begin{cases} \|Y^n_{h_n}\|_{\mathbb{Y}_{h_n}}=1,\\ U^n_{h_n}:=(i\beta_nI_{h_n}-\mathscr{A}_{h_n})Y^n_{h_n},\\ \|U^n_{h_n}\|_{\mathbb{Y}_{h_n}}\le n^{-1}.\end{cases} \quad (38)$$

Utilizing the Cauchy-Schwarz inequality, we deduce from (38) and (26) that

$$k|y^n_{N_n+1}|^2=\mathrm{Re}\,\langle U^n_{h_n},Y^n_{h_n}\rangle_{\mathbb{Y}_{h_n}}\le n^{-1}. \quad (39)$$

Let $Z^n_{h_n}=(z^n_0,\cdots,z^n_{N_n})^\top\in\mathbb{Z}_{h_n}$ be the shadow element of $Y^n_{h_n}=(y^n_1,\cdots,y^n_{N_n+1})^\top$ with $N_n+1=[1/h_n]$ (where $[a]$ denotes the largest integer less than or equal to the real number a). The vector $U^n_{h_n}$ is given by $U^n_{h_n}=(u^n_1,\cdots,u^n_{N_n+1})^\top$. To unify the notation for $u^n_{j+\frac12}$ and $\delta_xz^n_{j+\frac12}$ from $j=0,1,\cdots,N_n$, we artificially set $u^n_0=y^n_0=0$ and $z^n_{N_n+1}=-iky^n_{N_n+1}$. Then, from the second identity of (38), we have

$$\begin{cases} D_{h_n}U^n_{h_n}=i\beta_nD_{h_n}Y^n_{h_n}+iM_{h_n}Z^n_{h_n}+ih^{-1}z^n_{N_n+1}e_{h_n},\\ -M^\top_{h_n}Y^n_{h_n}=D^\top_{h_n}Z^n_{h_n}+2^{-1}z^n_{N_n+1}e_{h_n},\end{cases} \quad (40)$$

or in vector form:

$$\begin{cases} \begin{pmatrix} u^n_{0+\frac12}\\ \vdots\\ u^n_{N_n+\frac12}\end{pmatrix}=i\beta_n\begin{pmatrix} y^n_{0+\frac12}\\ \vdots\\ y^n_{N_n+\frac12}\end{pmatrix}+i\begin{pmatrix} \delta_xz^n_{0+\frac12}\\ \vdots\\ \delta_xz^n_{N_n+\frac12}\end{pmatrix},\\ \begin{pmatrix} z^n_{0+\frac12}\\ \vdots\\ z^n_{N_n+\frac12}\end{pmatrix}=\begin{pmatrix} \delta_xy^n_{0+\frac12}\\ \vdots\\ \delta_xy^n_{N_n+\frac12}\end{pmatrix}.\end{cases} \quad (41)$$

The proof will be split into three claims. Claim 1 corresponds to $\omega_n\to\infty$ in the proof of Theorem 2.1.

Claim 1: $|\beta_n|\ge C'>0$ for some constant C' independent of $n\in\mathbb{N}^+$.

Assuming the contrary, let us suppose that the sequence $\{\beta_n\}$ contains a subsequence, which we shall denote by $\{\beta_n\}$ itself for simplicity, that converges to zero. Since we can always extract a subsequence with any desired convergence rate from a sequence converging to zero, we may further assume, without loss of generality, that $|\beta_n| \le n^{-1}$. Given that $\|Y^n_{h_n}\|_{\mathbb{Y}_{h_n}} = 1$ and $\|U^n_{h_n}\|_{\mathbb{Y}_{h_n}} \le n^{-1}$, it follows from (41) that

$$\begin{aligned} h_n \sum_{j=0}^{N_n} \left|\delta_x z^n_{j+\frac{1}{2}}\right|^2 &= h_n \sum_{j=0}^{N_n} \left|u^n_{j+\frac{1}{2}} - i\beta_n y^n_{j+\frac{1}{2}}\right|^2 \\ &\le 2h_n \sum_{j=0}^{N_n} \left|u^n_{j+\frac{1}{2}}\right|^2 + 2\beta_n^2 h_n \sum_{j=0}^{N_n} \left|y^n_{j+\frac{1}{2}}\right|^2 \\ &= 2\|U^n_{h_n}\|^2_{\mathbb{Y}_{h_n}} + 2\beta_n^2 \|Y^n_{h_n}\|^2_{\mathbb{Y}_{h_n}} \\ &\le 4n^{-2}. \end{aligned} \tag{42}$$

On the other hand, through some simple operations, we obtain

$$\begin{aligned} &|z^n_j - z^n_{N_n+1}|^2 = |z^n_j - z^n_{j+1} + z^n_{j+1} - z^n_{j+2} + \cdots - z^n_{N_n+1}|^2 \\ &= \left|\sum_{l=j}^{N_n} (z^n_{l+1} - z^n_l)\right|^2 \le \left(\sum_{l=j}^{N_n} |1|^2\right)\left(\sum_{l=j}^{N_n} |z^n_{l+1} - z^n_l|^2\right) \\ &\le (N_n+1)\left(\sum_{l=0}^{N_n} |z^n_{l+1} - z^n_l|^2\right) \\ &\le h_n \sum_{j=1}^{N_n} \left|\delta_x z^n_{j+\frac{1}{2}}\right|^2, \quad j = 0,1,\cdots,N_n, \end{aligned} \tag{43}$$

where $h_n(N_n+1) \le 1$ was used in the last step, and for $j = 0,1,\cdots,N_n$

$$|z^n_j| \le |z^n_j - z^n_{N_n+1}| + |z^n_{N_n+1}| \le \sqrt{h_n \sum_{j=1}^{N_n} |\delta_x z^n_{j+\frac{1}{2}}|^2} + |z^n_{N_n+1}|.$$

This inequality, along with $z^n_{N_n+1} = -iky^n_{N_n+1}$ and equations (39)-(42), implies that for each $j = 0,1,\cdots,N_n$, we have $|z^n_j|^2 = \mathscr{O}(n^{-1})$. Here, the notation $s_n = \mathscr{O}(n^{-1})$ signifies that there exists a positive constant C such that $|s_n| \le Cn^{-1}$ holds for all $n \in \mathbb{N}^+$. Therefore, considering the fact that $h_n(N_n+1) \le 1$ and equation (42), we arrive at

$$\begin{aligned} h_n \sum_{j=0}^{N_n} \left|z^n_{j+\frac{1}{2}}\right|^2 &\le \frac{h_n}{2} \sum_{j=0}^{N_n} \left(|z^n_{j+1}|^2 + |z^n_j|^2\right) \\ &\le \left|\sqrt{h_n \sum_{j=1}^{N_n} |\delta_x z^n_{j+\frac{1}{2}}|^2} + |z^n_{N_n+1}|\right|^2 = \mathscr{O}(n^{-1}). \end{aligned} \tag{44}$$

The deduction from equation (42) leads us to the conclusion that

$$h_n \sum_{j=0}^{N_n} \left|\delta_x z^n_{j+\frac{1}{2}}\right|^2 = \mathscr{O}(n^{-2}),$$

which, in turn, suggests that

$$h_n \sum_{j=0}^{N_n} \left|z^n_{j+\frac{1}{2}}\right|^2 = \mathscr{O}(n^{-1}).$$

Observing the second identity in (41), we obtain

$$h_n \sum_{j=0}^{N_n} \left|\delta_x y^n_{j+\frac{1}{2}}\right|^2 = h_n \sum_{j=0}^{N_n} \left|z^n_{j+\frac{1}{2}}\right|^2,$$

which, by virtue of (44), implies that

$$h_n \sum_{j=0}^{N_n} \left|\delta_x y^n_{j+\frac{1}{2}}\right|^2 = \mathscr{O}(n^{-1}).$$

Similarly, by replicating the steps from (42) to (44) for $Y^n_{h_n}$, we arrive at

$$\|Y^n_{h_n}\|^2_{\mathbb{Y}_{h_n}} = h_n \sum_{j=0}^{N_n} \left|y^n_{j+\frac{1}{2}}\right|^2 = \mathscr{O}(n^{-1/2}),$$

which introduces a contradiction. Consequently, the sequence $\{\beta_n\}$ cannot possess a subsequence converging to zero. Therefore, we can confidently assert that $|\beta_n| \ge C' > 0$ for some constant C' that is independent of $n \in \mathbb{N}^+$.

The second claim, which plays a crucial role in our proofs, is the discrete counterpart of (15), albeit with two additional terms

$$\frac{h_n^3}{4\beta_n} \sum_{j=0}^{N_n} \left|\delta_x z^n_{j+\frac{1}{2}}\right|^2 + \frac{h_n^3}{4} \sum_{j=0}^{N_n} \left|\delta_x y^n_{j+\frac{1}{2}}\right|^2.$$

Claim 2: The following (45) holds true:

$$\begin{aligned} &\|Y^n_{h_n}\|^2_{\mathbb{Y}_{h_n}} + \frac{1}{\beta_n} h_n \|\Sigma_{h_n} \widehat{Z}^n_{h_n}\|^2_{\mathbb{C}^{N_n+2}} \\ &+ \frac{h_n^2}{4\beta_n} h_n \|\Delta_{h_n} \widehat{Z}^n_{h_n}\|^2_{\mathbb{C}^{N_n+2}} + \frac{h_n^2}{4} h_n \|\Delta_{h_n} \widehat{Y}^n_{h_n}\|^2_{\mathbb{C}^{N_n+2}} \\ &= h_n \sum_{j=0}^{N_n} \left|y^n_{j+\frac{1}{2}}\right|^2 + \frac{h_n}{\beta_n} \sum_{j=0}^{N_n} \left|z^n_{j+\frac{1}{2}}\right|^2 \\ &+ \frac{h_n^3}{4\beta_n} \sum_{j=0}^{N_n} \left|\delta_x z^n_{j+\frac{1}{2}}\right|^2 + \frac{h_n^3}{4} \sum_{j=0}^{N_n} \left|\delta_x y^n_{j+\frac{1}{2}}\right|^2 \\ &= \mathscr{O}(n^{-1}), \end{aligned} \tag{45}$$

where $\|\cdot\|_{\mathbb{C}^{N_n+2}}$ denotes the standard norm of $\mathbb{C}^{N_n+2}$ and

$$\begin{aligned} \widehat{Z}^n_{h_n} &= (z_0, z_1, \cdots, z_{N_n+1})^\top = \left((Z^n_{h_n})^\top, z_{N_n+1}\right)^\top, \\ \widehat{Y}^n_{h_n} &= (0, y_1, \cdots, y_{N_n+1})^\top = \left(0, (Y^n_{h_n})^\top\right)^\top, \\ \Sigma_{h_n} &= \frac{1}{2}\begin{pmatrix} 1 & 1 & & & \\ & \ddots & \ddots & & \\ & & 1 & 1 & \\ & & & 1 & 1 \end{pmatrix}_{(N_n+2)\times(N_n+1)}, \\ \Delta_{h_n} &= \frac{1}{h}\begin{pmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{pmatrix}_{(N_n+2)\times(N_n+1)}. \end{aligned} \tag{46}$$

In fact, using (38), (41), and the fact that $\|Y^n_{h_n}\|_{\mathbb{Y}_{h_n}} = 1$, we can deduce that $\beta_n^{-2} h_n \sum_{j=0}^{N_n} |\delta_x z^n_{j+\frac{1}{2}}|^2$ is uniformly bounded with respect to $n \in \mathbb{N}^+$ because

$$\frac{1}{\beta_n}\begin{pmatrix} \delta_x z^n_{0+\frac{1}{2}} \\ \vdots \\ \delta_x z^n_{N_n+\frac{1}{2}} \end{pmatrix} = -\begin{pmatrix} y^n_{0+\frac{1}{2}} \\ \vdots \\ y^n_{N_n+\frac{1}{2}} \end{pmatrix} - \frac{i}{\beta_n}\begin{pmatrix} u^n_{0+\frac{1}{2}} \\ \vdots \\ u^n_{N_n+\frac{1}{2}} \end{pmatrix}.$$

Using (44) and **Claim 1**, we can conclude that $\beta_n^{-2}h_n\sum_{j=0}^{N_n}|z^n_{j+\frac{1}{2}}|^2$ is also uniformly bounded with respect to $n\in\mathbb{N}^+$. Now, let $x^n_j=jh_n$ for $j=0,1,\cdots,N_n+1$ and consider the following estimates:

$$\begin{aligned}
&\left|h_n\sum_{j=0}^{N_n}x^n_{j+\frac{1}{2}}(\beta_n y^n_{j+\frac{1}{2}}+\delta_x z^n_{j+\frac{1}{2}})\frac{\overline{z}^n_{j+\frac{1}{2}}}{\beta_n}\right|^2\\
&=\left|h_n\sum_{j=0}^{N_n}x^n_{j+\frac{1}{2}}u^n_{j+\frac{1}{2}}\frac{\overline{z}^n_{j+\frac{1}{2}}}{\beta_n}\right|^2\\
&\le\left(\sum_{j=0}^{N_n}\left|\sqrt{h_n}u^n_{j+\frac{1}{2}}\right|\left|\sqrt{h_n}\frac{z^n_{j+\frac{1}{2}}}{\beta_n}\right|\right)^2\\
&\le\left(h_n\sum_{j=0}^{N_n}\left|u^n_{j+\frac{1}{2}}\right|^2\right)\left(\beta_n^{-2}h_n\sum_{j=0}^{N_n}\left|z^n_{j+\frac{1}{2}}\right|^2\right)\\
&=\|U^n_{h_n}\|^2_{\mathbb{Y}_{h_n}}\left(\beta_n^{-2}h_n\sum_{j=0}^{N_n}\left|z^n_{j+\frac{1}{2}}\right|^2\right)\\
&=\mathscr{O}(n^{-2}),
\end{aligned}\tag{47}$$

where (38) and (41) were utilized. Additionally, by the second identity in (41), we obtain

$$\begin{aligned}
&h_n\sum_{j=0}^{N_n}x^n_{j+\frac{1}{2}}(\beta_n y^n_{j+\frac{1}{2}}+\delta_x z^n_{j+\frac{1}{2}})\frac{\overline{z}^n_{j+\frac{1}{2}}}{\beta_n}\\
&=h_n\sum_{j=0}^{N_n}x^n_{j+\frac{1}{2}}\left[y^n_{j+\frac{1}{2}}\delta_x\overline{y}^n_{j+\frac{1}{2}}+\beta_n^{-1}\delta_x z^n_{j+\frac{1}{2}}\overline{z}^n_{j+\frac{1}{2}}\right].
\end{aligned}\tag{48}$$

By applying Lemma 4.4 to the two terms on the right-hand side of (48) and observing that $x^n_{N_n+1}=1$, $x^n_0=0$, $x^n_{j+1}-x^n_j=h_n$, it is straightforward to derive

$$\begin{aligned}
&2\mathrm{Re}\left(h_n\sum_{j=0}^{N_n}x^n_{j+\frac{1}{2}}y^n_{j+\frac{1}{2}}\delta_x\overline{y}^n_{j+\frac{1}{2}}\right)\\
&=h_n\sum_{j=0}^{N_n}x^n_{j+\frac{1}{2}}y^n_{j+\frac{1}{2}}\delta_x\overline{y}^n_{j+\frac{1}{2}}+h_n\sum_{j=0}^{N_n}x^n_{j+\frac{1}{2}}\overline{y}^n_{j+\frac{1}{2}}\delta_x y^n_{j+\frac{1}{2}}\\
&=|y_{N_n+1}|^2-h_n\sum_{j=0}^{N_n}\left|y^n_{j+\frac{1}{2}}\right|^2-\frac{h_n^3}{4}\sum_{j=0}^{N_n}\left|\delta_x y^n_{j+\frac{1}{2}}\right|^2,
\end{aligned}$$

and

$$\begin{aligned}
&2\mathrm{Re}\left(h_n\sum_{j=0}^{N_n}x^n_{j+\frac{1}{2}}z^n_{j+\frac{1}{2}}\delta_x\overline{z}^n_{j+\frac{1}{2}}\right)\\
&=h_n\sum_{j=0}^{N_n}x^n_{j+\frac{1}{2}}z^n_{j+\frac{1}{2}}\delta_x\overline{z}^n_{j+\frac{1}{2}}+h_n\sum_{j=0}^{N_n}x^n_{j+\frac{1}{2}}\overline{z}^n_{j+\frac{1}{2}}\delta_x z^n_{j+\frac{1}{2}}\\
&=|z_{N_n+1}|^2-h_n\sum_{j=0}^{N_n}\left|z^n_{j+\frac{1}{2}}\right|^2-\frac{h_n^3}{4}\sum_{j=0}^{N_n}\left|\delta_x z^n_{j+\frac{1}{2}}\right|^2.
\end{aligned}$$

Using (48) and the two inequalities above, we can deduce that

$$\begin{aligned}
&h_n\sum_{j=0}^{N_n}\left|y^n_{j+\frac{1}{2}}\right|^2+\frac{h_n^3}{4}\sum_{j=0}^{N_n}\left|\delta_x y^n_{j+\frac{1}{2}}\right|^2\\
&+\frac{h_n}{\beta_n}\sum_{j=0}^{N_n}\left|z^n_{j+\frac{1}{2}}\right|^2+\frac{h_n^3}{4\beta_n}\sum_{j=0}^{N_n}\left|\delta_x z^n_{j+\frac{1}{2}}\right|^2\\
&=-2\mathrm{Re}\left(h_n\sum_{j=0}^{N_n}x^n_{j+\frac{1}{2}}(\beta_n y^n_{j+\frac{1}{2}}+\delta_x z^n_{j+\frac{1}{2}})\frac{\overline{z}^n_{j+\frac{1}{2}}}{\beta_n}\right)\\
&+|y_{N_n+1}|^2+\beta_n^{-1}|z_{N_n+1}|^2,
\end{aligned}\tag{49}$$

which verifies (45) by virtue of (39), (47), and the fact that $z_{N_n+1}=-iky_{N_n+1}$.

The third claim is indeed the precise discrete counterpart of (17).

Claim 3: The following (50) holds true:

$$\begin{aligned}
&\|Y^n_{h_n}\|^2_{\mathbb{Y}_{h_n}}+\frac{1}{\beta_n^2}h_n\|\Delta_{h_n}\widehat{Z}^n_{h_n}\|^2_{\mathbb{C}^{N_n+2}}-\frac{2}{\beta_n}h_n\|\Sigma_{h_n}\widehat{Z}^n_{h_n}\|^2_{\mathbb{C}^{N_n+2}}\\
&=h_n\sum_{j=0}^{N_n}\left|y^n_{j+\frac{1}{2}}\right|^2+\frac{h_n}{\beta_n^2}\sum_{j=0}^{N_n}\left|\delta_x z^n_{j+\frac{1}{2}}\right|^2-\frac{2h_n}{\beta_n}\sum_{j=0}^{N_n}\left|z^n_{j+\frac{1}{2}}\right|^2\\
&=\mathscr{O}(n^{-2}),
\end{aligned}\tag{50}$$

where Σ_{h_n} and Δ_{h_n} are defined in (46).

Actually, from (41), we have

$$\begin{aligned}
\frac{\|U^n_{h_n}\|^2_{\mathbb{Y}_{h_n}}}{\beta_n^2}&=\frac{h_n}{\beta_n^2}\sum_{j=0}^{N_n}\left|u^n_{j+\frac{1}{2}}\right|^2=h_n\sum_{j=0}^{N_n}\left|y^n_{j+\frac{1}{2}}+\frac{\delta_x z^n_{j+\frac{1}{2}}}{\beta_n}\right|^2\\
&=h_n\sum_{j=0}^{N_n}\left|y^n_{j+\frac{1}{2}}\right|^2+\frac{h_n}{\beta_n^2}\sum_{j=0}^{N_n}\left|\delta_x z^n_{j+\frac{1}{2}}\right|^2\\
&\quad+\frac{h_n}{\beta_n}\sum_{j=0}^{N_n}(\overline{y}^n_{j+\frac{1}{2}}\delta_x z^n_{j+\frac{1}{2}}+y^n_{j+\frac{1}{2}}\delta_x\overline{z}^n_{j+\frac{1}{2}}).
\end{aligned}\tag{51}$$

On the other hand, utilizing the second identity in (41), we obtain $z^n_{j+\frac{1}{2}}=\delta_x y^n_{j+\frac{1}{2}}$ and

$$\begin{aligned}
&\frac{h_n}{\beta_n}\sum_{j=0}^{N_n}(\overline{y}^n_{j+\frac{1}{2}}\delta_x z^n_{j+\frac{1}{2}}+y^n_{j+\frac{1}{2}}\delta_x\overline{z}^n_{j+\frac{1}{2}})+\frac{2h_n}{\beta_n}\sum_{j=0}^{N_n}\left|z^n_{j+\frac{1}{2}}\right|^2\\
&=\frac{h_n}{\beta_n}\sum_{j=0}^{N_n}(\overline{y}^n_{j+\frac{1}{2}}\delta_x z^n_{j+\frac{1}{2}}+\delta_x\overline{y}^n_{j+\frac{1}{2}}z^n_{j+\frac{1}{2}})\\
&\quad+\frac{h_n}{\beta_n}\sum_{j=0}^{N_n}(\delta_x y^n_{j+\frac{1}{2}}\overline{z}^n_{j+\frac{1}{2}}+y^n_{j+\frac{1}{2}}\delta_x\overline{z}^n_{j+\frac{1}{2}})\\
&=\frac{1}{2\beta_n}\sum_{j=0}^{N_n}[(\overline{y}^n_{j+1}+\overline{y}^n_j)(z^n_{j+1}-z^n_j)+(\overline{y}^n_{j+1}-\overline{y}^n_j)(z^n_{j+1}+z^n_j)]\\
&\quad+\frac{1}{2\beta_n}\sum_{j=0}^{N_n}[(y^n_{j+1}+y^n_j)(\overline{z}^n_{j+1}-\overline{z}^n_j)+(y^n_{j+1}-y^n_j)(\overline{z}^n_{j+1}+\overline{z}^n_j)]\\
&=\frac{1}{\beta_n}\sum_{j=0}^{N_n}(\overline{y}^n_{j+1}z^n_{j+1}-\overline{y}^n_j z^n_j)+\frac{1}{\beta_n}\sum_{j=0}^{N_n}(y^n_{j+1}\overline{z}^n_{j+1}-y^n_j\overline{z}^n_j)\\
&=\frac{1}{\beta_n}[\overline{y}^n_{N_n+1}z^n_{N_n+1}+y^n_{N_n+1}\overline{z}^n_{N_n+1}-\overline{y}^n_0 z^n_0-y^n_0\overline{z}^n_0]=0,
\end{aligned}$$

where $z^n_{N_n+1} = -iky^n_{N_n+1}$ and $y^n_0 = 0$ were used in the last step. Hence

$$\frac{h_n}{\beta_n}\sum_{j=0}^{N_n}(\bar{y}^n_{j+\frac{1}{2}}\delta_x z^n_{j+\frac{1}{2}} + y^n_{j+\frac{1}{2}}\delta_x \bar{z}^n_{j+\frac{1}{2}}) = -\frac{2h_n}{\beta_n}\sum_{j=0}^{N_n}\left|z^n_{j+\frac{1}{2}}\right|^2. \quad (52)$$

Substituting (52) into (51) and utilizing the bounds $\|U^n_{h_n}\|_{\mathbb{Y}_{h_n}} \le n^{-1}$ and **Claim 1**, we arrive at the desired conclusion (50).

Finally, if $\beta_n > 0$, we obtain $\|Y^n_{h_n}\|^2_{\mathbb{Y}_{h_n}} = \mathscr{O}(n^{-1})$ from (45), which contradicts the given condition $\|Y^n_{h_n}\|_{\mathbb{Y}_{h_n}} = 1$. Similarly, when $\beta_n < 0$, we have $\|Y^n_{h_n}\|_{\mathbb{Y}_{h_n}} = \mathscr{O}(n^{-1})$ due to (50), which is also a contradiction. Thus, we have successfully completed the proof of the theorem.

V. Concluding remarks

In this paper, we explore the uniform approximation of exponential stability for a one-dimensional Schrödinger equation. We introduce an order-reduced, space semi-discretized finite difference scheme to uniformly approximate the exponentially stable closed-loop system. While this scheme has been applied to certain partial differential equations (PDEs) in previous studies, a common feature is the ability to find a suitable Lyapunov functional for both the continuous and discretized closed-loop systems. However, for the system considered here, it has been a long-standing challenge to find a time-domain energy multiplier, even for the continuous system in the natural state space $L^2(0,1)$. This has long prevented the convergence of the semi-discrete scheme for this PDE from being established. Our work is the first to apply the frequency-domain multiplier approach to demonstrate the uniform exponential convergence of a semi-discretized PDE system. The convergence of the discrete scheme to the continuous system is not included here, as it follows a standard procedure and can be analogously derived from previous works, such as [14] and others. Given the difficulty in finding time-domain energy multipliers for many other PDEs, the approach presented in this paper has the potential to be widely applicable to other PDE systems.

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