On convergence of non-linear extended state observer for multi-input multi-output systems with uncertainty

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Abstract: In this study, the convergence of non-linear extended state observer (ESO) for a class of multi-input multi-output non-linear systems with uncertainty is studied. The unknown part that comes from either the system itself or the external disturbance is considered as an augmented state. The state variable and augmented state are estimated simultaneously through the ESO, the set of all continuous functions from $\mathbb{R}^{n+1}$ to $\mathbb{R}$, the set of all continuous functions

1 Introduction

For an observable control system, the design and convergence of the state observer have been a big issue in understanding the system behaviour from the measurement as well as in the control design and fault diagnosis. There are huge literatures on this aspect. Examples can be found in [1–6]. Of special interest is the extended state observer (ESO) proposed in [7]. Roughly speaking, the ESO copes with the uncertainty coming from either the system itself or from the external disturbance. In this seminal idea, the uncertain part is considered as an augmented state and is estimated through the observer. The ESO is thus regarded as the major step towards the active disturbance rejection control [8]. The first ESO is designed in [7] as

\[
\begin{align*}
\dot{x}_1(t) &= \dot{x}_2(t) - \alpha_1 g_1(x_1(t) - y(t)) \\
\dot{x}_2(t) &= \dot{x}_3(t) - \alpha_2 g_2(x_1(t) - y(t)) \\
&\vdots \\
\dot{x}_n(t) &= \dot{x}_{n+1}(t) - \alpha_n g_n(x_1(t) - y(t)) + u(t) \\
\dot{x}_{n+1}(t) &= -\alpha_{n+1} g_{n+1}(x_1(t) - y(t))
\end{align*}
\]

which is for a general n-dimensional single input and single output (SISO) non-linear system of the following

\[
\begin{align*}
x(t) &= f(t, x(t), x(t), \ldots, x^{(n-1)}(t)) + w(t) + u(t) \\
y(t) &= x(t)
\end{align*}
\]

that can be written as

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= x_3(t) \\
&\vdots \\
\dot{x}_n(t) &= f(t, x_1(t), x_2(t), \ldots, x_n(t)) + w(t) + u(t) \\
y(t) &= x_1(t)
\end{align*}
\]

where $u \in C(\mathbb{R}, \mathbb{R})$, the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$, is the input (control), $y$ the output (measurement), $f \in C(\mathbb{R}^{n+1}, \mathbb{R})$, the set of all continuous functions from $\mathbb{R}^{n+1}$ to $\mathbb{R}$, a possibly unknown system function, and $w \in C(\mathbb{R}, \mathbb{R})$ the uncertain external disturbance. $f + w$ is called ‘total disturbance’, and $\alpha_i, i = 1, 2, \ldots, n + 1$ are regulable gain constants. The main idea of ESO is that for some appropriately chosen functions $g_i \in C(\mathbb{R}, \mathbb{R})$ (the linear functions are usually trivially to be the candidates), the state of the observer $\hat{x}_i$ for $i = 1, 2, \ldots, n$ and $\hat{x}_{n+1}$ can be, through regulating $\alpha_i$, considered as the approximations of the corresponding states $x_i$ for $i = 1, 2, \ldots, n$, and the total disturbance $f + w$, respectively. For the basic background of ESO, we refer to the elaborate paper [8]. The earlier similar works in dealing with disturbances with partial prior knowledge are the method under the name ‘external model’; see for instance [9, 10].

Although numerous numerical simulations and engineering practices (see e.g. [7, 8, 11, 12]) have witnessed the marvellous success of ESO, the study of its convergence is progressing slowly. The convergence of linear ESO for SISO...
system is investigated firstly in [13] and subsequently in [14]. The convergence of non-linear ESO for SISO system is available only very recently [15]. In this paper, we are concerned with the ESO for a class of multi-input multi-output (MIMO) non-linear systems as follows (see (3))

where \( n_i \in \mathbb{Z}, f_i \in C(\mathbb{R}^{n_1 + n_2 + \cdots + n_m + 1}, \mathbb{R}) \) represents the system function, \( w_i \in C(\mathbb{R}, \mathbb{R}) \) the external disturbance, \( u_i \) the control (input), \( y_i \) the observation (output), \( g_i \in C(\mathbb{R}^k, \mathbb{R}) \) the control (input), \( w_i \) and some \( w_i \in C(\mathbb{R}, \mathbb{R}) \), the corresponding ESO is not \( \varepsilon \)-exact for all \( Z \). This paper is organised as follows. In Section 2, we construct the ESO for system (3) where all \( f_i \) and \( w_i \) are unknown. In this case, the ESO is not only to estimate the state but also the augmented state \( f_i(x_1(t), \ldots, x_{n_1}(t), \ldots, x_{n_m}(t), w_1(t)) \). The convergence is presented rigorously. A special ESO where all \( f_i \) are known but all \( w_i \) are unknown is considered in Section 3. The difference of Section 3 with Section 2 lies in that we try to make use of the known information of \( f_i \) as much as possible in the design of the ESO. That is, in Section 3, the augmented state means only for the extended disturbance \( w_i \). Finally, in Section 4, as a practical application, the current control for permanent-magnet synchronous motor is studied numerically to indicate the applicability of the ESO.

### 2 Extended state observer for systems with total disturbance

We first transform (3) into a first-order system described by \( m \) number of subsystems of the first order differential equations (see (4))

\[
\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= x_3(t) \\
&\vdots \\
\dot{x}_m(t) &= x_{m+1}(t) + g_m(u_1(t), \ldots, u_k(t)) \\
y_i(t) &= x_i(t), \quad i = 1, 2, \ldots, m
\end{aligned}
\]

The corresponding ESO for system (4) is also composed of \( m \) subsystems (see (5))

\[
\begin{aligned}
\dot{\hat{x}}_{i,1}(t) &= \hat{x}_{i,2}(t) + e^{n_i-1} \phi_{i,1} \left( \frac{x_{1,1}(t) - \hat{x}_{1,1}(t)}{e^{n_i}} \right), \\
\dot{\hat{x}}_{i,2}(t) &= \hat{x}_{i,3}(t) + e^{n_i-2} \phi_{i,2} \left( \frac{x_{1,2}(t) - \hat{x}_{1,2}(t)}{e^{n_i}} \right) \\
&\vdots \\
\dot{\hat{x}}_{i,m}(t) &= \hat{x}_{i,m+1}(t) + \phi_{i,m} \left( \frac{x_{1,m}(t) - \hat{x}_{1,m}(t)}{e^{n_i}} \right) + g_i(u_1, u_2, \ldots, u_k) \\
\dot{\hat{x}}_{i,m+1}(t) &= \frac{1}{e^{n_i}} \phi_{i,m+1} \left( \frac{x_{1,m+1}(t) - \hat{x}_{1,m+1}(t)}{e^{n_i}} \right), \quad i = 1, 2, \ldots, m
\end{aligned}
\]
Theorem 1: Assume Assumptions (A1) and (A2). Then for any given initial values of (4) and (5), it has

(i) For any given \( a > 0 \)
\[
\lim_{t \to \infty} |x_{ij}(t) - \tilde{x}_{ij}(t)| = 0 \text{ uniformly for } t \in [a, \infty)
\]

(ii) For any \( \varepsilon \in (0, 1) \)
\[
|x_{ij}(t) - \tilde{x}_{ij}(t)| \leq K_\varepsilon e^{n_{i+1-j}}, \quad \forall t \geq \frac{2\lambda_{i,2}(n_i + 1)\varepsilon \ln \varepsilon^{-1}}{\lambda_{i,3}}
\]

where \( K_\varepsilon = \sqrt{\frac{\sum_{i=1}^n |x_{ij}(0) - \tilde{x}_{ij}(0)|^2}{\lambda_{i,1}}} \varepsilon x_{ij}, j = 1, 2, \ldots, n_i + 1, i = 1, 2, \ldots, m, \) are the solutions of (5), \( x_{ij}, j = 1, 2, \ldots, n_i, i = 1, 2, \ldots, m, \) the solutions of (4), \( e_{ij} = f(x_{1,i}, \ldots, x_{n_i}, \ldots, x_{m_i}, w_i) \) the augmented state, and \( K_\varepsilon \) are positive constants independent of \( \varepsilon \) but depending on initial values.

Proof: We first note that (see (6))

From Assumption (A1), \(|\Delta_i(t)| \leq M|\) is uniformly bounded for some \( M > 0 \) and all \( t \geq 0 \), \( i \in [1, 2, \ldots, m] \).

For every \( j = 1, 2, \ldots, n_i + 1, i = 1, 2, \ldots, m, \) set
\[
e_{ij}(t) = x_{ij}(t) - \tilde{x}_{ij}(t), \quad \eta_{ij}(t) = \frac{e_{ij}(t)}{e^{n_{i+1-j}}}, \quad \eta_i = (\eta_{1,i}, \ldots, \eta_{n_i,i})^T
\]

It follows from (4) and (5) that for every \( i \in [1, 2, \ldots, m], j \in [1, 2, \ldots, n_i], \eta_{ij} \) satisfies
\[
d\eta_{ij}(t) \leq \frac{d}{dt} \left( \frac{x_{ij}(t) - \tilde{x}_{ij}(t)}{e^{n_{i+1-j}}} \right) = \frac{x_{ij+1}(t) - \tilde{x}_{ij+1}(t)}{e^{n_{i+1-j}}} - \phi_j(\eta_{ij}(t)) = \eta_{ij+1}(t) - \phi_j(\eta_{ij}(t))
\]

and for every \( i \in [1, 2, \ldots, m], \)
\[
d\eta_{i,n_i+1}(t) \leq \varepsilon(\tilde{x}_{i,n_i+1}(t) - \tilde{x}_{i,n_i+1}(t)) = -\phi_{i,n_i+1}(\eta_{i,n_i+1}(t)) + \varepsilon \Delta_i(t)
\]

We then put all these equation together into the following differential equations satisfied by \( \eta_{ij} \).

\[
\begin{align*}
\dot{\eta}_{1,1}(t) &= \eta_{1,2}(t) - \phi_1(\eta_{1,1}(t)), \quad \eta_{1,1}(0) = e_{1,0}(0), \\
\dot{\eta}_{1,2}(t) &= \eta_{1,3}(t) - \phi_2(\eta_{1,2}(t)), \quad \eta_{1,2}(0) = e_{2,0}(0), \\
&\vdots \\
\dot{\eta}_{i,n_i+1}(t) &= \eta_{i,n_i+1}(t) - \phi_i(\eta_{i,n_i+1}(t)), \quad \eta_{i,n_i+1}(0) = e_{i,n_i}(0), \\
\dot{\eta}_{i,n_i+1}(t) &= -\phi_{i,n_i+1}(\eta_{i,n_i+1}(t)) + \varepsilon \Delta_i(t), \quad \eta_{i,n_i+1}(0) = e_{i,n_i+1}(0)
\end{align*}
\]

where \( e_{ij}(0) = x_{ij}(0) - \tilde{x}_{ij}(0) \) is the \( \varepsilon \)-independent initial value.

By Assumption (A2), we can find the derivative of \( V_i(\eta_j(t)) \) with respect to \( t \) along the solution of system (8) to be
\[
\frac{d}{dt} V_i(\eta_j(t)) = \sum_{j=1}^n \frac{\partial V_j}{\partial \eta_{ij}(t)} \left( \eta_{ij+1}(t) - \phi_j(\eta_{ij}(t)) \right)
\]

It then follows that
\[
\frac{d}{dt} V_i(\eta_j(t)) \leq \frac{-\lambda_{i,1}}{2\lambda_{i,2}} \sqrt{V_i(\eta_i(t))} + \frac{\sqrt{\lambda_{i,1}\varepsilon M \beta}}{2\lambda_{i,3}}
\]

By virtue of Assumption (A2) again, we have
\[
||\eta_j(t)||_{\mathbb{R}^{n+1}} \leq \sqrt{V_i(\eta_j(0))} \leq \sqrt{\lambda_{i,1} V_i(\eta_i(0))} e^{-\frac{\lambda_{i,1}}{2\lambda_{i,2}} t} + \frac{\varepsilon M \beta}{2\lambda_{i,1}} \int_0^t e^{-\frac{\lambda_{i,1}}{2\lambda_{i,2}} (t-s)} ds
\]

\[
\Delta_i(t) = \frac{d}{ds} \left. \left( f(x_{1,i}(s), \ldots, x_{n_i,i}(s), w_i(s)) \right) \right|_{s=\infty} = \sum_{j=1}^m \sum_{j=1}^{n_i} \frac{\partial f}{\partial x_{ij}}(x_{1,i}(t), \ldots, x_{n_i,i}(t), \ldots, x_{m_i,i}(t), w_i(t))
\]

\[
+ \sum_{j=1}^m g_i(\mu_i(t), \ldots, \nu_i(t)) \frac{\partial f}{\partial x_{i,j}}(x_{1,i}(t), \ldots, x_{n_i,i}(t), \ldots, x_{m_i,i}(t), w_i(t))
\]

\[
+ \frac{\partial f}{\partial w_i}(x_{1,i}(t), \ldots, x_{n_i,i}(t), \ldots, x_{m_i,i}(t), w_i(t))
\]
This together with (6) yields

$$\begin{align*}
|e_j(t)| &= e^{\lambda_{j+1}^{-1}t} \left| h_j \left( \frac{1}{\varepsilon} \right) \right| \\
&\leq e^{\lambda_{j+1}^{-1}t} \left\| \frac{1}{\varepsilon} \right\|_{\| \|_{\infty}} \\
&\leq e^{\lambda_{j+1}^{-1}t} \left( \sqrt{\lambda_{j+1} V(\eta_j(0))} e^{-\frac{\lambda_{j+1} t}{\lambda_{j+1}}} \right) \\
&+ \frac{\varepsilon M \beta_i}{2 \lambda_{j+1}} \int_0^t e^{-\frac{\lambda_{j+1} (t-s)}{\lambda_{j+1}}} ds \\
&\to 0 \text{ uniformly in } t \in [a, \infty) \text{ as } \varepsilon \to 0 \quad (12)
\end{align*}$$

which is just (i) of Theorem 1. (ii) of Theorem 1 can also follow from (12). In fact, from Assumption (A2), it follows that for any given $\varepsilon \in (0, 1)$, the first term of the right-hand side on the middle row of (12) can be estimated as

$$\begin{align*}
\sqrt{V(\eta_j(0))} e^{-\frac{\lambda_{j+1} t}{\lambda_{j+1}}} \\
&\leq \frac{\lambda_{j+1}}{\lambda_{j+1}} \left\| \eta_j(0) \right\|_{\| \|_{\infty}} e^{-\frac{\lambda_{j+1} t}{\lambda_{j+1}}} \\
&= \frac{\lambda_{j+1}}{\lambda_{j+1}} \left( \frac{x_j(0) - \hat{x}_j(0)}{\varepsilon} \right) \\
&\leq \frac{\lambda_{j+1}}{\lambda_{j+1}} \sum_{i=1}^{n_i} \left( \left| \eta_j(0) - \hat{x}_j(0) \right| \right) \\
&\leq \frac{\lambda_{j+1}}{\lambda_{j+1}} \sum_{i=1}^{n_i} \left( 1 - e^{-\frac{\lambda_{j+1} t}{\lambda_{j+1}}} \right) \\
&\leq \frac{\lambda_{j+1}}{\lambda_{j+1}} \sum_{i=1}^{n_i} \left( 1 - e^{-\frac{\lambda_{j+1} t}{\lambda_{j+1}}} \right) \geq 0
\end{align*}$$

where $x_j(0)$ and $\hat{x}_j(0)$ are initial values of system (4) and (5), respectively. The second term of the right-hand side on the middle row of (12) can be estimated as

$$\begin{align*}
\frac{1}{\varepsilon} \int_0^t e^{-\frac{\lambda_{j+1} (t-s)}{\lambda_{j+1}}} ds &= \frac{\lambda_{j+1}}{\lambda_{j+1}^2} \left( 1 - e^{-\frac{\lambda_{j+1} t}{\lambda_{j+1}}} \right) \\
&\leq \frac{\lambda_{j+1}}{\lambda_{j+1}^2} \left( 1 - e^{-\frac{\lambda_{j+1} t}{\lambda_{j+1}}} \right) \leq \frac{\lambda_{j+1}^3}{\lambda_{j+1}^2} \quad (14)
\end{align*}$$

(ii) of Theorem 1 then follow from (12) to (14). This completes the proof. \[\square\]

A typical example of ESO satisfying the conditions of Theorem 1 is the linear ESO. This is the case where all $\phi_i$ are linear functions: $\phi_i(r) = k_{i,r} r$ for all $r \in \mathbb{R}$, and $k_{i,r}$ is the real number such that the following matrix $E_i$ is Hurwitz

$$E_i = \begin{pmatrix}
-k_{i,1} & 1 & 0 & \cdots & 0 \\
-k_{i,2} & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-k_{i,n_i} & 0 & 0 & \cdots & 1 \\
-k_{i,n_i+1} & 0 & 0 & \cdots & 0
\end{pmatrix}$$

In this case, the ESO (5) is of the linear form

$$\begin{align*}
\dot{x}_{i,1}(t) &= \dot{x}_{i,2}(t) + \frac{1}{\varepsilon} k_{i,1} (x_{i,1}(t) - \hat{x}_{i,1}(t)) \\
\dot{x}_{i,2}(t) &= \dot{x}_{i,3}(t) + \frac{1}{\varepsilon} k_{i,2} (x_{i,1}(t) - \hat{x}_{i,1}(t)) \\
&\vdots \\
\dot{x}_{i,n_i}(t) &= \dot{x}_{i,n_i+1}(t) + \frac{1}{\varepsilon} k_{i,n_i} (x_{i,1}(t) - \hat{x}_{i,1}(t)) + g_i(u_1, u_2, \ldots, u_k) \\
\dot{x}_{i,n_i+1}(t) &= \frac{1}{\varepsilon} k_{i,n_i+1} (x_{i,1}(t) - \hat{x}_{i,1}(t)), \quad i = 1, 2, \ldots, m
\end{align*}$$

For the linear ESO (16), we have the following convergence result, which improves the corresponding result presented in [12] where the $a$ in the following Corollary 1 is required to be large.

Remark 1: It should be pointed out that the requirement for the boundedness of $\hat{w}_t$ is only arising from the fact that we want to estimate the augmented state. Otherwise, this requirement can be removed [15]. A typical external disturbance of finite sum of sinusoidal $w(t) = \sum a_j \sin \omega_j t$ satisfies this assumption.

Corollary 1: Suppose that all matrices $E_i$ in (15) are Hurwitz and Assumption (A1) is satisfied. Then for any given initial values of (4) and (16), the following conclusions hold.

(i) For every positive constant $a$

$$\lim_{t \to 0} |x_j(t) - \hat{x}_j(t)| = 0, \quad \text{uniformly for } t \in [a, \infty)$$

(ii) There exists an $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, there exists a $t_0 > 0$ such that

$$|x_j(t) - \hat{x}_j(t)| \leq K_j \varepsilon^{n_i+1} \quad t \in (t_0, \infty)$$

where $\hat{x}_j(t)$, $j = 1, 2, \ldots, n_i + 1$, $i = 1, 2, \ldots, m$, are the solution of (16), $x_j$, $j = 1, 2, \ldots, n_i + 1$, $i = 1, 2, \ldots, m$, the solution of (3), $x_{i,n_i+1} = f(x_1, \ldots, x_{i,1}, \ldots, x_{i,n_i+1}, w_i)$ the augmented state, and $K_j$ is positive number independent of $\varepsilon$ but depending on initial values.

Proof: By Theorem 1, we need only verify the Assumption (A2). To this end let

$$V(y) = (P_i y, y), \quad W_i(y) = (y, y), \quad \forall y \in \mathbb{R}^{n_i+1}$$

where $P_i$ is the positive-definite solution of the Lyapunov equation $P_i E_i + E_i^T P_i = -I_{n_i+1}$ for $n_i + 1$-dimensional identity matrix $I_{n_i+1}$. By basic linear algebra, it is easy to verify that

$$\sum_{j=1}^{n_i} \frac{\partial V_i}{\partial y_j} (y_j - k_{i,j} y_j) - \frac{\partial V_i}{\partial y_{n_i+1}} k_{i,n_i+1} y_j = -W_i(y)$$

and

$$\frac{\partial V_i}{\partial y_{n_i+1}} \leq 2 \lambda_{\max}(P_i) \|y\|^2_{\| \|_{\mbox{2}}}$$

where $\lambda_{\max}(P_i)$ and $\lambda_{\max}(P_i)$ denote the maximal and minimal eigenvalues of $P_i$, respectively. So Assumption (A2) is satisfied. The results then follow from Theorem 1. \[\square\]
In what follows of this section, we construct a special class of non-linear ESO and discuss its convergence. This is motivated from the convergence of homogeneous state observer for MIMO systems without uncertainty studied in [4, 6, 16–18].

Set \( \phi_i(r) = k_{ij}[r_i^{a_{ij}-(j-1)}] \) in (5), where \( a_i \in \{1, \cdots, 1\} \).

\( [r]^a = \text{sign}(r)|r|^a \), and \( k_{ij}, j = 1, 2, \cdots, n_i + 1 \) are constants such that every matrix \( E_i \) in (15) is Hurwitz. This reduces (5) into the following form (see (21))

In order to deal with the convergence of non-linear ESO (21), we introduce homogeneity and finite-time stability as follows.

Definition 1: A function \( V : \mathbb{R}^m \rightarrow \mathbb{R} \) is called homogeneous of degree \( \alpha \) with respect to \( \lambda_i \) for all \( i = 1, 2, \cdots, m \),

\[
V(\lambda_{i1}x_1, \lambda_{i2}x_2, \cdots, \lambda_{im}x_m) = \lambda^\alpha V(x_1, x_2, \cdots, x_m)
\]

for all \( \lambda > 0 \) and all \( (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m \).

A vector field \( g : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is called homogeneous of degree \( \alpha \) with respect to \( \lambda_i \) for all \( i = 1, 2, \cdots, n \),

\[
g(\lambda_{i1}x_1, \lambda_{i2}x_2, \cdots, \lambda_{im}x_m) = \lambda^\alpha g(x_1, x_2, \cdots, x_m)
\]

for all \( \lambda > 0 \) and all \( (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \), where \( g_i \) is the \( i \)-th component of \( g \).

Definition 2: The following system

\[
\dot{x}(t) = f(x(t)), \quad x(0) = x_0 \in \mathbb{R}^n
\]

is said to be globally finite-time stable, if it is Lyapunov stable, and for any \( x_0 \in \mathbb{R}^n \), there exists a \( T(x_0) > 0 \) such that the solution of (24) satisfies \( \lim_{t \to T(x_0)} x(t) = 0 \), and \( x(t) = 0 \) for all \( t \in [0, T(x_0), \infty) \).

The Lemma 1 below is Lemma 4.2 of [19].

Lemma 1: Let \( V_1, V_2 : \mathbb{R}^n \rightarrow \mathbb{R} \) be continuous functions, and be homogeneous of degree \( l_1 > 0, l_2 > 0 \) with respect to the same weights, respectively, and \( V_1 \) is positive definite. Then for any \( x \in \mathbb{R}^n \)

\[
\left( \min_{y \in V_1^{-1}(x)} V_2(y) \right)^{\frac{1}{2}} \leq \left( \max_{y \in V_1^{-1}(x)} V_2(y) \right)^{\frac{1}{2}}
\]

where \( V_1^{-1}(1) = \{ x \in \mathbb{R}^n \mid V_1(x) = 1 \} \). The Lemmas 2 and 3 below come from [5] directly.

Lemma 2: The vector field

\[
F_i(y) = \left( \begin{array}{c} y_2 + k_{i1}[y_1(t)]^{a_1} \\ y_3 + k_{i2}[y_1(t)]^{a_2} \\ \vdots \\ y_{n_i+1} + k_{in_i}[y_1(t)]^{a_{n_i}} \end{array} \right)
\]

is homogeneous of degree \( -d_i = a_i - 1 \) with respect to weights \( \{r_{ij} = (j-1)a_i - (j-2)\}_{j=1}^{n_i+1} \), where \( [r]^a = \text{sign}(r)|r|^a \), \( \forall r \in \mathbb{R} \).

Lemma 3: For some \( a_i \in (1, \frac{1}{n_i+1}, 1) \), if all matrices \( E_i \) in (15) are Hurwitz, then system \( \dot{y}(t) = F_i(y(t)) \) is globally finite-time stable.

Now, we are in a position to show the convergence of non-linear ESO (21).

Theorem 2: Suppose that every matrix \( E_i \) in (15) is Hurwitz and Assumptions (A1) is satisfied. Then there exists a constant \( \varepsilon_0 > 0 \), such that for any \( \varepsilon \in (0, \varepsilon_0) \), there exists a constant \( T_\varepsilon > 0 \) such that

(i) If \( a_i \in \left(1, \frac{1}{n_i+1}, 1\right) \), then

\[
|x_{ij}(t) - \hat{x}_{ij}(t)| \leq K_{ij} \varepsilon^{n_i+1-j} \varepsilon^{-\frac{n_i+1}{n_i+1-j}}, \quad \forall t \geq T_\varepsilon
\]

(ii) If \( a_i = 1 - \frac{1}{n_i+1} \), then

\[
|x_{ij}(t) - \hat{x}_{ij}(t)| = 0, \quad \forall t \geq T_\varepsilon
\]

where \( K_{ij} \) is positive constant independent of \( \varepsilon \) but depending on initial values. \( x_{ij}, \hat{x}_{ij}, j = 1, 2, \cdots, n_i \), \( i = 1, 2, \cdots, n_i + 1 \), are solutions of (4) and (21), respectively, \( \hat{x}_{i,n_i+1} = f(x_{1,1}, \cdots, x_{n_i,n_i}) \) is the augmented state.

Proof: For every \( i \in \{1, 2, \cdots, m\} \), let

\[
\eta_{ij}(t) = \frac{x_{ij}(\varepsilon t) - \hat{x}_{ij}(\varepsilon t)}{\varepsilon^{n_i+1-j}}, \quad j = 1, 2, \cdots, n_i + 1
\]

Then
A direct computation shows that $\eta_i = (\eta_{i,1}, \eta_{i,2}, \ldots, \eta_{i,n+1})^T$ satisfies

$$
\begin{align*}
\dot{\eta}_{i,1}(t) &= \eta_{i,2}(t) + k_{i,1}[\eta_{i,1}(t)]^{\lambda n} \\
\dot{\eta}_{i,2}(t) &= \eta_{i,3}(t) + k_{i,2}[\eta_{i,1}(t)]^{\lambda n-1} \\
&\vdots \\
\dot{\eta}_{i,n}(t) &= \eta_{i,n+1}(t) + k_{i,n}[\eta_{i,1}(t)]^{\lambda n-n_i} \\
\dot{\eta}_{i,n+1}(t) &= k_{i,n+1}[\eta_{i,1}(t)]^{\lambda n+n_i-1} + \varepsilon \Delta_i(t)
\end{align*}
$$

(28)

with $\Delta_i$ given by (5). From Lemmas 2 and 3, the error Equation (28) is a perturbed system of global finite-time stable system $\hat{y} = F_i(y)$, $y \in \mathbb{R}^{n+1}$. For the homogeneous global finite-time-stable system, it follows from Theorem 2 of [20] and Theorem 6.2 of [19], that there exists a positive-definite, radial unbounded, differentiable function $V_i : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $V_i(x)$ is homogeneous of degree $\gamma_i$ with respect to weights $\{r_{j,i}\}_{j=1}^n$, and the Lie derivative of $V_i$ along the vector fields $F_i$

$$
L_{F_i}V_i(x) = \sum_{j=1}^{n} \frac{\partial V_i}{\partial x_j} (\eta_{i,j+1} + k_{i,j}[\eta_{i,1}]^{\lambda n-1})
$$

is non-negative, where $\gamma_i \geq \max(d_i, r_{j,i})$. We note here that by radial unbounded for $V_i$ we mean $\lim_{|x| \rightarrow +\infty} V_i(x) = +\infty$, where $F_i$ is defined in Lemma 2.

From homogeneous of $V_i$, for any positive constant $\lambda$

$$
V_i(\lambda^{\alpha_{i,1}}x_1, \lambda^{\alpha_{i,2}}x_2, \ldots, \lambda^{\alpha_{i,n}}x_n) = \lambda^{\gamma_i} V_i(x_1, x_2, \ldots, x_n)
$$

(29)

Finding the derivatives of both sides of the above equation with respect to the arguments $x_i$ yields

$$
\lambda^{\alpha_{i,j}} \frac{\partial V_i}{\partial x_j}(\lambda^{\alpha_{i,1}}x_1, \lambda^{\alpha_{i,2}}x_2, \ldots, \lambda^{\alpha_{i,n}}x_n) = \lambda^{\gamma_i} \frac{\partial V_i}{\partial x_j}(x_1, x_2, \ldots, x_n)
$$

(30)

This shows that $\frac{\partial V_i}{\partial x_j}$ is homogeneous of degree $\gamma_i - r_{j,i}$ with respect to weights $\{r_{j,i}\}_{j=1}^n$.

Furthermore, the Lie derivative of $V_i$ along the vector field $F_i$

$$
L_{F_i}V_i(\lambda^{\alpha_{i,1}}x_1, \lambda^{\alpha_{i,2}}x_2, \ldots, \lambda^{\alpha_{i,n}}x_n) = \lambda^{\gamma_i} V_i(x_1, x_2, \ldots, x_n)
$$

(31)

So $L_{F_i}V_i$ is homogeneous of degree $\gamma_i - d_i$ with respect to weights $\{r_{j,i}\}_{j=1}^n$.

By Lemma 2, we have the inequalities as follows.

$$
\left| \frac{\partial V_i}{\partial x_{n+1}}(x) \right| \leq b_i V_i(x)^{\frac{\gamma_i+n_i}{n}}, \quad \forall x \in \mathbb{R}^{n+1}
$$

(32)

and

$$
L_{F_i} V_i(x) \leq -c_i V_i(x)^{\frac{\gamma_i+n_i}{n}}, \quad \forall x \in \mathbb{R}^{n+1}
$$

(33)

where $b_i, c_i$ are positive constants.

From Assumption (A1), there exist constants $M_i > 0$ such that $|\Delta_i(t)| \leq M_i$ for all $t > 0$ and $i \in \{1, 2, \ldots, m\}$. Now finding the derivative of $V_i$ along the solution of (28) gives

$$
\frac{dV_i(\eta_i(t))}{dt} = L_{F_i} V_i(\eta_i(t)) + \varepsilon \Delta_i(t) \frac{\partial V_i}{\partial x_{n+1}}
$$

$$
\leq -c_i V_i(\eta_i(t))^{\frac{\gamma_i+n_i}{n}} + \varepsilon M_i b_i V_i(\eta_i(t))^{\frac{\gamma_i+n_i}{n}}
$$

(34)

Let $\varepsilon_i = \frac{c_i}{M_i b_i}$. If $a_i = 1 - \frac{1}{1+n_i}$, then for any $\varepsilon \in (0, \varepsilon_i)$

$$
\frac{dV_i(\eta_i(t))}{dt} \leq -\frac{c_i}{2} V_i(\eta_i(t))^{\frac{\gamma_i+n_i}{n}}
$$

(35)

By theorem 4.2 of [21], there exists a $T_1 > 0$ such that $\eta_i(t) = 0$ for all $t \geq T_1$. This is (ii) for $T_1 = \max(T_1, T_2, \ldots, T_m)$.

When $a_i > 1 - \frac{1}{1+n_i}$, let

$$
A = \left\{ x \in \mathbb{R}^{n+1} \mid V_i(x) \geq \left( \frac{2M_i b_i}{c_i} \right)^{\frac{n}{\gamma_i+n_i}} \right\}
$$

For any $\varepsilon < \varepsilon_i$ and $\eta_i(t) \in A$, $V_i(\eta_i(t)) \geq \left( \frac{2M_i b_i}{c_i} \right)^{\frac{n}{\gamma_i+n_i}}$, it yields $M_i b_i \varepsilon \leq \frac{c_i}{2} (V_i(\eta_i(t)))^{-\frac{n}{\gamma_i+n_i}}$. This together with (34) leads to

$$
\frac{dV_i(\eta_i(t))}{dt} \leq -c_i (V_i(\eta_i(t)))^{\frac{\gamma_i+n_i}{n}} + \frac{c_i}{2} (V_i(\eta_i(t)))^{\frac{\gamma_i+n_i}{n}} (V_i(\eta_i(t)))^{\frac{\gamma_i+n_i}{n}}
$$

$$
= -\frac{c_i}{2} (V_i(\eta_i(t)))^{\frac{\gamma_i+n_i}{n}} < 0
$$

(36)

which shows that there exists a constant $T_1 > 0$ such that $\eta_i(t) \in A^c$ for all $t > T_1$.

Considering $|x_i|$ as the function of $\{x_1, x_2, \ldots, x_n\}$, it is easy to verify that $|x_i|$ is homogeneous of degree $r_{j,i}$ with respect to weights $\{r_{j,i}\}_{j=1}^n$. By Lemma 1, there exists a $L_{i,j} > 0$, such that

$$
|x_i| \leq L_{i,j} V_i(x)^{\frac{\gamma_i}{n}}, \quad \forall x \in \mathbb{R}^{n}
$$

(37)

This together with the fact $\eta_i(t) \in A^c$ for $t > T_i$ gives

$$
|\eta_i(t)| \leq K_{i,j} |x_i|^{\frac{\gamma_i}{n}}, \quad t > T_i
$$

(38)

where $K_{i,j} = j, 1, 2, \ldots, n_i$, $i = 1, 2, \ldots, m$ are positive constants. (ii) Then follows from (27) with $\varepsilon_i = \min(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)$, $T_i = \max(T_1, T_2, \ldots, T_m)$. The proof is complete. $\square$

**Remark 2:** We indicate several different aspects with the results of [15]. (a) Theorem 1 for MIMO is the generalisation of the counterpart in [15] for SISO. Actually, when $j = 1$, Theorem 1 is just Theorem 2.1 of [15]; (b) Theorem 2 gives a detailed analysis for non-linear ESO (21). When $j = 1$, it is reduced to Theorem 2.2 of [15] where only a
rough estimation is given as follows: for any $\sigma > 0$, there exists an $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$

$$|x(t) - \hat{x}(t)| \leq \sigma, \quad \forall t \in (t_0, \infty)$$

However, in (i) of Theorem 2 of this paper, the estimate is more elaborate; (c) Compared Theorem 1 with Theorem 2, we find that the power $n_1 + 1 - j + \frac{1}{(\epsilon \|x\|)^{\frac{1}{2}}}$ of $\varepsilon$ in Theorem 2 is larger than $n_1 + 2 - j$ in (ii) of Theorem 1. This shows, at least theoretically, that the non-linear ESO (21) converges faster than non-linear ESO (5) or linear ESO (16). Furthermore, if (ii) of Theorem 2 is the case, the finite time tracking is achieved.

3 ESO for systems with external disturbance only

In this section, we construct the ESO for MIMO system (4), in which $f(x_1, \ldots, x_{m}, \ldots, x_n)$, $x(t) = f(x_1, \ldots, x_n)$, and $f_1$ is known. Under this circumstance, we try to make use of information of $f_1$ as much as possible in designing the ESO, which is composed of following $m$ subsystems to estimate $x$ and $w$ (see (39)).

For the convergence of (39), we use Assumption (A4) below instead of Assumption (A1).

**Assumption (A4).** For every $i \in \{1, 2, \ldots, m\}$, $w_i, \hat{w}_i$ are uniformly bounded in $\mathbb{R}$, $f_i$ is Lipschitz continuous with Lipschitz constant $L_i$, that is

$$\left| f(x_1, \ldots, x_n) - f(y_1, \ldots, y_n) \right| \leq L_i \| x - y \|_{\Omega_{x}}$$

for all $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^{n_1 + \cdots + n_n}$. Moreover

$$\frac{\partial V_i}{\partial x_{n_i}} \leq \rho_i \| x \|_{\Omega_i}$$

where $x_i = (x_{i_1}, x_{i_2}, \ldots, x_{i_n})$, $\| \cdot \|_{\Omega}$ denotes the Euclidean norm of $\mathbb{R}$, $L_i, \rho_i$ are constants satisfying $L_i > L_1 + L_2 + \cdots + \rho_i$ and $V_i, W_i, \rho_1, \rho_2, \ldots, \rho_n$ are the constants in Assumption (A2).

**Theorem 3:** Under Assumptions (A2) and (A4), for any given initial values of (4) and (39), there exists a constant $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$, it has

(i) for any $a > 0$

$$\lim_{t \to a} |x(t) - \hat{x}(t)| = 0, \text{ uniformly for } t \in [a, \infty)$$

(ii) There exists an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, there exists a $t_0 > 0$, such that

$$|x(t) - \hat{x}(t)| \leq \sigma, \quad t \in (t_0, \infty)$$

where $i \in \{1, 2, \ldots, m\}$, $x_{i_1} = w_i$ is the augmented state of (4), $x_{i_2}, \ldots, x_{i_n}$ is the state of (4), $\hat{x}_{i_2}, \ldots, \hat{x}_{i_n}$, $\eta_1, \ldots, \eta_m$ is the state of (39), and $K_i$ is positive constant independent of $\varepsilon$ but depending on initial values.

**Proof:** For any $i \in \{1, 2, \ldots, m\}$, let $x_{i_1} = w_i$ and

$$\eta_i(t) = \frac{x_i(t) - \hat{x}_i(t)}{\varepsilon^{n_i - 1}}, \quad j = 1, 2, \ldots, n_i + 1 \quad (41)$$

A direct computation shows that $\eta_i(t)$ satisfies the following differential equation

$$\begin{aligned}
\dot{\eta}_{i_1}(t) &= \eta_{i_1}(t) - \phi_{i_1}(\eta_{i_1}(t)) \\
\dot{\eta}_{i_2}(t) &= \phi_{i_2}(\eta_{i_1}(t)) + \delta_{i_2}(t) \\
\dot{\eta}_{i_3}(t) &= \phi_{i_3}(\eta_{i_2}(t)) + \delta_{i_3}(t) \\
\ldots &= \ldots \\
\dot{\eta}_{i_{n_i+1}}(t) &= \phi_{i_{n_i+1}}(\eta_{i_{n_i+1}-1}(t)) + \delta_{i_{n_i+1}}(t) \\
\dot{\eta}_{i_{n_i+1}}(t) &= \phi_{i_{n_i+1}}(\eta_{i_{n_i+1}-1}(t)) + \delta_{i_i}(t)
\end{aligned}$$

where

$$\delta_{i_1}(t) = f_1(x_1(t), \ldots, x_{n_1}(t), \ldots, x_{n_m}(t))$$

$$\delta_{i_2}(t) = f_2(x_{i_2}(t), \ldots, x_{i_2-n_1+1}(t), \ldots, x_{n_m}(t))$$

$$\ldots$$

Set

$$\eta_i = (\eta_{i_1}, \ldots, \eta_{i_{n_i+1}})^T, \quad \eta = (\eta_{i_1}, \ldots, \eta_{i_{n_m+1}})^T,$$

$$V(\eta) = V_1(\eta_1) + V_2(\eta_2) + \cdots + V_m(\eta_m)$$
Finding the derivative of $V$ along the solution of error Equation (42) with respect to $t$ gives (43)

\[ \frac{d}{dt} \sqrt{V(\eta(t))} = \frac{1}{2} \sqrt{V(\eta(t))} \leq \frac{\Lambda}{2} \frac{d}{dt} \sqrt{V(\eta(t))} + \frac{\varepsilon \Gamma}{2} \]

This together with Assumption (A2) gives, for each $i \in \{1, 2, \ldots, m\}$, that

\[ \|\eta_i(t)\|_{z^{n_i+1}} \leq \sqrt{V(\eta(t))} \leq \sqrt{V(\eta(0))} e^{-\frac{\varepsilon}{\lambda_1} t} + \varepsilon \Gamma \int_0^t e^{-\frac{\varepsilon}{\lambda_1} s} ds \]

By (41), we get, for each $i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, n_i + 1\}$, that

\[ |x_{ij}(t) - \tilde{x}_{ij}(t)| = e^{x_{i+1, j}} |\eta_{ij}(t)| \leq e^{x_{i+1, j}} \frac{\lambda_{ij}}{\lambda_1} \left( \sqrt{V(\eta(0))} e^{-\frac{\varepsilon}{\lambda_1} t} + \varepsilon \Gamma \int_0^t e^{-\frac{\varepsilon}{\lambda_1} s} ds \right) \]

\[ \rightarrow 0 \text{ uniformly for } t \in [a, \infty) \text{ as } \varepsilon \rightarrow 0 \]

This is (i), (ii) follows also from the above inequality. This completes the proof. \( \square \)

4 Application to permanent magnet synchronous motor control

We point out that there are many practical problems that can be formulated into (3). A class of chemical reactors, distillation columns and fluidised bed in chemical control is discussed in [22]; a turbofan model is presented in [23]. The major contribution of this paper, different to those works mentioned, is that we give a mathematical proof of convergence. However, for more clear clarification of potential applications of the method discussed in this paper, we investigate the current control for permanent-magnet synchronous motor presented in [24] on page 152. Other example of [24] on page 107 on inverted pendulum subjected to a unknown disturbance force can be treated similarly.

The current control for permanent-magnet synchronous motors with parameter uncertainties is described by the following model

\[ \begin{align*}
    i_d &= \frac{1}{L} u_d - \frac{R}{L} i_d + \omega_e i_q \\
    i_q &= \frac{1}{L} u_q - \frac{R}{L} i_q - \lambda \omega_e
\end{align*} \]

where $i_d, i_q$ are $d$-axis and $p$-axis stator currents, respectively, $u_d, u_q$ represent $d$-axis and $p$-axis stator voltages, respectively, $R$ is the armature resistance, $L$ is the armature inductance, $\lambda$ is the flux linkage of parameter magnet, $\omega_e$ is the electrical angular velocity, $\omega_e = N_i \alpha_E$, $\alpha_E$ is the mechanical angular speed of the motor. The motion equation of the motor can be written as

\[ J \dot{\omega}_1 = \tau_e - \tau_l \]

where $\tau_e = K_l i_q$ is the generated motor torque, $\tau_l = B \dot{\omega}_1$ is the load torque, $N_i$ is the number of pole pairs of the motor, $K$ is the torque constant, $B$ is the coefficient of viscous friction.

The control purpose is to design control voltages $u_d$ and $u_q$ so that $i_d$ and $i_q$ track the desired current references $i_d^* \text{ and } i_q^*$, respectively.

In practice, the parameters $L, R, \lambda, J, B$ are usually not known exactly, and hence $f_1 = -\frac{R}{L} i_d + \omega_e i_q$, and $f_2 = -\frac{L}{J} \dot{\omega}_1 - \lambda \omega_e$ contain some uncertainties. So we consider $f_1, f_2$ as the total disturbances that can be estimated by the ESO. Strictly speaking, in order to apply Theorem 1, we need to check Assumption (A1) (Assumption (A2) is trivially satisfied if we use linear ESO). This is true for some
parameters. For instance, if $RN_iBJ > \lambda^2 LJ^2 + LN_i^2 K^2$, we can define the Lyapunov function

$$V(t) = \frac{1}{2} \left[ \dot{i}_d^2(t) + \dot{i}_q^2(t) + \omega^2(t) \right]$$

Then finding the derivative of $V$ along the solution of (48), we can obtain (see (49))

$$\dot{V}(t) = \frac{1}{L} i_d u_d - \frac{R}{L} i_d^2 + \omega d i_d + \frac{1}{L} \omega d i_d - \frac{R}{L} i_q^2 - \omega d i_q - \lambda i_d i_q + \frac{N_i K}{J} i_d \omega e - \frac{N_i B}{J} \omega q$$

$$\leq \frac{1}{L} i_d u_d - \frac{R}{L} i_d^2 + \frac{1}{L} \omega d i_d - \frac{R}{L} i_q^2 + \left( \frac{\lambda^2 J}{N_i B} i_d^2 + \frac{N_i B}{4 J} \omega^2 \right) + \left( \frac{N_i K^2}{2 J} i_d^2 + \frac{N_i B}{4 J} \omega^2 \right) - \frac{N_i B}{J} \omega^2 - \frac{R}{2L} i_d^2 + \frac{1}{2RL} i_d^2$$

$$+ \frac{1}{L} \omega d i_d - \frac{RN_i BJ - \lambda^2 LJ^2 - LN_i^2 K^2}{LN_i BJ} \omega^2 - \frac{N_i B}{2J} \omega^2 = \frac{RN_i BJ - \lambda^2 LJ^2 - LN_i^2 K^2}{2LN_i BJ} \omega^2$$

$$+ \frac{1}{2L(2RN_i BJ - \lambda^2 LJ^2 - LN_i^2 K^2)} i_d^2 - \frac{N_i B}{2J} \omega^2 \leq -K_1 V(t) + K_2 (u_d^2(t) + u_q^2(t))$$

(49)

**Fig. 1** Observer estimations by (50)
Fig. 2 Magnification of Figure 1

ESO for system (48) as follows

\[
\begin{align*}
\dot{x}_1 &= \dot{x}_2 + \frac{1}{\varepsilon}(i_d - \dot{x}_1) + u_d \\
\dot{x}_2 &= \frac{1}{\varepsilon}(i_d - \dot{x}_1) \\
\dot{x}_3 &= \dot{x}_4 + \frac{1}{\varepsilon}(i_q - \dot{x}_3) + u_q \\
\dot{x}_4 &= \frac{1}{\varepsilon}(i_q - \dot{x}_3)
\end{align*}
\] (50)

where \( \dot{x}_2 \) and \( \dot{x}_4 \) are used to compensate total disturbances \( f_1 = \frac{2}{L}i_d + \omega_e i_q, \) and \( f_2 = \frac{2}{L}i_q - \omega_e i_d - \lambda x_e, \) respectively.

Changing slightly the parameters \( R_J \) only used in [24] to satisfy \( RN_J > \lambda^2 L^2 + LN_J^2 K^2 \): \( L = 1.0, R = 2, J = 0.1, N_J = 4, \lambda = 0.001, B = 0.001, K = 3\lambda N_J/2, \) current references \( i_d^* = \sin(t), \) \( i_q^* = 1, \) gain \( \varepsilon = 0.0002 \) in (50), initial values \( i_d(0) = 1, i_q(0) = -0.5, \omega_e(0) = 0, \) \( \dot{x} = (0, 0, 0, 0), \) integral step \( h = 0.0001, \) we plot the numerical results of ESO (50) in Fig. 1. It is seen that the convergence is very fast: the solid-line curve and the dotted-line curve are almost coincident. To see them clearly, we magnify the Fig. 1 into Fig. 2. From both Figs. 1 and 2, we see that the extended state estimations by (50) are quite satisfactory. Since the initial conditions are different for original system and ESO, and the high gain in ESO as well, the peaking phenomena are observed in magnified Fig. 2. However, this can be effectively avoided in the feedback-loop. To see this point, we design the feedback control as

\[
\begin{align*}
u_d &= \text{sat}_M(-10(\dot{x}_1 - i_d^*) + (i_d^* - \dot{x}_2)) \\
u_q &= \text{sat}_M(-4(\dot{x}_1 - i_q^*) + (i_q^* - \dot{x}_4))
\end{align*}
\] (51)

where the saturation function \( \text{sat}_M \) is designed below to avoid possible damage of peak value of \( \dot{x}_i \) in the very beginning of the initial time [25]

\[
\text{sat}_M(r) = \begin{cases} r, & 0 \leq r \leq M \\ \frac{1}{2}r^2 + (M + 1)r - \frac{1}{2}M^2, & M < r \leq M + 1 \\ M + \frac{1}{2}, & r > M + 1 \end{cases}
\] (52)

The numerical results of \( i_d, i_d^* \) and \( i_q, i_q^* \) are plotted in Fig. 3. It is seen that both \( i_d \) and \( i_q \) tracks current references \( i_d^* \) and
$i^*$ are very satisfactory. More importantly, using saturated estimation in the feedback loop can avoid effectively the peak value problem. Actually, there are many other studies that have shown the effectiveness of peak value saturation for ESO like that in [8] and [22]. Finally, we indicate that using the same parameters as that in [24], we can also obtain the similar satisfactory results although the condition $RN_iBJ > \lambda^2 L J^2 + LN_i^2 K^2$ is not satisfied.

5 Concluding remarks

In this paper, we give the principle of designing of ESO for a class of MIMO non-linear systems to estimate not only the state but also the uncertainties from both dynamics of the system and the external disturbance. The convergence of the ESO is rigorously given. The results not only extend the corresponding results of [15] for SISO systems to this class of MIMO non-linear systems but also a special class of ESO motivated from the homogeneous state observer for MIMO systems without uncertainty is analysed in details, which is indigent in [15] even for SISO systems. Meanwhile, we indicate that the design of the ESO is very flexible. It can be non-linear like homogeneous one and linear one, for which the assumptions can be trivially checked. The only severe assumption is the prior estimate on the boundedness of the system with respect to bounded control and disturbance. However, this is also needed for SISO systems in [15]. Finally, in order to show the practical applicability of the method, we apply the result to the current control for permanent-magnet synchronous motor. The numerical simulation shows that the result is very satisfactory.

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7 References


