A Novel Extended State Observer for Output Tracking of MIMO Systems With Mismatched Uncertainty

Zhi-Liang Zhao and Bao-Zhu Guo

***Abstract*—In this paper, we develop a novel extended state observer (ESO), in terms of tracking error only, for output tracking of a class of multi-input multioutput systems with mismatched uncertainty. A novel ESO is constructed from the nonsmooth function "fal" to estimate both uncertainty and state of the system. An ESO-based output feedback controller is then designed to compensate (cancel) the uncertainty and to achieve the output tracking. The convergence of the closed-loop system is proved. The effectiveness of the proposed method is demonstrated by numerical results of trajectory tracking for a practical autonomous underwater vehicle model. We show that in the presence of measurement noise, this novel ESO leads to better performance than the linear ESO. Moreover, this type of ESO has much smaller peaking value than the linear ESO under the same tuning gain.**

***Index Terms*—Extended state observer (ESO), multi-input multioutput (MIMO) systems, nonlinear systems, output tracking, uncertainty.**

I. Introduction

As an emerging control technology, the active disturbance rejection control (ADRC), which was originally proposed by Han as an alternative of PID control [4], can effectively cope with vast uncertainty in control systems. The uncertainty, which is also referred to as "total disturbance" in ADRC, can include various internal uncertainties, external disturbances, and even complicated time-varying and nonlinear parts, which are hard to be dealt with by practitioners. This makes ADRC an almost model-free control strategy for many systems. Control practices in the last two decades show that in many situations ADRC may replace the dominant PID control in engineering [11].

It was indicated in [16] that the ADRC had been strictly tested by Parker Hannifin Parflex hose extrusion plant, where ADRC solution had been used to replace the well-tuned PID controller for eight months. It turned out that the product performance capability index (Cpk) was improved by more than 30% and the energy consumption was reduced by more than 50%. Very recently, some chips manufactures such as Texas Instruments and Freescale Semiconductor have replaced the PID by ADRC in their DSP control chips [17], [18]. In the last two decades, many industry applications by ADRC have been carried out in different contexts. Some important progresses leading to theoretical foundation of ADRC have also been made. We refer to a recent monography [6] and references therein.

The main feature of ADRC lies in its nature in dealing with uncertainty: an estimation and cancelation strategy. The key idea is to use an extended state observer (ESO) to estimate the total disturbance, which is then compensated (or canceled) in the feedback loop.

The big challenge for theoretical analysis of ADRC stems from the nonlinear ESO. As believed by Han [4], the properly designed nonlinear ESO would yield better performance than the linear ESO. Following numerous simulations, Han found the following nonlinear function "fal" for the ESO design:

$$\mathrm{fal}(\tau)=\begin{cases}\dfrac{\tau}{\delta^{1-\theta}}, & |\tau|\le\delta\\ |\tau|^{\theta}\,\mathrm{sign}(\tau)=[\tau]^{\theta}, & |\tau|>\delta\end{cases} \tag{1.1}$$

where $0<\theta<1$ and $\delta>0$ are constants.

For a class of second-order nonlinear systems, Han [4] constructed a fal-based triple-parameter-tuning ESO, a third-order system. This kind of ESO has been shown by numerical experiments that it is effective for system state and total disturbance estimation, with additional merits including peaking value reduction compared with the linear ESO. Very recently, we have proved the convergence of fal-based ESO for a class of nonlinear open-loop systems with uncertainty [14]. However, the convergence of the fal-based ESO in the feedback closed loop remains open. In this paper, we take up this challenge by developing a fal-based single-parameter-tuning ESO on output tracking for a class of multi-input multioutput (MIMO) systems with mismatched uncertainties, and by providing a convergence proof for the ESO-based closed-loop system.

The system considered in this paper is the following MIMO system with vast uncertainties:

$$\begin{cases}\dot{x}_{i1}(t)=x_{i2}(t)+f_{i1}(x_{i1}(t),w_{i1}(t))\\ \vdots\\ \dot{x}_{i(n_i-1)}(t)=x_{in_i}(t)\\ +f_{i(n_i-1)}(x_{i1}(t),\ldots,x_{i(n_i-1)}(t),w_{i(n_i-1)}(t))\\ \dot{x}_{in_i}(t)=f_{in_i}(x(t),\zeta(t),w_{in_i}(t))+\sum_{j=1}^{m}b_{ij}u_j(t)\\ \dot{\zeta}(t)=F_0(x(t),\zeta(t),w(t))\\ y_i(t)=C_{n_i}x_i(t)=x_{i1}(t),i=1,2,\ldots,m\end{cases} \tag{1.2}$$

where $\left(x^{\top}(t),\zeta^{\top}(t)\right)^{\top}$ is the system state with $\zeta(t)\in\mathbb{R}^s$ representing the zero dynamics, $x(t)=\left(x_1^{\top}(t),\ldots,x_m^{\top}(t)\right)^{\top}\in\mathbb{R}^{n_{\text{sum}}}$, $x_i(t)=$

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$(x_{i1}(t),\ldots,x_{in_i}(t))^\top \in \mathbb{R}^{n_i}$; $u(t)=(u_1(t),\ldots,u_m(t))^\top \in \mathbb{R}^m$ is the control input; $y(t)=(y_1(t),\ldots,y_m(t))^\top$ is the output; $w(t)$ and $w_{ij}(t)$ $(j=1,\ldots,n_i,\ i=1,\ldots,m)$ are the external disturbances; the nonlinear functions $F_0\in C^1(\mathbb{R}^{n_{\text{sum}}+s+1},\mathbb{R})$ and $f_{ij}\in C^{n_i+1-j}(\mathbb{R}^{j+1},\mathbb{R})$ $(j=1,\ldots,n_i, i=1,\ldots,m)$ are unknown functions; b_{ij}s are constants such that matrix $B=(b_{ij})_{m\times m}$ is invertible. The matrix C_{n_i} is defined as

$$C_{n_i}=(1,0,\ldots,0)_{1\times n_i}, i=1,2,\cdots,m.$$

From the above-mentioned assumption, we see that almost all of the system functions of (1.2), which cover the matched and mismatched vast uncertainties, are unknown. The purpose of control is to render the output $y(t)$ to track a reference signal $\mathscr{V}(t)=(v_1(t),\ldots,v_m(t))^\top$, i.e., $y(t)\to\mathscr{V}(t)$ as $t\to\infty$, and to guarantee the boundedness of the system state as well.

The main contribution of this paper is on the convergence of the fal-based ADRC for system (1.2), a long standing problem. We proceed as follows. In Section II, we construct a single-parameter-tuning fal-based ESO and an ESO-based output feedback controller. Section III is devoted to the proof of the convergence. In Section IV, we discuss a three-dimensional (3-D) trajectory tracking of autonomous underwater vehicles (AUV) to show the effectiveness of the proposed scheme. The numerical simulations are carried out to show the tracking performance of the proposed ADRC. It is shown that the proposed scheme is better than the linear ESO under the same tuning gain and measurement noise.

The following notation will be used throughout the paper:

$$\begin{aligned}
&n_{\text{sum}}=\sum_{i=1}^m n_i,\quad n_{\max}=\max_{1\le i\le m} n_i\\
&n^*=n_{\text{sum}}+m,\quad \tilde n=2n_{\text{sum}}+1\\
&\overline{(x_i)}_j(t)=(x_{i1}(t),\ldots,x_{ij}(t))\\
&\overline{w_{ij}}^k(t)=\left(w_{ij}(t),\dot w_{ij}(t),\ldots,w_{ij}^{(k-1)}(t)\right)\\
&\tilde v(t)=\left(v_1(t),\ldots,v_1^{(n_1-1)}(t),\ldots,v_m^{(n_m-1)}(t)\right)\\
&\tilde w_i(t)=(\overline{w_{i1}}^{n_i}(t),\ldots,w_{in_i}(t))^\top\\
&\tilde w(t)=\left(\tilde w_1^\top(t),\ldots,\tilde w_m^\top(t)\right)^\top\\
&\hat v(t)=(\tilde v^\top(t),\dot{\tilde v}^\top(t))^\top,\ \hat w(t)=(\tilde w^\top(t),\dot{\tilde w}^\top(t),w(t))^\top.
\end{aligned}$$

II. Control Design and Main Result

As indicated in Section I, all system functions in system (1.2) are supposed to be unknown. The output feedback controller is designed by using the error between the output and the reference only. We design this error-based feedback controller in three steps. The first step is to derive the tracking error equation, by using the system structure. The second step is to design a single-parameter-tuning fal-based ESO to estimate the state and the total disturbance. The output feedback controller is then designed in terms of ESO in the last step, where the total disturbance is canceled by its estimate.

Let the measured errors be denoted by

$$\begin{aligned}
e_{i1}(t) &= y_i(t)-v_i(t),\quad i=1,\ldots,m\\
e_{ij}(t) &= x_{ij}(t)-v_i^{(j-1)}(t)\\
&\quad+\phi_{ij}\left(\overline{(x_i)}_j(t),\overline{w_{i1}}^j(t),\ldots,w_{ij}(t)\right)
\end{aligned}\tag{2.1}$$

where $j=2,\ldots,n_i$, $i=1,2,\ldots,m$, $\phi_{ij}(\cdot)=f_{ij}(\cdot)+\tilde\phi_{ij}(\cdot)-v^{(j)}(t)$, and

$$\begin{aligned}
&\tilde\phi_{ij}\left(\overline{(x_i)}_j(t),\overline{w_{i1}}^j(t),\ldots,w_{ij}(t)\right)\\
&=\sum_{k=1}^{j-1}\frac{\partial\phi_{i(j-1)}\left(\overline{(x_i)}_{j-1}(t),\overline{w_{i1}}^{j-1}(t),\ldots,w_{i(j-1)}(t)\right)}{\partial x_{ik}}\\
&\cdot\left(x_{i(k+1)}(t)+f_{ik}\left(\overline{(x_i)}_k(t),w_{i(k)}(t)\right)\right)\\
&+\sum_{k=1}^{j-1}\sum_{h=0}^{j-1-k}w_{ik}^{(h+1)}(t)\\
&\cdot\frac{\partial\phi_{i(j-1)}\left(\overline{(x_i)}_{j-1}(t),\overline{w_{i1}}^{j-1}(t),\ldots,w_{i(j-1)}(t)\right)}{\partial w_{ik}^{(h)}}.
\end{aligned}\tag{2.2}$$

Let

$$\begin{aligned}
&\Phi_i(e(t),\tilde v(t),\zeta(t),\tilde w_i(t))=\phi_{in_i}\left(x(t),\zeta(t),\tilde w_i(t),v^{(j)}(t)\right)\\
&=\phi_{in_i}\left(e(t)+\tilde v(t),\zeta(t),\tilde w_i(t),v^{(j)}(t)\right)
\end{aligned}\tag{2.3}$$

and

$$e_i(t)=(e_{i1}(t),\ldots,e_{in_i}(t))^\top, e(t)=\left(e_1^\top(t),\ \ldots,e_m^\top(t)\right)^\top.\tag{2.4}$$

Then,

$$\begin{cases}
\dot e_i(t)=A_{n_i}e_i(t)+B_{n_i}\Big[\Phi_i(e(t),\tilde v(t),\zeta(t),\tilde w_i(t))\\
\quad+\sum_{j=1}^m b_{ij}u_j(t)\Big],\quad i=1,2,\ldots,m\\
\dot\zeta(t)=F_0(e(t)+\tilde v(t),\zeta(t),w(t))\\
y_e(t)=(e_{11}(t),\ldots,e_{m1}(t))^\top
\end{cases}\tag{2.5}$$

where

$$A_{n_i}=\begin{pmatrix}0 & I_{(n_i-1)\times(n_i-1)}\\ 0 & 0\end{pmatrix},\quad B_{n_i}=(0,\ldots,0,1)^\top\tag{2.6}$$

and $I_{(n_i-1)\times(n_i-1)}$ denotes the (n_i-1)-dimensional identity matrix. Other notations are presented at the end of Section I. By this formulation, the output tracking of (1.2) can be achieved by the output stabilization of e-subsystem of (2.5) with the measured output $y_e(t)$.

Now, the control objective is therefore to (practically) stabilize the e-subsystem of (2.5) while maintaining the ζ-subsystem being bounded. As previously mentioned, only the measured output error $y_e(t)=(e_{11}(t),\ldots,e_{m1}(t))$ could be used to construct the fal-based ESO in the feedback design in order to estimate the other states $(e_{i2}(t),\ldots,e_{in_i}(t))$ for $1\le i\le m$ and the total disturbances defined by

$$e_{i(n_i+1)}(t)\triangleq\Phi_i(e(t),\tilde v(t),\zeta(t),\tilde w_i(t)),\ i=1,,2,\ldots,m.\tag{2.7}$$

The fal-based single-parameter-tuning ESO is designed as follows:

$$\begin{aligned}
&\dot{\hat e}_i(t;r)=A_{n_i+1}\hat e_i(t;r)\\
&+\begin{pmatrix}\dfrac{k_{i1}}{r^{n_i-1}}g_1(r^{n_i}(e_{i1}(t)-\hat e_{i1}(t;r)))\\ \dfrac{k_{i2}}{r^{n_i-2}}g_2(r^{n_i}(e_{i1}(t)-\hat e_{i1}(t;r)))\\ \vdots\\ k_{i(n+1)}rg_{n_i+1}(r^{n_i}(e_{i1}(t)-\hat e_{i1}(t;r)))\end{pmatrix}^\top,\\
&i=1,2,\ldots,m
\end{aligned}\tag{2.8}$$

where $\hat{e}_i(t;r) = (\hat{e}_{i1}(t;r), \ldots, \hat{e}_{i(n_i+1)}(t;r)) \in \mathbb{R}^{n_i+1}$

$$g_j(\tau) = \text{fal}(\tau, \theta_j, 1),\ 1 \le j \le n_{\max} \tag{2.9}$$

the fal$(\cdot)$ is given in (1.1) with positive constant $\theta_j \in (0,1)$ to be specified later, and the design constants k_{ij}s are chosen so that the following matrices are Hurwitz:

$$K_i = \begin{pmatrix} k_{i1} & 1 & 0 & \cdots & 0 \\ k_{i2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_{in_i} & 0 & 0 & \cdots & 1 \\ k_{i(n_i+1)} & 0 & 0 & \cdots & 0 \end{pmatrix},\ i = 1, 2, \ldots, m. \tag{2.10}$$

We expect that for any $1 \le j \le n_i + 1, 1 \le i \le m$, $\hat{e}_{ij}(t;r)$ approximates $e_{ij}(t)$ as accurate as expected when r is tuned to be large enough. In particular, $\hat{e}_{i(n_i+1)}(t;r)$ gives an estimate of the total disturbance $e_{i(n_i+1)}(t)$. There is only one parameter r in (2.8) to be tuned according to the accuracy requirement and the variation of the total disturbance. Generally speaking, the more accurate the convergence of ESO needs to be and the faster the total disturbance changes, the larger the parameter r is required to be.

The ESO-based feedback controller is then designed as

$$\begin{cases} u_i^*(t;r) = \text{Sat}_{M_i}\left(\alpha_i^\top \bar{e}_i(t;r) - \hat{e}_{i(n_i+1)}(t;r)\right) \\ u(t;r) = B^{-1}u^*(t) = B^{-1}(u_1^*(t;r), \ldots, u_m^*(t;r))^\top \end{cases} \tag{2.11}$$

with $B = (b_{ij})_{m \times m}$

$$\begin{cases} \bar{e}_i(t;r) = (\hat{e}_{i1}(t;r), \ldots, \hat{e}_{in_i}(t;r))^\top \in \mathbb{R}^{n_i} \\ \alpha_i = (\alpha_{i1}, \ldots, \alpha_{in_i})^\top \in \mathbb{R}^{n_i},\ \ i = 1, 2 \ldots, m \end{cases} \tag{2.12}$$

and

$$\text{Sat}_{M_i}(\tau) = \begin{cases} \tau, & |\tau| \le M_i \\ M_i \text{sign}(\tau), & |\tau| > M_i. \end{cases} \tag{2.13}$$

The constants α_{ij}s are chosen so that the following matrices are Hurwitz:

$$\tilde{A}_{n_i} = A_{n_i} + B_{n_i}\alpha_i^\top,\ i = 1, \ldots, m \tag{2.14}$$

and M_is are specified later in (2.26). To emphasis the r-dependence of the controller, we shall denote the feedback controller as $u(t;r)$ after (2.11).

The following Lyapunov function is constructed for specifying the constants M_i in (2.11). Let

$$V(z) = \sum_{i=1}^{m} V_i(z_i),\ \ V_i(z_i) = z_i^\top P_i z_i \tag{2.15}$$

where $z = \left(z_1^\top, \ldots, z_m^\top\right)^\top \in \mathbb{R}^{n_{\text{sum}}}$, $z_i \in \mathbb{R}^{n_i}$, and P_i is the positive definite matrix solution to the Lyapunov equation

$$\tilde{A}_i^\top P_i + P_i \tilde{A}_i = -I_{n_i \times n_i}. \tag{2.16}$$

The matrices $\tilde{A}_{n_i}$s are given in (2.14). We can verify that

$$\bar{\lambda}_{\min}\|z\|^2 \le V(z) \le \bar{\lambda}_{\max}\|z\|^2 \tag{2.17}$$

where $\bar{\lambda}_{\min} = \min_{1\le i\le m} \lambda_{\min}(P_i)$, $\bar{\lambda}_{\max} = \max_{1\le i\le m} \lambda_{\max}(P_i)$, $\lambda_{\min}(\cdot)$, and $\lambda_{\max}(\cdot)$ are the minimal and maximal eigenvalues of the corresponding matrix, respectively. This renders the following sets compact:

$$\mathcal{A}_1 = \left\{z \in \mathbb{R}^{n_{\text{sum}}} \middle|\ V(z) \le R^*\right\}$$
$$\mathcal{A}_2 = \left\{z \in \mathbb{R}^{n_{\text{sum}}} \middle|\ V(z) \le R^* + 1\right\} \tag{2.18}$$

where $R^* = \sup_{z\in\mathcal{H}} V(z)$, $z = \left(z_1^\top, \ldots, z_m^\top\right)^\top \in \mathbb{R}^{n_{\text{sum}}}$, and $z_i = (z_{i1}, \ldots, z_{in_i})^\top \in \mathbb{R}^{n_i}$. Let $\mathcal{H}$ be an n_{sum}-dimensional closed hypercube

$$\mathcal{H} = \{z \in \mathbb{R}^{n_{\text{sum}}} \mid |z_{ij}| \le \beta_{ij} + 1\} \tag{2.19}$$

where $\beta_{ij} > 0$ is an upbound of the initial value $|e_{ij}(0)|$ of (2.5), which are r-independent constants. It is easy to verify that $\mathcal{H} \subset \mathcal{A}_1 \subset \mathcal{A}_2$.

Assumption A1: For the external disturbances and reference signal, we assume that there exist constants $\tilde{M}_1, \tilde{M}_2 > 0$ such that

$$\sup_{t\in[0,\infty)} \|\hat{v}(t)\| \le \tilde{M}_1,\ \ \sup_{t\in[0,\infty)} \|\hat{w}(t)\| \le \tilde{M}_2 \tag{2.20}$$

where $\hat{v}(t)$ and $\hat{w}(t)$ are defined at the end of Section I.

Assumption A2: There exist positive definite function $V_0 : \mathbb{R}^s \to \mathbb{R}$ and class $\mathcal{K}_\infty$ function $\chi : \mathbb{R}^{\tilde{n}} \to \mathbb{R}$ such that if $V_0(\nu) \ge \chi(z, \eta, \mu)$, then $L_{\tilde{F}_0(z,\eta,,\nu,\mu)} V_0(\nu) \le 0$, and

$$V_0(\zeta(0)) \le \sup_{(z,\eta,\mu)\in\mathcal{C}_1} \chi(z, \eta, \mu) \tag{2.21}$$

where

$$\mathcal{C}_1 = \mathcal{A}_2 \times \mathcal{U}_{n_{\text{sum}}}(\tilde{M}_1) \times \mathcal{U}_1(\tilde{M}_2) \tag{2.22}$$

and

$$\mathcal{U}_n(R) = \{\nu \in \mathbb{R}^n \mid \|\nu\| \le R\},\ \ n \in \mathbb{N}^+,\ R > 0. \tag{2.23}$$

Assumption A1 means that the external disturbances, reference signal, and their derivatives are all bounded, which are common assumption in control practice. This assumption can be relaxed to cover piecewise smooth reference signal and external disturbance. Assumption A2 means that the ζ-subsystem is input-to-state bounded, which is quite general.

Let

$$\mathcal{B} = \left\{\nu \in \mathbb{R}^s \ \middle|\ V_0(\zeta) \le \sup_{(z,\eta,\mu)\in\mathcal{C}_1} \chi(z, \eta, \mu)\right\} \tag{2.24}$$

and

$$\mathcal{C}_2 = \mathcal{A}_2 \times \mathcal{U}_{n_{\text{sum}}}(\tilde{M}_1) \times \mathcal{B} \times \mathcal{U}_1(\tilde{M}_2) \tag{2.25}$$

where $\mathcal{U}_{n_{\text{sum}}}(\tilde{M}_1)$ is defined in (2.23).

The constants M_i in (2.11) can now be specified as

$$M_{i1} = \sum_{j=1}^{n_i} |\alpha_{ij}|(\tilde{\beta}_{ij} + 1), \tilde{\beta}_{ij} = \sup_{z\in\mathcal{A}_2} |z_{ij}|$$
$$M_{i2} = \sup_{(z,\xi,\eta,\mu)\in\mathcal{C}_2} \Phi_i(z, \xi, \eta, \mu) + 1,\ M_i = M_{i1} + M_{i2}. \tag{2.26}$$

Now, we state Theorem II.1, the main result of this paper.

Theorem II.1: In ESO (2.8), let $\theta_j = j\theta - (j-1)$, $j = 1, 2, \ldots, n_{\max} + 1$. Suppose that Assumptions A1 and A2 are satisfied and α_{ij}s are chosen such that the matrices $\tilde{A}_{n_i}$s defined in (2.14) are Hurwitz. Then, there exist $\theta^* \in (0,1)$ and $r^* > 1$, so that if $\theta \in (\theta^*, 1)$ and $r \in (r^*, \infty)$, then

$$\left|\hat{e}_{ij}(t;r) - e_{ij}(t;r)\right| \le \Gamma_1 (1/r)^{n_i+2-j} \tag{2.27}$$

for any $t > t_r$, where t_r is an r-dependent constant with $\lim_{t\to\infty} t_r = 0$, and

$$\left|e_{ij}(t;r)\right| \le \Gamma_2/r \tag{2.28}$$

for any $t > \bar{t}_r$, where $\bar{t}_r$ is an r-dependent constant and Γ_1 and Γ_2 are r-independent constants. Furthermore, the system state $(x(t;r), \zeta(t;r))$ is uniformly bounded, independent of r.

We point out that the condition on the nonlinear functions in system (1.2) is no more than smoothness: All these functions are continuously differentiable in defined domains. Under additional assumptions like all $f_{ij}(\cdot)$ being Lipachitz continuous, other control methods such as high-gain observer based high-gain control [7, pp. 189–194], [12] can also make the error as small as expected after transient time. However, when some $e_{i(n_i+1)}(t)$ are large (even for large constant), both the gains in observer and controller must be large enough to suppress the uncertainty in high-gain control. The large tuning gains in observer and controller cause serious peaking value problem. This results in large control input in some time interval (usually near initial stage, or when the reference signal jumps), which may not be feasible in practical actuator. In this paper, although a large gain may be needed in fal-based ESO, the gain in controller is not required. Due to the saturation-like behavior of fal$(\cdot)$, the peaking value of switching ESO is much small even for large tuning gain. This fact will be illustrated in Section IV. In addition, the saturated controller can guarantee the control not exceeding the given value. Moreover, the switching ESO also enjoys merit of good performance under measurement noise, which is also illustrated at the end of Section IV.

III. Proof of Main Result

Since the proof is lengthy, we first briefly explain the main steps of the proof. First, in Section III-A, we show that the state $(e(t;r),\zeta(t;r))$ of system (2.5) lies in an r-independent compact set for $t\in[0,T]$, where T is independent of r as well. Also, some Lyapunov functions are constructed in this section. Second, a weighted error system is constructed and a bound of this weighted error system's initial state is found in Section III-B. Third, the ultimately boundedness of system state is analyzed in Section III-C, where the convergence of error $e_{ij}(t;r)-\hat{e}_{ij}(t;r)$ is therefore established. Finally, the proof of Theorem II.1 is presented in Section III-D.

A. Lyapunov Functions and Inequalities

In this section, we first show in the following Lemma III.1 that there exists $T>0$, which is independent of r such that for all $t\in[0,T]$, the state of error (2.5) stays in a compact set $\mathcal{A}_1\times\mathcal{B}$ defined in (2.18) and (2.24), which is also independent of r. The details are omitted.

Lemma III.1: Let $T=1/(\bar{M}_1+2\bar{M}_2)$ with

$$\bar{M}_1=\max_{1\le j\le n_i,\,1\le i\le m}\tilde{\beta}_{ij},\quad \bar{M}_2=\max_{1\le i\le m}M_i \tag{3.1}$$

where $\tilde{\beta}_{ij}$ and M_i are defined in (2.26). Then, for any $r>1$

$$\{e(t;r)|t\in[0,T]\}\subset\mathcal{A}_1,\quad \{\zeta(t;r)|t\in[0,T]\}\subset\mathcal{B} \tag{3.2}$$

where $\mathcal{A}_1$ *and* $\mathcal{B}$ are defined in (2.18) and (2.24), respectively.

In what follows, we present some Lyapunov functions and inequalities for analyzing the dynamic behavior of the error system later. The following auxiliary vector fields and systems are useful for later investigation. Let

$$\begin{aligned}
&F_i(\tilde{z}_i)=\left(F_{i1}(\tilde{z}_i),\ldots,F_{in_i}(\tilde{z}_i),F_{i(n_i+1)}(\tilde{z}_i)\right)^\top\\
&F_{ij}(\tilde{z}_i)=\tilde{z}_{i(j+1)}-k_{ij}[\tilde{z}_{i1}]^{\theta_j},\ j=1,2,\ldots,n_i\\
&F_{i(n_i+1)}(\tilde{z}_i)=-k_{i(n_i+1)}[\tilde{z}_{i1}]^{\theta_{n_i+1}}\\
&F(\tilde{z})=\left((F_i(\tilde{z}))^\top,\ldots,(F_m(\tilde{z}_m))^\top\right)^\top\\
&\tilde{z}=\left(\tilde{z}_1^\top,\ldots,\tilde{z}_m^\top\right)\in\mathbb{R}^{n^*}
\end{aligned} \tag{3.3}$$

and consider the system

$$\dot{\tilde{z}}_i(t)=F_i(\tilde{z}_i(t)),\quad \tilde{z}_i\in\mathbb{R}^{n_i+1},\ i=1,\ldots,m. \tag{3.4}$$

It can be verified that $F_i(\cdot)$ and system (3.4) are homogeneous of degree $d=\theta-1$ with weights $\{r_j=(j-1)\theta-(j-2)\}_{j=1}^{n_i+1}$. The details of weighted homogeneity can be found in [3].

Since the matrices K_is given in (2.10) are Hurwitz, it follows from [9] that there exists $\theta_1^*\in(n_{\max}/(n_{\max}+1),1)$ such that for any $(\theta_1^*,1)$, the system $\dot{\tilde{z}}_i(t)=F_i(\tilde{z}_i(t))$ is finite-time stable. The following Lemma III.2 is concluded from [10, Th. 2] and [3, Th. 6.2].

Lemma III.2: Let matrices K_is defined in (2.10) be Hurwitz. Then, for each $1\le i\le m$, there exists a positive definite, and radially unbounded Lyapunov function $V_{i\theta}:\mathbb{R}^{n_i+1}\to\mathbb{R}$, which is homogenous of degree $\gamma>1$ with weights $\{r_j\}_{j=1}^{n_i+1}$, such that the Lie derivative of $V_{i\theta}(\tilde{z})$ along the vector field $F_i(\tilde{z})$ is negative definite.

By [3, Lemma 4.2] and the homogeneity of $\frac{\partial V_{i\theta}(\tilde{z}_i)}{\partial\tilde{z}_{ij}}$, $L_{F_i}V_{i\theta}(\tilde{z}_i)$, and $|\tilde{z}_{ij}|$, we have

$$\begin{aligned}
&\left|\frac{\partial V_{i\theta}(\tilde{z}_i)}{\partial\tilde{z}_{ij}}\right|\le\bar{B}_1(V_{i\theta}(\tilde{z}_i))^{\frac{\gamma-r_j}{\gamma}}\\
&L_{F_i}V_{i\theta}(\tilde{z}_i)\le-\bar{B}_2(V_{i\theta}(\tilde{z}_i))^{\frac{\gamma+d}{\gamma}}\\
&|\tilde{z}_{ij}|\le\bar{B}_3(V_{i\theta}(\tilde{z}))^{\frac{r_j}{\gamma}},\tilde{z}_i\in\mathbb{R}^{n_i+1},\bar{B}_1,\bar{B}_2,\bar{B}_3>0.
\end{aligned} \tag{3.5}$$

Let further

$$V_\theta(\tilde{z}_1,\ldots,\tilde{z}_m)=\sum_{i=1}^m V_{i\theta}(\tilde{z}_i),\quad \tilde{z}_i\in\mathbb{R}^{n_i+1} \tag{3.6}$$

and

$$\begin{aligned}
&\tilde{V}_L(\tilde{z})=\sum_{i=1}^m\tilde{V}_{iL}(\tilde{z}_i),\quad \tilde{V}_{iL}(\tilde{z}_i)=\tilde{z}_i^\top\tilde{P}_i\tilde{z}_i\\
&\tilde{z}=\left(\tilde{z}_1^\top,\ldots,\tilde{z}_m^\top\right)^\top\in\mathbb{R}^{n^*},\quad \tilde{z}_i\in\mathbb{R}^i
\end{aligned} \tag{3.7}$$

where $\tilde{P}_i$ is the positive definite matrix solution to the Lyapunov equation

$$K_i^\top\tilde{P}_i+\tilde{P}K_i=-I_{(n_i+1)\times(n_i+1)}. \tag{3.8}$$

It is easy to verify that $\tilde{V}_L(\tilde{z})$ is continuous, positive definite. Furthermore, for all $\tilde{z}\in\mathbb{R}^{n^*}$

$$\lambda_{\min}\|\tilde{z}\|^2\le\tilde{V}_L(\tilde{z})\le\lambda_{\max}\|\tilde{z}\|^2 \tag{3.9}$$

where

$$\lambda_{\min}=\min_{1\le i\le m}\lambda_{\min}(\tilde{P}_i),\ \lambda_{\max}=\max_{1\le i\le m}\lambda_{\max}(\tilde{P}_i). \tag{3.10}$$

B. Bounds of Initial Value of the Weighted Error System

In this section, we construct a weighted error system and establish the initial state bounds for this error system.

Let

$$\tilde{e}_{ij}(t;r)=r^{n_i+1-j}(e_{ij}(t;r)-\hat{e}_{ij}(t;r)) \tag{3.11}$$

for $j=1,2,\ldots,n_i+1,\ i=1,\ldots,m$, and set

$$\begin{aligned}
&\tilde{e}_i(t;r)=\left(\tilde{e}_{i1}(t,r),\ldots,\tilde{e}_{i(n_i+1)}(t;r)\right)^\top\\
&\tilde{e}(t;r)=\left((\tilde{e}_1(t;r))^\top,\ldots,(\tilde{e}_m(t;r))^\top\right)^\top.
\end{aligned} \tag{3.12}$$

A straightforward computation shows that $\tilde{e}(t;r)$ satisfies the following equation:

$$\begin{aligned}
\frac{d\tilde{e}(t;r)}{dt}&=rG(\tilde{e}(t;r))+\Big((B_{n_1+1}\Delta_1(t;r))^\top,\\
&\ldots,(B_{n_m+1}\Delta_m(t;r))^\top\Big)^\top
\end{aligned} \tag{3.13}$$

where

$$G_i(\tilde{z}_i) = (G_{i1}(\tilde{z}_i), \ldots, G_{in_i}(\tilde{z}_i), G_{i(n_i+1)}(\tilde{z}_i))^\top$$
$$G_{ij}(\tilde{z}_i) = \tilde{z}_{i(j+1)} - k_{ij} g_j(\tilde{z}_{i1}),\ j = 1, \ldots, n_i$$
$$G_{i(n_i+1)}(\tilde{z}_i) = -k_{i(n_i+1)} g_{n_i+1}(\tilde{z}_{i1})$$
$$\tilde{z} = \left(\tilde{z}_1^\top, \ldots, \tilde{z}_m^\top\right)^\top,\ \tilde{z}_i \in \mathbb{R}^{n_i+1}. \tag{3.14}$$

By (3.13)

$$\dot{\tilde{e}}(t;r) = rF(\tilde{e}(t;r)) + r(G(\tilde{e}(t;r)) - F(\tilde{e}(t;r))) + \left((B_{n_1+1}\Delta_1(t;r))^\top, \ldots, (B_{n_m+1}\Delta_m(t;r))^\top\right)^\top \tag{3.15}$$

where $\Delta_i(t;r)$ is the derivative of the total disturbance $e_{i(n_i+1)}(t;r)$, which will be specified in (3.31) later. Now we estimate a bound of $\|F(\cdot) - G(\cdot)\|$. By the definitions of $F(\cdot)$ defined in (3.3) and $G(\cdot)$ defined in (3.14), for any $\tilde{z}_i = (\tilde{z}_{i1}, \ldots, \tilde{z}_{i(n_i+1)}) \in \mathbb{R}^{n_i+1}$, if $|\tilde{z}_{i1}| > 1$, then $G_i(\tilde{z}) = F_i(\tilde{z})$, and when $|\tilde{z}_{i1}| \le 1$

$$|G_{ij}(\tilde{z}_i) - F_{ij}(\tilde{z}_i)| = k_{ij}(\tilde{z}_{i1} - [\tilde{z}_{i1}]^{\theta_j})$$
$$j = 1, 2, \ldots, n_i,\ i = 1, 2, \ldots, m. \tag{3.16}$$

Now, we suppose that θ_j satisfies condition of Theorem II.1. Then, we can verify that

$$\lim_{\theta\to 1} \theta_j = \lim_{\theta\to 1}(j\theta - (j-1)) = 1$$
$$j = 1, 2, \ldots, n_i + 1,\ i = 1, 2, \ldots, m. \tag{3.17}$$

This yields

$$\lim_{\theta\to 1} \max_{\tau\in[-1,1]} \left|[\tau]^{\theta_i} - \tau\right| = 0. \tag{3.18}$$

Let

$$\tilde{\delta}_1 = \bar{B}_2 / \left(2\bar{B}_1 \sum_{i=1}^{m}\sum_{j=1}^{n_i+1} |k_{ij}|\right). \tag{3.19}$$

By (3.18), there exists $\theta_2^* \in [\theta_1^*, 1)$ such that for any $\theta \in [\theta_2^*, 1)$

$$\max_{1\le i\le n_{\max}+1, |\tau|\le 1} \left|\tau - [\tau]^{\theta_i}\right| < \tilde{\delta}_1. \tag{3.20}$$

Define

$$V_{\min}(\tilde{z}) = \min_{\theta\in[\theta_2^*, 1]} V_\theta(\tilde{z}),\quad \tilde{z} \in \mathbb{R}^{n^*}. \tag{3.21}$$

By the properties of $V_\theta(\cdot)$, $V_{\min}(\cdot)$ is continuous, positive definite, and radially unbounded. Therefore, there exist class $\mathcal{K}_\infty$ functions [2] $\kappa_j (j = 1, 2) : [0, \infty) \to [0, \infty)$, such that for all $\tilde{z} \in \mathbb{R}^{n^*}$

$$\kappa_1(\|\tilde{z}\|) \le V_{\min}(\tilde{z}) \le \kappa_2(\|\tilde{z}\|). \tag{3.22}$$

Now we introduce auxiliary weighted homogeneous functions $\Upsilon_i : \mathbb{R}^{n_i+1} \to \mathbb{R} (i = 1, 2, \ldots, m)$ to estimate an initial state bound of system (3.13)

$$\Upsilon(\tilde{z}_{i1}, \ldots, \tilde{z}_{i(n_i+1)}) = |\tilde{z}_{i1}|^{\frac{\gamma}{r_1}} + \cdots + |z_{i(n_i+1)}|^{\frac{\gamma}{r_{n_i+1}}}. \tag{3.23}$$

It is easy to verify that $\Upsilon_i(\tilde{z}_i)$ is positive definite, homogeneous of degree γ with weights $\{r_j\}_{j=1}^{n_i+1}$. By [3, Lemma 4.2], there exists $c_1 > 0$ such that $V_{i\theta}(\tilde{z}_i) \le c_1 \Upsilon_i(\tilde{z}_i)$ for all $\tilde{z}_i \in \mathbb{R}^{n_i+1}$. This together with (3.11) and (3.23) yields

$$V_{i\theta}(\tilde{e}_i(0;r)) \le c_1 \textstyle\sum_{j=1}^{n_i+1} |e_{ij}(0) - \hat{e}_{ij}(0)|^{\frac{\gamma}{r_j}} r^{\frac{(n_i+1-j)\gamma}{r_j}} \tag{3.24}$$

where $e_{ij}(0)$ and $\hat{e}_{ij}(0)$ $(j = 1, 2, \ldots, n_i + 1,\ i = 1, 2, \ldots, m)$ are r-independent initial values.

For any $\theta > n_{\max}/(n_{\max} + 1)$, we have

$$n_i \ge \frac{n_i - (j-1)}{r_j} \ge \frac{n_i - j}{r_{j+1}},\quad j = 2, \ldots, n_i - 1.$$

Hence, for any $r \ge 1$

$$r^{\frac{(n_i+1-j)\gamma}{r_j}} \le r^{n_i\gamma} \le r^{n_{\max}\gamma},\quad i = 2, \ldots, m. \tag{3.25}$$

Therefore, for any $r \ge 1$

$$V_\theta(\tilde{e}(0;r)) \le A r^{n_{\max}\gamma}$$
$$A = c_1 \sum_{i=1}^{m}\sum_{j=1}^{n_i} |e_{ij}(0) - \hat{e}_{ij}(0)|^{\frac{\gamma}{r_j}}. \tag{3.26}$$

Let

$$\mathscr{C}_{r\theta} = \left\{\tilde{z} \in \mathbb{R}^{n^*} \,\middle|\, V_\theta(\tilde{z}) \le A r^{n_{\max}\gamma}\right\}. \tag{3.27}$$

It is obvious that for any $r > 1$, $\tilde{e}(0;r) \in \mathscr{C}_{r\theta}$. Define

$$\mathscr{D}_2 = \left\{\tilde{z} \in \mathbb{R}^{n^*} | V_{\min}(\tilde{z}) \le 1\right\}$$
$$\mathscr{D}_3 = \left\{\tilde{z} \in \mathbb{R}^{n^*} | \tilde{V}_L(\tilde{z}) \le \bar{b}\right\} \tag{3.28}$$

where $\bar{b} = \min\left\{\lambda_{\min}\kappa_2^{-1}(1/2), \lambda_{\min}/4\right\}$ and $\lambda_{\min}$ is given in (3.10). For every $\tilde{z} \in \mathscr{D}_3$

$$V_{\min}(\tilde{z}) \le \kappa_2(\|\tilde{z}\|) \le \kappa_2(\bar{b}/\lambda_{\min}) \le 1/2 < 1 \tag{3.29}$$

and as a result, $\mathscr{D}_3 \subset \mathscr{D}_2$, $\mathscr{D}_2 - \mathscr{D}_3 \ne \emptyset$. Furthermore, for any $\tilde{z} \in \mathscr{D}_3$

$$|\tilde{z}_{i1}| \le \|\tilde{z}\| \le \sqrt{\tilde{V}_L(\tilde{z})/\lambda_{\min}} \le 1/2 < 1. \tag{3.30}$$

C. Ultimately Boundedness of the System State

In this section, we first establish the convergence of the fal-based single-parameter-tuning ESO (2.8) in Proposition III.1. Then, we are able to develop the ultimately boundedness of the system state.

First of all, we analyze the derivatives of the total disturbances. Let

$$\begin{aligned}\Delta_i(t;r) =&\ \dot{e}_{i(n_i+1)}(t;r) \\ &= \frac{d\Phi_i(e(t;r), \tilde{v}(t), \zeta(t;r), \tilde{w}_i(t))}{dt} \\ &= \sum_{i=1}^{m} \nabla_{e_i}\Phi_i(e(t;r), \tilde{v}(t), \zeta(t;r), \tilde{w}_i(t)).\end{aligned} \tag{3.31}$$

By the smoothness of $F_0(\cdot)$, $f_{ij}(\cdot)$, for any $r > 1$, $\Delta_i(t;r)$ is continuous with respect to t. In the following, we will prove that if $\Delta_i(t;r)$ is bounded by an r-independent constant in a finite interval $[0, \bar{T}](\bar{T} > 0)$ or infinite interval $[0, \infty)$, then there exists $t_r > 0$ such that for $t > t_r$, the observer error $e_{ij}(t;r) - \hat{e}_{ij}(t;r)$ can be as small as expected as long as r is large. From [14], we can obtain the following proposition.

Proposition III.1: 1) For any $r > 1$ and each $i = 1, \ldots, m$, if $\Delta_i(t;r)$ defined in (3.31) is bounded by an r-independent constant $\tilde{M}_i$ over $[0, \bar{T}]$, then there exist $\theta^* > 0$ and $r^* > 1$ such that for any $\theta \in (\theta^*, 1)$ and $r > r^*$

$$|e_{ij}(t;r) - \hat{e}_{ij}(t;r)| \le \Gamma\,(1/r)^{n_i+2-j}\quad \forall\, t \in (t_r, \bar{T}) \tag{3.32}$$

where Γ is an r-independent constant and $t_r \in (0, \bar{T})$ is an r-dependent constant satisfying $\lim_{r\to\infty} t_r = 0$.

2) For any $r > 1$ and $i = 1, \ldots, m$, if $\Delta_i(t;r)$ defined in (3.31) is bounded by r-independent constant over $[0, \infty)$, then there exist $\theta^* > 0$ and $r^* > 1$ such that for any $\theta \in (\theta^*, 1),\ r > r^*$

$$|e_{ij}(t;r) - \hat{e}_{ij}(t;r)| \le \Gamma\,(1/r)^{n_i+2-j}\quad \forall\, t \in (t_r, \infty) \tag{3.33}$$

where Γ is an r-independent constant and $t_r > 0$ is an r-dependent constant satisfying $\lim_{r\to\infty} t_r = 0$.

Proposition III.2: There exists $r^* > 1$ such that for any $\theta \in (\theta^*, 1)$ and $r > r^*$, the solution of (2.5) satisfies that

$$\{e(t;r)|\ t \in [0,\infty)\} \subset \mathcal{A}_2$$
$$\{\zeta(t;r)|\ t \in [0,\infty)\} \subset \mathcal{B} \tag{3.34}$$

where $\mathcal{A}_2$ is defined in (2.18) and $\mathcal{B}$ in (2.24).

Proof: From Lemma III.1, for any $r > 1$

$$\{e(t;r)|\ t \in [0,T]\} \subset \mathcal{A}_1 \subset \mathcal{A}_2^\circ. \tag{3.35}$$

If for any $t \in [0,\infty)$, $e(t;r) \in \mathcal{A}_2$, then by Assumption A2, $\{\zeta(t;r)|\ t \in [0,\infty)\} \subset \mathcal{B}$. So, we only need to prove that there exists $r^* > 1$ such that for all $r > r^*$, $\{e(t;r)|\ t \in [0,\infty)\} \subset \mathcal{A}_2$. We prove this assertion by contradiction. If it is not true, then for any $r > 1$, there exist $T_{2r} > T_{1r} > T$ such that

$$e(T_{1r};r) \in \partial\mathcal{A}_1,\ e(T_{2r};r) \in \partial\mathcal{A}_2$$
$$\{e(t;r)|t \in [T_{1r}, T_{2r}]\} \subset \mathcal{A}_2 - \mathcal{A}_1^\circ$$
$$\{e(t;r)|t \in [0, T_{2r}]\} \subset \mathcal{A}_2. \tag{3.36}$$

This together with Assumption A1 and Lemma III.1 concludes that

$$(e(t;r), \tilde{v}(t), \zeta(t,r), \tilde{w}(t)) \in \mathcal{C}_2\ \forall\, t \in [0, T_{2r}] \tag{3.37}$$

where $\mathcal{C}_2$ is defined in (2.25).

From the continuity of $\Delta_i(t;r)$ and (3.37), there exists an r-independent constant $\hat{M} > 0$ such that $|\Delta_i(t;r)| \le \hat{M}$ for any $t \in [0, T_{2r}]$. Let

$$\hat{\delta}_1 = \min\left\{1, \min_{z\in\mathcal{A}_2-\mathcal{A}_1^\circ} \|z\|/N\right\} \tag{3.38}$$

with

$$N = 2\max_{1\le i\le m} \lambda_{\max}(P_i)\left(\sum_{i=1}^{n_i}|\alpha_i| + 1\right). \tag{3.39}$$

By Proposition III.1, there exists $r^* > 1$ such that

$$|e_{ij}(t;r) - \hat{e}_{ij}(t;r)| \le \hat{\delta}_1\quad \forall\, r > r^*,\ t \in [T, T_2] \tag{3.40}$$

where $\hat{\delta}_1$ is given in (3.38). So, for each $1 \le j \le n_i$

$$|\hat{e}_{ij}(t;r)| \le |e_{ij}(t;r)| + \hat{\delta}_1 \le \tilde{\beta}_{ij} + 1 \tag{3.41}$$

and

$$|\hat{e}_{i(n_i+1)}(t;r)| \le |e_{i(n_i+1)}(t;r)| + 1 \le M_{i2} \tag{3.42}$$

where $\tilde{\beta}_{ij}$ and M_{i2} are given in (2.26). By (2.7), (2.26), (3.36), (3.41), and (3.42), for any $t \in [T, T_{2r}]$

$$\left|\alpha_i^\top \hat{e}_i(t;r) - \hat{e}_{i(n_i+1)}(t;r)\right| \le M_i. \tag{3.43}$$

Therefore

$$\dot{e}_i(t;r) == \tilde{A}_{n_i} e_i(t;r) + B_{n_i}\Big[e_{i(n_i+1)}(t;r) - \hat{e}_{i(n_i+1)}(t;r)$$
$$+ \alpha_i^\top(\bar{e}_i(t;r) - e_i(t;r))\Big] \tag{3.44}$$

for all $t \in [T, T_{2r}]$, $\quad i = 1, 2, \ldots, m$.

Finding the derivative of the Lyapunov function $V(\cdot)$ defined in (2.15) along the solution of (3.44) to obtain, for any $r > r^*$ and $t \in [T_{1r}, T_{2r}]$, such that

$$\left.\frac{dV(e(t;r))}{dt}\right|_{(3.44)} \le \|e(t;r)\|\left(-\min_{z\in\mathcal{A}_2-\mathcal{A}_1^\circ}\|z\| + N\hat{\delta}_1\right) \le 0$$

which implies that $V(e(T_{1r};r)) \ge V(e(T_{2r};r))$. While by (3.36) and (2.18), we have $V(e(T_{2r};r)) = V(e(T_{1r};r)) + 1$. This is a contradiction. ■

D. Proof of Theorem II.1

From Proposition III.2, there exists $\tilde{r}_1^* > 1$ such that for all $r \in (\tilde{r}_1^*, \infty)$ and $t \in [0,\infty)$

$$e(t;r) \in \mathcal{A}_2,\quad \zeta(t;r) \in \mathcal{B} \tag{3.45}$$

and hence for $r \in (\tilde{r}_1^*, \infty)$ and $t \in [0,\infty)$

$$(e(t;r), \tilde{v}(t), \zeta(t;r), \tilde{w}(t)) \in \mathcal{C}_2. \tag{3.46}$$

From the smoothness of $f_{ij}(\cdot)$ and $F_0(\cdot)$, all partial derivatives of $\Phi_i(\cdot)$ are continuous. This together with (3.31) and (3.45) concludes that there exists an r-independent constant $\hat{M}_1 > 0$ such that for each $i = 1, 2, \ldots, m$

$$\Delta_i(t;r) \le \hat{M}_1\quad \forall\, t \in [0,\infty),\ r \in (\tilde{r}_1^*, \infty). \tag{3.47}$$

By virtue of Proposition III.1, there exists $\tilde{r}_2^* > \tilde{r}_1^*$ so that the first assertion of Theorem II.1, i.e., (2.27) holds true with $\Gamma_1/r < \hat{\delta}_1$ for any $r \in (\tilde{r}_2^*, \infty)$ and $t \in (t_r, \infty)$, where $\hat{\delta}_1$ is given in (3.38) and t_r is an r-dependent constant satisfying $\lim_{r\to\infty} t_r = 0$.

In what follows, we will prove the second assertion of Theorem II.1, i.e., (2.28). Similarly with (3.44), since for any $r \in (\tilde{r}_2^*, \infty)$ and $t \in (0,\infty)$, (3.41), (3.42), and (3.46) hold true, hence

$$\dot{e}_i(t;r) = \tilde{A}_{n_i} e_i(t;r) + B_{n_i}\Big[e_{i(n_i+1)}(t;r)$$
$$- \hat{e}_{i(n_i+1)}(t;r) + \alpha_i^\top(\bar{e}_i(t;r) - e_i(t;r))\Big] \tag{3.48}$$

for all $r \in (\tilde{r}_2^*, \infty)$ and $t \in [t_r, \infty)$, $\quad i = 1, 2, \ldots, m$.

Taking (2.27) into account and finding the derivative of the Lyapunov function $V(\cdot)$ defined in (2.15) along the solution of (3.48) to obtain, for any $r > \tilde{r}_2^*$ and $t \in [t_r, \infty)$, that

$$\left.\frac{dV(e(t;r))}{dt}\right|_{(3.48)} \le -2\lambda_1 V(e(t;r)) + \frac{2\lambda_2}{r}\sqrt{V(e(t;r))} \tag{3.49}$$

with

$$\lambda_1 = \frac{1}{2\max_{1\le i\le m}\lambda_{\max}(P_i)}$$
$$\lambda_2 = \frac{N\Gamma_1}{\sqrt{\min_{1\le i\le m}\lambda_{\min}(P_i)}}. \tag{3.50}$$

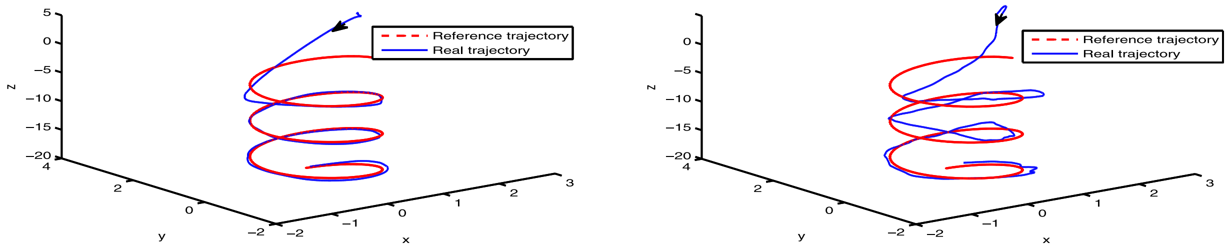
By (3.49), we conclude that for any $t > t_r$ and $r > \tilde{r}_2^*$, if $V(e(t;r)) \ne 0$, then

$$\left.\frac{d\sqrt{V(e(t;r))}}{dt}\right|_{(3.48)} \le -\lambda_1\sqrt{V(e(t;r))} + \frac{\lambda_2}{r}. \tag{3.51}$$

Applying the comparison principle of the ordinary differential equations again, we obtain

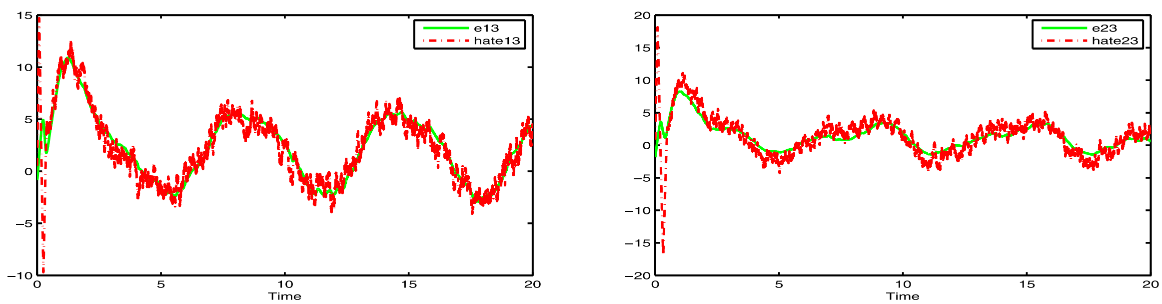
$$\sqrt{V(e(t;r))} \le \sqrt{V(e(t_{1r};r))}\exp(-\lambda_1(t - t_{1r}))$$
$$+ \frac{\lambda_2}{r}\int_{t_{1r}}^{t}\exp(-\lambda_1(t-s))ds. \tag{3.52}$$

Equation (2.28) can then be deduced from (3.52) and the boundedness of $e(t;r)$. ■

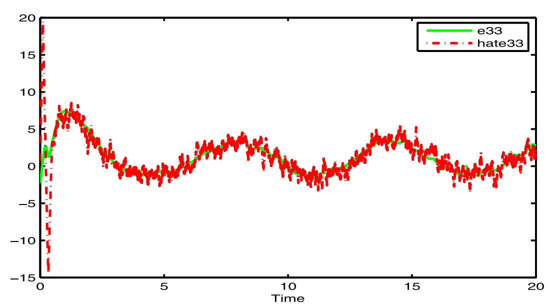


(a) (b)

Fig. 1. 3-D trajectory tracking with measurement noise. (a) 3-D trajectory tracking by switching ESO with measurement noise. (b) 3-D trajectory tracking by linear ESO with measurement noise.



(a) (b)



(c)

Fig. 2. Total disturbance estimation results by switching ESO. (a) $e_{13}(t)$ and $\hat{e}_{13}(t;80)$. (b) $e_{23}(t)$ and $\hat{e}_{23}(t;80)$. (c) $e_{33}(t)$ and $\hat{e}_{33}(t;80)$.

IV. Application to 3-D Trajectory Tracking of AUV

The dynamics of the AUV can be modeled as follows [13]:

$$\begin{cases} \dot{\mathbf{x}}(t) = J(\mathbf{x}(t))\mathbf{v}(t) \\ M\dot{\mathbf{v}}(t) + C(\mathbf{v}(t))\mathbf{v}(t) + D(\mathbf{v}(t))\mathbf{v}(t) \\ +\mathbf{g}(\mathbf{x}(t)) + \mathbf{d}(t) = \mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{x}(t) \end{cases} \tag{4.1}$$

where

$$\mathbf{x}(t) = (x(t), y(t), z(t), \phi(t), \vartheta(t), \psi(t))^\top \tag{4.2}$$

denotes the vehicle location and orientation in the earth-fixed frame, $\mathbf{v}(t)$ the vector of vehicle's velocity in the body-fixed frame, and $\mathbf{y}(t)$ the output. In (4.1), set $\vartheta = 0, \phi = 0, \psi = 0$ to be fixed. The control purpose is to drive the AUV to track the 3-D reference trace $v(t) = (v_1(t), v_2(t), v_3(t))^\top$. The feedback controller is designed as (2.11) based on ESO (2.8) with $m = 3, n_1 = n_2 = n_3 = 2, M_i = 10$, $\alpha_i = (-4, -2)^\top, k_{i1} = k_{i2} = 3, k_{i3} = 1, i = 1, 2, 3$, and $r = 80$. We use Eular integration method in numerical simulations, where the integral step is taken 0.001, and the reference trajectory is chosen as $(\cos t, \sin t, -t)^\top$. The known control magnification matrix $B = I_{3\times 3}$, and the unknown functions $f_{i2}(t)$ are chosen as

$$\begin{aligned} f_{12}(t) =&\ x_{22}(t)x_{12}(t) + (1 + \sin(x_{12}(t)))x_{22}(t) \\ &+ 2x_{12}(t) + 2 + \sin t \\ f_{22}(t) =&\ -x_{12}(t)x_{32}(t) + (1 + \sin(x_{22}(t)))x_{32}(t) \\ &+ 2x_{22}(t) + \cos 2t \\ f_{32}(t) =&\ x_{32}(t) + e^{-t}\sin(t + \pi/4). \end{aligned} \tag{4.3}$$

Suppose that the output error $(e_{11}(t), e_{21}(t), e_{31}(t))$ is contaminated by the noise, that is, in ESO (2.8), $e_{i1}(t)$ is replaced by $e_{i1}(t) + 0.001\mathcal{N}(t)$, where $\mathcal{N}(t)$ is the standard Gaussian noise produced by the MATLAB command "randn." The numerical result by switching ESO is plotted in Fig. 1(a) and linear ESO in Fig. 1(b).

From Fig. 1, we can see that under the same noise and other parameters and system functions, the tracking effect by switching ESO is much better than linear ESO. The total disturbance estimation by switching ESO is plotted in Fig. 2, and that by linear ESO in Fig. 3.

It can be seen from Figs. 2 and 3 that the switching ESO can tolerate much more measurement noise than the linear ESO under the same tuning parameter and design parameters. In addition, the peaking value

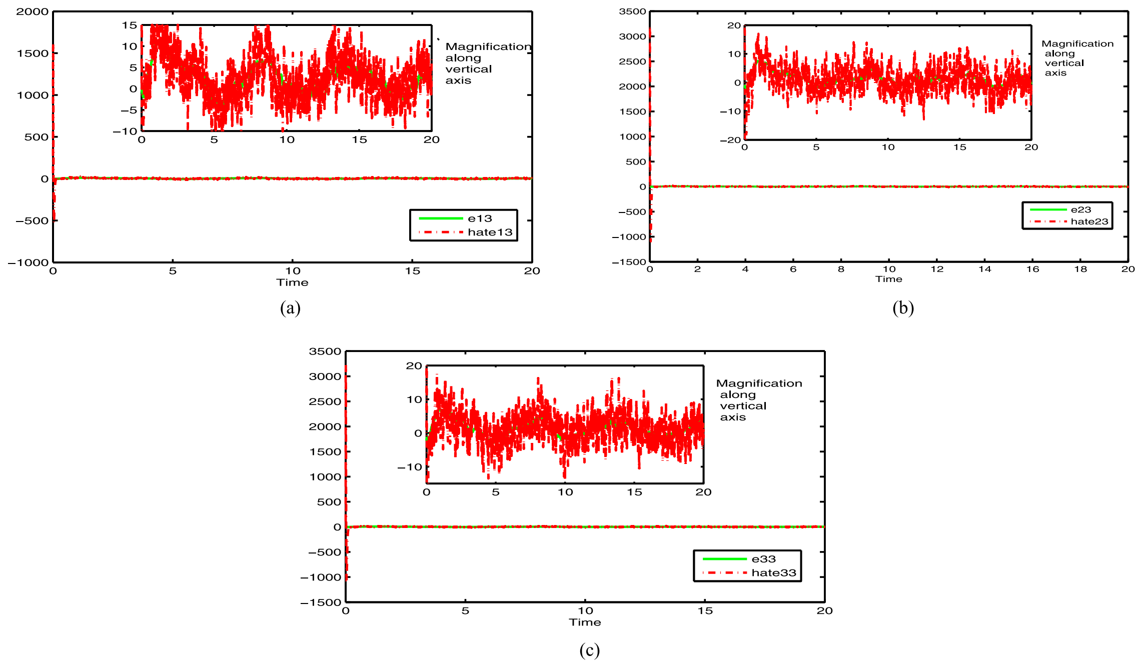


Fig. 3. Total disturbance estimation by linear ESO. (a) $e_{13}(t)$ and $\hat{e}_{13}(t; 80)$. (b) $e_{23}(t)$ and $\hat{e}_{23}(t; 80)$. (c) $e_{33}(t)$ and $\hat{e}_{33}(t; 80)$.

of switching ESO (less than 20) is much smaller than that of the linear ESO (near or great than 2000).

To end this section, we would emphasis again on the advantages of switching ESO for measurement noise tolerance and peaking value reduction. The reason behind the superb performance of the switching ESO is attributed to the saturation like behavior of fal$(\cdot)$. Suppose that the output of the system is contaminated by the noise $\mathcal{N}(t)$, that is, $y_i(t) = x_{i1}(t) + \mathcal{N}(t)$. The linear ESO is sensitive to the measurement noise for large r because the noise is magnified to be $r^j \mathcal{N}(t)$ in the jth equation of $\hat{e}_i$-subsystem of linear ESO [6]. The magnification coefficient of $\mathcal{N}(t)$ in $\hat{e}_i$-subsystem of switching ESO (2.8) is $r^{n_i \theta_j - n_i + j}$, which can be much smaller than r^j for large r. For instance for $n_i = 2$, $r = 80$, $\theta = 0.7$, which are the parameters used in the numerical simulation, in the third equation of switching ESO, the r-related term is $r^{n_i \theta_j - n_i + j} = 80^{1.2} \approx 192.1799$, whereas the corresponding term in the third-order linear ESO is $r^3 = 512000$.

A novel high-gain observer was proposed recently in [1] where good performance in the presence of measurement noise is obtained by increasing the order of the observer and limiting the power of the gain. The key idea here is similar [1] because the saturation-like behavior of fal$(\cdot)$ can also limit the power of the gain.

V. Conclusion

In this paper, we propose a fal-based, single-parameter-tuning ESO. The output tracking controller of mismatched uncertain MIMO systems is designed by using the fal-based ESO and saturation functions. The convergence of the fal-based ESO and the output tracking are established. Numerical simulations for a 3-D trajectory tracking of AUV confirm the effectiveness of the proposed method. Compared with the linear ESO under the same tuning gain, the fal-based ESO allows smaller peaking value. Furthermore, the latter also leads to better performance in the presence of measurement noise.

References

[1] D. Astolfi and L. Marconi, “A high-gain nonlinear observer with limited gain power,” *IEEE Trans. Automat. Control*, vol. 60, no. 11, pp. 3059–3064, Nov. 2015.

[2] A. Bacciotti and L. Rosier, *Lyapunov Functions and Stability in Control Theory*. Berlin, Germany: Springer-Verlag, 2005.

[3] S. P. Bhat and D.S. Bernstein, “Geometric homogeneity with applications to finite-time stability,” *Math. Control Signals Syst.*, vol. 17, pp. 101–127, 2005.

[4] J. Q. Han, “From PID to active disturbance rejection control,” *IEEE Trans. Ind. Electron.*, vol. 56, no. 3, pp. 900–906, Mar. 2009.

[5] J. P. Gauthier, H. Hammouri, and S. Othman, “A simple observer for nonlinear systems applications to bioreactors,” *IEEE Trans. Automat. Control*, vol. 37, no. 6, pp. 875–880, Jun. 1992.

[6] B. Z. Guo and Z. L. Zhao, *Active Disturbance Rejection Control for Nonlinear Systems: An Introduction*, Hoboken, NJ, USA: Wiley, 2016.

[7] A. Isidori, *Nolinear Control Systems*, 2nd ed. London, U.K.: Springer-Verlag, 1995.

[8] H. K. Khalil and L. Praly, “High-gain observers in nonlinear feedback control,” *Int. J. Robust Nonlinear Control*, vol. 24, pp. 993–1015, 2014.

[9] W. Perruquetti, T. Floquet, and E. Moulay, “Finite-time observers: Application to secure communication,” *IEEE Trans. Automat. Control*, vol. 53, no. 1, pp. 356–360, Feb. 2008.

[10] L. Rosier, “Homogeneous Lyapunov function for homogeneous continuous vector field,” *Syst. Control Lett.*, vol. 19, pp. 467–473, 1992.

[11] G. J. Silva, A. Datta, and S. P. Bhattacharrya, “New results on the synthesis of PID controllers,” *IEEE Trans. Automat. Control*, vol. 47, no. 2, pp. 241–252, Feb. 2002.

[12] A. Teel and L. Praly, “Tools for semiglobal stabilization by partial state and output feedback,” *SIAM J. Control Optim.*, vol. 33, pp. 1443–1488, 1995.

[13] L. J. Zhang, X. Qi, and Y. J. Pang, “Adaptive output feedback control based on DRFNN for AUV,” *Ocean Eng.*, vol. 36, pp. 716–722, 2009.

[14] Z. L. Zhao and B.Z. Guo, “A nonlinear extended state observer based on fractional power functions,” *Automatica*, vol. 81, pp. 286–296, 2017.

[15] Q. Zheng, L. Guo, and Z. Q. Gao, “On stability analysis of active disturbance rejection control for nonlinear time-varying plants with unknown dynamics,” in *Proc. IEEE Conf. Decis. Control*, 2007, pp. 3501–3506.

[16] Q. Zheng and Z. Gao, “An energy saving, factory-validated disturbance decoupling control design for extrusion processes,” in *Proc. World Congr. Intell. Control Autom.*, 2012, pp. 2891–2896.

[17] “TMS320F28069M, TMS320F28068M InstaSPINTM-MOTION Software,” *Technical Reference Manual, Literature No. SPRUHJ0A, Texas Instruments*, Dallas, TX, USA, Apr. 2013, Revised Nov. 2013.

[18] Kinetis Motor Suite, Freescale Semiconductor Inc., Austin, TX, USA, 2015. [Online]. Available: http://cache.freescale.com/files/soft_dev_tools/doc/fact_sheet/KINTMOTSUITFS.pdf. Accessed on: Jun. 5, 2015.