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Brief paper

# A nonlinear extended state observer based on fractional power functions ☆

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ABSTRACT

In this paper, we investigate a nonlinear extended state observer (ESO) constructed from piece-wise smooth functions consisted of linear and fractional power functions. This structure of ESO was first proposed in the 1990's and has been widely used in active disturbance rejection control for engineering controls. Its convergence, however, has remained an open problem up to this day. The main objective of this paper is to provide a convergence theory with explicit error estimation. The performances of this type ESO are studied by numerical simulation and compared with linear ESO. The numerical results show that the ESO proposed in this paper enjoys the advantages of smaller peaking value and better measurement noise tolerance.



## 1. Introduction

Due to its ability to deal with vast internal and external uncertainty, the active disturbance rejection control (ADRC) (Han, 2009) is becoming an emerging technology in control engineering. The last two decades have witnessed ADRC's success in many industrial applications including DC–DC power converter (Sun & Gao, 2005), flight vehicles control (Xia & Fu, 2013), Gasoline Engines (Xue et al., 2015), hydraulic systems control (Yao, Jiao, & Ma, 2014). The ADRC's characteristics of energy saving has also been demonstrated. For example, a 30% improvement in product performance capability index (Cpk) and 50% reduction in energy consumption were concluded in the test conducted in Parker Hannifin Parflex hose extrusion plant for over a period of eight months (Zheng & Gao, 2012).

The extended state observer (ESO) is central to ADRC. Note that the effect of the so-called "total disturbance" of system, which may contain internal uncertainty, external disturbance, and anything that is hard to model or deal with, can be exhibited in the observable measured output. Through a properly designed ESO, the "total disturbance" can be estimated. Then, it can be naturally canceled in the feedback loop.

A first ESO was proposed by J.Q. Han in late 1980's (Han, 2009) where there are multiple tuning parameters to be tuned to estimate system state and total disturbance. For easy use, Gao (2003) proposed a one-parameter tuning linear ESO in terms of bandwidth, where the high-gain approach is incorporated. The convergence of linear ESO, also known as extended high-gain observer in other context (Praly & Jiang, 2004; Freidovich & Khalil, 2008), is discussed in Zheng, Gao, and Gao (2007). Other types of linear ESO are subsequently proposed for various systems such as control and disturbance unmatched systems (Li, Yang, Chen, & Chen, 2012), and the system without a prior knowledge of nominal control parameter (Jiang, Huang, & Guo, 2015). Very recently, a linear ESO with adaptive gain is investigated in Xue et al. (2015).

In addition to these ESO aforementioned, the nonlinear function commonly used in ESO in practice is of the following form:

$$\mathrm{fal}(\tau,\alpha,\delta)=\begin{cases}\dfrac{\tau}{\delta^{1-\alpha}}, & |\tau|\le\delta,\\ |\tau|^{\alpha}\mathrm{sign}(\tau), & |\tau|>\delta,\end{cases} \tag{1.1}$$

where $0<\alpha<1$ and $\delta>0$ are constants. Based on numerous computer simulations and engineering practices, Han (2009) claimed that the ESO with nonlinear function of type (1.1) is quite effective for state and "total disturbance" estimation, leading to good performance including small peaking value. For nonlinear

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ESO on the other hand, although some progresses have been made in recent papers (Guo & Zhao, 2011; Zhao & Guo, 2015), none of them considers the nonlinear function (1.1). A fundamental theoretical question for this type of ESO is how to choose $\alpha$ and $\delta$ so that the convergence is guaranteed. Despite its importance, little is done toward answering this question since (1.1) was proposed. In this paper, we aim at providing an answer to this question by investigating convergence of ESO constructed from nonlinear function (1.1).

For the sake of exposition, we suppose in this paper that $\delta = 1$ since other cases can be similarly dealt with. Consider the following lower triangle nonlinear system with vast uncertainty:

$$\begin{cases} \dot{x}_1(t) = x_2(t) + \phi_1(t, u(t), x_1(t)), \\ \qquad \vdots \\ \dot{x}_{n-1}(t) = x_n(t) + \phi_{n-1}(t, u(t), x_1(t), \dots, x_{n-1}(t)), \\ \dot{x}_n(t) = f(t, x(t), w(t)) + \phi_n(t, u(t), x(t)), \\ y(t) = x_1(t), \end{cases} \tag{1.2}$$

where $x(t) = (x_1(t), \dots, x_n(t)) \in \mathbb{R}^n$ is the state, $\phi_i \in C(\mathbb{R}^{i+2}, \mathbb{R})$ are known system functions, $f \in C(\mathbb{R}^{n+2}, \mathbb{R})$ is an unknown system function, $y(t)$ is the measured output, $u(t)$ is the control input, $w(t)$ is the external disturbance. The "total disturbance" or "extended state" is denoted by

$$x_{n+1}(t) \triangleq f(t, x(t), w(t)). \tag{1.3}$$

We propose the following ESO for system (1.2):

$$\begin{cases} \dot{\hat{x}}_1(t; r) = \hat{x}_2(t; r) + \dfrac{k_1}{r^{n-1}} \mathscr{G}_1(r^n(x_1(t) - \hat{x}_1(t; r))) \\ \qquad + \phi_1(t, u(t), x_1(t)), \\ \qquad \vdots \\ \dot{\hat{x}}_n(t; r) = \hat{x}_{n+1}(t; r) + k_n \mathscr{G}_n(r^n(x_1(t) - \hat{x}_1(t; r))) \\ \qquad + \phi_n(t, u(t), x_1(t), \hat{x}_2(t; r), \dots, \hat{x}_n(t; r)), \\ \dot{\hat{x}}_{n+1}(t; r) = r k_{n+1} \mathscr{G}_{n+1}(r^n(x_1(t) - \hat{x}_1(t; r))), \end{cases} \tag{1.4}$$

where $r$ is a constant gain, $k_i$'s are constants to be chosen so that the following matrix is Hurwitz:

$$K = \begin{pmatrix} -k_1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_n & 0 & 0 & \cdots & 1 \\ -k_{n+1} & 0 & 0 & \cdots & 0 \end{pmatrix}_{(n+1)\times(n+1)}, \tag{1.5}$$

and $\{\mathscr{G}_i\}_{i=1}^n$ is of the form:

$$\mathscr{G}_i(\tau) = \mathrm{fal}(\tau, \theta_i, 1) \tag{1.6}$$

with $\theta_i \in (0, 1)$, $i = 1, 2, \dots, n+1$, being positive constants to be specified later.

The remaining part of the paper is organized as follows. In Section 2, we present convergence result of the ESO with fractional power function $\mathscr{G}_i(\cdot)$'s defined in (1.6). State observer reduced from ESO is also introduced. Since the proof of the main result is lengthy and needs some mathematical techniques, it is carried out separately in Section 3. The numerical simulations are presented in Section 4 to demonstrate the convergence as well as other properties including peaking value reduction and measurement noise tolerance.

## 2. Main results

In this section, we present the convergence of ESO (1.4) based on fractional power function (1.6). To this purpose, we make some basic assumptions on the plant.

**Assumption A1.** All the functions including the disturbance $w(t)$ and its derivative $\dot{w}(t)$, and the solution of (1.2) are supposed to be uniformly bounded. The unknown function is supposed to be $f \in C^1(\mathbb{R}^{n+2}, \mathbb{R})$ and there exists continuous function $\tilde{f} : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $|f(t, \xi)| + \left|\frac{\partial f(t,\xi)}{\partial t}\right| \le \tilde{f}(\xi), \quad \forall\, t \in [0, \infty), \quad \xi \in \mathbb{R}^{n+1}$. For the known functions $\phi_i \in C(\mathbb{R}^{i+2}, \mathbb{R})$, there exist continuous bounded function $\mathscr{L} \in C(\mathbb{R}^2, \mathbb{R})$ and continuous functions $\tilde{\phi}_i \in \mathbb{C}(\mathbb{R}^i, \mathbb{R})$ such that

$$\begin{cases} |\phi_i(t, u, \nu_1, \nu_2, \dots, \nu_i) - \phi_i(t, u, \nu_1, \tilde{\nu}_2, \dots, \tilde{\nu}_i)| \\ \quad \le \mathscr{L}(t, u) \|(\nu_2 - \tilde{\nu}_2, \dots, \nu_i - \tilde{\nu}_i)\|^{\alpha_i}, \\ |\phi_i(t, u, \nu_1, \dots, \nu_i)| \le \tilde{\phi}_i(\nu_1, \dots, \nu_i), \\ \alpha_i \in (0, 1],\ \nu_i, \tilde{\nu}_i \in \mathbb{R},\ i = 1, 2, \dots, m. \end{cases} \tag{2.1}$$

**Remark 2.1.** It is important to stress that we focus only on convergence of ESO for open loop system. The boundedness of state is used for estimation of state-dependent total disturbance. If the "total disturbance" is state-independent or only the state is estimated, the boundedness of the state can be removed, see Theorem 2.1 and Corollary 2.1 later. In addition, the state is bounded in most practical control systems such as those for faults diagnosis (Yan, Tian, Shi, & Wang, 2008). Finally, since the ESO is designed for control purpose, in case that the system is not bounded, we can also use feedback to make the system bounded, which will be investigated in the forthcoming paper.

Let

$$\alpha = \max_{1\le i\le n} (n+1-i)(1-\alpha_i), \qquad \alpha^* = \min_{1\le i\le n} \alpha_i. \tag{2.2}$$

The main result is stated as Theorem 2.1.

**Theorem 2.1.** *Suppose that in system (1.2), $\alpha_i \in (0, 1]$, $\alpha < 1$, and Assumption A1 holds. Let $\theta_i = i\theta - (i-1)$, $i = 1, 2, \dots, n+1$ in ESO (1.4). Then there exist $\theta^* \in (n/(n+1), 1)$ and $r^* > 0$ such that for any $\theta \in [\theta^*, 1)$, $r > r^*$, and any initial state $(x_{10}, x_{20}, \dots, x_{n0})$ of system (1.2) and initial state $(\hat{x}_{10}, \hat{x}_{20}, \dots, \hat{x}_{n0}, \hat{x}_{(n+1)0})$ of ESO (1.4), the observer errors satisfy, for any $t > t_r$, $i = 1, 2, \dots, n+1$, that*

$$|x_i(t) - \hat{x}_i(t; r)| \le \Gamma (1/r)^{n+1-i+\frac{1}{(1-\alpha)(2-\alpha^*)}}, \tag{2.3}$$

*where $t_r > 0$ is $r$-dependent and satisfies $\lim_{r\to\infty} t_r = 0$, $x_{n+1}(t)$ is the total disturbance defined in (1.3), and $\Gamma$ is an $r$-independent constant defined in (3.56).*

*Moreover, if the "total disturbance" (1.3) is independent of the state, that is, $f(\cdot) = w(t)$, then (2.3) holds without assuming the boundedness of the system state.*

We first point out two features of ESO (1.4) where $\mathscr{G}_i(\tau)'s$ play the role of somehow saturation-like behaviors. The other two merits of peaking value reduction and noise tolerance will be discussed at the end of Section 4.

It is seen from (2.3) that the error between the state of ESO (1.4) and state of system (1.2) including total disturbance can be made as small as desired by tuning gain parameter $r$ to be large enough. In fact, (2.3) together with $\lim_{r\to\infty} t_r = 0$ implies that for any $T > 0$, $i = 1, 2, \dots, n+1$,

$$\lim_{r\to\infty} \sup_{t\in[T,\infty)} |x_i(t) - \hat{x}_i(t; r)| = 0. \tag{2.4}$$

Generally speaking, to make state of ESO approximate state and total disturbance to an acceptable small error, the gain parameter $r$ should be tuned according to variation speed of the total disturbance: The smaller the variation speed of the total disturbance, the smaller tuning parameter $r$. If the total disturbance is not varying with time (a constant: $f(\cdot) = \bar{d} \in \mathbb{R}$), then a small tuning gain can guarantee asymptotic convergence.

**Corollary 2.1.** *Suppose that in system (1.2), (2.1) holds with $\alpha_i \in (0,1]$ and $\alpha < 1, f(\cdot) = \bar{d}$. Let $\theta_i = i\theta - (i-1), i = 1, 2, \ldots, n+1$ in ESO (1.4). Then there exist $\theta^* \in (0,1)$ and $r^* > 0$, such that for any $\theta \in [\theta^*, 1)$, $r > r^*$, initial state $(x_{10}, x_{20}, \ldots, x_{n0})$ of system (1.2), and initial state $(\hat{x}_{10}, \hat{x}_{20}, \ldots, \hat{x}_{n0}, \hat{x}_{(n+1)0})$ of ESO (1.4), the observer errors satisfy*

(i) *If $\alpha^* < 1$, then for any $t > t_r,\ i = 1, 2, \ldots, n+1$,*

$$|x_i(t) - \hat{x}_i(t,r)| < \tilde{\Gamma}(1/r)^{n+1-i+\frac{1}{(1-\alpha)(2-\alpha^*)}}, \tag{2.5}$$

*where $t_r > 0$ is an $r$-dependent constant satisfying $\lim_{t\to\infty} t_r = 0$, $x_{n+1}(t) = \bar{d}$, and $\tilde{\Gamma}$ is given by (3.61);*

(ii) *If $\alpha^* = 1$, then*

$$\lim_{t\to\infty} |x_i(t) - \hat{x}_i(t,r)| = 0, \quad i = 1, 2, \ldots, n+1, \tag{2.6}$$

*where $\alpha$ and $\alpha^*$ are defined in (2.2) and $x_{n+1}(t) = \bar{d}$.*

When the uncertainty term $f(\cdot) \equiv 0$, the total disturbance estimation is not necessary and ESO (1.4) is therefore reduced to a state observer by removing the $(n+1)$th equation in (1.4). In this case, we can prove that the observer error satisfies

(i) If $\alpha^* < 1$, then $\forall\ t > t_r,\ i = 1, 2, \ldots, n$,

$$|x_i(t) - \hat{x}_i(t,r)| < \hat{\Gamma}(1/r)^{n-i+\frac{1}{(1-\alpha)(2-\alpha^*)}}, \tag{2.7}$$

where $t_r$ is an $r$-dependent constant satisfying $\lim_{t\to\infty} t_r = 0$, and $\hat{\Gamma}$ is a constant independent of $r$.

(ii) If $\alpha^* = 1$, then

$$\lim_{t\to\infty} |x_i(t) - \hat{x}_i(t,r)| = 0, \quad i = 1, 2, \ldots, n, \tag{2.8}$$

where $\alpha$ and $\alpha^*$ are defined in (2.2). In this case, we do not need to suppose that the solution is bounded.

The main contribution of this paper is proving convergence for "fal$(\cdot)$" based ESO under the Hölder continuous assumption on a class of open-loop systems with vast uncertainty, a long standing problem in ADRC. It is indicated numerically that the advantages of ESO constructed from "fal$(\cdot)$" are smaller peaking value and better performance under measurement noise. Recently, some nonlinear functions such as homogeneous functions are adopted in observer design for Hölder continuous nonlinear systems such as Andrieu, Praly, and Astolfi (2008), Levant (2003) and Yang and Lin (2004). However, to the best of our knowledge, no observer is constructed from the piecewise smooth function composed of fractional power function and linear function like fal$(\cdot)$, and no observer is designed for the general Hölder continuous nonlinear system satisfying (2.1) up to date.

## 3. Proof of main results

The difficulty in proving Theorem 2.1 lies in the fact that ESO (1.4) switches between the linear ESO (Guo & Zhao, 2011, Corollary 2.1) and fractional power ESO (Guo & Zhao, 2011, Corollary 2.2). It is intractable to examine the switching times on the two switching hypersurfaces $\{(z_1, z_2, \ldots, z_{n+1}) \in \mathbb{R}^{n+1} |\ z_1 = 1\}$ and $\{(z_1, z_2, \ldots, z_{n+1}) \in \mathbb{R}^{n+1} |\ z_1 = -1\}$. And it seems impossible to derive the exact solution of the nonlinear ESO.

The main steps of the proof are outlined below:

(1) Let

$$\eta_i(t;r) = r^{n+1-i}(x_i(t) - \hat{x}_i(t;r)), \quad i = 1, 2, \ldots, n+1 \tag{3.1}$$

be the re-scaled errors between solutions of system (1.2) and ESO (1.4). A straightforward computation shows that

$$\eta(t;r) = (\eta_1(t;r), \eta_2(t;r), \ldots, \eta_{n+1}(t;r))$$

satisfies the following equation:

$$\frac{d\eta(t;r)}{dt} = r\mathscr{G}(\eta(t;r)) + \Phi(t;r), \tag{3.2}$$

where $\mathscr{G}(\cdot)$ is given in (3.4) and

$$\begin{aligned} &\Phi(t;r) = (\Phi_1(t;r), \ldots, \Phi_{n+1}(t;r))^\top \\ &\Phi_i(t;r) = r^{n+1-i}(\phi_i(t, u(t), x_1(t), \ldots, x_i(t)) \\ &\quad - \phi_i(t, u(t), x_1(t), \ldots, \hat{x}_i(t;r))), \quad i = 1, 2, \ldots, n, \\ &\Phi_{n+1}(t;r) = \dot{x}_{n+1}(t). \end{aligned} \tag{3.3}$$

(2) Construct two Lyapunov functions $V_\theta(\cdot)$ and $V_L(\cdot)$ according to fractional power and linear functions in fractional power ESO, respectively.

(3) By analyzing the derivative of $V_\theta(\cdot)$ and $V_L(\cdot)$ along the error Eq. (3.2), we show that the state of (3.2) enters and will be staying in a compact set that contains the original state and lies in the domain between two switching hypersurfaces. This step is the most difficult part.

(4) The proof of the theorem is accomplished by analyzing the derivative of $V_L(\cdot)$ along the error Eq. (3.2).

In what follows, we give the detailed proof of the theorem. First of all, we introduce two auxiliary vector fields and an auxiliary system. Let

$$\begin{aligned} \mathscr{G}(z) &= (z_2 - k_1\mathscr{G}_1(z_1), \ldots, -k_{n+1}\mathscr{G}_{n+1}(z_1))^\top, \\ F(z) &= (F_1(z), \ldots, F_{n+1}(z))^\top, \\ &= (z_2 - k_1[z_1]^{\theta_1}, \ldots, -k_{n+1}[z_1]^{\theta_{n+1}})^\top, \end{aligned} \tag{3.4}$$

and

$$\dot{z}(t) = F(z(t)), \quad z = (z_1, z_2, \ldots, z_{n+1})^\top \in \mathbb{R}^{n+1}. \tag{3.5}$$

Now we show that system (3.5) is homogeneous of degree $d = \theta - 1$ with weights $\{r_i = (i-1)\theta - (i-2)\}_{i=1}^{n+1}$. The details of weighted homogeneity can be found in Bhat and Bernstein (2005).

Let matrix $K$ in (1.5) be Hurwitz. By Bhat and Bernstein (2005), Perruquetti, Floquet, and Moulay (2008), and Rosier (1992), there exists a positive definite, and radially unbounded Lyapunov function $V_\theta : \mathbb{R}^{n+1} \to \mathbb{R}$ which is homogeneous of degree $\gamma > 1$ with weights $\{r_i\}_{i=1}^{n+1}$. Moreover, the Lie derivative of $V_\theta(z)$ along the vector field $F(z)$ is negative definite.

It follows from the homogeneity of $V_\theta(\cdot)$ that $L_F(V_\theta(z))$ and $\frac{\partial V_\theta(z)}{\partial z_i}$ are also weighted homogeneous functions. Let $\chi_i(z_1, z_2, \ldots, z_{(n+1)}) = |z_i|$. For any $\lambda > 0$, $\chi_i(\lambda^{r_1}z_1, \lambda^{r_2}z_2, \ldots, \lambda^{r_{n+1}}z_{n+1}) = \lambda^{r_i}|z_i| = \lambda^{r_i}\chi_i(z_1, z_2, \ldots, z_{n+1})$. Hence $\chi_i(z_1, z_2, \ldots, z_{(n+1)}) = |z_i|$ is homogeneous of degree $r_i$ with weights $\{r_i\}_{i=1}^{n+1}$.

**Lemma 3.1** (*Lemma 4.2 of Bhat & Bernstein, 2005*). *Suppose that $V_1(z)$ and $V_2(z)$ are continuous real-valued functions over $\mathbb{R}^{n+1}$, homogeneous of degrees $d_1$ and $d_2$ respectively with weights $\{r_i\}_{i=1}^{n+1}$. Assume that $V_1(z)$ is positive definite. Then, for every $z \in \mathbb{R}^{n+1}$,*

$$\begin{aligned} &\left(\min_{z\in\{\nu\in\mathbb{R}^{n+1}|\ V_1(\nu)=1\}} V_2(z)\right)(V_1(z))^{\frac{d_2}{d_1}} \le V_2(z) \\ &\le \left(\max_{z\in\{\nu\in\mathbb{R}^{n+1}|\ V_1(\nu)=1\}} V_2(z)\right)(V_1(z))^{\frac{d_2}{d_1}}. \end{aligned} \tag{3.6}$$

By (3.6), together with the homogeneity of $\frac{\partial V_\theta(z)}{\partial z_i}$, $L_F V_\theta(z)$, and $|z_i|$, we have

$$\begin{aligned} &\left|\frac{\partial V_\theta(z)}{\partial z_i}\right| \le B_1(V_\theta(z))^{\frac{\gamma - r_i}{\gamma}}, \\ &(L_F(V_\theta))(z) \le -B_2(V_\theta(z))^{\frac{\gamma+d}{\gamma}}, \\ &|z_i| \le B_3(V_\theta(z))^{\frac{r_i}{\gamma}}, \quad z \in \mathbb{R}^{n+1},\ B_i > 1. \end{aligned} \tag{3.7}$$

Let

$$V_L(z) = z^\top P z, \quad z \in \mathbb{R}^{n+1}, \tag{3.8}$$

and

$$V_{\max}(z) = \max_{\theta\in[\theta_1^*,1]} V_\theta(z), \qquad V_{\min}(z) = \min_{\theta\in[\theta_1^*,1]} V_\theta(z). \tag{3.9}$$

By the properties of $V_\theta(z)$, $V_{\max}(z)$ and $V_{\min}(z)$ are continuous, positive definite and radially unbounded. Therefore there exist class $\mathcal{K}_\infty$ functions (Khalil, 2002) $\kappa_{\theta i}, \kappa_i, \tilde{\kappa}_i (i = 1, 2) : [0,\infty) \to [0,\infty)$ such that

$$\begin{aligned} &\kappa_{\theta 1}(\|z\|) \le V_\theta(z) \le \kappa_{\theta 2}(\|z\|), \\ &\kappa_1(\|z\|) \le V_{\max}(z) \le \kappa_2(\|z\|), \\ &\tilde{\kappa}_1(\|z\|) \le V_{\min}(z) \le \tilde{\kappa}_2(\|z\|), \quad \forall\, z \in \mathbb{R}^{n+1}. \end{aligned} \tag{3.10}$$

In addition, the Lyapunov function $V_L(z)$ satisfies

$$\lambda_{\min}(P)\|z\|^2 \le V_L(z) \le \lambda_{\max}(P)\|z\|^2. \tag{3.11}$$

The above arguments are about properties of the constructed Lyapunov functions. Now we estimate the bound of norm of $\Phi(t;r)$ given by (3.3). From now on, we shall use $\underline{e_i}$ to denote an $i$-dimensional vector $\underline{e_i} = (e_1, e_2, \ldots, e_i)$, $e_i \in \mathbb{R}$, $i \in \mathbb{N}^+$. By Assumption A1, there exists $M_1$ such that $|\dot{x}_{n+1}(t)| \le M_1$. If $f(\cdot) = w(t)$, $|\dot{x}_{n+1}(t)|$ is also uniformly bounded without requiring the boundedness of system state and control input. By using Assumption A1 again, there exists $M_2 > 0$ such that $|\mathcal{L}(t, u(t))| \le M_2$ for all $t \in [0,\infty)$. Hence for $r > 1$,

$$\begin{aligned} &r^{n+1-i}|\phi_i(t, u(t), x_1(t), x_2(t), \ldots, x_i(t)) \\ &\quad - \phi_i(t, u(t), x_1(t), \hat{x}_2(t;r), \ldots, \hat{x}_i(t;r))| \\ &\le M_2 r^{(n+1-i)(1-\alpha_i)} \|\underline{\eta_i(t;r)}\|^{\alpha_i}. \end{aligned} \tag{3.12}$$

This, together with (3.3), yields

$$\|\Phi(t;r)\| \le M_1 + M_2 \sum_{i=1}^{n} r^{(n+1-i)(1-\alpha_i)} \|\eta(t;r)\|^{\alpha_i}. \tag{3.13}$$

Now we estimate the bound of $\|F(\cdot) - \mathscr{G}(\cdot)\|$. By the definition of $\mathscr{G}_i$ defined in (1.6), for any $e = (e_1, e_2, \ldots, e_{n+1}) \in \mathbb{R}^{n+1}$, if $|e_1| > 1$, then $\mathscr{G}(e) = F(e)$, and when $|e_1| \le 1$,

$$\|\mathscr{G}(e) - F(e)\| \le \max_{1\le i\le n+1, |e_1|\le 1} \left| e_1 - [e_1]^{\theta_i} \right| \|\underline{k_{n+1}}\|, \tag{3.14}$$

where $F(e)$ and $\mathscr{G}(e)$ are defined in (3.4). Hence (3.14) are valid on whole $\mathbb{R}^{n+1}$.

By

$$\lim_{\theta\to 1} \max_{\tau\in[-1,1]} \left| [\tau]^{\theta_i} - \tau \right| = 0, \tag{3.15}$$

for

$$\tilde{\delta}_1 = B_2/(2B_1(n+1)\|\underline{k_{n+1}}\|), \tag{3.16}$$

there exists $\theta_2^* \in [\theta_1^*, 1)$ such that for any $\theta \in [\theta_2^*, 1)$,

$$\max_{1\le i\le n+1, |e_1|\le 1} \left| e_1 - [e_1]^{\theta_i} \right| < \tilde{\delta}_1. \tag{3.17}$$

Now we introduce the following auxiliary weighted homogeneous function $\Psi : \mathbb{R}^{n+1} \to \mathbb{R}$ to estimate the bound of $V_\theta(\eta(0,r))$:

$$\Psi(z_1, z_2, \ldots, z_{n+1}) = |z_1|^{\frac{\gamma}{r_1}} + \cdots + |z_{n+1}|^{\frac{\gamma}{r_{n+1}}}. \tag{3.18}$$

It is easy to verify that $\Psi(z)$ is positive definite, homogeneous of degree $\gamma$ with weights $\{r_i\}_{i=1}^{n+1}$. By Lemma 3.1, there exists $c_1 > 0$ such that $V_\theta(e) \le c_1\Psi(e)$ for all $e \in \mathbb{R}^{n+1}$ and $\theta > (n-1)/n$. Therefore

$$V_\theta(\eta(0;r)) \le A r^{n\gamma}, \quad A = c_1 \sum_{i=1}^{n+1} |x_1(0) - \hat{x}_1(0)|^{\frac{\gamma}{r_i}}. \tag{3.19}$$

Let

$$\mathscr{D}_1 = \left\{ e \in \mathbb{R}^{n+1} \,\middle|\, V_\theta(e) \le A r^{n\gamma} \right\}, \tag{3.20}$$

$$\mathscr{D}_2 = \left\{ e \in \mathbb{R}^{n+1} \,\middle|\, V_{\min}(e) \le \max\{\tilde{\kappa}_2(1), 1\} \right\}, \tag{3.21}$$

and

$$\mathscr{D}_3 = \left\{ e \in \mathbb{R}^{n+1} \,\middle|\, V_L(e) \le \min\left\{ 1, \frac{\lambda_{\min}(P)}{2} \right\} \right\}, \tag{3.22}$$

where $V_L(e)$ is defined by (3.8) and $V_{\min}(e)$ by (3.9). It is seen that $\mathscr{D}_3 \subset \mathcal{U}(1/2) \subset \mathcal{U}(1) \subset \mathscr{D}_2$, $\mathcal{U}(\rho) = \{e \in \mathbb{R}^{n+1} | \; \|e\| \le \rho\}$, $\rho > 0$. It is also seen from (3.19) that $\eta(0;r) \in \mathscr{D}_1$. Since $\lim_{r\to\infty} \mathscr{D}_1 = \mathbb{R}^{n+1}$ and $\mathscr{D}_2$ is an $r$-independent compact set, there exists $r_1^* > 0$ such that $\mathscr{D}_2 \subset \mathscr{D}_1$ and $\mathscr{D}_2 \ne \mathscr{D}_1$ for any $r > r_1^*$. Let $\eta(t;r)$ be defined in (3.1). By (3.2),

$$\dot{\eta}(t;r) = rF(\eta(t;r)) + r(\mathscr{G}(\eta(t;r)) - F(\eta(t;r))) + \Phi(t;r), \tag{3.23}$$

where $\Phi(t;r)$ is given by (3.3).

The proof of Theorem 2.1 is mainly based on ultimate boundedness of the state of system (3.23) by the compact set $\mathscr{D}_3$ defined in (3.22).

**Proposition 3.1.** *Let $\eta(t;r)$ be the solution of system (3.23). Then there exists $\hat{r}^* > 1$ such that*

$$\eta(t;r) \in \mathscr{D}_3, \quad \forall\, r > \hat{r}^*,\; t > t_{2r}, \tag{3.24}$$

*where $t_{2r}$ is an $r$-dependent constant and $t_{2r} \to 0$ as $r \to \infty$.*

**Proof.** Finding the derivative of $V_\theta(\eta(t;r))$ with respect to $t$ along the error system (3.23), we obtain

$$\begin{aligned} \left.\frac{dV_\theta(\eta(t;r))}{dt}\right|_{(3.23)} &= r(L_F(V_\theta))(\eta(t;r)) \\ &+ rL_{(\mathscr{G}-F)}V_\theta(\eta(t;r)) + \sum_{i=1}^{n+1} \frac{\partial V_\theta}{\partial \eta_i}(\eta(t;r))\Phi_i(t;r). \end{aligned} \tag{3.25}$$

It follows from (3.3), (3.12), and (3.7) that

$$\begin{aligned} \left.\frac{dV_\theta(\eta(t;r))}{dt}\right|_{(3.23)} &\le -rB_2(V_\theta(\eta(t;r)))^{\frac{\gamma+d}{\gamma}} \\ &+ r\tilde{\delta}_1\|\underline{k_{n+1}}\|B_1 \sum_{i=1}^{n+1} V_\theta(\eta(t;r))^{\frac{\gamma-r_i}{\gamma}} \\ &+ B_1 M_2 r^{\alpha} \sum_{i=1}^{n} (V_\theta(\eta(t;r)))^{\frac{\gamma+\alpha_i r_i - r_i}{\gamma}} \\ &+ B_1 M_1 (V_\theta(\eta(t;r)))^{\frac{\gamma-r_{n+1}}{\gamma}}. \end{aligned} \tag{3.26}$$

The proof will be accomplished by splitting separately into three different cases.

*Case* 1: *For any $i = 1, 2, \ldots, n$, $\alpha_i < 1$ and $\alpha = \max(n+1-i)(1-\alpha_i) < 1$.*

For any $\theta \in ((i-1)/i, 1)$, it is easy to verify that $1 - \theta \le (i-1)\theta - (i-2) = r_i$, and hence

$$(\gamma - 1 + \theta)/\gamma \ge (\gamma - r_i)/\gamma. \tag{3.27}$$

For any $\theta \in ((1+(1-\alpha_i)(i-2))/(1+(1-\alpha_i)(i-1)), 1)$, we can obtain $1-\theta \le (1-\alpha_i)((i-1)\theta-(i-2)) = (1-\alpha_i)r_i$. It then follows that

$$(\gamma-1+\theta)/\gamma \ge (\gamma+(\alpha_i-1)r_i)/\gamma. \tag{3.28}$$

Let

$$\mathscr{D}_\theta = \left\{e \in \mathbb{R}^{n+1} |\ V_\theta(e) \le 1\right\}. \tag{3.29}$$

By the definition of $\mathscr{D}_\theta$, $V_\theta(e) > 1$ for any $e \in \mathscr{D}_1 - \mathscr{D}_\theta$. This combines (3.26), (3.27), and (3.28) to obtain that if $\eta(t;r) \in \mathscr{D}_1 - \mathscr{D}_\theta$,

$$\left.\frac{dV_\theta(\eta(t;r))}{dt}\right|_{(3.23)} \le -(rB_2 - r\tilde{\delta}_1 nB_1\|\underline{k_{n+1}}\| - B_1M_1 - (nB_1B_3M_2)r^{\alpha})(V_\theta(\eta(t;r)))^{\frac{\gamma+d}{\gamma}}. \tag{3.30}$$

Let

$$r_2^* = \max\left\{\frac{8B_1M_1}{B_2}, \ \left(\frac{8nB_1B_3M_2}{B_2}\right)^{\frac{1}{1-\alpha}}\right\}.$$

By (3.16) and (3.30), it follows that for any $\theta \in [\theta_2^*, 1)$ and $r > r_2^*$,

$$\begin{aligned}\left.\frac{dV_\theta(\eta(t;r))}{dt}\right|_{(3.23)} &\le -\left(rB_2 - \frac{rB_2}{2} - \frac{rB_2}{8} - \frac{rB_2}{8}\right)(V_\theta(\eta(t;r)))^{\frac{\gamma+d}{\gamma}}\\ &\le -\frac{r}{4}(V_\theta(\eta(t;r)))^{\frac{\gamma+d}{\gamma}}\\ &\le -\frac{r}{4}\min_{e\in\mathscr{D}_1-\mathscr{D}_\theta}(V_\theta(e))^{\frac{\gamma+d}{\gamma}} < 0.\end{aligned} \tag{3.31}$$

*Case* 2: $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$. In this case, we can obtain that

$$\left.\frac{dV_\theta(\eta(t;r))}{dt}\right|_{(3.23)} \le -rB_2(V_\theta(\eta(t;r)))^{\frac{\gamma+d}{\gamma}} + r\tilde{\delta}_1B_1\|\underline{k_{n+1}}\|\sum_{i=1}^{n+1}(V_\theta(\eta(t;r)))^{\frac{\gamma-r_i}{\gamma}} + nB_1B_3M_2V_\theta(\eta(t;r)) + B_1M_1(V_\theta(\eta(t;r)))^{\frac{\gamma-r_{n+1}}{\gamma}}. \tag{3.32}$$

Similarly with *Case 1*, when $\eta(t;r) \in \mathscr{D}_1 - \mathscr{D}_\theta$, we can obtain

$$\left.\frac{dV_\theta(\eta(t;r))}{dt}\right|_{(3.23)} \le nB_1B_3M_2V_\theta(\eta(t;r)) - (rB_2 + r\tilde{\delta}_1nB_1\|\underline{k_{n+1}}\| + B_1M_1)(V_\theta(\eta(t;r)))^{\frac{\gamma+d}{\gamma}}. \tag{3.33}$$

For any $r > r_2^*$,

$$\frac{dV_\theta(\eta(t;r))}{dt} \le -\frac{3r}{8}(V_\theta(\eta(t;r)))^{\frac{\gamma+d}{\gamma}} + nB_1B_3M_2V_\theta(\eta(t;r)) \tag{3.34}$$

For any $e = (e_1, e_2, \ldots, e_{n+1})^\top \in \mathscr{D}_1$, it follows from (3.20) that $V_\theta(e) \le Ar^{n\gamma}$.

Let

$$\theta^* = \max\left\{\theta_2^*, \ \frac{n-1}{n}\right\},$$
$$r_3^* = \max\left\{r_2^*, \ \left(\frac{8nB_1B_3M_2A^{(1-\theta)/\gamma}}{B_2}\right)^{\frac{1}{n\theta-(n-1)}}\right\}. \tag{3.35}$$

For any $\theta \in (\theta^*, 1)$, $1-n(1-\theta) = n\theta-(n-1) > 0$. For any $r > r_3^*$, a simple computation shows that $B_2r^{1-n(1-\theta)} > 8nB_1B_3M_2A^{(1-\theta)/\gamma}$. This, together with $V_\theta(e) \le Ar^{n\gamma}, \ \forall\ e \in \mathscr{D}_1$, shows that if $\eta(t;r) \in \mathscr{D}_1$, then

$$B_2r \ge 8nB_1B_3M_2(V_\theta(t;r))^{\frac{1-\theta}{\gamma}}. \tag{3.36}$$

Since $d = \theta - 1$, we have arrived at

$$\frac{rB_2}{8}(V_\theta(t;r))^{\frac{\gamma+d}{\gamma}} \ge nB_1B_3M_2V_\theta(t;r). \tag{3.37}$$

This together with (3.34) concludes that if $\eta(t;r) \in \mathscr{D}_1 - \mathscr{D}_\theta$, then (3.31) holds true for any $r > r_3^*$,

*Case* 3: $\alpha_{i_1} = \alpha_{i_2} = \cdots = \alpha_{i_m} = 1, \alpha_{i_j} \in (0,1), j = m+1, \ldots, n$. In this case, similarly with *Cases* 1 *and* 2, we can also prove that there exists $r_4^* > 0$ such that (3.31) holds true for $\eta(t;r) \in \mathscr{D}_1 - \mathscr{D}_\theta$, for any $r > r_4^*$,

Let $r_5^* = \max\{r_3^*, r_4^*\}$. With the same arguments above, we can also show that for any $r > r_5^*$, (3.31) holds true if $\eta(t;r) \in \mathscr{D}_1 - \mathscr{D}_\theta$, no matter $\alpha_i < 1$ or $\alpha_i = 1$, or even some of these numbers are less than one and some equal to one.

From (3.31), we know that for any $\theta \in [\theta_2^*, 1)$ and $r > r^*$, $V_\theta(\eta(t;r))$ is strictly decreasing in $\mathscr{D}_1 - \mathscr{D}_\theta$. So there exists $t_{1r} > 0$ such that $\{\eta(t;r)|\ t > t_{1r}\} \subset \mathscr{D}_\theta \subset \mathscr{D}_2$. In what follows, we estimate the time $t_{1r}$. Let $\zeta(t)$ be a nonnegative function and satisfy the following initial value problem:

$$\dot{\zeta}(t) = -\frac{r}{4}(\zeta(t))^{\frac{\gamma+d}{\gamma}}, \qquad \zeta(0) = V_\theta(\eta(0;r)).$$

A direct computation shows that

$$\zeta = \begin{cases}\left(-\frac{|d|r}{4\gamma}t + (\zeta(0))^{\frac{|d|}{\gamma}}\right)^{\frac{\gamma}{|d|}}, & t \le \frac{4\gamma}{|d|}\frac{(\zeta(0))^{\frac{|d|}{\gamma}}}{r},\\ 0, & t > \frac{4\gamma}{|d|}\frac{(\zeta(0))^{\frac{|d|}{\gamma}}}{r}.\end{cases} \tag{3.38}$$

By (3.31) and applying the comparison principle of ordinary differential equations, we have

$$t_{1r} \le \frac{4\gamma}{|d|}r(\zeta(0))^{\frac{|d|}{\gamma}} \le \frac{4\gamma}{1-\theta}A^{\frac{1-\theta}{\gamma}}\left(\frac{1}{r}\right)^{n\theta-(n-1)}. \tag{3.39}$$

Let

$$\tau^* = \max_{e\in\mathscr{D}_2}|e_1| < \infty,$$
$$\tilde{\delta}_2 = \frac{\min_{e\in\mathscr{D}_2}\|e\|}{4\lambda_{\max}(P)\|\underline{k_{n+1}}\|\max_{e\in\mathscr{D}_2}\|e\|}. \tag{3.40}$$

For every $\tau \in [-\tau^*, \tau^*]$ and each $1 \le i \le n+1$, since $|\mathscr{G}_i(\tau)-\tau|$ is continuous in $\theta$, $\max_{i=1,\ldots,n+1,\tau\in[-\tau^*,\tau^*]}|\mathscr{G}_i(\tau)-\tau|$ is $\theta$-continuous as well. Considering $\max_{i=1,\ldots,n+1,\tau\in[-\tau^*,\tau^*]}|\mathscr{G}_i(\tau)-\tau| = 0$ for $\theta = 1$, we can derive that there exists $\theta_3^* \in [\theta_2^*, 1)$ such that for any $\theta \in [\theta_3^*, 1)$,

$$\max_{i=1,\ldots,n+1,\tau\in[-\tau^*,\tau^*]}|\mathscr{G}_i(\tau)-\tau| < \tilde{\delta}_2. \tag{3.41}$$

Next, finding the derivative of $V_L(\eta(t;r))$ along the solution $\eta(t;r)$ of (3.23) in the compact set $\mathscr{D}_2$, where $V_L(\cdot)$ is defined by (3.8), yields

$$\left.\frac{dV_L(\eta(t;r))}{dt}\right|_{(3.23)} \le 2\lambda_{\max}(P)\|\Phi(t;r)\|\,\|\eta(t;r)\| + 2r\lambda_{\max}(P)\|\mathscr{G}(\eta(t;r)) - K\eta(t;r)\|\,\|\eta(t;r)\| - r\|\eta(t;r)\|^2. \tag{3.42}$$

This together with (3.13) and (3.41) gives that for any $\theta \in [\theta_3^*, 1)$, $r > r_5^*$, and $t > t_{1r}$, if $\eta(t;r) \in \mathscr{D}_2 - \mathscr{D}_3$, then

$$\begin{aligned}\left.\frac{dV_L(\eta(t;r))}{dt}\right|_{(3.23)} &\le -r \min_{e\in\mathscr{D}_2-\mathscr{D}_3} \|e\|^2 + 2M_1\lambda_{\max}(P) \max_{e\in\mathscr{D}_2-\mathscr{D}_3} \|e\| \\ &+ 2r\tilde{\delta}_2\lambda_{\max}(P)\|\underline{k_{n+1}}\| \max_{e\in\mathscr{D}_2-\mathscr{D}_3} \|e\| \\ &+ 2r^{\alpha}M_2\lambda_{\max}(P) \max_{e\in\mathscr{D}_2-\mathscr{D}_3} \sum_{i=1}^{n} \|e\|^{\alpha_i+1}, \end{aligned} \tag{3.43}$$

where $\alpha$ is given in (2.2). Let

$$r_6* = \max\left\{ r_5^*, \frac{16M_1\lambda_{\max}(P) \max_{e\in\mathscr{D}_2-\mathscr{D}_3} \|e\|}{\min_{e\in\mathscr{D}_2-\mathscr{D}_3} \|e\|}, \times \left( \frac{16M_2\lambda_{\max}(P) \max_{e\in\mathscr{D}_2-\mathscr{D}_3} \sum_{i=1}^{n} \|e\|^{\alpha_i+1}}{\min_{e\in\mathscr{D}_2-\mathscr{D}_3} \|e\|^2} \right)^{\frac{1}{1-\alpha}} \right\}. \tag{3.44}$$

Notice (3.40). For any $\theta \in [\theta_3^*, 1)$, $r > r_6^*$, and $t > t_{1r}$, if $\eta(t;r) \in \mathscr{D}_2 - \mathscr{D}_3$, the derivative of $V_L(\eta(t;r))$ along the solution $\eta(t;r)$ of (3.23) is estimated as

$$\begin{aligned}\left.\frac{dV_L(\eta(t;r))}{dt}\right|_{(3.23)} &\le -r \min_{e\in\mathscr{D}_2-\mathscr{D}_3} \|e\|^2 + \frac{r}{2} \min_{e\in\mathscr{D}_2-\mathscr{D}_3} \|e\|^2 \\ &+ \frac{r}{4} \min_{e\in\mathscr{D}_2-\mathscr{D}_3} \|e\|^2 < -\frac{r}{4} \min_{e\in\mathscr{D}_2-\mathscr{D}_3} \|e\|^2 < 0. \end{aligned} \tag{3.45}$$

This, with the definition of $\mathscr{D}_2$, shows that there exists

$$t_{2r} = t_{1r} + \frac{8\lambda_{\max}(P)(\tilde{\kappa}_1^{-1}(\max\{\tilde{\kappa}_2(1),\ 1\}))^2}{\min_{e\in\mathscr{D}_2-\mathscr{D}_3} \|e\|^2} \frac{1}{r}$$

such that $\{\eta(t;r) | \ t > t_{2r}\} \subset \mathscr{D}_3$ for all $\theta \in [\theta_3^*, 1)$ and $r > r_6^*$. By (3.39)

$$\lim_{r\to\infty} t_{2r} \le \lim_{t\to\infty} \left( \frac{4\gamma}{1-\theta} A^{\frac{1-\theta}{\gamma}} (1/r)^{n\theta-(n-1)} + \frac{8\lambda_{\max}(P)(\tilde{\kappa}_1^{-1}(\max\{\tilde{\kappa}_2(1),\ 1\}))^2}{\min_{e\in\mathscr{D}_2-\mathscr{D}_3} \|e\|^2} \frac{1}{r} \right) = 0. \tag{3.46}$$

This completes the proof of the proposition. □

**Proof of Theorem 2.1.** For any $\forall\ e = (e_1, \ldots, e_{n+1}) \in \mathscr{D}_3$, by definition of $\mathscr{D}_3$, we have

$$|e_1| \le \|e\| \le \frac{V_L(e)}{\lambda_{\min}(P)} \le 1/2 < 1. \tag{3.47}$$

This together with the definition of $\mathscr{G}_i$ shows that $\mathscr{G}_i(e) = e$ for all $e \in \mathscr{D}_3, i = 1, 2, \ldots, n+1$. Therefore, $G(e) = Ke$ for all $e \in \mathscr{D}_3$. We thus conclude that for any $\theta \in [\theta_3^*, 1)$, $r > r_6^*$, and $t > t_{2r}$, (3.23) can be rewritten as

$$\dot{\eta}(t;r) = rK\eta(t;r) + \Phi(t;r). \tag{3.48}$$

Finding the derivative of $V_L(\eta(t;r))$ along the solution $\eta(t;r)$ of (3.48) yields

$$\left.\frac{dV_L(\eta(t;r))}{dt}\right|_{(3.48)} \le -r\|\eta(t;r)\|^2 + 2\lambda_{\max}(P) \times \|\eta(t;r)\| \left( M_1 + M_2 \sum_{i=1}^{n} r^{(n+1-i)(1-\alpha_i)} \|\eta(t;r)\|^{\alpha_i} \right). \tag{3.49}$$

From (2.2) and assumptions on $\alpha$ and $\alpha^*$, we have $0 \le (n+1-i)(1-\alpha_i) \le \alpha < 1$ and $0 < \alpha^* \le \alpha_i \le 1$, where $i = 1, 2, \ldots, n$. Hence for any $r > 1$ and $i \in \{1, 2, \ldots, m\}$, $r^{(n+1-i)(1-\alpha_i)} \le r^{\alpha}$. By Proposition 3.1, for any $t > t_{2r}$, $\eta(t;r) \in \mathscr{D}_3$. By (3.47), $\|\eta(t;r)\| < 1$. Hence $\|\eta(t;r)\| \le \|\eta(t;r)\|^{\alpha^*}$ and $\|\eta(t;r)\|^{\alpha_i} \le \|\eta(t;r)\|^{\alpha^*}$, $i = 1, 2, \ldots, n$. Therefore,

$$\left.\frac{dV_L(\eta(t;r))}{dt}\right|_{(3.48)} \le -\frac{r}{\lambda_{\max}(P)} V_L(\eta(t;r)) + \frac{2r^{\alpha}(M_1+nM_2)\lambda_{\max}(P)}{(\lambda_{\min}(P))^{\alpha^*/2}} V_L^{\frac{\alpha^*}{2}}(\eta(t;r)). \tag{3.50}$$

If

$$V_L(\eta(t;r)) > \left( \frac{4(M_1+nM_2)(\lambda_{\max}(P))^2}{(\lambda_{\min}(P))^{\alpha^*/2}} \right)^{\frac{2}{2-\alpha^*}} \left(\frac{1}{r}\right)^{\frac{2}{(1-\alpha)(2-\alpha^*)}}, \tag{3.51}$$

then

$$\left.\frac{dV_L(\eta(t;r))}{dt}\right|_{(3.48)} < -\frac{r}{2\lambda_{\max}(P)} V_L(\eta(t;r)).$$

For any $t > t_{2r}$. By the comparison principle of the ordinary differential equations, we have

$$V_L(\eta(t;r)) \le \exp\left( -\frac{r}{2\lambda_{\max}(P)} (t - t_{2r}) \right) V_L(\eta(t_{2r};r)).$$

From Proposition 3.1, $\eta(t;r) \in \mathscr{D}_3$ for any $t > t_{2r}$, and hence $V_L(\eta(t;r)) \le 1$. So if $V_L(\eta(t;r))$ satisfies (3.51), then for any $t > t_r = t_{2r} + 1/r^{1/2}$,

$$\begin{aligned} V_L(\eta(t;r)) &\le \exp\left( -\frac{r}{2\lambda_{\max}(P)} (t - t_{2r}) \right) \\ &\le \exp\left( -\frac{r^{1/2}}{2\lambda_{\max}(P)} \right). \end{aligned} \tag{3.52}$$

Since

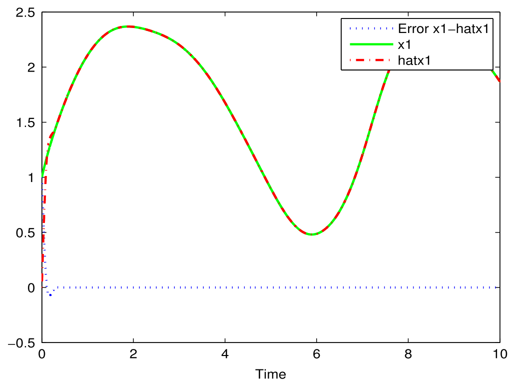
$$\lim_{r\to\infty} r^{\frac{2}{(1-\alpha)(2-\alpha^*)}} \exp\left( -\frac{r^{1/2}}{2\lambda_{\max}(P)} \right) = 0, \tag{3.53}$$

there exists $r_7^* > r_6^*$ such that for any $\theta \in [\theta_3^*, 1)$, $r > r_7^*$, and $t > t_r$,

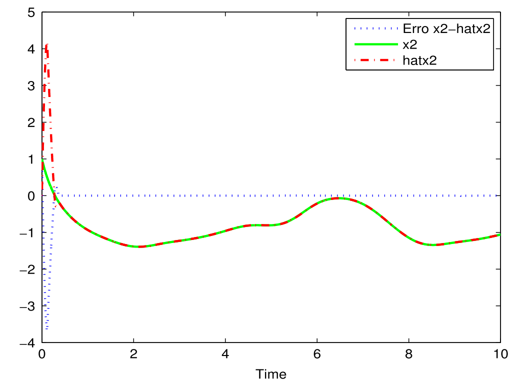
$$\begin{aligned} V_L(\eta(t;r)) &\le e^{-\frac{r^{1/2}}{2\lambda_{\max}(P)}} \\ &\le \left( \frac{4(M_1+nM_2)(\lambda_{\max}(P))^2}{(\lambda_{\min}(P))^{\alpha^*/2}} \right)^{\frac{2}{2-\alpha^*}} \left(\frac{1}{r}\right)^{\frac{2}{(1-\alpha)(2-\alpha^*)}}, \end{aligned} \tag{3.54}$$

and so

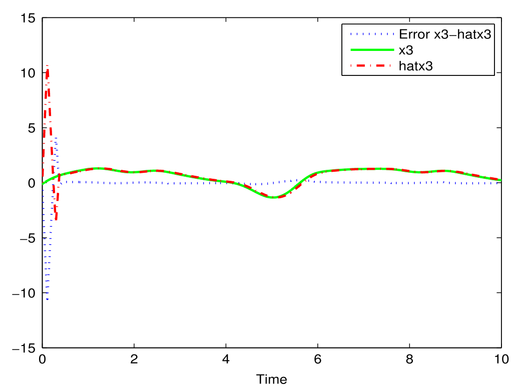
$$\|\eta(t;r)\| \le \left( \frac{V_L(\eta(t;r))}{\lambda_{\min}(P)} \right)^{1/2} \le \Gamma \left(\frac{1}{r}\right)^{\frac{1}{(1-\alpha)(2-\alpha^*)}}, \tag{3.55}$$




(a) $x_1(t)$ and $\hat{x}_1(t; 50)$.




(b) $x_2(t)$ and $\hat{x}_2(t; 50)$.




(c) $x_3(t)$ and $\hat{x}_3(t; 50)$.

**Fig. 1.** The numerical results for system (4.1) by fractional power ESO.

where

$$\Gamma = \frac{1}{\sqrt{\lambda_{\min}(P)}}\left(\frac{4(M_1+nM_2)(\lambda_{\max}(P))^2}{(\lambda_{\min}(P))^{\alpha^*/2}}\right)^{\frac{1}{2-\alpha^*}}. \tag{3.56}$$

The transient time $t_r$ satisfies that

$$\lim_{r\to\infty} t_r \le \lim_{r\to\infty}\left(\frac{4\gamma}{1-\theta}A^{\frac{1-\theta}{\gamma}}\left(\frac{1}{r}\right)^{n\theta-(n-1)} + \frac{4\kappa_{2\theta}\circ\tilde{\kappa}_1^{-1}(\max\{\tilde{\kappa}_2(1),\ 1\})}{\min\limits_{e\in\mathscr{D}_2-\mathscr{D}_3}\|e\|^2}\frac{1}{r}+\frac{1}{r^{1/2}}\right) = 0. \tag{3.57}$$

Theorem 2.1 then follows by combining (3.55), (3.1) and (3.57). □

**Proof of Corollary 2.1.** Let

$\eta(t; r) = (\eta_1(t; r), \ldots, \eta_{n+1}(t; r))$

and let $\eta_i(t; r)$ be defined by (3.1) with $x_{n+1}(t) = \bar{d}$, and let the Lyapunov function $V_L(z)$ be defined by (3.8). We can easily verify that (3.23) also holds. Similarly with (3.49) in the proof of Theorem 2.1, we can obtain that there exist $\theta^* \in (0, 1)$, $r_1^* > 1$, and $t_{1r} > 0$ ($\lim_{r\to\infty} t_{1r} = 0$) such that $V_L(\eta(t; r)) < 1$ for any $\theta \in [\theta^*, 1)$, $r > r_1^*$, and $t > t_{1r}$. Hence the derivative of $V_L(\eta(t; r))$ along the solution $\eta(t; r)$ of (3.23) satisfies

$$\left.\frac{dV_L(\eta(t;r))}{dt}\right|_{(3.48)} \le -\frac{r}{\lambda_{\max}(P)}V_L(\eta(t;r)) + \frac{2r^{\alpha}nM_2\lambda_{\max}(P)}{\sqrt{(\lambda_{\min}(P))^{1+\alpha^*}}}(V_L(\eta(t;r)))^{\frac{1+\alpha^*}{2}}. \tag{3.58}$$

If $\alpha^* < 1$ and

$$V_L(\eta(t;r)) > \left(\frac{4nM_2\lambda_{\max}(P)^2}{\sqrt{\lambda_{\min}(P)^{1+\alpha^*}}}\right)^{\frac{2}{1+\alpha^*}}\left(\frac{1}{r}\right)^{\frac{2}{(1-\alpha)(1-\alpha^*)}}, \tag{3.59}$$

then

$$\left.\frac{dV_L(\eta(t;r))}{dt}\right|_{(3.48)} \le -\frac{r}{2\lambda_{\max}(P)}V_L(\eta(t;r)).$$

By the comparison principle of ordinary equation, we have

$$\begin{aligned} V_L(\eta(t;r)) &\le \exp\left(-\frac{r}{2\lambda_{\max}(P)}(t-t_{1r})\right)V_L(\eta(t_{1r};r)) \\ &\le \exp\left(-\frac{r^{1/2}}{2\lambda_{\max}(P)}\right),\quad t > t_r = t_{1r}+\frac{1}{r^{1/2}}. \end{aligned}$$

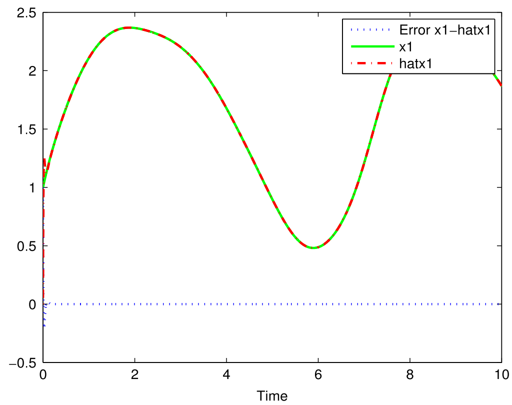
Considering (3.53), there exists $r^* > r_1^*$ such that for any $r > r^*$

$$\begin{aligned} V_L(\eta(t;r)) &\le \exp\left(-\frac{r^{1/2}}{2\lambda_{\max}(P)}\right) \\ &\le \left(\frac{4nM_2(\lambda_{\max}(P))^2}{\sqrt{\lambda_{\min}(P)^{1+\alpha^*}}}\right)^{\frac{2}{1+\alpha^*}}\left(\frac{1}{r}\right)^{\frac{2}{(1-\alpha)(1-\alpha^*)}}, \end{aligned} \tag{3.60}$$

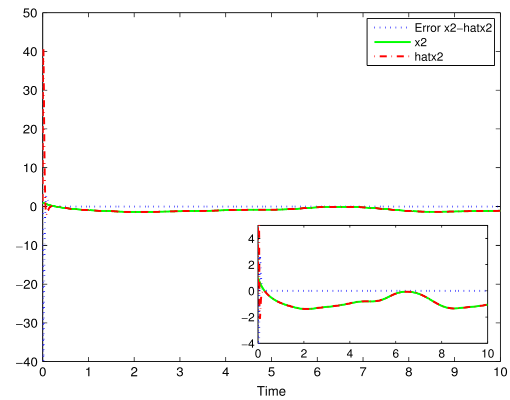
and hence

$$\|\eta(t;r)\| \le \frac{\sqrt{V_L(\eta(t;r))}}{\sqrt{\lambda_{\min}(P)}} \le \tilde{\Gamma}\left(\frac{1}{r}\right)^{\frac{1}{(1-\alpha)(1-\alpha^*)}},$$

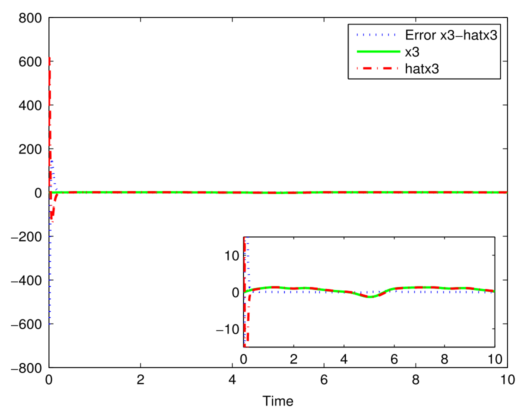
$$\tilde{\Gamma} = \frac{(4nM_2(\lambda_{\max}(P))^2)^{\frac{1}{1+\alpha^*}}}{\lambda_{\min}(P)}. \tag{3.61}$$




(a) $x_1(t)$ and $\hat{x}_1(t;50)$.




(b) $x_2(t)$ and $\hat{x}_2(t;50)$.




(c) $x_3(t)$ and $\hat{x}_3(t;50)$.

**Fig. 2.** The numerical results for system (4.1) by linear ESO.

If $\alpha_i^*=1$ then $\alpha=0$, and so

$$\left.\frac{dV_L(\eta(t;r))}{dt}\right|_{(3.48)} \leq -\frac{(r-2nM_2\lambda_{\max}(P))}{\lambda_{\max}(P)}V_L(\eta(t;r))<0. \quad (3.62)$$

This gives $\lim_{t\to\infty}\|\eta(t;r)\|=0$. □

## 4. Numerical simulations

**Example 4.1.** Consider the following uncertain nonlinear system:

$$\begin{cases}\dot{x}_1(t)=x_2(t)+\phi_1(t,u(t),x_1(t)),\\ \dot{x}_2(t)=f(t,x(t),w(t))+\phi_2(t,u(t),x_1(t),x_2(t)),\end{cases} \quad (4.1)$$

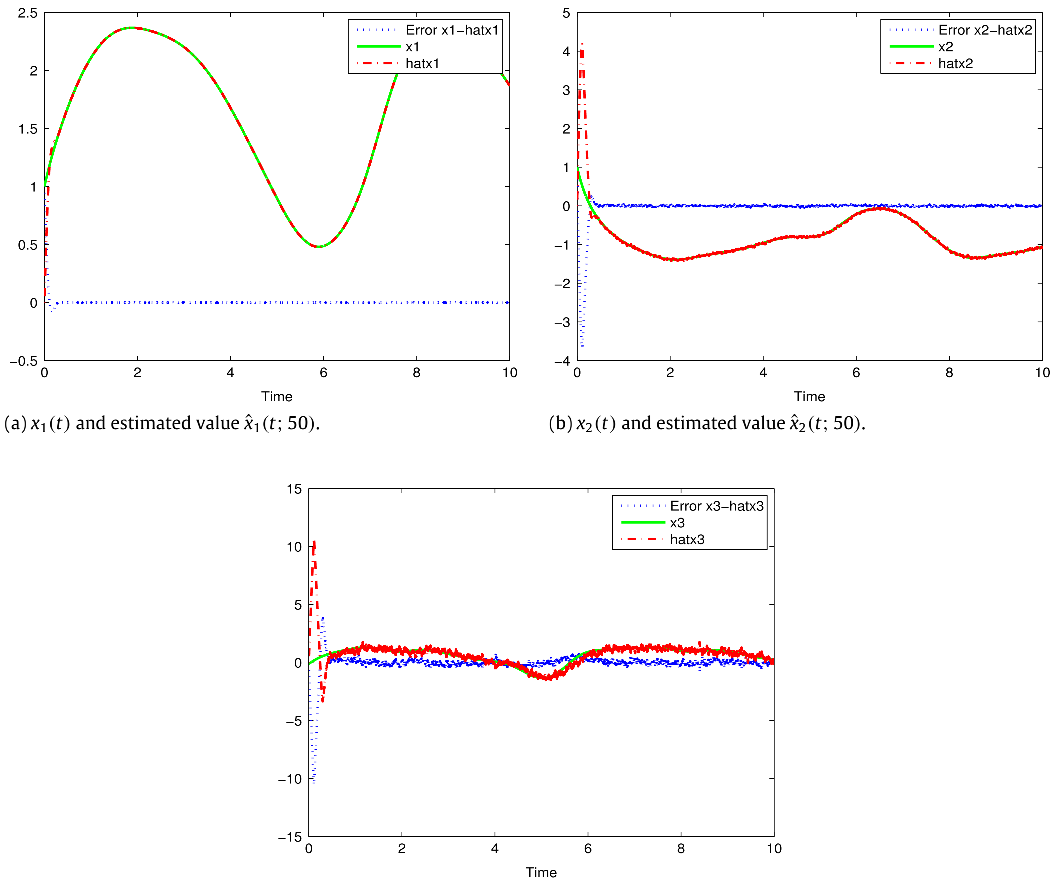
where $\phi_1(t,u,x_1) = (1+\sin t)\sin x_1$ and $\phi_2(t,u,x_1,x_2) = (1+\sin(t))\sin x_2-4x_2+u$ are known functions, $u(t)$ the control input, and $y(t)=x_1(t)$ the output. The total disturbance $x_3(t)\triangleq f(t,x(t),w(t))$ is completely unknown, with $w(t)$ the external disturbance.

In numerical simulations, we take $w(t)$ and $f(\cdot)$ in system (4.1) as $w(t)=\sin(2t+1)$, $f(t,x,w)=\sin t+w+\cos(x_1+x_2+w)$. The solution of system (4.1) may not be bounded. However, we apply direct output feedback $u(t)=-2y(t)+v(t)$ where $v(t)$ is the new control input and for simplicity, we just take $v(t)=0$. In this case, system (4.1) satisfies Assumption A1. Since the total disturbance is uniformly bounded, the solution of system (4.1) is bounded as well.

In fractional power ESO (1.4), let $n=2$, the nonlinear functions $\mathscr{G}_i(\cdot)$ are chosen as (1.6) with $\theta_1=0.7$, $\theta_2=0.4$, and $\theta_3=0.1$. The Euler integral method is adopted with step size of 0.001, the initial value of the system state is $(1,1)$, and the initial state of the ESO is $(0,0,0)$. The numerical results for fractional power ESO are plotted in Fig. 1, where we use tuning parameter $r=50$ for the simulation. The three different curves in Fig. 1(a) are $x_1(t)-\hat{x}_1(t;50)$, $x_1(t)$, and $\hat{x}_1(t,50)$ respectively. The curves in Fig. 1(b) are $x_2(t)-\hat{x}_2(t;50)$, $x_2(t)$, and $\hat{x}_2(t,50)$. and Fig. 1(c) shows $x_3(t)-\hat{x}_3(t;50)$, $x_3(t)$, and $\hat{x}_3(t,50)$. The small figure on the right bottom of Fig. 1(c) is magnification of Fig. 1(c) along the vertical axis. It is seen from Fig. 1 that convergence of fractional power ESO is satisfactory.

Next we present numerical results for system (1.2) with state and total disturbance under linear ESO (that is in (1.4), $g_i(\tau)=\tau$). With the same tuning parameter $r=50$, the numerical results of linear ESO are plotted in Fig. 2. we can see that the linear ESO can also satisfactorily estimate the state and total disturbance. However, comparing Figs. 1 and 2, it is evident that the peaking value of fractional power ESO is much smaller than that of linear ESO. Precisely, we can see that the peaking value of $\hat{x}_3(t;50)$ of fractional power ESO is about 10, whereas the peaking value of $\hat{x}_3(t;50)$ of linear ESO reaches almost 600.

Now we give numerical simulations in the presence of measurement noise. Suppose that the output $y(t)$ is contaminated by the noise $0.002\mathcal{N}(t)$, where $\mathcal{N}(t)$ is the standard Gaussian noise generated by the Matlab program command "randn". Let the other functions and parameters be the same as in Fig. 1. The numerical results of ESO with fractional power functions $\mathscr{G}_i'$s are plotted in Fig. 3. Finally, using the same parameters as in Fig. 3, the numerical results of linear ESO with measurement noise $0.002\mathcal{N}(t)$ are plotted in Fig. 4.




(a) $x_1(t)$ and estimated value $\hat{x}_1(t; 50)$.

(b) $x_2(t)$ and estimated value $\hat{x}_2(t; 50)$.

(c) $x_3(t)$ and estimated value $\hat{x}_3(t; 50)$.

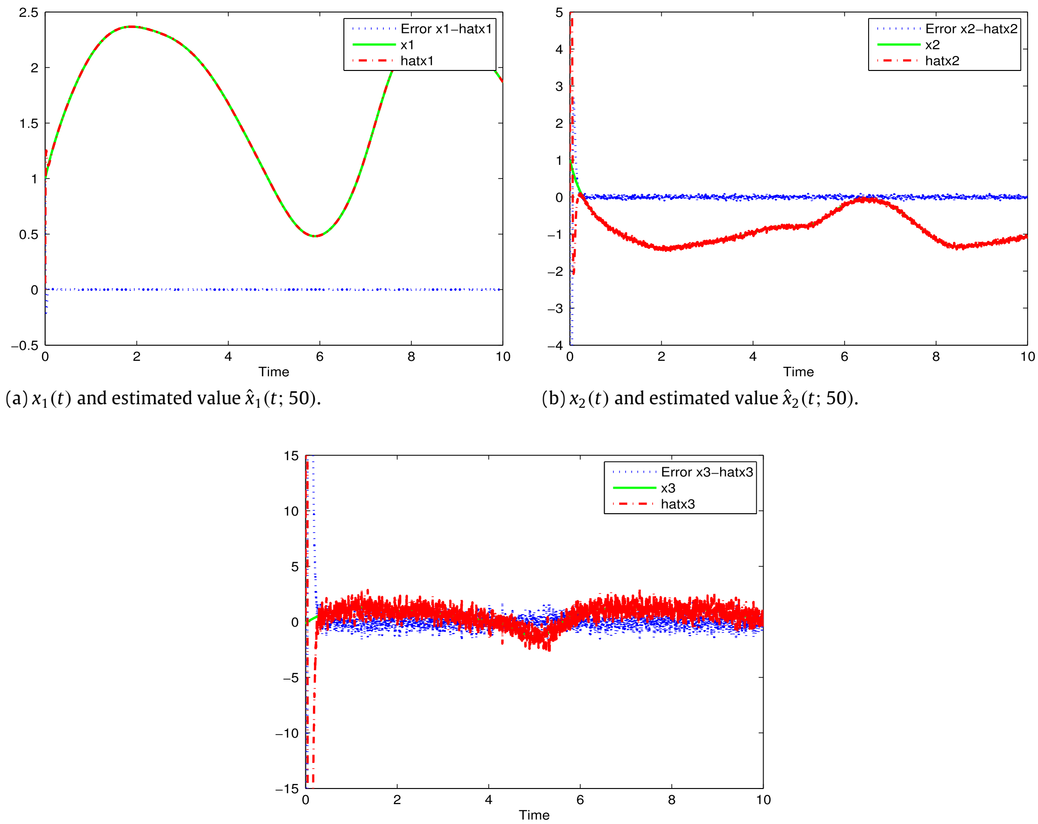
**Fig. 3.** The numerical results for system (4.1) with measurement contaminated by random noise by fractional power ESO.

To conclude this section, we discuss the advantage in terms of peaking reduction and measurement noise sensitivity of fractional power ESO. It can be shown that peaking often occurs in the initial stage, which is caused by the high-tuning gain and the non-zero (possibly very large) error $x_1(0) - \hat{x}_1(0)$. Specifically, the $i$th equation of ESO contains a term $\frac{k_i}{r^{n-i}}\mathcal{G}_i(r^n(x_1(t) - \hat{x}_1(t; r)))$ which can be very large for some functions $\mathcal{G}_i(\cdot)$'s. These large terms in the right hand of ESO in the beginning force the state of ESO to have large chattering. For instance, when $\mathcal{G}_i(\tau) = \tau$, that is, ESO (1.4) is linear ESO, the term $\frac{k_i}{r^{n-i}}\mathcal{G}_i(r^n(x_1(t) - \hat{x}_1(t; r)))$ becomes $r^i(x_1(0) - \hat{x}_1(0))$ which is large when $x_1(0) - \hat{x}_1(0) \neq 0$ and $r$ is large.

Nonlinear function $\mathcal{G}_i(\cdot)$'s defined by (1.6) possess saturation-like behavior for large $\tau$. This is the main reason for ESO based on $\mathcal{G}_i(\cdot)$'s defined by (1.6) to have smaller peaking value than linear ESO under the same tuning parameter $r$. Actually, for ESO with $\mathcal{G}_i$'s defined by (1.6), the $r$-dependent term at initial time is $r^{n\theta_i - n+i}|x_1(0) - \hat{x}_1(0)|^{\theta_i}$. Since $\theta_i \in (0, 1)$, we have $n\theta_i - n + i < i$ and hence $r^{n\theta_i-n+i}$ can be smaller than $r^i$ in linear ESO for large $r$. For example, in ESO (1.4), let $n = 2$, $r = 50$, $\theta = 0.7$ (hence $\theta_3 = 0.1$), $x_1(0) - \hat{x}_1(0)$=1. In the third equation of (1.4), the $r$-related term is $r^{n\theta_i-n+i}|x_1(0) - \hat{x}_1(0)| = 50^{1.2} \approx 109.3362$, whereas the corresponding term in the third order linear ESO is $r^3|x_1(0) - \hat{x}_1(0)| = 125\,000$. In Figs. 1 and 2, we use these data for comparing the peaking value numerically. The peaking value reduction for fractional power ESO with $\mathcal{G}_i$'s defined by (1.6) compared over linear ESO is significant.

When the output of the system is contaminated by noise $n(t)$, that is, $y(t) = x_1(t) + \mathcal{N}(t)$, the linear ESO is sensitive to the measurement noise for large $r$ because the noise is magnified to become $r^i\mathcal{N}(t)$ in the $i$th equation of linear ESO. Given the credit of the saturation function of $\mathcal{G}_i's$ again, the magnification coefficient of $\mathcal{N}(t)$ in ESO with $\mathcal{G}_i$'s defined by (1.6) is $r^{n\theta_i-n+i}$ which is much smaller than $r^i$ for large $r$. This advantage is also shown in Figs. 3 and 4.

Finally, we point out that a novel high-gain observer is proposed recently in Astolfi and Marconi (2015) where good performance in the presence of measurement noise is obtained by increasing order of the observer and limiting the power of the gain. The idea here is similar to Astolfi and Marconi (2015) because our nonlinear functions $\mathcal{G}_i(\cdot)$'s in ESO (1.4) have saturation-like behavior. When $|r^n(x_1(t)-\hat{x}_1(t; r))| > 1$, after being saturated by $\mathcal{G}_i(\cdot)$'s, the power of gain $r$ becomes smaller. The small power can then help reduce the peaking value and guarantee good performance in the presence of measurement noise. When $|r^n(x_1(t) - \hat{x}_1(t; r))| \leq 1$, the power of the gain $r$ is large and render good tracking accuracy, which are the same as the power of the gain in linear ESO or traditional high-gain observers. The differences of observer in Astolfi and Marconi (2015) and ESO in this paper are (a) the observer in Astolfi and Marconi (2015) is for the state estimation only whereas ESO here estimates not only the system state but also the "total disturbance"; (b) for an $n$th order system, the observer in Astolfi and Marconi (2015) is the $(2n - 2)$th order whereas ESO here is $(n+1)$th order; (c) the power of gain in Astolfi and Marconi (2015) is reduced in the whole process whereas in this paper, it is reduced only when $r^n|x_1(t)-\hat{x}_1(t)| > 1$; (d) it is pointed out by the authors that the observer in Astolfi and Marconi (2015) is not effective for peaking value problem whereas ESO in this paper could reduce the peaking value.




(a) $x_1(t)$ and estimated value $\hat{x}_1(t; 50)$.

(b) $x_2(t)$ and estimated value $\hat{x}_2(t; 50)$.

(c) $x_3(t)$ and estimated value $\hat{x}_3(t; 50)$.

**Fig. 4.** The numerical results for system (4.1) with measurement contaminated by random noise by linear ESO.

## 5. Concluding remarks

In this paper, we investigate the capability of the ESO constructed from fractional power functions by providing basic principle of choosing the parameters to ensure convergence, a long-standing problem in active disturbance rejection control. This type of ESO has been widely used since the introduction of active disturbance rejection control to solve engineering problems in the last two decades. It has been shown that this type of ESO has the advantages of smaller peaking value and good performance in the presence of measurement noise when compared with the popular linear ESO. This paper provided a comprehensive mathematical investigation for this problem. It establishes the convergence for this type of ESO. This work represents one step forward in building convergence theory for active disturbance rejection control based on fractional power extended state observer.

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