Output Regulation for a Wave Equation With Unknown Exosystem

Ren-Xi Zhao and Bao-Zhu Guo, *Senior Member, IEEE*

***Abstract*—In this article, the output regulation for a 1-D wave equation where the disturbances generated from an unknown finite-dimensional exosystem appear in all possible channels is studied. The system is first transformed into a new system for which the disturbance appears in tracking error only. An adaptive observer approach is adopted in investigation to estimate all possible unknown frequencies that have entered into a transformed new system. By the estimates of the unknown frequencies, we are able to design a tracking-error-based feedback control to achieve output regulation and disturbance rejection for this partial differential equations (PDEs) in two different cases. In the first case, the derivative of the tracking error is allowed to be used in the control design, which leads to the exponential convergence of the tracking error. In the second case, the tracking error is solely used and the asymptotic convergence is achieved. A remarkable characteristic of the problem lies in the fact that the control operator is unbounded and is noncollocated with the regulated output, which represents a difficult situation for output regulation on PDEs. The proposed approach is potentially applicable to other PDEs.**

***Index Terms*—Adaptive internal model, disturbance rejection, output regulation, wave equation.**

I. Introduction

OUTPUT regulation is one of the most fundamental problems in the control theory. The main objective of the problem is designing a tracking error feedback control to regulate the output to track the reference signal asymptotically in the presence of disturbance. If both the reference signal and the disturbance are generated from a linear autonomous system, which is called exosystem, the problem can be solved perfectly for linear time invariant systems by the internal model principle developed in the 1970s from [2] and [5]. The internal model principle has been applied to nonlinear lumped parameter systems [12] and abstract distributed parameter systems first on systems with bounded control and observation operators [1], [25] and later on systems with unbounded control and observation operators [24], [26]. In particular, the systems considered in [24] and [26] are a large class of abstract linear infinite-dimensional systems called well-posed and regular linear systems. On the other hand, some progresses on output tracking from partial differential equation (PDE) point of view have also been made over the years. In [3] and [4], the backstepping method was applied to the output regulation of parabolic PDEs. In [3], the regulated output is different from the measured output, which has rarely been discussed abstractly for infinite-dimensional systems yet is important in engineering applications. An output tracking problem for 1-D wave equation were considered in [9] by means of the adaptive control method, where the disturbance and reference signals are sinusoidal signals with known frequencies. A recent article [8] proposed an observer-based control by the trajectory planning approach for output regulation of a wave equation. All these problems in [8] and [9] have been treated systematically by an observer-based internal model principle in recent works [6], [7]. However, in all these articles aforementioned, the frequencies of sinusoidal disturbances were supposed to be known. To the best of our knowledge, only a few studies have been carried out for the output tracking of the infinite-dimensional systems with unknown frequencies like those in [29] and [30] where the control and observation operators were assumed to be bounded, and the systems were assumed to be transformable into canonical form.

On the other hand, there are many works attributed to output regulation for systems described by finite-dimensional linear system [ordinary differential equations (ODEs)] with unknown exosystem. When the frequencies of the sinusoidal disturbances are unknown but the number of frequencies is known, solutions were presented in terms of adaptive internal model in [19]. Later studies focused on the case where the number of the frequencies is also unknown, which can be found in [20], [21], and [22]. In particular, for the minimum phase linear systems, [20] proposed an elegant solution. For the nonminimum phase systems, some progress has also been made in [21] and [22] where the adaptive algorithms were proposed to estimate the number of unknown frequencies. A recent progress was made in [23] where the disturbance rejection problem was solved for stable plant with both the exosystem and plant being unknown.

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Ren-Xi Zhao is with the Key Laboratory of System and Control, Academy of Mathematics and Systems Science, Academia Sinica, Beijing 100190, China, and also with the School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China (e-mail: zhaorenxi@amss.ac.cn).

Bao-Zhu Guo is with the School of Mathematics and Physics, North China Electric Power University, Beijing 102206, China, also with the Key Laboratory of System and Control, Academy of Mathematics and Systems Science, Academia Sinica, Beijing 100190, China, and also with the School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China (e-mail: bzguo@iss.ac.cn).

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Motivated from works aforementioned for lumped parameter systems, we proposed an adaptive internal model for a 1-D heat equation in [10] where only the asymptotic convergence could be achieved when the number of frequencies of the exosystem was unknown. Since heat equation has naturally high regularity and is not very typical infinite-dimensional system for it has only finitely many unstable poles, we continue to use the adaptive internal model approach to solve noncollocated output tracking problems for a PDE system described by 1-D wave equation where the matrix of the exosystem is unknown. The wave equation is essentially infinite dimensional for its infinitely many unstable poles and has many engineering applications in wharf gantry cranes carrying cargo in marine industry [14], container cranes in port automation, and flexible links in gantry robots. In particular, an output tracking problem of the anticollocatedly disturbed cage in an ascending cable elevator was studied in [28], where the ascending cable elevator was modeled as a wave equation, and the system was disturbed by sinusoidal disturbances with unknown frequencies. However, in [28], the unknown sinusoidal disturbances entered one channel only in the system, whereas we allow the disturbances entering all possible channels. Other engineering problems of output tracking of the wave equation can be found in piezoelectric stack actuators [18] and moving string systems with tip payload [13]. Furthermore, compared with asymptotic convergence in [10], we pursuit exponential convergence for tracking error in this article by using a new adaptive observer to estimate the number and frequencies of the sinusoidal disturbances. In addition, this article is different from [10] in other two aspects. First, to achieve exponential convergence, we use additionally the derivative of the tracking error in the control design, which is likewise the PD control, and hence, is not a standard tracking error feedback problem. At the same time, the convergence of the derivative of the tracking error is also considered. The approach used in this article can make the convergence of [10] from asymptotic convergence to exponential convergence. In the second case, we also consider solely the tracking error feedback. In this case, since the observation operator is compact, only asymptotic convergence can be achieved (see [11]). This makes our control design much trickier than [10].

The system that we are concerned with is described by the following 1-D wave equation:

$$\begin{cases} y_{tt}(x,t) = y_{xx}(x,t) + \Delta(x)w_1(t), & x\in(0,1), t>0\\ y_x(0,t) = w_2(t), & t\geq 0\\ y_x(1,t) = u(t) + w_3(t), & t\geq 0\\ y(x,0) = y_0(x), y_t(x,0) = y_1(x), & x\in(0,1)\\ Y(t) = y(0,t), & t\geq 0 \end{cases} \tag{1}$$

where $u(t)$ is the control input, $Y(t)$ is the performance output to be regulated, $\Delta(\cdot)\in L^2((0,1);\mathbb{R})$ is an unknown function, and $w_i(t) (i = 1,2,3)$ represent the external disturbances.

The disturbances $w_i(\cdot)$ and the reference signal $r(\cdot)$ are assumed to be generated by an exosystem of the form

$$\begin{cases} \dot{v}(t) = Sv(t),\ v(0) = v_0\in\mathbb{R}^n\\ w_i(t) = F_iv(t),\ i = 1,2,3\\ r(t) = F_4v(t) \end{cases} \tag{2}$$

where all $S\in\mathbb{R}^{n\times n}$, $F_i\in\mathbb{R}^{1\times n}$ for $i = 1,2,3,4$ and the initial value v_0 are unknown. We consider system (1) in the standard energy state space $H = H^1(0,1)\times L^2(0,1)$. The tracking error is denoted by $e(t) = Y(t) - r(t)$. The control aim is to design a tracking error feedback control so that

$$\lim_{t\to\infty}|e(t)| = \lim_{t\to\infty}|Y(t) - r(t)| = 0. \tag{3}$$

The following assumption is made throughout this article.

Assumption A: The spectrum of S is $\{0, \pm j\omega_i, 1\leq i\leq r\}$ with $n = 2r+1$, where $\omega_1 < \omega_2 < \ldots < \omega_r$ are positive distinct unknown parameters. It is supposed that the unknown r has an upper bound: $r\leq m$ for a known positive integer m.

From Assumption A, the general solution of the exosystem (2) contains no more than m sinusoidal functions with unknown frequencies depending on the eigenvalues of S. More precisely, the reference signal and the disturbance signals are of the forms

$$r(t) = \sum_{i=1}^{m}(a_{i4}\cos\omega_i t + b_{i4}\sin\omega_i t) + c_4$$

$$w_j(t) = \sum_{i=1}^{m}(a_{ij}\cos\omega_i t + b_{ij}\sin\omega_i t) + c_j,\ j = 1,2,3$$

for some unknown coefficients $\{\omega_i\}$, $\{a_{ij}\}$, $\{b_{ij}\}$, and $\{c_j\}$.

Compared with ODE problems developed in [19], [21], and [22] where a key step is to convert the coupled systems of plant and exosystem into an observable canonical form and design an adaptive observer for the coupled systems, this article is technically to separate the PDE part from the exosystem through a transformation and design observers for PDE and ODE part separately. This is because there is no observable canonical form for infinite-dimensional systems.

The rest of this article is organized as follows. In Section II, we discuss the case where both the tracking error $e(t)$ and its derivative $\dot{e}(t)$ can be used to the control design. Since both $e(t)$ and $\dot{e}(t)$ are used in the control design, by the internal model principle, we are only able to regulate $e(t)$ independently. For $\dot{e}(t)$, we consider it as the derivative of $e(t)$ not as an independent output. In this sense, we regulate both $e(t)$ and $\dot{e}(t)$ simultaneously that

$$\lim_{t\to\infty}|e(t)| = 0,\ \int_0^\infty e^{\beta t}|\dot{e}(t)|^2\,dt < \infty \text{ for some } \beta > 0.$$

The first convergence is exponentially and the second convergence is quite weak like those in [24] but stronger than [26] because $\dot{e}(t)$ is usually very unbounded for general initial states in the state space. Section III is devoted to the case where only tracking error $e(t)$ is measurable and we achieve $\lim_{t\to\infty}|e(t)| = 0$, which might be the best result to be expected since the observation operator is compact in this case. Some numerical simulations are presented in Section IV for illustration, and finally, Section V concludes this article.

A. Notations and Useful Lemmas

Throughout this article, $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote the norm and the inner product of $L^2(0,1)$. The $H = H^1(0,1)\times L^2(0,1)$ is a Hilbert space with the inner product given by

$$\langle (f_1, g_1), (f_2, g_2)\rangle_H = \int_0^1 [f_1'(x)\overline{f_2'(x)} + g_1(x)\overline{g_2(x)}]dx + f_1(0)\overline{f_2(0)}.$$

In order to analyze the stability of coupled PDE systems, we need the following Lemma 1.1.

Lemma 1.1: (See [31, lemma 2.1]) Let A be the generator of an exponentially stable operator semigroup e^{At} on the Hilbert space X. Assume that $B_i \in \mathscr{L}(U_i, X_{-1}), i = 1, 2, \ldots, n$, are admissible control operators for e^{At} with control Hilbert space U_i, where $X_{-1} = [D(A^*)]'$ is the dual space of $D(A^*)$ with the pivot space X. Then, the initial problem

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{n} B_i u_i(t),\ x(0) = x_0 \in X,\ u_i \in L^2_{loc}([0,\infty); U_i)$$

admits a unique solution $x \in C([0,\infty); X)$, and if $u_i \in L^\infty([0,\infty); U_i), i = 1, 2, \ldots, n$, then $x(\cdot)$ is bounded. If for each index i, either $u_i \in L^2(0,\infty; U_i)$ or $\lim_{t\to\infty} \|u_i(t)\|_{U_i} = 0$, then $\lim_{t\to\infty} x(t) = 0$. Moreover, if there exist constants $M_0, \mu_0, \alpha > 0$ such that $\|u_i(t)\|_U \le M_0 e^{-\mu_0 t}$ or $e^{\alpha\cdot} u_i \in L^2(0,\infty; U_i)$, then $\|x(t)\| \le M e^{-\mu t}$ for some $M, \mu > 0$.

There are similar conclusions for linear time-varying systems, which will be used later for analysis of the convergence of the adaptive observer.

Lemma 1.2: (See [16, ch. 9]) Let $A(\cdot) \in L^\infty(0,\infty; \mathbb{R}^{n\times n})$, $f \in L^1_{loc}([0,\infty); \mathbb{R}^n)$, and the origin of the nominal system $\dot{z}(t) = A(t)z(t),\ z(0) = z_0 \in \mathbb{R}^n$ is exponentially stable. Then, the initial problem

$$\dot{x}(t) = A(t)x(t) + f(t),\ x(0) = x_0 \in \mathbb{R}^n$$

admits a unique solution $x \in C([0,\infty); \mathbb{R}^n)$. Moreover

1) if $\lim_{t\to\infty} f(t) = 0$, then $\lim_{t\to\infty} x(t) = 0$;

2) if there exist constants $M_0, \mu_0 > 0$ such that $\|f(t)\| \le M_0 e^{-\mu_0 t}$, then $\|x(t)\| \le M e^{-\mu t}$ for some $M, \mu > 0$.

Lemma 1.3: (See [15, Lemma 4.8.3]) Let $f, g : [0,\infty) \to \mathbb{R}^n$ be piecewise continuous, and $f(\cdot)$ be persistently exciting (PE), that is, there exist constants $\alpha_0, \alpha_1, T_0 > 0$ such that

$$\alpha_1 I \ge \frac{1}{T_0}\int_t^{t+T_0} f(\tau)f^\top(\tau)d\tau \ge \alpha_0 I \forall t \ge 0.$$

If $\lim_{t\to\infty} g(t) = 0$ or $g \in L^2(0,\infty; \mathbb{R}^n)$, then $f(\cdot) + g(\cdot)$ is PE also.

II. Regulation of Both Tracking Error and Its Derivative

In this section, we suppose that both the tracking error $e(t)$ and its derivative $\dot{e}(t)$ are measurable, aiming at designing an error feedback regulator to regulate both tracking error and its derivative. To simplify the structure of the coupled system (1) and (2), we introduce a transformation for the system (1)

$$z(x,t) = y(x,t) + g(x)v(t) \tag{4}$$

where $g : [0,1] \to \mathbb{R}^{1\times(2r+1)}$ satisfies

$$\begin{cases} g''(x) = g(x)S^2 + F_1\Delta(x) \\ g'(0) = g(0)[c_0 + c_1 S] - F_2 + c_0 F_4 + c_1 F_4 S, \quad c_0, c_1 > 0 \\ g'(1) = -F_3. \end{cases} \tag{5}$$

Lemma 2.1: The boundary value problem (5) admits a unique solution $g^\top \in H^2((0,1); \mathbb{R}^{2r+1})$.

Proof: Let $g_1(\cdot)$ be the solution of the following boundary value problem:

$$\begin{cases} g_1''(x) = 0 \\ g_1'(0) = g_1(0)[c_0 + c_1 S] - F_2 + c_0 F_4 + c_1 F_4 S \\ g_1'(1) = -F_3 \end{cases} \tag{6}$$

which obviously admits a unique solution $g_1 \in C^\infty((0,1); \mathbb{R}^{2r+1})$. Consider the following boundary value problem:

$$\begin{cases} h''(x) = h(x)S^2 + (g_1(x)S^2 + F_1\Delta(x)) \\ h'(0) = h(0)[c_0 + c_1 S] \\ h'(1) = 0. \end{cases} \tag{7}$$

Let $\{\psi_i\}_{i=1}^{2r+1}$ be eigenvectors of S corresponding to the eigenvalues $\{\lambda_i\}_{i=1}^{2r+1}$, respectively. Right multiply by ψ_i in (7) to obtain

$$\begin{cases} h_i''(x) = h_i(x)\lambda_i^2 + \Delta_i(x) \\ h_i'(0) = h_i(0)[c_0 + c_1\lambda_i] \\ h_i'(1) = 0 \end{cases} \tag{8}$$

where $h_i(x) = h(x)\psi_i$, and $\Delta_i(x) = (g_1(x)S^2 + F_1\Delta(x))\psi_i$. For $\lambda_i \ne 0$, the solution of (8) can be found as

$$h_{2i-1}(x) = k_{1i}e^{\lambda_i x} + k_{2i}e^{-\lambda_i x} + \frac{1}{2\lambda_i}\int_0^x e^{\lambda_i(x-\xi)}\Delta_i(\xi)\,d\xi - \frac{1}{2\lambda_i}\int_0^x e^{-\lambda_i(x-\xi)}\Delta_i(\xi)d\xi$$

where k_{1i} and k_{2i} are determined by

$$\begin{cases} [-c_0 - c_1\lambda_i + \lambda_i]k_{1i} - [c_0 + c_1\lambda_i + \lambda_i]k_{2i} = 0 \\ \lambda_i e^{\lambda_i}k_{1i} - \lambda_i e^{-\lambda_i}k_{2i} \\ = \frac{1}{2}\int_0^1 e^{\lambda_i(1-\xi)}\Delta_i(\xi)\,d\xi + \frac{1}{2}\int_0^1 e^{-\lambda_i(1-\xi)}\Delta_i(\xi)\,d\xi. \end{cases} \tag{9}$$

It is a trivial exercise to check that the determinant of (9) for unknown variables k_{1i} and k_{2i}, that is, the determinant of the matrix $\begin{pmatrix} -c_0 - c_1\lambda_i + \lambda_i & -[c_0 + c_1\lambda_i + \lambda_i] \\ \lambda_i e^{\lambda_i} & -\lambda_i e^{-\lambda_i} \end{pmatrix}$ is nonzero. For the eigenvalue $\lambda_i = 0$, we have

$$\begin{cases} h_i''(x) = \Delta_i(x) \\ h_i'(0) = c_0 h_i(0),\ h_i'(1) = 0 \end{cases}$$

which has solution

$$h_i(x) = -\left(x + \frac{1}{c_0}\right)\int_0^1 \Delta_i(\xi)\,d\xi + \int_0^x (x-\xi)\Delta_i(\xi)\,d\xi.$$

This shows that the solution of (7) always exists for any $c_0, c_1 > 0$ and

$$h(x) = (h_1(x), \dots, h_{2r+1}(x))[\psi_1, \dots, \psi_{2r+1}]^{-1} \in H^2((0,1); \mathbb{R}^{2r+1}).$$

Therefore, $g(x) = g_1(x) + h(x)$ is the unique solution of (5). ■

The extended system of $(z(\cdot,\cdot), v(\cdot))$ is then governed by

$$\begin{cases} z_{tt}(x,t) = z_{xx}(x,t) \\ z_x(0,t) = c_0[z(0,t) - e(t)] + c_1[z_t(0,t) - \dot{e}(t)] \\ z_x(1,t) = u(t) \\ \dot{v}(t) = Sv(t) \\ e(t) = z(0,t) - (g(0) + F_4)v(t) \\ \dot{e}(t) = z_t(0,t) - (g(0) + F_4)Sv(t). \end{cases} \quad (10)$$

A remarkable feature of the system (10) is that the disturbance appears in tracking error only. In addition, the system (10) is composed of two decoupled subsystems of the PDE-subsystem and the ODE-subsystem. The PDE-subsystem of (10) has damping at $x = 0$. This is the reason why we make the transformation (4).

By Assumption A, the term $(g(0) + F_4)v(t)$ contains the sinusoids of no more than m distinct frequencies, which can be expressed without loss of generality as

$$(g(0)+F_4)v(t) = \sum_{i=1}^{l}(A_i \cos\omega_i t + B_i \sin\omega_i t) + C, \; l \le r \le m \quad (11)$$

where A_i, B_i, and C are unknown parameters and $A_i^2 + B_i^2 > 0, i = 1, \dots, l$.

Lemma 2.2: The $(g(0) + F_4)v(t)$ can be generated by the exosystem of the following:

$$\begin{cases} \dot{d}(t) = S_c(\theta)d(t) = A_c d(t) - \sum_{i=1}^{m} \theta_i E_{2i} d_1(t) \\ (g(0) + F_4)v(t) = d_1(t) \end{cases} \quad (12)$$

where $d(t) = (d_1(t), d_2(t), \dots, d_{2m+1}(t))^\top \in \mathbb{R}^{2m+1}$

$$A_c = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, S_c(\theta) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ -\theta_1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ -\theta_m & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

E_{2i} is the $2i$th column of the $(2m + 1) \times (2m + 1)$ identity matrix, and $\theta = [\theta_1, \dots, \theta_m]^\top = [\theta_1, \dots, \theta_l, 0, \dots, 0]^\top \in \mathbb{R}^m$ with $\theta_1, \dots, \theta_l$ being chosen so that

$$s^{2\,l} + \theta_1\, s^{2(l-1)} + \cdots + \theta_l \triangleq \prod_{i=1}^{l}(s^2 + \omega_i^2). \quad (13)$$

Proof: By (11), $(g(0) + F_4)v(t)$ can be generated by the following exosystem:

$$\begin{cases} \dot{\eta}(t) = S_\eta \eta(t), \; \eta(t) \in \mathbb{R}^{2l+1} \\ (g(0) + F_4)v(t) = \gamma_\eta \eta(t) \end{cases}$$

where

$$\begin{cases} S_\eta = \text{blockdiag}\{\omega_1 S_0, \omega_2 S_0, \dots, \omega_l S_0, 0_{1\times 1}\} \\ S_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ \gamma_\eta = [1, 0, \dots, 1, 0, 1] \\ \eta(0) = (A_1, B_1, \dots, A_l, B_l, C)^\top. \end{cases}$$

It is a trivial exercise that the pair (S_η, γ_η) is observable that guarantees that there exists a coordinate transformation

$$\eta^E(t) = T_1 \eta(t), \; \eta^E(t) = (\eta_1^E(t), \dots, \eta_{2l+1}^E(t))^\top$$

where T_1 is a nonsingular $(2l + 1) \times (2l + 1)$ matrix, to convert the observable pair (S_η, γ_η) into a canonical form

$$\begin{cases} \dot{\eta}^E(t) = S_E(\theta)\eta^E(t) \\ (g(0) + F_4)v(t) = \eta_1^E(t) \end{cases}$$

with

$$S_E(\theta) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ -\theta_1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ -\theta_l & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Since the characteristic polynomial of S_η is the same as S_E, the $\theta_1, \dots, \theta_l$ can be chosen such that

$$s^{2l+1} + \theta_1\, s^{2l-1} + \cdots + \theta_{l-1}s^3 + \theta_l s \triangleq s\prod_{i=1}^{l}(s^2 + \omega_i^2). \quad (14)$$

Next, let $T_2 = [I_{2l+1}\; 0_{(2l+1)\times(2m-2\,l)}]^\top$ and $d(t) = T_2\eta^E(t)$. A direct computation shows that $d(\cdot)$ satisfies (12). ■

We therefore write $(z(\cdot,t), d(\cdot))$ to be governed by

$$\begin{cases} z_{tt}(x,t) = z_{xx}(x,t) \\ z_x(0,t) = c_0[z(0,t) - e(t)] + c_1[z_t(0,t) - \dot{e}(t)] \\ z_x(1,t) = u(t) \\ \dot{d}(t) = S_c(\theta)d(t) \\ e(t) = z(0,t) - C_c d(t) \\ \dot{e}(t) = z_t(0,t) - C_c S_c(\theta)d(t) \end{cases} \quad (15)$$

where $C_c = [1, 0, \dots, 0] \in \mathbb{R}^{1\times(2m+1)}$. From now on, we only need to design error feedback control for the transformed system (15), which is much simpler than systems (1) and (2).

A. Feedforward Control Design

In this subsection, we design a feedforward control for the system (15). Let $f_0(x,\theta) = f_0(x) \in \mathbb{R}^{1\times(2m+1)}$ be the solution of the following equation:

$$\begin{cases} f_0''(x) = f_0(x)S_c^2(\theta) \\ f_0'(0) = c_0 C_c + c_1 C_c S_c(\theta) \\ f_0(0) = C_c. \end{cases} \quad (16)$$

Lemma 2.3: The initial value problem (16) admits a unique solution, which is continuously differentiable with respect to x and the parameter θ.

Proof: It is clear that (16) admits a unique solution

$$(f_0(x,\theta), f_0'(x,\theta)) = (f_0(0), f_0'(0))e^{\begin{pmatrix} 0 & S_c^2(\theta) \\ I & 0 \end{pmatrix} x}.$$

Let $F_1(x,\theta) = f_0(x,\theta) - f_0(0), F_2(x,\theta) = f_0'(x,\theta) - f_0'(0)$. Then, (F_1, F_2) is governed by

$$\begin{cases} F_1'(x) = F_2(x) + c_0C_c + c_1C_cS_c(\theta) \\ F_2'(x) = (F_1(x) + C_c)S_c^2(\theta) \\ F_1(0) = F_2(0) = 0. \end{cases} \tag{17}$$

Let $F = (F_1, F_2)$. We can write (17) as

$$F' = \left(F_2 + c_0C_c + c_1C_cS_c(\theta), (F_1 + C_c)S_c^2(\theta)\right) := \mathbb{F}(F,\theta).$$

It is easy to check that $\mathbb{F}(F,\theta)$ is continuously differentiable with respect to the components F and θ. Therefore, the solution of (16) or (17) is continuously differentiable with respect to (x,θ). ■

Let $\varepsilon(x,t) = z(x,t) - f_0(x)d(t)$. Then

$$\begin{cases} \varepsilon_{tt}(x,t) = \varepsilon_{xx}(x,t) \\ \varepsilon_x(0,t) = 0,\ \varepsilon_x(1,t) = u(t) - f_0'(1)d(t) \\ \dot{d}(t) = S_c(\theta)d(t) \\ e(t) = \varepsilon(0,t). \end{cases} \tag{18}$$

It can be seen that the output regulation problem of $z(\cdot,t) \to f_0(\cdot)d(t)$ has been transformed into a stabilization problem of $\varepsilon(\cdot,t) \to 0$ $(t \to \infty)$ in H. We can thus design naturally a feedforward control as follows:

$$\begin{aligned} u(t) &= -c_2\varepsilon(1,t) - c_3\varepsilon_t(1,t) + f_0'(1,\theta)d(t) \\ &= -c_2z(1,t) - c_3z_t(1,t) + f_0'(1,\theta)d(t) \\ &\quad + c_2f_0(1,\theta)d(t) + c_3f_0(1,\theta)S_c(\theta)d(t),\ c_2, c_3 > 0 \end{aligned} \tag{19}$$

and the closed-loop of the system (18) under control (19) reads

$$\begin{cases} \varepsilon_{tt}(x,t) = \varepsilon_{xx}(x,t) \\ \varepsilon_x(0,t) = 0, \\ \varepsilon_x(1,t) = -c_2\varepsilon(1,t) - c_3\varepsilon_t(1,t). \end{cases} \tag{20}$$

Set $\tilde{\varepsilon}(x,t) = \varepsilon(1-x,t)$. Then, $\tilde{\varepsilon}(\cdot,\cdot)$ satisfies

$$\begin{cases} \tilde{\varepsilon}_{tt}(x,t) = \tilde{\varepsilon}_{xx}(x,t) \\ \tilde{\varepsilon}_x(0,t) = c_2\tilde{\varepsilon}(0,t) + c_3\tilde{\varepsilon}_t(0,t) \\ \tilde{\varepsilon}_x(1,t) = 0. \end{cases} \tag{21}$$

Lemma 2.4: System (21), and hence, the system (20) is exponentially stable in $H, \tilde{\varepsilon}(0,\cdot), \tilde{\varepsilon}(1,\cdot) \in C([0,\infty);\mathbb{R})$ and

$$\lim_{t\to\infty} |\tilde{\varepsilon}(0,t)| = 0,\ \lim_{t\to\infty} |\tilde{\varepsilon}(1,t)| = 0 \tag{22}$$

exponentially. Moreover

$$\int_0^\infty e^{\tilde{\alpha}t}|\tilde{\varepsilon}_t(1,t)|^2 dt < \infty. \tag{23}$$

for some $\tilde{\alpha} > 0$.

Proof: The existence of the C_0-semigroup solution to (21) is straightforward and we omit the details here. We only prove in the real space H because the imaginary part is exactly the same. Define the energy of system (21) as

$$E(t) = \frac{1}{2}\int_0^1 \left[\tilde{\varepsilon}_x^2(x,t) + \tilde{\varepsilon}_t^2(x,t)\right] dx + \frac{c_2}{2}\tilde{\varepsilon}^2(0,t).$$

The derivative of $E(t)$ along (21) satisfies

$$\dot{E}(t) = -c_3\tilde{\varepsilon}_t^2(0,t). \tag{24}$$

We construct the energy multiplier as

$$\rho(t) = \int_0^1 (x-1)\tilde{\varepsilon}_x(x,t)\tilde{\varepsilon}_t(x,t)dx + 2p\tilde{\varepsilon}(0,t)\int_0^1 \tilde{\varepsilon}_t(x,t)dx + pc_3\tilde{\varepsilon}^2(0,t)$$

where p is a positive real to be determined later. Obviously, $|\rho(t)| \le C_1E(t)$ for some constant $C_1 > 0$. Next, it is found that

$$\begin{aligned} \dot{\rho}(t) &= \frac{1}{2}\tilde{\varepsilon}_t^2(0,t) + \frac{1}{2}\tilde{\varepsilon}_x^2(0,t) - \frac{1}{2}\int_0^1 \left[\tilde{\varepsilon}_x^2(x,t) + \tilde{\varepsilon}_t^2(x,t)\right] dx \\ &\quad + 2p\tilde{\varepsilon}_t(0,t)\int_0^1 \tilde{\varepsilon}_t(x,t)dx - 2pc_2\tilde{\varepsilon}^2(0,t) \\ &\le \left(\frac{1}{2} + c_3^2\right)\tilde{\varepsilon}_t^2(0,t) + c_2^2\tilde{\varepsilon}^2(0,t) \\ &\quad - \frac{1}{2}\int_0^1 \left[\tilde{\varepsilon}_x^2(x,t) + \tilde{\varepsilon}_t^2(x,t)\right] dx \\ &\quad + 4p^2\tilde{\varepsilon}_t^2(0,t) + \frac{1}{4}\int_0^1 \tilde{\varepsilon}_t^2(x,t)dx - 2pc_2\tilde{\varepsilon}^2(0,t) \\ &\le C_2\tilde{\varepsilon}_t^2(0,t) - C_3E(t) \end{aligned}$$

where p is chosen so that $p \ge \frac{c_2}{2}$, and $C_i > 0, i = 2,3$ are constants. Let

$$L(t) = E(t) + \frac{\varepsilon_0}{C_1}\rho(t), \varepsilon_0 > 0.$$

Then

$$(1-\varepsilon_0)E(t) \le L(t) \le (1+\varepsilon_0)E(t), \dot{L}(t) \le -\frac{C_3\varepsilon_0}{C_1(1+\varepsilon_0)}L(t)$$

for all sufficiently small $\varepsilon_0 > 0$. This shows that

$$E(t) \le \frac{1+\varepsilon_0}{1-\varepsilon_0}e^{-\frac{C_3\varepsilon_0}{C_1(1+\varepsilon_0)}t}E(0). \tag{25}$$

This, together with the Sobolev embedding theorem, advises the exponential stability (22). Finally, let

$$\xi(t) = e^{\tilde{\alpha}t}\int_0^1 x\tilde{\varepsilon}_x(x,t)\tilde{\varepsilon}_t(x,t)dx$$

where $\tilde{\alpha} < \frac{C_3}{1+\varepsilon_0}$. Then, $\xi(t)$ decays exponentially to zero as $t \to \infty$. Since it is well known that $\tilde{\varepsilon}_t(1,\cdot) \in L^2_{loc}([0,\infty);\mathbb{R})$, we can write

$$\dot{\xi}(t) = \tilde{\alpha}\xi(t) + \frac{1}{2}e^{\tilde{\alpha}t}\tilde{\varepsilon}_t^2(1,t) - \frac{1}{2}e^{\tilde{\alpha}t}\int_0^1 [\tilde{\varepsilon}_x^2(x,t) + \tilde{\varepsilon}_t^2(x,t)]dx$$

which implies that $\xi(t)$ is absolutely continuous. Integrating over $(0,\infty)$ from both sides of the aforementioned equality and taking (25) into account, we arrive at

$$\int_0^\infty e^{\tilde{\alpha}t}\tilde{\varepsilon}_t^2(1,t)dt<\infty$$

which is (23). ■

B. Error-Based Observer Design

In this subsection, we design an observer for the system (15) to recover the state $(z(\cdot,t),d(t))$ and estimate online the θ by the output measurement $(e(t),\dot{e}(t))$. Since the system (15) may have unknown initial value, an observer for the z-subsystem of (15) is just a copy of the plant as

$$\begin{cases}\hat{z}_{tt}(x,t)=\hat{z}_{xx}(x,t)\\ \hat{z}_x(0,t)=c_0[\hat{z}(0,t)-e(t)]+c_1[\hat{z}_t(0,t)-\dot{e}(t)]\\ \hat{z}_x(1,t)=u(t)\\ (\hat{z}(\cdot,0),\hat{z}_t(\cdot,0))=(\hat{z}_0(\cdot),\hat{z}_1(\cdot))\in H.\end{cases}\tag{26}$$

Define the observer error $\tilde{z}(x,t)=z(x,t)-\hat{z}(x,t)$. Then

$$\begin{cases}\tilde{z}_{tt}(x,t)=\tilde{z}_{xx}(x,t)\\ \tilde{z}_x(0,t)=c_0\tilde{z}(0,t)+c_1\tilde{z}_t(0,t)\\ \tilde{z}_x(1,t)=0\end{cases}\tag{27}$$

which is similar to (21), and hence, is exponentially stable in H by virtue of Lemma 2.4. In particular, $\tilde{z}(0,\cdot),\tilde{z}(1,\cdot)\in C([0,\infty);\mathbb{R})$, $\lim_{t\to\infty}|\tilde{z}(0,t)|=0$, $\lim_{t\to\infty}|\tilde{z}(1,t)|=0$ exponentially, and

$$\int_0^\infty e^{\alpha t}|\tilde{z}_t(1,t)|^2dt<\infty \text{ for some } \alpha>0.$$

We next introduce a known function

$$y_d(t)=-e(t)+\hat{z}(0,t)=C_cd(t)-\tilde{z}(0,t)$$

and consider the following system:

$$\begin{cases}\dot{d}(t)=S_c(\theta)d(t)=A_cd(t)-\sum_{i=1}^m\theta_iE_{2i}d_1(t)\\ y_d(t)=C_cd(t)-\tilde{z}(0,t).\end{cases}\tag{28}$$

Motivated by [21], we first introduce two cascaded filters to detect the number of the exosystem's frequencies

$$\begin{cases}\dot{\xi}_i(t)=\Gamma\xi_i(t)-[0\ I_{2m}]E_{2i}y_d(t), & \xi_i(t)\in\mathbb{R}^{2m}\\ \mu_i(t)=[1\ 0\ \dots\ 0]\xi_i(t), & 1\le i\le m\\ \dot{\Omega}(t)=-\lambda_b\Omega(t)+\lambda_c\mu(t)\mu(t)^\top, & \Omega(t)\in\mathbb{R}^{m\times m}\\ \Omega_i(t)=[I_i,0_{i\times(m-i)}]\Omega(t)[I_i,0_{i\times(m-i)}]^\top, & 1\le i\le m\end{cases}\tag{29}$$

where $\mu(t)=[\mu_1(t),\dots,\mu_m(t)]^\top,\lambda_b,\lambda_c>0$, and $b=[b_1,\dots,b_{2m}]^\top$ is chosen so that Γ is a Hurwitz matrix

$$\Gamma=\begin{bmatrix}-b_1 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ -b_{2m-1} & 0 & \cdots & 1\\ -b_{2m} & 0 & \cdots & 0\end{bmatrix}.$$

By means of the change of coordinates

$$\begin{bmatrix}\chi_1(t)\\ \phi(t)\end{bmatrix}=d(t)-\begin{bmatrix}0\\ \sum_{i=1}^m\xi_i(t)\theta_i+bC_cd(t)\end{bmatrix}\tag{30}$$

with $\phi(t)\in\mathbb{R}^{2m},\chi_1(t)=C_cd(t)\in\mathbb{R}$, we obtain

$$\begin{cases}\dot{\chi}_1(t)=\phi_1(t)+b_1\chi_1(t)+\theta^\top\mu(t)\\ \dot{\phi}(t)=\Gamma\phi(t)+\beta\chi_1(t)-\tilde{z}(0,t)M_2\theta\end{cases}$$

where M_2 is a constant $2m\times m$ matrix, and β is a $2m\times 1$ matrix

$$\beta=\begin{bmatrix}b_2-b_1^2,\cdots, & b_{2m}-b_{2m-1}b_1, & -b_{2m}b_1\end{bmatrix}^\top.$$

Motivated by [22], we design an adaptive observer for (28) according to the output $y_d(t)$ as follows:

$$\begin{cases}\dot{\hat{\chi}}_1(t)=\hat{\phi}_1(t)+b_1y_d(t)+\sum_{i=1}^m\mu_i(t)\hat{\theta}_i(t)+k_o(y_d(t)-\hat{\chi}_1(t))\\ \dot{\hat{\phi}}(t)=\Gamma\hat{\phi}(t)+\beta y_d,\ \hat{\phi}(t)\in\mathbb{R}^{2m}\\ \hat{d}(t)=\begin{bmatrix}\hat{\chi}_1(t)\\ \hat{\phi}(t)\end{bmatrix}+\begin{bmatrix}0\\ \sum_{i=1}^m\xi_i(t)\hat{\theta}_i(t)+b\hat{\chi}_1(t)\end{bmatrix}\end{cases}\tag{31}$$

with the parameter adaptive law

$$\begin{cases}\dot{\hat{\theta}}_i(t)=g\mu_i(t)(y_d(t)-\hat{\chi}_1(t)),\ e^{-|\det(\Omega_i)|^{1/i}t}\le\frac{1}{2}\\ \dot{\hat{\theta}}_i(t)=g\mu_i(t)(y_d(t)-\hat{\chi}_1(t))-\alpha\hat{\theta}_i(t),\text{otherwise}\end{cases}\tag{32}$$

for $1\le i\le m$, where $\alpha,g,k_0>0$ can be arbitrarily chosen. Recall that $\theta=[\theta_1,\dots,\theta_m]^\top=[\theta_1,\dots,\theta_l,0,\dots,0]^\top\in\mathbb{R}^m$ with $\theta_1,\dots,\theta_l$ being the coefficients of the polynomial (13).

Lemma 2.5: For any initial state

$$(\hat{\chi}_1(0),\hat{\phi}(0),\hat{\theta}(0),\Omega(0),\{\xi_i(0)\}_{i=1}^m)\in\mathbb{R}\times\mathbb{R}^{2m}\times\mathbb{R}^m\times\mathbb{S}_+^m\times\mathbb{R}^{2m\times m},\text{ where }\mathbb{S}_+^m=\{A\in\mathbb{R}^{m\times m}: A\text{ is positive definite}\}$$

there hold

$$\lim_{t\to\infty}e^{-|\det(\Omega_i)|^{1/i}\cdot t}=\begin{cases}0, & i=1,\dots,l\\ 1, & i=l+1,\dots,m\end{cases}$$

exponentially, and

$$\lim_{t\to\infty}\|\hat{\theta}(t)-\theta\|=0,\quad \lim_{t\to\infty}\|\hat{d}(t)-d(t)\|=0\tag{33}$$

exponentially.

Proof: Set $\mu_i(t)=\mu_{ip}(t)+\mu_{ie}(t)$, where $\mu_{ip}(\cdot)$ is the solution to

$$\begin{cases}\dot{\xi}_{ip}(t)=\Gamma\xi_{ip}(t)-[0\ I_{2m}]E_{2i}d_1(t)\\ \mu_{ip}(t)=[1\ 0\ \dots\ 0]\xi_{ip}(t),\ i=1,\dots,m\end{cases}\tag{34}$$

and $\mu_{ie}(\cdot)$ is governed by

$$\begin{cases}\dot{\xi}_{ie}(t)=\Gamma\xi_{ie}(t)+[0\ I_{2m}]E_{2i}\tilde{z}(0,t)\\ \mu_{ie}(0)=\mu_i(0)-\mu_{ip}(0)\\ \mu_{ie}(t)=[1\ 0\ \dots\ 0]\xi_{ie}(t),\ i=1,\dots,m.\end{cases}\tag{35}$$

Since $d_1(\cdot)$ is bounded and Γ is Hurwitz, $\mu_{ip}(\cdot)$ is bounded. By [15, Th. 5.2.1], the vector $[\mu_{1p}(t),\dots,\mu_{lp}(t)]^\top$ is PE (but $[\mu_{1p}(t),\dots,\mu_{kp}(t)]^\top,k\ge l+1$ is not) because $d_1(\cdot)$ contains the sinusoids of l distinct frequencies. For the system (35), since Γ is Hurwitz, and $\lim_{t\to\infty}|\tilde{z}(0,t)|=0$ exponentially, we conclude that $\lim_{t\to\infty}|\mu_{ie}(t)|=0$ exponentially. By Lemma 1.3 and $\mu_i(t)=\mu_{ip}(t)+\mu_{ie}(t)$, the vector $[\mu_1(t),\ldots,\mu_l(t)]^\top$ is also PE, that is, there exist $K,T>0$ such that for all $t\geq 0$ and $i=1,\ldots,l$

$$\int_t^{t+T}[\mu_1(s),\ldots,\mu_i(s)]^\top[\mu_1(s),\ldots,\mu_i(s)]\,ds\geq KI_i.$$

From (29), it is seen that

$$\Omega_i(t)=e^{-\lambda_b t}\Omega_i(0)+\lambda_c\int_0^t e^{-\lambda_b(t-s)}[\mu_1(s),\ldots,\mu_i(s)]^\top[\mu_1(s),\ldots,\mu_i(s)]ds. \tag{36}$$

Since $\Omega(0)$ is positive definite, it follows from a similar argument of [21, lemma 3.1] that $|\det(\Omega_i(t))|^{1/i}\geq\delta_m>0\,\forall t\geq 0$, $i=1,\ldots,l$, where δ_m is a positive number and that $|\det(\Omega_i)|^{1/i}, i=1+1,\ldots,m$ tend to zero exponentially as time goes to infinity. Therefore

$$\lim_{t\to\infty}e^{-|\det(\Omega_i)|^{1/i}\cdot t}=\begin{cases}0, & i=1,\ldots,l,\\ 1, & i=l+1,\ldots,m\end{cases}$$

exponentially. Hence, there exists some $T_p>0$ such that for all $t\geq T_p$

$$\dot{\hat\theta}_i(t)=g\mu_i(t)(y_d(t)-\hat\chi_1(t)),\qquad 1\leq i\leq l$$

$$\dot{\hat\theta}_i(t)=g\mu_i(t)(y_d(t)-\hat\chi_1(t))-\alpha\hat\theta_i(t),\quad l+1\leq i\leq m.$$

Define the error variables $\tilde\chi_1(t)=\chi_1(t)-\hat\chi_1(t), \tilde\phi(t)=\phi(t)-\hat\phi(t), \tilde\theta(t)=\theta-\hat\theta(t)$. Then

$$\begin{cases}\dot{\tilde\phi}(t)=\Gamma\tilde\phi(t)+\tilde z(0,t)(\beta-M_2\theta)\\ \dot{\tilde\chi}_1(t)=-k_0\tilde\chi_1(t)+\tilde\phi_1(t)+\mu(t)^\top\tilde\theta(t)+\tilde z(0,t)(b_1+k_0)\\ \dot{\tilde\theta}_i(t)=-g\mu_i(t)\tilde\chi_1(t)+g\mu_i(t)\tilde z(0,t),\ 1\leq i\leq l\\ \dot{\tilde\theta}_i(t)=-g\mu_i(t)\tilde\chi_1(t)-\alpha\tilde\theta_i(t)\\ \qquad +g\mu_i(t)\tilde z(0,t),\ l+1\leq i\leq m.\end{cases} \tag{37}$$

Since Γ is Hurwitz and $\lim_{t\to\infty}|\tilde z(0,t)|=0$ exponentially, we conclude that $\lim_{t\to\infty}\|\tilde\phi(t)\|=0$ exponentially. Now, we consider the last three equations of the system (37) as perturbations of the following nominal system:

$$\begin{cases}\dot{\tilde\chi}_1(t)=-k_0\tilde\chi_1(t)+\mu(t)^\top\tilde\theta(t)\\ \dot{\tilde\theta}_i(t)=-g\mu_i(t)\tilde\chi_1(t), & 1\leq i\leq l\\ \dot{\tilde\theta}_i(t)=-g\mu_i(t)\tilde\chi_1(t)-\alpha\tilde\theta_i(t), & l+1\leq i\leq m.\end{cases} \tag{38}$$

Define the Lyapunov function as

$$V(t)=\frac{1}{2}\left(\tilde\chi_1^2+\frac{1}{g}\tilde\theta^\top\tilde\theta+p\left(Q\tilde\theta[l]-\mu[l]\tilde\chi_1\right)^\top\times\left(Q\tilde\theta[l]-\mu[l]\tilde\chi_1\right)\right)$$

where p is a positive real to be determined later, $\mu[l]=[\mu_1,\ldots,\mu_l]^\top, \theta[l]=[\theta_1,\ldots,\theta_l]^\top$, and $Q(t)$ is generated by

$$\begin{cases}\dot Q=-Q+\mu[l]\mu^\top[l],\ Q(0)=e^{-T}KI_l\\ \text{Recall that }\displaystyle\int_t^{t+T}\mu[l](s)\mu^\top[l](s)\,ds\geq KI_l\,\forall t\geq 0.\end{cases} \tag{39}$$

Since $\mu[l](t)$ is bounded, we can write

$$\|\mu[l](t)\|\leq\mu_M\quad\forall t\geq 0. \tag{40}$$

From (39) and (40), it follows that

$$Ke^{-2\,T}I\leq Q(t)\leq\mu_M^2 I\ \ \forall t\geq 0. \tag{41}$$

The time derivative of $V(t)$, in the light of (38), is found to be

$$\begin{aligned}\dot V(t)&=-k_0\tilde\chi_1^2+\mu^\top\tilde\theta\chi_1-\mu^\top\tilde\theta\chi_1-\sum_{i=l+1}^m\frac{\alpha}{g}\tilde\theta_i^2\\ &\quad+p\left(Q\tilde\theta[l]-\mu[l]\tilde\chi_1\right)^\top\left(\dot Q\tilde\theta[l]-gQ\mu[l]\tilde\chi_1\right)\\ &\quad+p\left(Q\tilde\theta[l]-\mu[l]\tilde\chi_1\right)^\top\left(k_0\mu[l]\tilde\chi_1-\mu[l]\mu^\top\tilde\theta-\dot\mu[l]\tilde\chi_1\right)\\ &=-k_0\tilde\chi_1^2-\sum_{i=l+1}^m\frac{\alpha}{g}\tilde\theta_i^2-p\left\|Q\tilde\theta[l]-\mu[l]\tilde\chi_1\right\|^2\\ &\quad+p\left(Q\tilde\theta[l]-\mu[l]\tilde\chi_1\right)^\top\Big\{(k_0-1)\mu[l]\tilde\chi_1\\ &\quad-\mu[l]\sum_{i=l+1}^m\mu_i\tilde\theta_i-gQ\mu[l]\tilde\chi_1-\dot\mu[l]\tilde\chi_1\Big\}\end{aligned} \tag{42}$$

for all $t>T_p$. Since μ_i and $\dot\mu_i$ are bounded, by choosing p sufficiently small, we conclude that the origin of the nominal system (38) is exponentially stable. By Lemma 1.2, $\lim_{t\to\infty}\|\hat\theta(t)-\theta\|=0$ and $\lim_{t\to\infty}|\hat\chi_1(t)-\chi_1(t)|=0$ exponentially. Since $\lim_{t\to\infty}\|\tilde\phi(t)\|=0$ exponentially, it then follows from (31) and (30) that

$$\tilde d(t)=d(t)-\hat d(t)=\begin{bmatrix}\tilde\chi_1(t)\\ \tilde\phi(t)\end{bmatrix}+\begin{bmatrix}0\\ \sum_{i=1}^m\xi_i(t)\tilde\theta_i(t)+b\tilde\chi_1(t)\end{bmatrix}$$

which is obviously tend to 0 exponentially as $t\to\infty$ by boundedness of $\xi_i(t)$. ∎

Remark 1: When θ is known, we can design a standard Luenberger observer

$$\dot{\hat d}(t)=S_c(\theta)\hat d(t)+L(y_d(t)-C_c\hat d(t))$$

for (28) and there also holds $\lim_{t\to\infty}\tilde d(t)=0$ exponentially.

Remark 2: Different from the adaptive internal model method for output regulation problem of ODEs discussed in [19], [21], and [22] where the observer design was split into four steps:

1) Convert the plant into an observable canonical form;
2) Convert the exosystem into an observable canonical form;
3) Convert the coupled system of plant and exosystem into an observable canonical form;
4) Design an adaptive observer for the coupled system of plant and exosystem.

In this article, through transformation (4), the PDE part (plant) is separated from the exosystem, and the observers for PDE-part and ODE-part can be designed separately.

C. Error-Based Feedback Control Design

By (19), (31), and (26), we design, therefore, naturally a tracking error feedback control as follows:

$$\begin{cases} u(t) = -c_2\hat{z}(1,t) - c_3\hat{z}_t(1,t) \\ +f_0'(1,\hat{\theta})\hat{d}(t) + c_2 f_0(1,\hat{\theta})\hat{d}(t) + c_3 f_0(1,\hat{\theta})S_c(\hat{\theta})\hat{d}(t) \\ \hat{z}_{tt}(x,t) = \hat{z}_{xx}(x,t) \\ \hat{z}_x(0,t) = c_0[\hat{z}(0,t) - e(t)] + c_1[\hat{z}_t(0,t) - \dot{e}(t)] \\ \hat{z}_x(1,t) = u(t) \\ y_d(t) = -e(t) + \hat{z}(0,t) \\ \dot{\hat{\chi}}_1(t) = \hat{\phi}_1(t) + b_1 y_d(t) + \sum_{i=1}^m \mu_i(t)\hat{\theta}_i(t) + k_o(y_d(t) - \hat{\chi}_1(t)) \\ \dot{\hat{\phi}}(t) = \Gamma\hat{\phi}(t) + \beta y_d, \quad \hat{\phi}(t) \in \mathbb{R}^{2m} \\ \hat{d}(t) = \begin{bmatrix} \hat{\chi}_1(t) \\ \hat{\phi}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \sum_{i=1}^m \xi_i(t)\hat{\theta}_i(t) + b\hat{\chi}_1(t) \end{bmatrix} \\ \dot{\hat{\theta}}_i(t) = g\mu_i(t)(y_d(t) - \hat{\chi}_1(t)), \quad e^{-|\det(\Omega_i)|^{1/i}\cdot t} \le \frac{1}{2} \\ \dot{\hat{\theta}}_i(t) = g\mu_i(t)(y_d(t) - \hat{\chi}_1(t)) - \alpha\hat{\theta}_i(t), \quad \text{otherwise} \\ \dot{\xi}_i(t) = \Gamma\xi_i(t) - [0\ I_{2m}]E_{2i}y_d(t), \quad \xi_i(t) \in \mathbb{R}^{2m} \\ \dot{\Omega}(t) = -\lambda_b\Omega(t) + \lambda_c\mu(t)\mu(t)^\top, \quad \Omega(t) \in \mathbb{R}^{m\times m} \end{cases} \tag{43}$$

which is obtained with replacements of the states in feedforward control (19) with their estimates from observers (26) and (31).

Remark 3: In the control (43), $f_0(x,\theta)$ determined by the initial value problem (16) plays an important role. However, for different PDEs, $f_0(x,\theta)$ is not always a solution to an initial value problem like (16). For example, in (1), when the performance output is $Y(t) = y(1,t)$, then $f_0(x) = f_0(x,\theta)$ is changed to be a solution of a two-point boundary value problem of the following:

$$\begin{cases} f_0''(x) = f_0(x)S_c^2(\theta) \\ f_0'(0) = 0 \\ f_0(1) = C_c. \end{cases} \tag{44}$$

Nevertheless, our method can still be applied because the solutions to (44) can be expressed as

$$(f_0(x,\theta), f_0'(x,\theta)) = C_c\left\{\begin{bmatrix} I & 0\end{bmatrix} e^{A_0(\theta)}\begin{bmatrix} I \\ 0\end{bmatrix}\right\}^{-1}\begin{bmatrix} I & 0\end{bmatrix} e^{A_0(\theta)x} \tag{45}$$

where $A_0(\theta) = \begin{pmatrix} 0 & S_c^2(\theta) \\ I & 0\end{pmatrix}$. Actually, from the first two equations of (44), $(f_0(x,\theta), f_0'(x,\theta)) = (f_0(0,\theta), 0)e^{A_0(\theta)x}$. Since (44) admits a unique solution under some proper conditions, $f_0(0,\theta)$ is uniquely determined by $f_0(1) = C_c$, i.e., the linear equation $f_0(0,\theta)\begin{bmatrix} I & 0\end{bmatrix} e^{A_0(\theta)}\begin{bmatrix} I \\ 0\end{bmatrix} = C_c$ admits a unique solution, which implies in turn that $\begin{bmatrix} I & 0\end{bmatrix} e^{A_0(\theta)}\begin{bmatrix} I \\ 0\end{bmatrix}$ is invertible, and hence, $(f_0(x,\theta), f_0'(x,\theta))$ can be written as (45). Similar treatment can be done for other PDEs.

D. Well-Posedness and Stability of the Closed-Loop System

The closed loop of the system (1) under control (43) is

$$\begin{cases} y_{tt}(x,t) = y_{xx}(x,t) + \Delta(x)F_1 v(t) \\ y_x(0,t) = F_2 v(t) \\ y_x(1,t) = u(t) + F_3 v(t) \\ \dot{v}(t) = Sv(t) \\ e(t) = y(0,t) - F_4 v(t) \\ u(t) = -c_2\hat{z}(1,t) - c_3\hat{z}_t(1,t) \\ +f_0'(1,\hat{\theta})\hat{d}(t) + c_2 f_0(1,\hat{\theta})\hat{d}(t) + c_3 f_0(1,\hat{\theta})S_c(\hat{\theta})\hat{d}(t) \\ \hat{z}_{tt}(x,t) = \hat{z}_{xx}(x,t) \\ \hat{z}_x(0,t) = c_0[\hat{z}(0,t) - e(t)] + c_1[\hat{z}_t(0,t) - \dot{e}(t)] \\ \hat{z}_x(1,t) = u(t) \\ y_d(t) = -e(t) + \hat{z}(0,t) \\ \dot{\hat{\chi}}_1(t) = \hat{\phi}_1(t) + b_1 y_d(t) + \sum_{i=1}^m \mu_i(t)\hat{\theta}_i(t) + k_o(y_d(t) - \hat{\chi}_1(t)) \\ \dot{\hat{\phi}}(t) = \Gamma\hat{\phi}(t) + \beta y_d, \quad \hat{\phi}(t) \in \mathbb{R}^{2m} \\ \hat{d}(t) = \begin{bmatrix} \hat{\chi}_1(t) \\ \hat{\phi}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \sum_{i=1}^m \xi_i(t)\hat{\theta}_i(t) + b\hat{\chi}_1(t) \end{bmatrix} \end{cases} \tag{46}$$

with the parameter adaptive law

$$\begin{cases} \dot{\hat{\theta}}_i(t) = g\mu_i(t)(y_d(t) - \hat{\chi}_1(t)), \quad e^{-|\det(\Omega_i)|^{1/i}\cdot t} \le \frac{1}{2} \\ \dot{\hat{\theta}}_i(t) = g\mu_i(t)(y_d(t) - \hat{\chi}_1(t)) - \alpha\hat{\theta}_i(t), \text{ otherwise} \end{cases}$$

for $1 \le i \le m$, and filter

$$\begin{cases} \dot{\xi}_i(t) = \Gamma\xi_i(t) - [0\ I_{2m}]E_{2i}y_d(t), \quad \xi_i(t) \in \mathbb{R}^{2m} \\ \mu_i(t) = [1\ 0\ \dots\ 0]\xi_i(t), \quad 1 \le i \le m \\ \dot{\Omega}(t) = -\lambda_b\Omega(t) + \lambda_c\mu(t)\mu(t)^\top, \quad \mu(t) = [\mu_1, \dots, \mu_m(t)]^\top \\ \Omega_i(t) = [I_i, 0_{i\times(m-i)}]\Omega(t)[I_i, 0_{i\times(m-i)}]^\top, \quad 1 \le i \le m. \end{cases}$$

We consider system (46) in the state space $\mathscr{H} = H^2 \times \mathbb{R} \times \mathbb{R}^{2m} \times \mathbb{R}^m \times \mathbb{S}_+^m \times \mathbb{R}^{2m\times m}$.

Theorem 2.1: Suppose that $c_0, c_1, c_2, c_3 > 0$. Then, for any unknown coefficients F_1, F_2, F_3, F_4, S, unknown function $\Delta(\cdot)$ and any initial state

$$(y(\cdot,0), y_t(\cdot,0), \hat{z}(\cdot,0), \hat{z}_t(\cdot,0) \\ \hat{\chi}_1(0), \hat{\phi}(0), \hat{\theta}(0), \Omega(0), \{\xi_i(0)\}_{i=1}^m) \in \mathscr{H}$$

the closed-loop system (46) admits a unique bounded solution $(y, y_t, \hat{z}, \hat{z}_t, \hat{\chi}_1, \hat{\phi}, \hat{\theta}) \in C([0,\infty); \mathscr{H})$ such that $\lim_{t\to\infty}|e(t)| = 0$ exponentially, and

$$\int_0^\infty e^{\beta t}|\dot{e}(t)|^2\, dt < \infty.$$

Proof: Using the variables $\varepsilon(x,t), \tilde{z}(x,t), \tilde{\chi}_1(t), \tilde{\phi}(t)$, and $\tilde{\theta}(t)$ given by (18), (27), and (37), the closed-loop system (46) is equivalent to

$$\begin{cases}\varepsilon_{tt}(x,t)=\varepsilon_{xx}(x,t)\\ \varepsilon_x(0,t)=0\\ \varepsilon_x(1,t)=-c_2\varepsilon(1,t)-c_3\varepsilon_t(1,t)\\ +\mathscr{P}(t)+c_2\tilde{z}(1,t)+c_3\tilde{z}_t(1,t)\\ \tilde{z}_{tt}(x,t)=\tilde{z}_{xx}(x,t)\\ \tilde{z}_x(0,t)=c_0\tilde{z}(0,t)+c_1\tilde{z}_t(0,t)\\ \tilde{z}_x(1,t)=0\\ y_d(t)=C_cd(t)-\tilde{z}(0,t)\\ \dot{\hat{\chi}}_1(t)=\hat{\phi}_1(t)+b_1y_d(t)+\sum_{i=1}^m\mu_i(t)\hat{\theta}_i(t)\\ +k_o(y_d(t)-\hat{\chi}_1(t))\\ \dot{\hat{\phi}}(t)=\Gamma\hat{\phi}(t)+\beta y_d\\ \dot{\hat{\theta}}_i(t)=g\mu_i(t)(y_d(t)-\hat{\chi}_1(t)),\quad e^{-|\det(\Omega_i)|^{1/i}\cdot t}\leq\frac{1}{2}\\ \dot{\hat{\theta}}_i(t)=g\mu_i(t)(y_d(t)-\hat{\chi}_1(t))-\alpha\hat{\theta}_i(t),\ \text{otherwise}\\ \dot{\xi}_i(t)=\Gamma\xi_i(t)-[0\ I_{2\,m}]E_{2i}y_d(t),\quad \xi_i(t)\in\mathbb{R}^{2\,m}\\ \dot{\Omega}(t)=-\lambda_b\Omega(t)+\lambda_c\mu(t)\mu(t)^\top,\quad \Omega(t)\in\mathbb{R}^{m\times m}\\ e(t)=\varepsilon(0,t)\end{cases}\tag{47}$$

where

$$\mathscr{P}(t):=-f_0'(1,\theta)d(t)-c_2f_0(1,\theta)d(t)-c_3f_0(1,\theta)S_c(\theta)d(t)+f_0'(1,\hat{\theta})\hat{d}(t)+c_2f_0(1,\hat{\theta})\hat{d}(t)+c_3f_0(1,\hat{\theta})S_c(\hat{\theta})\hat{d}(t).\tag{48}$$

The existence and stability of the solution of $(\tilde{z},\hat{\chi}_1,\hat{\phi},\hat{\theta},\{\xi_i\}_{i=1}^m,\Omega)$ part of (47) has been shown in Lemmas 2.4 and 2.5. We only need to consider the ε-part of system (47), which can be written abstractly as

$$\frac{d}{dt}(\varepsilon(\cdot,t),\varepsilon_t(\cdot,t))=\mathbb{A}(\varepsilon(\cdot,t),\varepsilon_t(\cdot,t))+\mathscr{B}(\mathscr{P}(t)+c_2\tilde{z}(1,t)+c_3\tilde{z}_t(1,t))$$

where the operator $\mathbb{A}:D(\mathbb{A})(\subset H)\to H$ is defined by

$$\begin{cases}\mathbb{A}(f,g)=(g,f'')\\ D(\mathbb{A})=\{(f,g)\in H^2(0,1)\times H^1(0,1)|\\ f'(0)=0,f'(1)=-c_2f(1)-c_3g(1)\}\end{cases}$$

and $\mathscr{B}=(0,\delta(\cdot-1))$. By Lemma 2.4, $\mathbb{A}$ generates an exponentially stable C_0-semigroup on H. It is well known that $\mathscr{B}$ is admissible for $e^{\mathbb{A}t}$. It follows from Lemma 2.4 that $e^{\alpha/2\cdot}\tilde{z}_t(1,\cdot)\in L^2(0,\infty),\tilde{z}(1,\cdot)\in C([0,\infty);\mathbb{R})$ and $\lim_{t\to\infty}\tilde{z}(1,t)=0$ exponentially.

Furthermore, we also have $\mathscr{P}(\cdot)\in C([0,\infty);\mathbb{R})$ and $\lim_{t\to\infty}\mathscr{P}(t)=0$ exponentially. Actually, to this end, it suffices to show that $f_0'(1,\hat{\theta}(t)),f_0(1,\hat{\theta}(t)),S_c(\hat{\theta}(t))$ are continuous in t, and

$$\begin{cases}\lim_{t\to\infty}\|f_0(1,\hat{\theta})-f_0(1,\theta)\|=0\\ \lim_{t\to\infty}\|f_0'(1,\hat{\theta})-f_0'(1,\theta)\|=0\\ \lim_{t\to\infty}\|S_c(\hat{\theta})-S_c(\theta)\|=0\end{cases}$$

all are exponentially. Since $\theta(\cdot)\in C([0,\infty);\mathbb{R}^m)$, and $\lim_{t\to\infty}\|\hat{\theta}(t)-\theta\|=0$ exponentially, we see that $\hat{\theta}(t)$ is bounded. Now, suppose that $\theta,\hat{\theta}(t)\in[-M,M]^m$ for some $M>0$. By Lemma 2.3, $f_0'(1,\theta),f_0(1,\theta)$, and $S_c(\theta)$ are continuously differentiable with respect to the parameter θ, and hence, they are Lipschitz continuous over $[-M,M]^m$. We can thus get immediately the continuity of $f_0'(1,\hat{\theta}(t)),f_0(1,\hat{\theta}(t)),S_c(\hat{\theta}(t))$ with respect to time, and

$$\begin{cases}\lim\limits_{t\to\infty}\|f_0(1,\hat{\theta})-f_0(1,\theta)\|\leq\lim\limits_{t\to\infty}L_1\|\hat{\theta}(t)-\theta\|=0\\ \lim\limits_{t\to\infty}\|f_0'(1,\hat{\theta})-f_0'(1,\theta)\|\leq\lim\limits_{t\to\infty}L_2\|\hat{\theta}(t)-\theta\|=0\\ \lim\limits_{t\to\infty}\|S_c(\hat{\theta})-S_c(\theta)\|\leq\lim\limits_{t\to\infty}L_3\|\hat{\theta}(t)-\theta\|=0\end{cases}$$

all are exponentially too. It then follows from Lemma 1.1 that the ε-part of (47) admits a unique solution $(\varepsilon,\varepsilon_t)\in C([0,\infty);H)$, and

$$\|(\varepsilon(\cdot,t),\varepsilon_t(\cdot,t))\|_H\leq M_1e^{-\omega_1t}$$

for some $M_1,\omega_1>0$. Therefore, the transformations $y(x,t)=\varepsilon(x,t)+f_0(x)d(t)-g(x)v(t),\hat{z}(x,t)=\varepsilon(x,t)+f_0(x)d(t)-\tilde{z}(x,t)$ imply that $(y,y_t,\hat{z},\hat{z}_t)$ are well defined in $C([0,\infty);H^2)\cap L^\infty(0,\infty;H^2)$. By the Sobolev trace theorem,

$$\lim_{t\to\infty}|\varepsilon(0,t)|=0\tag{49}$$

exponentially. Define

$$\rho_1(t)=2\int_0^1(x-1)\varepsilon_t(x,t)\varepsilon_x(x,t)dx$$

which satisfies $|\rho_1(t)|\leq\|(\varepsilon(\cdot,t),\varepsilon_t(\cdot,t))\|_H^2$. Finding $\dot{\rho}_1(t)$ and performing integration by parts, we obtain

$$\begin{aligned}\dot{\rho}_1(t)&=2\int_0^1(x-1)\varepsilon_t(x,t)\varepsilon_{xt}(x,t)dx\\ &\quad+2\int_0^1(x-1)\varepsilon_{xx}(x,t)\varepsilon_x(x,t)dx\\ &=\varepsilon_t^2(0,t)+\varepsilon_x^2(0,t)-\int_0^1[\varepsilon_t^2(x,t)+\varepsilon_x^2(x,t)]dx\\ &\geq\varepsilon_t^2(0,t)-\|(\varepsilon(\cdot,t),\varepsilon_t(\cdot,t))\|_H^2.\end{aligned}$$

For any $0<\beta<2\omega_1$, it has

$$\begin{aligned}\int_0^\infty e^{\beta t}\varepsilon_t^2(0,t)\,dt&\leq\int_0^\infty e^{\beta t}\|(\varepsilon(\cdot,t),\varepsilon_t(\cdot,t))\|_H^2\,dt\\ &\quad+|\rho_1(0)|+\beta\int_0^\infty e^{\beta t}|\rho_1(t)|\,dt\\ &<\infty.\end{aligned}$$

This completes the proof of the theorem. ∎

Remark 4: When θ is known, the control (43) becomes

$$\begin{cases} u(t) = -c_2\hat{z}(1,t) - c_3\hat{z}_t(1,t) \\ +f_0'(1,\theta)\hat{d}(t) + c_2 f_0(1,\theta)\hat{d}(t) + c_3 f_0(1,\theta)S_c(\theta)\hat{d}(t) \\ \hat{z}_{tt}(x,t) = \hat{z}_{xx}(x,t) \\ \hat{z}_x(0,t) = c_0[\hat{z}(0,t) - e(t)] + c_1[\hat{z}_t(0,t) - \dot{e}(t)] \\ \hat{z}_x(1,t) = u(t) \\ y_d(t) = -e(t) + \hat{z}(0,t) \\ \dot{\hat{d}}(t) = S_c(\theta)\hat{d}(t) + L(y_d(t) - C_c\hat{d}(t)). \end{cases}$$

In this case, $\lim_{t\to\infty} e(t) = 0$ exponentially which leads to an internal model principle discussed in [7].

III. Regulation of Tracking Error Only

In this section, we deal with the case where only the displacement signal $e(\cdot)$ can be measured. We show that the closed-loop system in this case is only internally asymptotically stable and the tracking error is also asymptotically convergent. This is because the measured signal, corresponding to a compact output operator, is too weak to make closed-loop exponentially stable (see [11]).

Motivated from [7] for asymptotic stabilization of wave equation, we introduce

$$\dot{Z}_1(t) = -k_1[Z_1(t) - e(t)],\ k_1 > 0,\ Z_1(0) = Z_{10} \in \mathbb{R} \quad (50)$$

which is completely determined by the tracking error. Similarly with the last section, we introduce a transformation for the system (1)

$$\begin{cases} z(x,t) = y(x,t) + g(x)v(t) \\ Z_2(t) = Z_1(t) + g_z v(t) \end{cases}$$

where $g : [0,1] \to \mathbb{R}^{1\times(2r+1)}, g_v \in \mathbb{R}^{1\times(2r+1)}$ satisfies

$$\begin{cases} g''(x) = g(x)S^2 + F_1\Delta(x) \\ g'(0) = -F_2 + (k_1+k_2)[F_4 + g(0)] - k_1 g_z, \quad k_2 > 0 \\ g'(1) = -F_3 \\ g_z S = -k_1[g_z - g(0) - F_4]. \end{cases} \quad (51)$$

Lemma 3.1: The boundary value problem (51) admits a unique solution $(g^\top, g_z^\top) \in H^2((0,1);\mathbb{R}^{2r+1}) \times \mathbb{R}^{2r+1}$.

Proof: According to the fourth equation of (51), we can see that $g_z = k_1[F_4 + g(0)][S + k_1]^{-1}$. The rest of proof is very similar to the proof of Lemma 2.1. ∎

The extended system of $(z(\cdot,\cdot), Z_2(\cdot), v(\cdot))$ is then governed by

$$\begin{cases} z_{tt}(x,t) = z_{xx}(x,t) \\ z_x(0,t) = (k_1+k_2)[z(0,t) - e(t)] - k_1[Z_2(t) - Z_1(t)] \\ z_x(1,t) = u(t) \\ \dot{Z}_2(t) = -k_1(Z_2(t) - z(0,t)) \\ \dot{v}(t) = Sv(t) \\ e(t) = z(0,t) - (g(0) + F_4)v(t). \end{cases}$$

By Assumption A, the term $(g(0) + F_4)v(t)$ contains the sinusoids of no more than m distinct frequencies, which can be expressed, without loss of generality, as

$$(g(0)+F_4)v(t) = \sum_{i=1}^{l}(A_i\cos\omega_i t + B_i\sin\omega_i t) + C, \quad l \le r \le m$$

where $A_i, B_i,$ and C are unknown parameters and $A_i^2 + B_i^2 > 0, i = 1,\ldots,l$.

Same as the previous section, $(g(0) + F_4)v(t)$ can be generated by the exosystem of the following:

$$\begin{cases} \dot{d}(t) = S_c(\theta)d(t) = A_c d(t) - \sum_{i=1}^{m}\theta_i E_{2i} d_1(t) \\ (g(0) + F_4)v(t) = d_1(t) \end{cases}$$

where $d(t) = (d_1(t), d_2(t), \ldots, d_{2m+1}(t))^\top \in \mathbb{R}^{2m+1}$, A_c and $S_c(\theta)$ are defined by Lemma 2.2. The E_{2i} is the $2i$th column of the $(2m+1)\times(2m+1)$ identity matrix, and $\theta = [\theta_1,\ldots,\theta_l,0,\ldots,0]^\top \in \mathbb{R}^m$ with $\theta_1,\ldots,\theta_l$ being defined by (13).

We therefore write $(z(\cdot,\cdot), Z_2(\cdot), d(\cdot))$ as

$$\begin{cases} z_{tt}(x,t) = z_{xx}(x,t) \\ z_x(0,t) = (k_1+k_2)[z(0,t) - e(t)] - k_1[Z_2(t) - Z_1(t)] \\ z_x(1,t) = u(t) \\ \dot{Z}_2(t) = -k_1(Z_2(t) - z(0,t)) \\ \dot{d}(t) = S_c(\theta)d(t) \\ e(t) = z(0,t) - C_c d(t) \end{cases} \quad (52)$$

where $C_c = [1,0,\ldots,0] \in \mathbb{R}^{1\times(2m+1)}$. From now on, we only need to design tracking error feedback control for the transformed system (52).

A. Feedforward Control Design

In this subsection, we design a feedforward control for the system (52). Let $f_0(x,\theta) = f_0(x) \in \mathbb{R}^{1\times(2m+1)}$ be the solution of the following equation:

$$\begin{cases} f_0''(x) = f_0(x)S_c^2(\theta) \\ f_0'(0) = (k_1+k_2)C_c - k_1^2 C_c[S_c(\theta) + k_1]^{-1} \\ f_0(0) = C_c \end{cases}$$

which admits a unique solution

$$(f_0(x,\theta), f_0'(x,\theta)) = (f_0(0), f_0'(0))e^{\begin{pmatrix} 0 & S_c^2(\theta) \\ I & 0 \end{pmatrix}x}.$$

Let $\varepsilon(x,t) = z(x,t) - f_0(x)d(t)$. Then, the $\varepsilon(\cdot,\cdot)$ is governed by

$$\begin{cases} \varepsilon_{tt}(x,t) = \varepsilon_{xx}(x,t) \\ \varepsilon_x(0,t) = 0,\ \varepsilon_x(1,t) = u(t) - f_0'(1)d(t) \\ \dot{d}(t) = S_c(\theta)d(t) \\ e(t) = \varepsilon(0,t). \end{cases} \quad (53)$$

Similar to the previous section, we can then naturally design a feedforward control of the following:

$$u(t) = -c_2\varepsilon(1,t) - c_3\varepsilon_t(1,t) + f_0'(1,\theta)d(t)$$

$$= -c_2 z(1,t) - c_3 z_t(1,t) + f_0'(1,\theta)d(t) + c_2 f_0(1,\theta)d(t) + c_3 f_0(1,\theta)S_c(\theta)d(t). \tag{54}$$

B. Error-Based Observer Design

In this subsection, we design an observer for the system (52) to recover the state $(z(\cdot,t), Z_2(t), d(t))$ and estimate online the θ by the output measurement $e(t)$. Once again, an observer for the z-subsystem of (52) is also a direct copy of the plant as

$$\begin{cases} \hat{z}_{tt}(x,t) = \hat{z}_{xx}(x,t) \\ \hat{z}_x(0,t) = (k_1+k_2)[\hat{z}(0,t) - e(t)] - k_1[\hat{Z}_2(t) - Z_1(t)] \\ \hat{z}_x(1,t) = u(t) \\ \dot{\hat{Z}}_2(t) = -k_1(\hat{Z}_2(t) - \hat{z}(0,t)) \\ (\hat{z}(\cdot,0), \hat{z}_t(\cdot,0), \hat{Z}_2(0)) = (\hat{z}_0(\cdot), \hat{z}_1(\cdot), Z_{20}) \in H \times \mathbb{R}. \end{cases} \tag{55}$$

Define the observer error as $\tilde{z}(x,t) = z(x,t) - \hat{z}(x,t)$ and $\tilde{Z}_2(t) = Z_2(t) - \hat{Z}_2(t)$. Then

$$\begin{cases} \tilde{z}_{tt}(x,t) = \tilde{z}_{xx}(x,t) \\ \tilde{z}_x(0,t) = (k_1+k_2)\tilde{z}(0,t) - k_1\tilde{Z}_2(t) \\ \tilde{z}_x(1,t) = 0 \\ \dot{\tilde{Z}}_2(t) = -k_1(\tilde{Z}_2(t) - \tilde{z}(0,t)) \end{cases} \tag{56}$$

which is asymptotically stable in $H \times \mathbb{R}$ claimed by [7, Lemma 2.1]. By the Sobolev trace theorem, $\tilde{z}(0,\cdot), \tilde{z}(1,\cdot) \in C([0,\infty);\mathbb{R})$ and

$$\lim_{t\to\infty} |\tilde{z}(0,t)| = \lim_{t\to\infty} |\tilde{z}(1,t)| = 0.$$

We can thus introduce a known function

$$y_d(t) = -e(t) + \hat{z}(0,t) = C_c d(t) - \tilde{z}(0,t)$$

and consider the following ODE system:

$$\begin{cases} \dot{d}(t) = S_c(\theta)d(t) = A_c d(t) - \sum_{i=1}^m \theta_i E_{2i} d_1(t) \\ y_d(t) = C_c d(t) - \tilde{z}(0,t). \end{cases} \tag{57}$$

It is now the time for us to design an adaptive observer for (57) according to the output $y_d(t)$.

Lemma 3.2: For any initial state

$$(\hat{\chi}_1(0), \hat{\phi}(0), \hat{\theta}(0), \Omega(0), \{\xi_i(0)\}_{i=1}^m) \in \mathbb{R} \times \mathbb{R}^{2m} \times \mathbb{R}^m \times \mathbb{S}_+^m \times \mathbb{R}^{2m\times m}$$

there hold

$$\lim_{t\to\infty} \|\hat{\theta}(t) - \theta\| = 0, \quad \lim_{t\to\infty} \|\hat{d}(t) - d(t)\| = 0$$

where $\hat{d}(\cdot)$ and $\xi_i(\cdot), \Omega_i(\cdot)$ are updated by the same adaptive observer and filter given by (31) and (29), respectively, but the adaptation dynamics for the estimates $\hat{\theta}_i(t), i = 1, 2, \ldots, m$ are given by

$$\begin{cases} \dot{\hat{\theta}}_i(t) = g\mu_i(t)(y_d(t) - \hat{\chi}_1(t)), \; |\det(\Omega_i)|^{1/i} \geq \delta > 0 \\ \dot{\hat{\theta}}_i(t) = g\mu_i(t)(y_d(t) - \hat{\chi}_1(t)) - \alpha\hat{\theta}_i(t), \text{otherwise} \end{cases} \tag{58}$$

where δ is a small threshold, and $\alpha, g > 0$ can be arbitrarily chosen.

Proof: We still set $\mu_i(t) = \mu_{ip}(t) + \mu_{ie}(t)$ where $\mu_{ip}(\cdot)$ is the solution to

$$\begin{cases} \dot{\xi}_{ip}(t) = \Gamma\xi_{ip}(t) - [0 \; I_{2m}]E_{2i}d_1(t) \\ \mu_{ip}(t) = [1 \; 0 \; \ldots \; 0]\xi_{ip}(t), \quad i = 1,\ldots,m \end{cases} \tag{59}$$

and $\mu_{ie}(\cdot)$ is governed by

$$\begin{cases} \dot{\xi}_{ie}(t) = \Gamma\xi_{ie}(t) + [0 \; I_{2m}]E_{2i}\tilde{z}(0,t) \\ \mu_{ie}(0) = \mu_i(0) - \mu_{ip}(0) \\ \mu_{ie}(t) = [1 \; 0 \; \ldots \; 0]\xi_{ie}(t), \quad i = 1,\ldots,m. \end{cases} \tag{60}$$

By the proof of Lemma 2.5, the vector $[\mu_{1p}(t),\ldots,\mu_{lp}(t)]^\top$ is PE (but $[\mu_{1p}(t),\ldots,\mu_{kp}(t)]^\top, k \geq l+1$ is not). For the system (60), since Γ is Hurwitz and $\lim_{t\to\infty}|\tilde{z}(0,t)| = 0$, we conclude that $\lim_{t\to\infty}|\mu_{ie}(t)| = 0$. By Lemma 1.3, and $\mu_i(t) = \mu_{ip}(t) + \mu_{ie}(t)$, the vector $[\mu_1(t),\ldots,\mu_l(t)]^\top$ is PE (but $[\mu_1(t),\ldots,\mu_k(t)]^\top, k \geq l+1$ is not). It follows from (36) and a similar argument of [21, lemma 3.1] that $|\det(\Omega_i(t))|^{1/i} \geq \delta_m > 0 \quad \forall t \geq 0, \; i = 1,\ldots,l$, where δ_m is a positive number and $|\det(\Omega_i)|^{1/i}, i = 1+1,\ldots,m$ tend to zero as time goes to infinity. By choosing δ sufficiently small, there exists some $T > 0$ such that for all $t \geq T$

$$\dot{\hat{\theta}}_i(t) = g\mu_i(t)(y_d(t) - \hat{\chi}_1(t)), \qquad 1 \leq i \leq l$$
$$\dot{\hat{\theta}}_i(t) = g\mu_i(t)(y_d(t) - \hat{\chi}_1(t)) - \alpha\hat{\theta}_i(t), \quad l+1 \leq i \leq m.$$

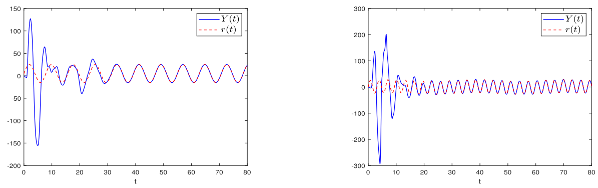
Define the error variables $\tilde{\chi}_1(t) = \chi_1(t) - \hat{\chi}_1(t), \tilde{\phi}(t) = \phi(t) - \hat{\phi}(t), \tilde{\theta}(t) = \theta - \hat{\theta}(t)$, where the definitions of χ_1, ϕ are the same as in the previous section [see (30)]. Then

$$\begin{cases} \dot{\tilde{\phi}}(t) = \Gamma\tilde{\phi}(t) + \tilde{z}(0,t)(\beta - M_2\theta) \\ \dot{\tilde{\chi}}_1(t) = -k_0\tilde{\chi}_1(t) + \tilde{\phi}_1(t) + \mu(t)^\top\tilde{\theta}(t) + \tilde{z}(0,t)(b_1+k_0) \\ \dot{\tilde{\theta}}_i(t) = -g\mu_i(t)\tilde{\chi}_1(t) + g\mu_i(t)\tilde{z}(0,t), \quad 1 \leq i \leq l \\ \dot{\tilde{\theta}}_i(t) = -g\mu_i(t)\tilde{\chi}_1(t) - \alpha\tilde{\theta}_i(t) + g\mu_i(t)\tilde{z}(0,t), \; l+1 \leq i \leq m. \end{cases} \tag{61}$$

Since Γ is Hurwitz and $\lim_{t\to\infty}|\tilde{z}(0,t)| = 0$, we conclude that $\lim_{t\to\infty}\|\tilde{\phi}(t)\| = 0$. Now we consider the last three equations of the system (61) as perturbations of the nominal system (38), whose origin is, by the proof of Lemma 2.5, exponentially stable. By Lemma 1.2, it is seen that $\lim_{t\to\infty}\|\hat{\theta}(t) - \theta\| = 0$ and $\lim_{t\to\infty}|\hat{\chi}_1(t) - \chi_1(t)| = 0$. Since $\lim_{t\to\infty}\|\tilde{\phi}(t)\| = 0$, same as the previous section,

$$\tilde{d}(t) = \begin{bmatrix} \tilde{\chi}_1(t) \\ \tilde{\phi}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \sum_{i=1}^m \xi_i(t)\tilde{\theta}_i(t) + b\tilde{\chi}_1(t) \end{bmatrix}$$

is obviously tending to 0 as $t \to \infty$ by boundedness of $\xi_i(t)$. ■



(a) (b)

Fig. 1. Tracking performance of $Y(t)$. (a) First simulation. (b) Second simulation.

C. Tracking Error Feedback Control Design

By (54), we can therefore design naturally a tracking error feedback control as

$$\begin{cases} u(t) = -c_2\hat{z}(1,t) - c_3\hat{z}_t(1,t) \\ + f_0'(1,\hat{\theta})\hat{d}(t) + c_2 f_0(1,\hat{\theta})\hat{d}(t) + c_3 f_0(1,\hat{\theta})S_c(\hat{\theta})\hat{d}(t) \\ \hat{z}_{tt}(x,t) = \hat{z}_{xx}(x,t) \\ \hat{z}_x(0,t) = (k_1+k_2)[\hat{z}(0,t) - e(t)] - k_1[\hat{Z}_2(t) - Z_1(t)] \\ \hat{z}_x(1,t) = u(t) \\ \dot{\hat{Z}}_2(t) = -k_1(\hat{Z}_2(t) - \hat{z}(0,t)), \quad \dot{Z}_1(t) = -k_1[Z_1(t) - e(t)] \\ y_d(t) = -e(t) + \hat{z}(0,t) \\ \dot{\hat{\chi}}_1(t) = \hat{\phi}_1(t) + b_1 y_d(t) + \sum_{i=1}^m \mu_i(t)\hat{\theta}_i(t) + k_o(y_d(t) - \hat{\chi}_1(t)) \\ \dot{\hat{\phi}}(t) = \Gamma\hat{\phi}(t) + \beta y_d, \quad \hat{\phi}(t) \in \mathbb{R}^{2\,m} \\ \hat{d}(t) = \begin{bmatrix} \hat{\chi}_1(t) \\ \hat{\phi}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \sum_{i=1}^m \xi_i(t)\hat{\theta}_i(t) + b\hat{\chi}_1(t) \end{bmatrix} \\ \dot{\hat{\theta}}_i(t) = g\mu_i(t)(y_d(t) - \hat{\chi}_1(t)), \quad |\det(\Omega_i)|^{1/i} \geq \delta \\ \dot{\hat{\theta}}_i(t) = g\mu_i(t)(y_d(t) - \hat{\chi}_1(t)) - \alpha\hat{\theta}_i(t), \quad \text{otherwise} \\ \dot{\xi}_i(t) = \Gamma\xi_i(t) - [0\ I_{2\,m}]E_{2i}y_d(t), \quad \xi_i(t) \in \mathbb{R}^{2\,m} \\ \dot{\Omega}(t) = -\lambda_b\Omega(t) + \lambda_c\mu(t)\mu(t)^\top, \quad \Omega(t) \in \mathbb{R}^{m\times m} \end{cases} \tag{62}$$

where

$$(f_0(1,\hat{\theta}), f_0'(1,\hat{\theta})) = (C_c, (k_1+k_2)C_c - k_1^2 C_c[S_c(\hat{\theta}) + k_1]^{-1}) \times e^{\begin{pmatrix} 0 & S_c^2(\hat{\theta}) \\ I & 0 \end{pmatrix}}. \tag{63}$$

Since $S_c(\hat{\theta}) + k_1$ might be singular, we need to replace $(S_c(\hat{\theta}) + k_1)^{-1}$ in (63) with

$$\mathscr{T}(\hat{\theta}) = \text{adj}[S_c(\hat{\theta}) + k_1]\frac{\det[S_c(\hat{\theta}) + k_1]}{\text{sat}(\det^2[S_c(\hat{\theta}) + k_1])}$$

where

$$\text{sat}(a) = \begin{cases} a, & \text{if } a \geq \delta_0 \\ \delta_0, & \text{if } a < \delta_0 \end{cases}$$

with $\delta_0 > 0$ being a small threshold and $\text{adj}[S_c(\hat{\theta}) + k_1]$ is the adjoint matrix of $S_c(\hat{\theta}) + k_1$.

D. Well-Posedness and Stability of the Closed-Loop System

The close loop of the system (1) under control (62) is

$$\begin{cases} y_{tt}(x,t) = y_{xx}(x,t) + \Delta(x)F_1 v(t) \\ y_x(0,t) = F_2 v(t) \\ y_x(1,t) = u(t) + F_3 v(t) \\ \dot{v}(t) = Sv(t) \\ e(t) = y(0,t) - F_4 v(t) \\ u(t) = -c_2\hat{z}(1,t) - c_3\hat{z}_t(1,t) \\ + f_0'(1,\hat{\theta})\hat{d}(t) + c_2 f_0(1,\hat{\theta})\hat{d}(t) + c_3 f_0(1,\hat{\theta})S_c(\hat{\theta})\hat{d}(t) \\ \hat{z}_{tt}(x,t) = \hat{z}_{xx}(x,t) \\ \hat{z}_x(0,t) = (k_1+k_2)[\hat{z}(0,t) - e(t)] - k_1[\hat{Z}_2(t) - Z_1(t)] \\ \hat{z}_x(1,t) = u(t) \\ \dot{\hat{Z}}_2(t) = -k_1(\hat{Z}_2(t) - \hat{z}(0,t)), \dot{Z}_1(t) = -k_1[Z_1(t) - e(t)] \\ y_d(t) = -e(t) + \hat{z}(0,t) \\ \dot{\hat{\chi}}_1(t) = \hat{\phi}_1(t) + b_1 y_d(t) + \sum_{i=1}^m \mu_i(t)\hat{\theta}_i(t) + k_o(y_d(t) - \hat{\chi}_1(t)) \\ \dot{\hat{\phi}}(t) = \Gamma\hat{\phi}(t) + \beta y_d, \quad \hat{\phi}(t) \in \mathbb{R}^{2\,m} \\ \hat{d}(t) = \begin{bmatrix} \hat{\chi}_1(t) \\ \hat{\phi}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \sum_{i=1}^m \xi_i(t)\hat{\theta}_i(t) + b\hat{\chi}_1(t) \end{bmatrix} \end{cases} \tag{64}$$

with parameter adaptive law

$$\begin{cases} \dot{\hat{\theta}}_i(t) = g\mu_i(t)(y_d(t) - \hat{\chi}_1(t)), \ |\det(\Omega_i)|^{1/i} \geq \delta > 0 \\ \dot{\hat{\theta}}_i(t) = g\mu_i(t)(y_d(t) - \hat{\chi}_1(t)) - \alpha\hat{\theta}_i(t), \ \text{otherwise} \end{cases}$$

for $1 \leq i \leq m$, and filter

$$\begin{cases} \dot{\xi}_i(t) = \Gamma\xi_i(t) - [0\ I_{2\,m}]E_{2i}y_d(t), \xi_i(t) \in \mathbb{R}^{2\,m} \\ \mu_i(t) = [1\ 0\ \ldots\ 0]\xi_i(t), \ 1 \leq i \leq m \\ \dot{\Omega}(t) = -\lambda_b\Omega(t) + \lambda_c\mu(t)\mu(t)^\top, \mu(t) = [\mu_1, \ldots, \mu_m(t)]^\top \\ \Omega_i(t) = [I_i, 0_{i\times(m-i)}]\Omega(t)[I_i, 0_{i\times(m-i)}]^\top, \ 1 \leq i \leq m. \end{cases}$$

We consider system (64) in the state space $\mathscr{H}_2 = H^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^{2\,m} \times \mathbb{R}^m \times \mathbb{S}_+^m \times \mathbb{R}^{2m\times m}$.

Theorem 3.1: Suppose that $k_1, k_2, c_2, c_3 > 0$. For any unknown coefficients F_1, F_2, F_3, F_4, S, unknown function $\Delta(\cdot)$ and any initial state

$$(y(\cdot,0), y_t(\cdot,0), \hat{z}(\cdot,0), \hat{z}_t(\cdot,0), \hat{Z}_2(0), Z_1(0), \hat{\chi}_1(0), \hat{\phi}(0), \hat{\theta}(0), \Omega(0), \{\xi_i(0)\}_{i=1}^m) \in \mathscr{H}_2$$

the closed-loop system (64) admits a unique bounded solution in $C([0,\infty);\mathscr{H}_2)$ such that

$$\lim_{t\to\infty} |e(t)| = 0.$$

Proof: Using the variables $\varepsilon(x,t), \tilde{z}(x,t), \tilde{Z}_2(t)$ given by (53) and (56). Define $\hat{\varepsilon}(x,t) = \varepsilon(x,t) - \tilde{z}(x,t)$. Then, $\hat{\varepsilon}(x,t)$ is governed by

$$\begin{cases} \hat{\varepsilon}_{tt}(x,t) = \hat{\varepsilon}_{xx}(x,t) \\ \hat{\varepsilon}_x(0,t) = -(k_1+k_2)\tilde{z}(0,t) + k_1\tilde{Z}_2(t) \\ \hat{\varepsilon}_x(1,t) = -c_2\hat{\varepsilon}(1,t) - c_3\hat{\varepsilon}_t(1,t) + \mathscr{P}(t) \\ e(t) = \tilde{z}(0,t) + \hat{\varepsilon}(0,t) \end{cases} \tag{65}$$

where

$$\mathscr{P}(t) := -f_0'(1,\theta)d(t) - c_2 f_0(1,\theta)d(t) - c_3 f_0(1,\theta)S_c(\theta)d(t) + f_0'(1,\hat{\theta})\hat{d}(t) + c_2 f_0(1,\hat{\theta})\hat{d}(t) + c_3 f_0(1,\hat{\theta})S_c(\hat{\theta})\hat{d}(t).$$

The system (65) can be written abstractly as

$$\frac{d}{dt}(\hat{\varepsilon}(\cdot,t),\hat{\varepsilon}_t(\cdot,t)) = \mathbb{A}(\hat{\varepsilon}(\cdot,t),\hat{\varepsilon}_t(\cdot,t)) + \mathscr{B}\mathscr{P}(t) + \mathscr{B}_1(-(k_1+k_2)\tilde{z}(0,t) + k_1\tilde{Z}_2(t))$$

where the operator $\mathbb{A} : D(\mathbb{A})(\subset H) \to H$ is defined in the proof of Theorem 2.1, and $\mathscr{B} = (0, \delta(\cdot - 1)), \mathscr{B}_1 = (0, -\delta(\cdot))$. It is well known that $\mathscr{B}$ and $\mathscr{B}_1$ are admissible for $e^{\mathbb{A}t}$. It follows from (56) that $\tilde{z}(0,\cdot), \tilde{Z}_2(\cdot) \in C([0,\infty);\mathbb{R})$ and $\lim_{t\to\infty} \tilde{z}(0,t) = \lim_{t\to\infty} \tilde{Z}_2(t) = 0$. Moreover, we also have $\mathscr{P} \in C([0,\infty);\mathbb{R})$ and $\lim_{t\to\infty} \mathscr{P}(t) = 0$. By Lemma 1.1, the $\hat{\varepsilon}$-system admits a unique solution $(\hat{\varepsilon},\hat{\varepsilon}_t) \in C([0,\infty);H)$, and

$$\lim_{t\to\infty} \|(\hat{\varepsilon}(\cdot,t),\hat{\varepsilon}_t(\cdot,t))\|_H = 0.$$

Therefore, the transformations $y(x,t) = \hat{\varepsilon}(x,t) + \tilde{z}(x,t) + f_0(x)d(t) - g(x)v(t), \hat{z}(x,t) = \hat{\varepsilon}(x,t) + f_0(x)d(t)$ imply that $(y, y_t, \hat{z}, \hat{z}_t)$ are well-defined in $C([0,\infty);H^2) \cap L^\infty(0,\infty;H^2)$. By the Sobolev trace embedding theorem

$$\lim_{t\to\infty} |\hat{\varepsilon}(0,t)| \le \lim_{t\to\infty} \|\hat{\varepsilon}(\cdot,t)\|_{H^1(0,1)} \le \lim_{t\to\infty} \|(\hat{\varepsilon}(\cdot,t),\hat{\varepsilon}_t(\cdot,t))\|_H = 0.$$

This, together with (65), gives

$$\lim_{t\to\infty} |e(t)| = \lim_{t\to\infty} |\tilde{z}(0,t) + \hat{\varepsilon}(0,t)| = 0.$$

■

Remark 5: When θ is known, the control (62) becomes

$$\begin{cases} u(t) = -c_2\hat{z}(1,t) - c_3\hat{z}_t(1,t) \\ +f_0'(1,\theta)\hat{d}(t) + c_2 f_0(1,\theta)\hat{d}(t) + c_3 f_0(1,\theta)S_c(\theta)\hat{d}(t) \\ \hat{z}_{tt}(x,t) = \hat{z}_{xx}(x,t) \\ \hat{z}_x(0,t) = (k_1+k_2)[\hat{z}(0,t) - e(t)] - k_1[\hat{Z}_2(t) - Z_1(t)] \\ \hat{z}_x(1,t) = u(t) \\ \dot{\hat{Z}}_2(t) = -k_1(\hat{Z}_2(t) - \hat{z}(0,t)), \dot{Z}_1(t) = -k_1[Z_1(t) - e(t)] \\ y_d(t) = -e(t) + \hat{z}(0,t) \\ \dot{\hat{d}}(t) = S_c(\theta)\hat{d}(t) + L(y_d(t) - C_c\hat{d}(t)). \end{cases}$$

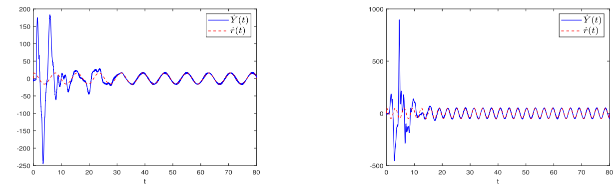
In this case, $\lim_{t\to\infty} e(t) = 0$, which leads to an internal model principle discussed in [7].

IV. Numerical Simulations

In this section, we demonstrate some numerical simulations for illustration. Consider the closed-loop system (46) with the disturbances $w_1(t) = w_2(t) = w_3(t) = 5\cos(\frac{4}{5}t)$, and the reference signal $r(t)$ being

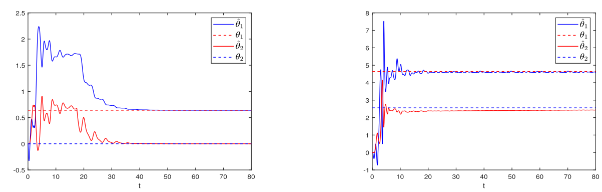
$$\begin{cases} r(t) = 20\sin(\frac{4}{5}t) + 5, & \text{in the first simulation} \\ r(t) = 25\sin(2t) + 3\sin(\frac{1}{5}t), & \text{in the second simulation.} \end{cases}$$

For the controller, we choose $m = 2$ so that two unknown parameters θ_1 and θ_2 are needed. Numerical simulations have been carried out with parameters $\Delta(x) = x/5$, $c_0 = c_1 = c_2 = c_3 = 2$, $\lambda_b = \lambda_c = 1$, $b = [4,6,4,1]^\top$, $\alpha = g = k_0 = 1$, and all the initial states of the plant and controller except $\Omega(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are set to be zero. The simulations are implemented using finite difference with 50 points on $[0,1]$.



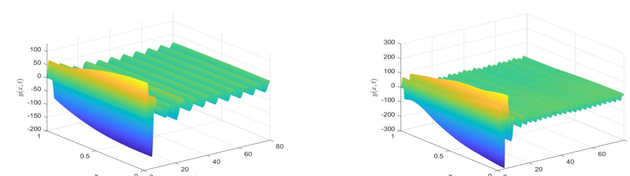
(a) (b)

Fig. 2. Tracking performance of $\dot{Y}(t)$. (a) First simulation. (b) Second simulation.



(a) (b)

Fig. 3. Frequency estimation. (a) First simulation. (b) Second simulation.



(a) (b)

Fig. 4. Evolution of the plant $y(x,t)$. (a) First simulation. (b) Second simulation.

Figs. 1 and 2 plot the tracking performances of the $Y(t)$ and $\dot{Y}(t)$. Fig. 3 displays the frequency estimation of $\hat{\theta}(t)$. From the figures, we see that both frequency estimation and tracking performance are satisfactorily. Fig. 4 shows that the state of plant is bounded.

It is worth noting that in the second simulation, the number of frequencies entering the system is $l = 3$, which is greater than $m = 2$, which is not in line with Assumption A, but the output regulation of the system is approximately still valid in this case, which shows that our controller may estimate the dominant two (2 and $\frac{4}{5}$) of the three frequencies with good robustness.

V. Conclusion

This article develops output regulation for a hyperbolic PDE system with disturbances being generated from a completely unknown exosystem. The system is described by 1-D wave equation where the control operator is unbounded and is non-collocated with the regulated output, which is a difficult situation in output regulation of PDEs. Motivated from adaptive estimation of frequencies of sinusoidal signals in signal process and adaptive internal model principle for lumped parameter systems, we develop an adaptive internal model principle for the PDE discussed. Two different cases are investigated. In the first case, the derivative of the tracking error is allowed to be used in the control design, which is somehow PD control. We can achieve the exponential convergence for the tracking error. The derivative of the tracking error is also shown to be convergent in a generalized sense. The second case is solely the tracking error feedback, for which, only the asymptotic convergence is guaranteed due to the observation operator being compact. All the estimations are in real time and the control is actually robust to disturbances in all possible channels. Numerical simulations validate the theoretical results.

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Ren-Xi Zhao received the B.Sc. degree in mathematics from the Central South University, Changsha, China, in 2019. He is currently working toward the Ph.D. degree with the Academy of Mathematics and Systems Science, Academia Sinica, Beijing, China.

His research interests include control theory of infinite-dimensional systems.

Bao-Zhu Guo (Senior Member, IEEE) received the Ph.D. degree in applied mathematics from the Chinese University of Hong Kong, Hong Kong, in 1991.

Since 2000, he has been with the Academy of Mathematics and Systems Science, the Chinese Academy of Sciences, Beijing, China, where he is a Research Professor in mathematical system theory. His research interests include theory of infinite-dimensional systems and active disturbance rejection control.

Dr. Guo was the recipient of the One Hundred Talent Program from the Chinese Academy of Sciences in 1999, and the National Science Fund for Distinguished Young Scholars in 2003.