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Brief paper

Output regulation for a heat equation with unknown exosystem ☆

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ABSTRACT

In this paper, we consider output regulation for a 1-d heat equation where the disturbances generated from an unknown finite-dimensional exosystem enter all possible channels. We adopt adaptive observer internal model approach which has been well developed for lumped parameter systems over two decades to estimate all possible unknown frequencies that have entered into a transformed system. By the estimates of the unknown frequencies, we are able to design a tracking error based feedback control to achieve output regulation and disturbance rejection for this PDE. A significance of the problem lies in the fact that both the control and observation operators are unbounded. The proposed approach is potentially applicable to other PDEs.

1. Introduction

Output regulation is one of the most important problems in control theory, which aims at designing a tracking error feedback control to regulate output to track asymptotically reference signal in the presence of disturbance. If both the reference signal and the disturbance are generated from a linear autonomous system which is called exosystem, the problem can be solved perfectly for linear time invariant systems by the internal model principle developed in the 1970s by Davison (1976) and Francis and Wonham (1976). The internal model principle has been applied later on to nonlinear finite-dimensional systems (Huang, 2004) and even abstract infinite-dimensional systems (Natarajan & Benstman, 2016; Natarajan, Gilliam, & Weiss, 2014; Paunonen & Pohjolainen, 2010; Rebarber & Weiss, 2003; Schumacher, 1983; Xu & Dubljevic, 2017).

However, the theory for abstract linear infinite-dimensional systems is difficult to be applied directly to systems described by partial differential equations (PDEs) unless both the control and observation operators are bounded. Usually, the abstract setting is discussed in a broader sense (Paunonen, 2017) for which some abstract conditions are hard to be checked for PDEs. In addition, we found recently that in observer-based internal model principle, the PDE approach and abstract setting design are not always coincident. For this reason, some progresses on output tracking from PDE point of view have also been made over the years like Deutscher (2015, 2016), Guo, Zhou, and Krstic (2020) and Guo and Jin (2020). The problems of Guo and Jin (2020), Guo et al. (2020) have been solved recently in Guo and Meng (2020, 2021a, 2021b) by means of the observer-based internal model principle with robustness, less restriction and fast convergence. However, in all these papers aforementioned, the frequencies of the harmonic disturbances were supposed to be known. To the best of our knowledge, only a few studies have been carried out for the output tracking of the infinite-dimensional systems with unknown frequencies like those in Wang, Ji, and Sheng (2014) and Wang, Ji, and Wang (2014) where the control and observation operators were assumed to be bounded.

On the other hand, there are many works attributed to online estimation of the frequencies for finite sum of the sinusoid signals and output regulation for systems described by ordinary differential equations (ODEs) with unknown exosystem. The main stream is represented by a series of works from Marino and Tomei (2002, 2003, 2007), to Marino and Tomei (2013, 2017), over two decades. In this paper, adopted the methods from Marino and Tomei (2017) and Kim and Shim (2015), we propose an adaptive internal model based control method to solve an output tracking problem for a PDE system described by a 1-d heat equation where the exosystem is not necessarily known, which means that the frequencies of the sinusoidal signals that appear in the reference and disturbances can be unknown. In addition, both the control and observation operators are unbounded, which has potential applicability to other PDEs.

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The system that we consider in this paper is described by the following heat equation:

$$\begin{cases} w_t(x,t) = w_{xx}(x,t) + F(x)p(t), x \in (0,1), t > 0, \\ w_x(0,t) = Np(t), t \geq 0, \\ w_x(1,t) = u(t) + Dp(t), t \geq 0, \\ w(x,0) = w_0(x), x \in [0,1], \\ y_c(t) = w(0,t), \quad t \geq 0, \end{cases} \tag{1}$$

where $u(\cdot)$ is the control, and $y_c(\cdot)$ which is non-collocated with control is the output to be regulated, the $F(\cdot) \in L^\infty(0,1;\mathbb{R}^{1\times n})$, $N \in \mathbb{R}^{1\times n}$, and $D \in \mathbb{R}^{1\times n}$ are unknown coefficients of the in-domain and boundary disturbances, $w_0(\cdot)$ is the initial state. The disturbance $p(\cdot)$ is produced from the following exosystem:

$$\begin{cases} \dot{p}(t) = Gp(t), t > 0, \\ p(0) = p_0, \end{cases} \tag{2}$$

where the unknown $p(t) \in \mathbb{R}^n$. It is assumed that both the matrix $G \in \mathbb{R}^{n\times n}$ and the initial value p_0 are unknown. We consider system (1) in the usual state space $H = L^2(0,1)$.

Denote the reference trajectory by

$$y_{ref}(t) = Mp(t), \tag{3}$$

where $M \in \mathbb{R}^{1\times n}$ is also unknown, and the tracking error is denoted by $y_e(t) = y_c(t) - y_{ref}(t)$. The control objective is to design a tracking error feedback control so that

$$\lim_{t\to\infty} |y_e(t)| = \lim_{t\to\infty} |y_c(t) - y_{ref}(t)| = 0. \tag{4}$$

The following assumption is made throughout the paper.

Assumption 1.1. The spectrum of G is either $\{\pm j\omega_i, 1 \leq i \leq r\}$ with $n = 2r$ or $\{0, \pm j\omega_i, 1 \leq i \leq r\}$ with $n = 2r+1$, where $\omega_1 < \omega_2 < \cdots < \omega_r$ are positive distinct unknown parameters. It is supposed that r has an upper bound: $r \leq m$ for a known positive integer m.

By Assumption 1.1, the general solution of the exosystem (2) includes steplike functions and sinusoidal functions with unknown frequencies, which typically arise in applications. Define

$$w^r(x,t) = \Gamma(x)p(t) \text{ and } u_r(t) = \gamma p(t), \tag{5}$$

which satisfy

$$\begin{cases} w_t^r(x,t) = w_{xx}^r(x,t) + F(x)p(t), \\ w_x^r(0,t) = Np(t), \\ w_x^r(1,t) = u_r(t) + Dp(t), \\ w^r(0,t) = Mp(t), \end{cases} \tag{6}$$

that is, $w^r(x,t)$ and $u_r(t)$ are the reference signals of $w(x,t)$ and $u(t)$. The coefficients $\Gamma(\cdot)$ and γ are determined by the following regulator equation:

$$\begin{cases} \Gamma''(x) = \Gamma(x)G - F(x), \\ \Gamma'(0) = N, \\ \Gamma(0) = M, \\ \gamma = \Gamma'(1) - D, \end{cases} \tag{7}$$

which admits a unique solution. Obviously, the state regulation error $\varepsilon(x,t) = w(x,t) - w^r(x,t)$ satisfies

$$\begin{cases} \varepsilon_t(x,t) = \varepsilon_{xx}(x,t), \\ \varepsilon_x(0,t) = 0, \\ \varepsilon_x(1,t) = u(t) - \gamma p(t), \\ y_e(t) = \varepsilon(0,t). \end{cases} \tag{8}$$

We proceed as follows. In Section 2, we consider a special case of $r = 1$ to display simply the approach. Section 3 is devoted to the case of $r \geq 1$. In Section 4, we demonstrate some numerical simulations for illustration, followed up by concluding remarks in Section 5.

2. Main results for $r = 1$

In order to show clearly about our control design approach, we consider, in this section, the case of $r = 1$, $n = 2$. The case of $r \geq 1$ will be discussed in next section. The following assumption is convenient for the discussion in this section although it is not essential and will be removed in next section.

Assumption 2.1. The pair (G, γ) is observable and the initial value $p(0)$ excites all oscillatory modes of the exosystem.

By Assumptions 1.1 and 2.1, we may write $u_r(t)$ as

$$u_r(t) = A\cos\omega t + B\sin\omega t, \tag{9}$$

where A, B, ω are unknown parameters with $A^2 + B^2 > 0$. Hence $u_r(t)$ can be described by the exosystem of the following:

$$\begin{cases} \dot{\eta}(t) = G_c\eta(t), \\ u_r(t) = \gamma p(t) = \gamma_c\eta(t), \end{cases} \tag{10}$$

where $\gamma_c = [1, 0]$, $\eta(0) = (A, B)^\top$, and G_c is a 2×2 matrix:

$$G_c = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}.$$

We design naturally a feedforward control for system (8) as follows:

$$u(t) = -\alpha_2\varepsilon(1,t) + \gamma_c\eta(t), \ \ \alpha_2 > 0, \tag{11}$$

and the closed-loop of system (8) under control (11) reads

$$\begin{cases} \varepsilon_t(x,t) = \varepsilon_{xx}(x,t), \\ \varepsilon_x(0,t) = 0, \\ \varepsilon_x(1,t) = -\alpha_2\varepsilon(1,t), \end{cases} \tag{12}$$

which, by lemma 1.1 of Guo and Meng (2020), is exponentially stable and $|y_e(t)| = |\varepsilon(0,t)|$ converges to zero exponentially as $t\to\infty$.

In the rest of this section, we are devoted to design a suitable observer to estimate $(\varepsilon(1,t), \eta(t))$ in (11). To this purpose, we introduce a transform:

$$z(x,t) = \varepsilon(x,t) + g(x)\eta(t), \tag{13}$$

where $g(x) = (g_1(x), g_2(x))$ satisfies

$$\begin{cases} g''(x) = g(x)G_c, \\ g'(0) = \alpha_1 g(0), \ \ \alpha_1 > 0, \\ g'(1) = \gamma_c. \end{cases} \tag{14}$$

The extended system of $(z(\cdot,\cdot), \eta(\cdot))$ is then governed by

$$\begin{cases} z_t(x,t) = z_{xx}(x,t), \\ z_x(0,t) = \alpha_1 z(0,t) - \alpha_1 y_e(t), \\ z_x(1,t) = u(t), \\ \dot{\eta}(t) = G_c\eta(t), \\ y_e(t) = z(0,t) - g(0)\eta(t). \end{cases} \tag{15}$$

It is seen that the z-subsystem in (15) has damping at $x = 0$. The existence of the solution to (14) is guaranteed by the following Lemma 2.1.

Lemma 2.1. *The boundary value problem (14) admits a unique solution.*

Proof. Let $w_1 = (1, i)^\top$ and $w_2 = (1, -i)^\top$ be the eigenvectors of G_c corresponding to the eigenvalues $i\omega$ and $-i\omega$ respectively, which will be used throughout this section. Right multiply by w_1 in (14) to obtain

$$\begin{cases} g_a''(x) = i\omega g_a(x), \\ g_a'(0) = \alpha_1 g_a(0), \ \ g_a'(1) = 1, \end{cases} \tag{16}$$

where $g_a(x) = g(x)w_1$. Then, the solution of (16) can be found as

$$g_a(x) = \frac{(\beta+\alpha_1)e^{\beta x} + (\beta-\alpha_1)e^{-\beta x}}{\beta(\beta+\alpha_1)e^{\beta} - \beta(\beta-\alpha_1)e^{-\beta}}, \tag{17}$$

where $\beta = \sqrt{i\omega}$. It is easy to check that the denominator of (17) is non-zero. For the eigenvalue $-i\omega$, we can similarly obtain

$$g_b(x) = \frac{(\beta^*+\alpha_1)e^{\beta^* x} + (\beta^*-\alpha_1)e^{-\beta^* x}}{\beta^*(\beta^*+\alpha_1)e^{\beta^*} - \beta^*(\beta^*-\alpha_1)e^{-\beta^*}}, \tag{18}$$

where $g_b(x) = g(x)w_2$, $\beta^* = \sqrt{-i\omega}$. Therefore, the solution of (14) always exists for any $\alpha_1 > 0$ that $g(x) = (g_a(x), g_b(x))[w_1, w_2]^{-1}$. ■

Now, since the initial value of (15) is unknown, we design an observer for z-subsystem of (15) as follows:

$$\begin{cases} \hat{z}_t(x,t) = \hat{z}_{xx}(x,t), \\ \hat{z}_x(0,t) = \alpha_1\hat{z}(0,t) - \alpha_1 y_e(t), \\ \hat{z}_x(1,t) = u(t). \end{cases} \tag{19}$$

The observer error $\tilde{z}(x,t) = z(x,t) - \hat{z}(x,t)$ satisfies

$$\begin{cases} \tilde{z}_t(x,t) = \tilde{z}_{xx}(x,t), \\ \tilde{z}_x(0,t) = \alpha_1\tilde{z}(0,t), \\ \tilde{z}_x(1,t) = 0, \end{cases} \tag{20}$$

which is, similar to (12), exponentially stable in H.

Lemma 2.2. *Let $\tilde{z}(\cdot,\cdot)$ be the solution of (20) in $H = L^2(0,1)$. Then, $\tilde{z}(0,\cdot), \tilde{z}(1,\cdot) \in L^2(0,T)$ for any $T > 0$. Moreover, there are $M^*, \omega^* > 0$, such that*

$$|\tilde{z}(0,t)| + |\tilde{z}(1,t)| \le M^* e^{-\omega^* t}\|\tilde{z}(\cdot,0)\|, \forall t \ge \varepsilon, \tag{21}$$

for any $\varepsilon > 0$.

Proof. We only discuss $\tilde{z}(1,t)$ since the counterpart for $\tilde{z}(0,t)$ is similar. From the proof of lemma 1.1 of Guo and Meng (2020), the solution of (20) can be represented as

$$\begin{cases} \tilde{z}(x,t) = \sum_{n=0}^{\infty} b_n e^{\mu_n t} g_n(x), \\ \|\tilde{z}(\cdot,0)\|^2 = \sum_{n=0}^{\infty} b_n^2 < \infty, \end{cases} \tag{22}$$

where (there is a typo in (9) of Guo and Meng (2020))

$$\begin{cases} \mu_n = -2\alpha_1 - (n\pi)^2 + \mathscr{O}(n^{-1}) < 0, \\ g_n(x) = \cos n\pi x + \mathscr{O}(n^{-1}), \end{cases} \tag{23}$$

with $\{g_n(x)\}$ being an orthonormal basis for H. First, (21) comes from

$$\begin{aligned} |\tilde{z}(1,t)| &\le \sum_{n=0}^{\infty} |b_n g_n(1)| e^{\mu_n t} \\ &\le C\left(\sum_{n=0}^{\infty} e^{2\mu_n t}\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} b_n^2\right)^{\frac{1}{2}} \\ &\le L_0 e^{-\omega_0 t}\|\tilde{z}(\cdot,0)\|, \forall t \ge \varepsilon \end{aligned}$$

for some $L_0, \omega_0 > 0$. Next,

$$\begin{aligned} &\int_0^T \tilde{z}^2(1,s)\,ds = \int_0^T \left(\sum_{n=0}^{\infty} b_n e^{\mu_n s} g_n(1)\right)^2 ds \\ &\le C^2\left(\int_0^T \sum_{n=0}^{\infty} e^{2\mu_n s}\,ds\right)\left(\sum_{n=0}^{\infty} b_n^2\right) \\ &\le C^2\left(\sum_{n=0}^{\infty} \frac{1}{-2\mu_n}\right)\left(\sum_{n=0}^{\infty} b_n^2\right) < C_1\|\tilde{z}(\cdot,0)\|^2 \end{aligned}$$

for some $C_1 > 0$. This shows that $\tilde{z}(1,\cdot) \in L^2(0,T)$, for any $T > 0$. ■

Since from (15), $y_e(t) = z(0,t) - g(0)\eta(t)$ and hence $g(0)\eta(t) = -y_e(t) + z(0,t)$, we define an approximation of $g(0)\eta(t)$ by a known function $y_d(t) = \hat{z}(0,t) - y_e(t) = g(0)\eta(t) - \tilde{z}(0,t)$ where $\tilde{z}(0,t)$ comes from (20). Consider the following system:

$$\begin{cases} \dot{\eta}(t) = G_c\eta(t), \\ y_d(t) = g(0)\eta(t) - \tilde{z}(0,t). \end{cases} \tag{24}$$

We shall design an adaptive observer according to $y_d(t)$. For this purpose, we need the following Lemma 2.3.

Lemma 2.3. *The pair $(G_c, g(0))$ is observable for every $\omega \in (0,+\infty)$.*

Proof. It is known that $(G_c, g(0))$ is observable if and only if $(G_0, g^*(0))$ is observable, where $G_0 = J^{-1}G_cJ = \text{diag}\{i\omega, -i\omega\}$, $g^*(0) = g(0)J = (g_a(0), g_b(0)), J = [w_1, w_2]$. It is easy to show that $(G_0, g^*(0))$ is observable if and only if $g_a(0) \neq 0$ and $g_b(0) \neq 0$ which are true for every $\omega \in (0,+\infty)$ by the expressions (17) and (18). ■

Lemma 2.3 guarantees that there exists a coordinate transformation:

$$d(t) = T\eta(t),\ \ d(t) = (d_1(t), d_2(t))^\top \in \mathbb{R}^2, \tag{25}$$

where T is nonsingular for all $\omega \in (0,+\infty)$, which transforms the observable pair $(G_c, g(0))$ into an canonical form:

$$\begin{cases} \dot{d}(t) = S_c(\theta)d(t), \\ y_d(t) = \gamma_c d(t) - \tilde{z}(0,t), \end{cases} \tag{26}$$

with $\theta = \omega^2$ and

$$\gamma_c = g(0)T^{-1},\ \ S_c(\theta) = TG_cT^{-1} = \begin{bmatrix} 0 & 1 \\ -\theta & 0 \end{bmatrix}. \tag{27}$$

Lemma 2.4. *There exists an adaptive observer for (26). Precisely, for any $(\xi(0), \hat{\chi}_1(0), \hat{\phi}(0), \hat{\theta}(0)) \in \mathbb{R}^4$, the following adaptive observer:*

$$\begin{cases} \dot{\xi}(t) = -\lambda\xi(t) - y_d(t), \\ \dot{\hat{\chi}}_1(t) = \hat{\phi}(t) + \lambda y_d(t) + \hat{\theta}(t)\xi(t) + k_0(y_d(t) - \hat{\chi}_1(t)), \\ \dot{\hat{\phi}}(t) = -\lambda\hat{\phi}(t) - \lambda^2 y_d(t), \\ \dot{\hat{\theta}}(t) = g\xi(t)(y_d(t) - \hat{\chi}_1(t)), \\ \hat{d}_1(t) = \hat{\chi}_1(t), \\ \hat{d}_2(t) = \hat{\phi}(t) + \xi(t)\hat{\theta}(t) + \lambda\hat{\chi}_1(t), \end{cases} \tag{28}$$

with $g > 0, \lambda > 0, k_0 > \frac{1}{4\lambda}$ satisfies

$$\lim_{t\to\infty} |\hat{\theta}(t) - \theta| = 0 \text{ and } \lim_{t\to\infty} \|\hat{d}(t) - d(t)\|_{\mathbb{R}^2} = 0$$

exponentially.

Proof. Since by Lemma 2.3, $d_1(t)$ contains one sinusoid signal, the proof is very similar to theorem 2.1 of Marino and Tomei (2017) and we omit the details due to page limitation. ■

Let $f_0(x,\theta) = f_0(x) \in \mathbb{R}^{1\times 2}$ be the solution of the following equation

$$\begin{cases} f_0''(x) = f_0(x)S_c(\theta), \\ f_0'(0) = \alpha_1\gamma_c, \\ f_0(0) = \gamma_c, \end{cases} \tag{29}$$

which is an initial value problem of an ordinary differential equation. Hence, the solution of (29) is continuously differentiable with respect to the parameters θ. By (27), it is easily to check that $f_0(x)T = g(x)$ which results in

$$\varepsilon(x,t) = z(x,t) - g(x)\eta(t) = z(x,t) - f_0(x)d(t), \tag{30}$$

and

$$\gamma_c\eta(t) = g'(1)\eta(t) = f_0'(1,\theta)d(t). \tag{31}$$

By Lemma 2.4 and (11), we can design naturally an error feedback control as follows:

$$u(t) = -\alpha_2\hat{z}(1,t) + f_0'(1,\hat{\theta})\hat{d}(t) + \alpha_2 f_0(1,\hat{\theta})\hat{d}(t). \tag{32}$$

Lemma 2.5. *For any functions $a(\cdot), b(\cdot) \in C[0,+\infty)\cap L^\infty[0,+\infty)$ and $\hat{a}(\cdot), \hat{b}(\cdot) \in C[0,+\infty)$, if $|a(t)-\hat{a}(t)|$ and $|b(t)-\hat{b}(t)|$ converge exponentially to zero as $t \to +\infty$, then so does for $|a(t)b(t) - \hat{a}(t)\hat{b}(t)|$ as $t \to +\infty$.*

Proof. The proof is trivial and we omit the details. ■

Lemma 2.6. *The error feedback control $u(t) = -\alpha_2\hat{z}(1,t) + f_0'(1,\hat{\theta})\hat{d}(t) + \alpha_2 f_0(1,\hat{\theta})\hat{d}(t)$ converges exponentially to $-\alpha_2\varepsilon(1,t) + \gamma_c\eta(t)$ as $t \to \infty$.*

Proof. Since $|\theta - \hat{\theta}(t)|$ converges exponentially to zero as $t \to +\infty$, we may suppose that $|\theta - \hat{\theta}(t)| \le Ce^{-\beta t}$ for some constants $C, \beta > 0$, which implies that $\hat{\theta}(t)$ is bounded. Suppose that $\hat{\theta}(t), \theta \in [-M, M]$ for some $M > 0$. Let $\tilde{u}(t) = u(t) - (-\alpha_2\varepsilon(1,t) + \gamma_c\eta(t))$. Then,

$$\begin{aligned}\tilde{u}(t) &= \alpha_2\tilde{z}(1,t) + f_0'(1,\hat{\theta})\hat{d}(t) + \alpha_2 f_0(1,\hat{\theta})\hat{d}(t) \\ &\quad - f_0'(1,\theta)d(t) - \alpha_2 f_0(1,\theta)d(t).\end{aligned} \tag{33}$$

By Lemma 2.5 and Lemma 2.2, it suffices to prove $\lim_{t\to\infty} \|f_0(1,\hat{\theta}(t)) - f_0(1,\theta)\| = 0$ and $\lim_{t\to\infty} \|f_0'(1,\hat{\theta}(t)) - f_0'(1,\theta)\| = 0$ exponentially. Since $f_0(1,\theta), f_0'(1,\theta)$ are continuously differentiable with respect to the parameter θ, they are Lipschitz continuous functions over the domain $[-M, M]$. Therefore,

$$\|f_0(1,\hat{\theta}(t)) - f_0(1,\theta(t))\| \le L_1|\hat{\theta}(t) - \theta| \le L_1Ce^{-\beta t},$$

and

$$\|f_0'(1,\hat{\theta}(t)) - f_0'(1,\theta)\| \le L_2|\hat{\theta}(t) - \theta| \le L_2Ce^{-\beta t},$$

for some constants $L_1, L_2, \beta > 0$. ■

Finally, we write the close-loop of system (1) under the feedback control (32) as follows:

$$\begin{cases} w_t(x,t) = w_{xx}(x,t) + F(x)p(t), \\ w_x(0,t) = Np(t), \\ w_x(1,t) = -\alpha_2\hat{z}(1,t) + f_0'(1,\hat{\theta})\hat{d}(t) \\ \qquad\qquad +\alpha_2 f_0(1,\hat{\theta})\hat{d}(t) + Dp(t), \\ \dot{p}(t) = Gp(t), \\ y_e(t) = w(0,t) - Mp(t), \\ \hat{z}_t(x,t) = \hat{z}_{xx}(x,t), \\ \hat{z}_x(0,t) = \alpha_1\hat{z}(0,t) - \alpha_1 y_e(t), \\ \hat{z}_x(1,t) = -\alpha_2\hat{z}(1,t) + f_0'(1,\hat{\theta})\hat{d}(t) + \alpha_2 f_0(1,\hat{\theta})\hat{d}(t), \\ y_d(t) = -y_e(t) + \hat{z}(0,t), \\ \dot{\xi}(t) = -\lambda\xi(t) - y_d(t), \\ \dot{\hat{\chi}}_1(t) = \hat{\phi}(t) + \lambda y_d(t) + \hat{\theta}(t)\xi(t) + k_0(y_d(t) - \hat{\chi}_1(t)), \\ \dot{\hat{\phi}}(t) = -\lambda\hat{\phi}(t) - \lambda^2 y_d(t), \\ \dot{\hat{\theta}}(t) = g\xi(t)(y_d(t) - \hat{\chi}_1(t)), \\ \hat{d}_1(t) = \hat{\chi}_1(t), \\ \hat{d}_2(t) = \hat{\phi}(t) + \xi(t)\hat{\theta}(t) + \lambda\hat{\chi}_1(t). \end{cases} \tag{34}$$

Theorem 2.1. *Suppose that $\alpha_1, \alpha_2 > 0$ and Assumption 2.1 holds. For any unknown coefficients $F(\cdot), M, N, D, \omega$ and any initial state $(w(\cdot,0), \hat{z}(\cdot,0), \xi(\cdot), \hat{\chi}_1(0), \hat{\phi}(0), \hat{\theta}(0)) \in (L^2(0,1))^2 \times \mathbb{R}^4$, the output tracking of the closed-loop system (34) is guaranteed that*

$$\lim_{t\to\infty} |y_e(t)| = 0 \tag{35}$$

exponentially.

Proof. The ε-system (8) under feedback control (32) now reads

$$\begin{cases} \varepsilon_t(x,t) = \varepsilon_{xx}(x,t), \\ \varepsilon_x(0,t) = 0, \\ \varepsilon_x(1,t) = u(t) - \gamma_c\eta(t) = -\alpha_2\varepsilon(1,t) + \tilde{u}(t), \\ y_e(t) = \varepsilon(0,t). \end{cases} \tag{36}$$

By Lemma 2.2, the $\tilde{u}(\cdot)$ defined by (33) satisfies $\tilde{u}(\cdot) \in L^2(0,T)$ for any $T > 0$. System (36) can be written abstractly as

$$\dot{\varepsilon}(\cdot,t) = \mathbb{A}\varepsilon(\cdot,t) + \delta(x-1)\tilde{u}(t),$$

where the operator $\mathbb{A} : D(\mathbb{A})(\subset H) \to H$ is defined by

$$\begin{cases} \mathbb{A}f(x) = f''(x), \\ D(\mathbb{A}) = \{f(x) \in H^2(0,1)| \\ \qquad\qquad f'(0) = 0, f'(1) = -\alpha_2 f(1)\}. \end{cases} \tag{37}$$

Once again, from the proof of lemma 1.1 of Guo and Meng (2020), $\varepsilon \in C(0,\infty; H)$, we can write the solution of (36) as

$$\begin{aligned}\varepsilon(x,t) &= \sum_{n=0}^{\infty}\langle\phi_n(\cdot), \varepsilon(\cdot,\varepsilon_0)\rangle_H e^{\lambda_n(t-\varepsilon_0)}\phi_n(x) \\ &\quad + \int_{\varepsilon_0}^t \sum_{n=0}^{\infty}\phi_n(1)\phi_n(x)e^{\lambda_n(t-s)}\tilde{u}(s)ds, \\ &= I_1(x,t) + I_2(x,t)\end{aligned} \tag{38}$$

for any given $\varepsilon_0 > 0$, where

$$\begin{cases} \lambda_n = -2\alpha_2 - (n\pi)^2 + \mathcal{O}(n^{-1}) < 0, \\ \phi_n(x) = \cos n\pi x + \mathcal{O}(n^{-1}), \end{cases} \tag{39}$$

with $\{\phi_n(x)\}$ being an orthonormal basis for H. Then, $\varepsilon(0,t) = I_1(0,t) + I_2(0,t)$. Same to the proof of Lemma 2.2, $I_1(0,t)$ satisfies

$$|I_1(0,t)| \le C_2 e^{\lambda_0 t}\|\varepsilon(\cdot,\varepsilon_0)\|, \forall t \ge \varepsilon_0, \tag{40}$$

for some constants C_2 independent of the initial value. As for the second term, by Lemma 2.6, we may suppose $|\tilde{u}(t)| \le Ce^{-\mu t}$ for $t \ge \varepsilon_0$, where $C > 0$ and $0 < \mu < -\lambda_0$. Then,

$$\begin{aligned}\left|\int_{\varepsilon_0}^t e^{\lambda_n(t-s)}\tilde{u}(s)\,ds\right| &\le \frac{C}{-\lambda_n - \mu}[e^{-\mu t} - e^{\lambda_n t}] \\ &\le \frac{C}{-\lambda_n - \mu}e^{-\mu t}, \forall t \ge \varepsilon_0.\end{aligned}$$

Since $|\phi_n(0)\phi_n(1)| \le C_0$ for some constant C_0 and all $n = 0, 1, \ldots$, we have

$$|I_2(0,t)| \le \sum_{n=0}^{\infty}\frac{C_0C}{-\lambda_n - \mu}e^{-\mu t} \le C_3e^{-\mu t}, \forall t \ge \varepsilon_0,$$

which leads to

$$\lim_{t\to\infty}|\varepsilon(0,t)| = \lim_{t\to\infty}|y_e(t)| = 0$$

exponentially. ■

Remark 2.1. The proof Theorem 2.1 corrected an inappropriate proof of theorem 3.2 of Guo and Meng (2020).

3. Main results for $r \geq 1$

In this section, we deal with the general case of $r \geq 1$ and $n = 2r + 1$ without Assumption 2.1 which means the number of frequencies is unknown yet has a known upper bound m under Assumption 1.1. We consider only the case of $n = 2r + 1$, since the solution to the problem with $n = 2r$ follows the same steps. Similarly with the last section, we introduce a transformation for system (8):

$$z(x,t) = \varepsilon(x,t) + h(x)p(t), \tag{41}$$

where $h(x) \in \mathbb{R}^{1\times(2r+1)}$ satisfies

$$\begin{cases} h''(x) = h(x)G, \\ h'(0) = \alpha_1 h(0),\ \alpha_1 > 0, \\ h'(1) = \gamma. \end{cases} \tag{42}$$

The extended system of $(z(\cdot,\cdot), p(\cdot))$ is then governed by

$$\begin{cases} z_t(x,t) = z_{xx}(x,t), \\ z_x(0,t) = \alpha_1 z(0,t) - \alpha_1 y_e(t), \\ z_x(1,t) = u(t), \\ \dot{p}(t) = Gp(t), \\ y_e(t) = z(0,t) - h(0)p(t). \end{cases} \tag{43}$$

It is seen that the z-subsystem in (43) has damping at $x = 0$. The proof for the existence of the solution to (42) is straightforward and we omit the details here. By Assumption 1.1, the term $h(0)p(t)$ contains the sinusoids of no more than m distinct frequencies, which can be expressed without loss of generality as

$$h(0)p(t) = \sum_{i=1}^{l}(A_i \cos\omega_i t + B_i \sin\omega_i t) + C,\ l \leq r \leq m, \tag{44}$$

where A_i, B_i, C are unknown parameters and $A_i^2 + B_i^2 > 0, i = 1, \dots, l$.

Lemma 3.1. *The $h(0)p(t)$ can be generated by exosystem of the following:*

$$\begin{cases} \dot{d}(t) = S_c(\theta)d(t) = A_c d(t) - \sum_{i=1}^{m}\theta_i E_{2i} d_1(t), \\ h(0)p(t) = d_1(t) \end{cases} \tag{45}$$

where $d(t) = (d_1(t), d_2(t), \dots, d_{2m+1}(t))^\top \in \mathbb{R}^{2m+1}$,

$$A_c = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix},\ S_c(\theta) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ -\theta_1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ -\theta_m & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

E_{2i} is the $2i$-th column of the $(2m+1)\times(2m+1)$ identity matrix, and $\theta = [\theta_1, \dots, \theta_l, 0, \dots, 0]^\top \in \mathbb{R}^m$ with $\theta_1, \dots, \theta_l$ being chosen so that

$$s^{2l} + \theta_1 s^{2(l-1)} + \cdots + \theta_l \triangleq \prod_{i=1}^{l}(s^2 + \omega_i^2). \tag{46}$$

Proof. We can consider $h(0)p(t)$ to be generated by the following exosystem:

$$\begin{cases} \dot{\eta}(t) = G_\eta \eta(t),\ \eta(t) \in \mathbb{R}^{2l+1}, \\ h(0)p(t) = \gamma_\eta \eta(t), \end{cases} \tag{47}$$

where

$$\begin{cases} G_\eta = \mathrm{diag}\{G(\omega_1), G(\omega_2), \dots, G(\omega_l), 0_{1\times 1}\}, \\ G(\omega_i) = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{bmatrix}, \\ \gamma_\eta = [1, 0, \dots, 1, 0, 1], \\ \eta(0) = (A_1, B_1, \dots, A_l, B_l, C)^\top. \end{cases} \tag{48}$$

It is a trivial exercise that the pair (G_η, γ_η) is observable which guarantees that there exists a coordinate transformation:

$$\eta^E(t) = T_1\eta(t),\ \eta^E(t) = (\eta_1^E(t), \dots, \eta_{2l+1}^E(t))^\top, \tag{49}$$

where T_1 is a nonsingular $(2l+1)\times(2l+1)$ matrix, which converts the observable pair (G_η, γ_η) into an canonical form:

$$\begin{cases} \dot{\eta}^E(t) = G_E(\theta)\eta^E(t), \\ h(0)p(t) = \eta_1^E(t), \end{cases} \tag{50}$$

with

$$G_E(\theta) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ -\theta_1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ -\theta_l & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Since the characteristic polynomial of G_η is the same as G_E, we can see that $\theta_1, \dots, \theta_l$ can be chosen such that

$$s^{2l+1} + \theta_1 s^{2l-1} + \cdots + \theta_{l-1}s^3 + \theta_l s \triangleq s\prod_{i=1}^{l}(s^2 + \omega_i^2). \tag{51}$$

Next, let $T_2 = [I_{2l+1}\ 0_{(2m-2l)\times(2l+1)}]^\top$, and $d(t) = T_2\eta^E(t)$. A direct computation shows that $d(\cdot)$ satisfies (45). ■

We therefore write $(z(\cdot,t), d(\cdot))$ as governed by

$$\begin{cases} z_t(x,t) = z_{xx}(x,t), \\ z_x(0,t) = \alpha_1 z(0,t) - \alpha_1 y_e(t), \\ z_x(1,t) = u(t), \\ \dot{d}(t) = S_c(\theta)d(t), \\ y_e(t) = z(0,t) - C_c d(t), \end{cases} \tag{52}$$

where $C_c = [1, 0, \dots, 0] \in \mathbb{R}^{1\times(2m+1)}$.

3.1. Error-based observer design

We can design an observer for the z-subsystem in (52) as

$$\begin{cases} \hat{z}_t(x,t) = \hat{z}_{xx}(x,t), \\ \hat{z}_x(0,t) = \alpha_1\hat{z}(0,t) - \alpha_1 y_e(t), \\ \hat{z}_x(1,t) = u(t). \end{cases} \tag{53}$$

Define the observer errors to be $\tilde{z}(x,t) = z(x,t) - \hat{z}(x,t)$. Then,

$$\begin{cases} \tilde{z}_t(x,t) = \tilde{z}_{xx}(x,t), \\ \tilde{z}_x(0,t) = \alpha_1\tilde{z}(0,t), \\ \tilde{z}_x(1,t) = 0, \end{cases} \tag{54}$$

which is, as mentioned in last section, exponentially stable in H and

$$\lim_{t\to\infty}\left|\tilde{z}(0,t)\right| = 0,\ \lim_{t\to\infty}\left|\tilde{z}(1,t)\right| = 0$$

exponentially. We can thus introduce a known function

$$y_d(t) = -y_e(t) + \hat{z}(0,t) = C_c d(t) - \tilde{z}(0,t),$$

and consider the following system:

$$\begin{cases}\dot{d}(t)=S_c(\theta)d(t),\\ y_d(t)=C_cd(t)-\tilde{z}(0,t).\end{cases} \tag{55}$$

Once again, we design an adaptive observer for (55) according to the output $y_d(t)$. The design of adaptive observer (57) in Lemma 3.2 is inspired by Kim and Shim (2015).

Lemma 3.2. *For any initial state $(\Xi(t),\hat{d}(0),\hat{\theta}(0))\in\mathbb{R}^{(2m+1)\times m}\times\mathbb{R}^{2m+1}\times\mathbb{R}^m$, there hold*

$$\lim_{t\to\infty}\|\hat{\theta}(t)-\theta\|=0,\quad \lim_{t\to\infty}\|\hat{d}(t)-d(t)\|=0, \tag{56}$$

where $\hat{\theta}(\cdot)$ and $\hat{d}(\cdot)$ are updated by the following adaptive observer for (55):

$$\begin{cases}\dot{\hat{d}}(t)=A_c\hat{d}(t)-By_d(t)\hat{\theta}(t)\\ \qquad\qquad +L(y_d(t)-C_c\hat{d}(t))+\Xi(t)\dot{\hat{\theta}}(t),\\ \dot{\hat{\theta}}(t)=\lambda_a\Xi(t)^\top C_c^\top(y_d(t)-C_c\hat{d}(t))\\ -\lambda_a diag(e^{-\det^2[\Omega_1(t)]t},\ldots,e^{-\det^2[\Omega_m(t)]t})\cdot\hat{\theta}(t),\\ \dot{\Xi}(t)=(A_c-LC_c)\Xi(t)-By_d(t),\\ \dot{\Omega}(t)=-\lambda_b\Omega(t)+\lambda_c\Xi(t)^\top C_c^\top C_c\Xi(t),\end{cases} \tag{57}$$

with $\Xi(t)\in\mathbb{R}^{(2m+1)\times m}$, $\Omega(t)\in\mathbb{R}^{m\times m}$, $\lambda_a,\lambda_b,\lambda_c>0$, $B=(E_2,\ldots,E_{2m})$, $\Omega_i(t)=[I_i,0_{i\times(m-i)}]\Omega(t)[I_i,0_{i\times(m-i)}]^\top$. The observer gain $L\in\mathbb{R}^{(2m+1)\times 1}$ is chosen so that A_c-LC_c is Hurwitz, and the initial $\Omega(0)$ is any positive definite symmetric matrix.

Proof. Let $\Xi_i(t)$ be the ith column of $\Xi(t)$ and let $\mu_i(t)$ be the first element of $\Xi_i(t)$. In addition, let $\mu(t)=[\mu_1(t),\ldots,\mu_m(t)]^\top$. Then, $\mu_i(t)=C_c\Xi_i(t)$, $\mu(t)=\Xi^\top(t)C_c^\top$. By (57),

$$\begin{cases}\dot{\Xi}_i(t)=(A_c-LC_c)\Xi_i(t)-E_{2i}y_d(t),\\ \mu_i(t)=C_c\Xi_i(t).\end{cases} \tag{58}$$

Set $\mu_i(t)=\mu_{ip}(t)+\mu_{ie}(t)$ where $\mu_{ip}(\cdot)$ is the solution to

$$\begin{cases}\dot{\Xi}_{ip}(t)=(A_c-LC_c)\Xi_{ip}(t)-E_{2i}d_1(t),\\ \mu_{ip}(t)=C_c\Xi_{ip}(t),\ \ i=1,\ldots,m,\end{cases} \tag{59}$$

and $\mu_{ip}(\cdot)$ is governed by

$$\begin{cases}\dot{\Xi}_{ie}(t)=(A_c-LC_c)\Xi_{ie}(t)+E_{2i}\tilde{z}(0,t),\\ \mu_i(0)=\mu_{ip}(0)+\mu_{ie}(0)\\ \mu_{ie}(t)=C_c\Xi_{ie}(t),\ \ i=1,\ldots,m.\end{cases} \tag{60}$$

Since $d_1(\cdot)$ is bounded and A_c-LC_c is Hurwitz, $\mu_{ip}(\cdot)$ is bounded as well. By theorem 5.2.1 of Ioannou and Sun (1996), the vector $[\mu_{1p}(t),\ldots,\mu_{lp}(t)]^\top$ is persistently exciting (PE) (but $[\mu_{1p}(t),\ldots,\mu_{kp}(t)]^\top, k\ge l+1$ is not) because $d_1(\cdot)$ contains the sinusoids of l distinct frequencies. For system (60), since A_c-LC_c is Hurwitz, and $\lim_{t\to\infty}|\tilde{z}(0,t)|=0$ exponentially, we conclude that $\lim_{t\to\infty}|\mu_{ie}(t)|=0$ exponentially. By lemma 2.6.6 of Sastry and Bodson (1989) and $\mu_i(t)=\mu_{ip}(t)+\mu_{ie}(t)$, the vector $[\mu_1(t),\ldots,\mu_l(t)]^\top$ is also PE. Similarly to lemma 2 of Kim and Shim (2015), we can prove that

$$\lim_{t\to\infty}e^{-\det^2[\Omega_i(t)]t}=\begin{cases}0, & i=1,\ldots,l,\\ 1, & i=l+1,\ldots,m,\end{cases} \tag{61}$$

and

$$\lim_{t\to\infty}\mathrm{diag}\left(e^{-\det^2[\Omega_1(t)]t},\ldots,e^{-\det^2[\Omega_m(t)]t}\right)\cdot\theta \triangleq \lim_{t\to\infty}D(t)\theta=0 \tag{62}$$

exponentially. Now let $\tilde{d}(t)=d(t)-\hat{d}(t)$ and $\tilde{\theta}(t)=\theta-\hat{\theta}(t)$, which satisfy

$$\begin{cases}\dot{\tilde{d}}(t)=(A_c-LC_c)\tilde{d}(t)+\Xi(t)\dot{\tilde{\theta}}(t)\\ \qquad -By_d(t)\tilde{\theta}(t)+(L-B\theta)\tilde{z}(0,t),\\ \dot{\tilde{\theta}}(t)=-\lambda_a\Xi(t)^\top C_c^\top(C_c\tilde{d}(t)-\tilde{z}(0,t))\\ \qquad +\lambda_aD(t)(\theta-\tilde{\theta}(t)).\end{cases} \tag{63}$$

Let $\phi(t)=\tilde{d}(t)-\Xi(t)\tilde{\theta}(t)$, $\phi(t)=(\phi_1(t),\ldots,\phi_{2m+1}(t))^\top\in\mathbb{R}^{2m+1}$. Then, we have

$$\begin{aligned}\dot{\phi}(t)&=\dot{\tilde{d}}(t)-\Xi(t)\dot{\tilde{\theta}}(t)-\dot{\Xi}(t)\tilde{\theta}(t)\\ &=(A_c-LC_c)\phi(t)+(L-B\theta)\tilde{z}(0,t),\end{aligned} \tag{64}$$

and

$$\begin{aligned}&\dot{\tilde{\theta}}(t)=-\lambda_a\Xi(t)^\top C_c^\top(\phi_1(t)\\ &+C_c\Xi(t)\tilde{\theta}(t)-\tilde{z}(0,t))+\lambda_aD(t)(\theta-\tilde{\theta}(t))\\ &=-\lambda_a\left(\mu(t)\mu(t)^\top+D(t)\right)\tilde{\theta}(t)\\ &-\lambda_a\mu(t)(\phi_1(t)-\tilde{z}(0,t))+\lambda_aD(t)\theta.\end{aligned} \tag{65}$$

For system (64), since A_c-LC_c is Hurwitz and $\lim_{t\to\infty}|\tilde{z}(0,t)|=0$ exponentially, we conclude that $\lim_{t\to\infty}|\phi(t)|=0$ exponentially. Similarly with lemma 2 of Kim and Shim (2015), we can prove that

$$\lim_{t\to\infty}\tilde{\theta}(t)=0, \tag{66}$$

which is the first limit in (56). Since $\tilde{d}(t)=\phi(t)+\Xi(t)\tilde{\theta}(t)$ and $\Xi(\cdot)$ is bounded, we obtain the second limit in (56):

$$\lim_{t\to\infty}\tilde{d}(t)=0. \tag{67}$$

■

3.2. Feedforward controller design

Let $f_0(x,\theta)=f_0(x)\in\mathbb{R}^{1\times(2m+1)}$ be the solution of the following equation

$$\begin{cases}f_0''(x)=f_0(x)S_c(\theta),\\ f_0'(0)=\alpha_1C_c,\\ f_0(0)=C_c,\end{cases} \tag{68}$$

which is an initial value problem of an ordinary differential equation and hence the solution of (68) is continuously differentiable with respect to the parameters θ. Let $w^c(x,t)=z(x,t)-f_0(x)d(t)$. Then, the $w^c(\cdot,\cdot)$ is governed by

$$\begin{cases}w_t^c(x,t)=w_{xx}^c(x,t),\\ w_x^c(0,t)=0,\\ w_x^c(1,t)=u(t)-f_0'(1,\theta)d(t),\\ y_e(t)=w^c(0,t).\end{cases} \tag{69}$$

We can then naturally design a feedforward control of the following:

$$\begin{aligned}u(t)&=-\alpha_2w_c(1,t)+f_0'(1,\theta)d(t)\\ &=-\alpha_2z(1,t)+f_0'(1,\theta)d(t)+\alpha_2f_0(1,\theta)d(t).\end{aligned} \tag{70}$$

3.3. Error-based feedback controller design

By (70), we can therefore design naturally a tracking error feedback control:

$$u(t)=-\alpha_2\hat{z}(1,t)+f_0'(1,\hat{\theta})\hat{d}(t)+\alpha_2f_0(1,\hat{\theta})\hat{d}(t). \tag{71}$$

The close-loop of system (1) under control (71) is

$$\begin{cases} w_t(x,t) = w_{xx}(x,t) + F(x)p(t), \\ w_x(0,t) = Np(t), \\ w_x(1,t) = -\alpha_2 \hat{z}(1,t) + f_0'(1,\hat{\theta})\hat{d}(t) \\ \qquad\qquad +\alpha_2 f_0(1,\hat{\theta})\hat{d}(t) + Dp(t), \\ \dot{p}(t) = Gp(t), \\ y_e(t) = w(0,t) - Mp(t), \\ \hat{z}_t(x,t) = \hat{z}_{xx}(x,t), \\ \hat{z}_x(0,t) = \alpha_1 \hat{z}(0,t) - \alpha_1 y_e(t), \\ \hat{z}_x(1,t) = -\alpha_2 \hat{z}(1,t) + f_0'(1,\hat{\theta})\hat{d}(t) + \alpha_2 f_0(1,\hat{\theta})\hat{d}(t), \\ y_d(t) = -y_e(t) + \hat{z}(0,t), \\ \dot{\hat{d}}(t) = A_c\hat{d}(t) - By_d(t)\hat{\theta}(t) + L(y_d(t) - C_c\hat{d}(t)) \\ \qquad +\Xi(t)\dot{\hat{\theta}}(t), \\ \dot{\hat{\theta}}(t) = \lambda_a \Xi(t)^\top C_c^\top (y_d(t) - C_c\hat{d}(t)) \\ -\lambda_a \mathrm{diag}(e^{-\det^2[\Omega_1(t)]t}, \ldots, e^{-\det^2[\Omega_m(t)]t}) \cdot \hat{\theta}(t), \\ \dot{\Xi}(t) = (A_c - LC_c)\Xi(t) - By_d(t), \quad \Xi(t) \in \mathbb{R}^{(2m+1)\times m}, \\ \dot{\Omega}(t) = -\lambda_b \Omega(t) + \lambda_c \Xi(t)^\top C_c^\top C_c \Xi(t), \quad \Omega(t) \in \mathbb{R}^{m\times m}. \end{cases} \tag{72}$$

Theorem 3.1. *Suppose that $\alpha_1, \alpha_2 > 0, \lambda_a, \lambda_b, \lambda_c, L, \Omega(0)$ are chosen as in Lemma 3.2. For any unknown coefficients $F(\cdot), M, N, D, G$ and any initial state $(w(\cdot,0), \hat{z}(\cdot,0), \Xi(0), \hat{d}(0), \hat{\theta}(0)) \in (L^2(0,1))^2 \times \mathbb{R}^{(2m+1)\times m} \times \mathbb{R}^{2m+1} \times \mathbb{R}^m$, the output tracking of the closed-loop system (72) is guaranteed that*

$$\lim_{t\to\infty} |y_e(t)| = 0. \tag{73}$$

Proof. The w^c-system (69) under control (71) now reads

$$\begin{cases} w_t^c(x,t) = w_{xx}^c(x,t), \\ w_x^c(0,t) = 0, \\ w_x^c(1,t) = -\alpha_2 w^c(1,t) + \tilde{u}(t), \\ y_e(t) = w^c(0,t), \end{cases} \tag{74}$$

where

$$\begin{aligned} \tilde{u}(t) &= \alpha_2 \tilde{z}(1,t) + f_0'(1,\hat{\theta})\hat{d}(t) + \alpha_2 f_0(1,\hat{\theta})\hat{d}(t) \\ &\quad - f_0'(1,\theta)d(t) - \alpha_2 f_0(1,\theta)d(t). \end{aligned} \tag{75}$$

By Lemma 2.2, $\tilde{u}(\cdot) \in L^2(0,T)$, for any $T > 0$. We claim that $\lim_{t\to\infty} |\tilde{u}(t)| = 0$. To this end, it suffices to prove

$$\lim_{t\to\infty} \|f_0(1,\hat{\theta}(t)) - f_0(1,\theta)\| = 0, \tag{76}$$

and

$$\lim_{t\to\infty} \|f_0'(1,\hat{\theta}(t)) - f_0'(1,\theta)\| = 0. \tag{77}$$

However, both convergence are guaranteed because $\|\hat{\theta}(t) - \theta\| \to 0 (t \to \infty)$ and $f_0(1,\theta), f_0'(1,\theta)$ are continuously differentiable with respect to the parameter θ, and hence they are Lipschitz continuous functions over some finite domain. System (74) can be written abstractly as

$$\dot{w}^c(\cdot,t) = \mathbb{A}w^c(\cdot,t) + \delta(x-1)\tilde{u}(t),$$

where the operator $\mathbb{A} : D(\mathbb{A})(\subset H) \to H$ is defined by (37), which generates an exponentially stable C_0-semigroup on H. Since $\lim_{t\to\infty} |\tilde{u}(t)| = 0$, and $\delta(x-1)$ is admissible for $e^{\mathbb{A}t}$, we conclude immediately that

$$\lim_{t\to\infty} \|w^c(\cdot,t)\| = 0.$$

Therefore, both $w(x,t) = w^c(x,t) + f_0(x)d(t) + (\Gamma(x) - h(x))p(t)$ and $\hat{z}(x,t) = w^c(x,t) + f_0(x)d(t) - \tilde{z}(x,t)$ are bounded in H with respect to time. The remaining is the proof of $\lim_{t\to\infty} |y_e(t)| = 0$. Similarly to (38), we can write the solution of (74) as

$$\begin{aligned} w^c(x,t) &= \sum_{n=0}^{\infty} a_n e^{\lambda_n t}\phi_n(x) \\ &\quad + \int_0^t \sum_{n=0}^{\infty} \phi_n(1)\phi_n(x)e^{\lambda_n(t-s)}\tilde{u}(s)ds, \\ &= I_1(x,t) + I_2(x,t), \end{aligned} \tag{78}$$

where $\sum_{n=0}^{\infty} a_n^2 = \|w^c(\cdot,0)\|^2$, and $\lambda_n, \phi_n(x)$ are defined by (39), and hence $w^c(0,t) = I_1(0,t) + I_2(0,t)$. Similarly to (40), there holds

$$|I_1(0,t)| \le C_2 e^{\lambda_0 t}\|w^c(\cdot,0)\|, \forall t \ge \varepsilon > 0 \tag{79}$$

for some $\varepsilon > 0$. As for the second term, since $\lim_{t\to\infty} |\tilde{u}(t)| = 0$, for any given $\sigma > 0$, there exists $t_0 > 0$ such that $|\tilde{u}(t)| \le \sigma, t \ge t_0$. Hence,

$$\begin{aligned} &\left|\int_0^t e^{\lambda_n(t-s)}\tilde{u}(s)\,ds\right| \\ &\le \left|\int_{t_0}^t e^{\lambda_n(t-s)}\sigma\,ds\right| + \left|\int_0^{t_0} e^{\lambda_n(t-s)}\tilde{u}(s)\,ds\right| \\ &\le \frac{\sigma}{-\lambda_n} + \left(\int_0^{t_0} \tilde{u}^2(s)\,ds\right)^{\frac{1}{2}} \left(\int_0^{t_0} e^{2\lambda_n(t-s)}\,ds\right)^{\frac{1}{2}} \\ &\le \frac{\sigma}{-\lambda_n} + \|\tilde{u}\|_{L^2(0,t_0)} \left(\frac{1}{-2\lambda_n}\right)^{\frac{1}{2}} e^{\lambda_n(t-t_0)}. \end{aligned}$$

Since $|\phi_n(0)\phi_n(1)| \le C_0$ for some constant C_0 and all $n = 0, 1, \ldots$, we have

$$\begin{aligned} |I_2(0,t)| &\le \sum_{n=0}^{\infty} \frac{C_0\sigma}{-\lambda_n} + \sum_{n=0}^{\infty} \frac{C_0\|\tilde{u}\|_{L^2(0,t_0)}}{(-2\lambda_n)^{\frac{1}{2}}} e^{\lambda_n(t-t_0)} \\ &\le L_1\sigma + C_0\|\tilde{u}\|_{L^2(0,t_0)} \left(\sum_{n=0}^{\infty} e^{2\lambda_n(t-t_0)}\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} \frac{1}{-2\lambda_n}\right)^{\frac{1}{2}} \\ &\le L_1\sigma + L_{t_0}e^{\lambda_0(t-t_0)}, t > t_0. \end{aligned} \tag{80}$$

which leads to $\lim_{t\to\infty} |w^c(0,t)| \le L_1\sigma$. By the arbitrariness of σ, we have $w^c(0,t) \to 0$ as $t \to \infty$. ■

Remark 3.1. Compared with the previous section, where the tracking error converges exponentially to zero, we only obtain the asymptotic convergence of the tracking error $y_e(t)$ here due to unknown number of the frequencies.

4. Numerical simulation

As an illustrating example, we consider the following system:

$$\begin{cases} w_t(x,t) = w_{xx}(x,t), \\ w_x(0,t) = 10\sin 0.2t, \quad w_x(1,t) = u(t), \\ y_e(t) = w(0,t) - 10\sin t, \\ w(x,0) = 10. \end{cases} \tag{81}$$

The parameters of the controller are chosen as $m = 2, \alpha_1 = \alpha_2 = \lambda_a = \lambda_b = \lambda_c = 1, L = [4,6,4,1]^\top$, and

$$\hat{z}(x,0) = 1, \ (\Xi(0), \hat{d}(0), \hat{\theta}(0)) = 0, \ \Omega(0) = I_2. \tag{82}$$

Fig. 1(a) plots the tracking performance of $w(0,t)$. It is obvious that $w(0,t)$ tracks $y_{ref}(t)$ well after $t \ge 30$. Fig. 1(b) and Fig. 1(c)

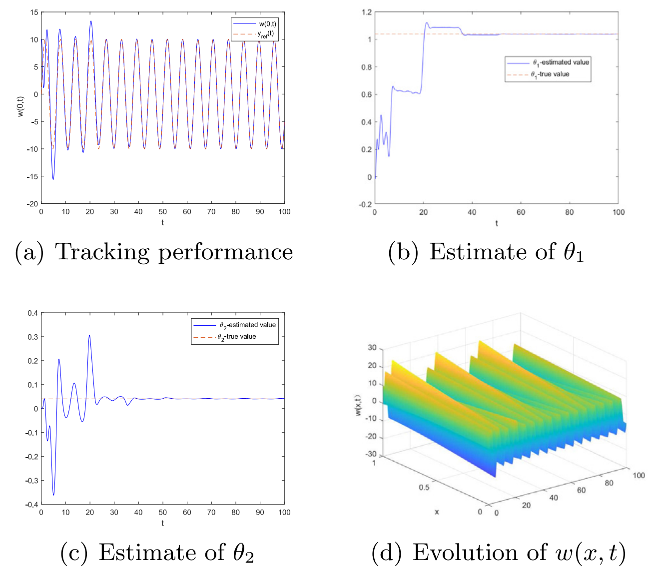

Fig. 1. Tracking performance, frequency estimate and evolution of $w(x,t)$ for system (81)–(82).

display the tracking performance of $\hat{\theta}(t)$ from which we can find that $\hat{\theta}(t)$ tends to θ satisfactorily. Fig. 1(d) shows the w-part of system (81) and (82) is bounded. In order to verify the robustness of the control (82), a second set of simulation has been carried out for the following system where only one frequency has really entered into the system and thus $\mu(t) = [\mu_1(t), \mu_2(t)]^\top$ is not PE:

$$\begin{cases} w_t(x,t) = w_{xx}(x,t), \\ w_x(0,t) = 0, \;\; w_x(1,t) = u(t), \\ y_e(t) = w(0,t) - 10\sin t, \\ w(x,0) = 10. \end{cases} \tag{83}$$

However, as plotted in Fig. 2, the same controller (82) can also regulate the closed-loop system (82) and (83).

5. Concluding remarks

This paper is a first effort to develop output regulation for a boundary controlled PDE system where the disturbance is generated from a completely unknown exosystem. The system is described by a 1-d heat equation where the control and observation operators are unbounded, which represents a difficult situation in output regulation of PDEs. Motivated from adaptive estimation of frequencies of sinusoid signals in signal process and adaptive internal model for lumped parameter systems, we develop an adaptive internal model for output regulation of this PDE system. All the estimations are in real time and the control is robust to disturbances in all possible channels. Numerical simulations validate the theoretical results. When the number of the unknown frequencies is available in a transformed system, the convergence can be exponential while the number is unknown, only asymptotic convergence can be achieved. Some preliminary studies show that the approach is applicable to other 1-d PDEs.

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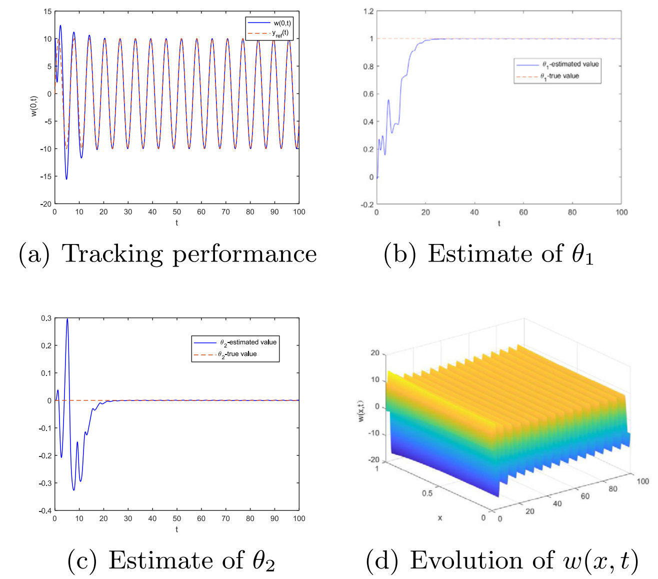

Fig. 2. Tracking performance, frequency estimate and evolution of $w(x,t)$ for system (83)–(82).

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