SOME COMPACT CLASSES OF OPEN SETS UNDER HAUSDORFF DISTANCE AND APPLICATION TO SHAPE OPTIMIZATION∗

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Abstract. In this paper, we introduce three new classes of open sets in a general Euclidean space $\mathbb{R}^N$. It is shown that every class of open sets is compact under the Hausdorff distance. The result is then applied to a shape optimization problem of elliptic equation. The existence of the optimal solution is presented.

Key words. Laplacian, eigenvalue, eigenfunction, shape optimization

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1. Introduction. The existence of the optimal solution is a big issue in shape optimization. Many approaches to achieving existence are available in the literature. For the regularity assumption for the boundary of an unknown domain, we refer to [4, 6, 10, 11, 13, 14, 15]. The results by certain capacitary constraints can be found in [2, 12]. In [2, 3, 5, 7], the generalized perimeter and constraints or the penalty terms constructed from generalized perimeter and constraints are used in dealing with existence, where for the second case conditions on the dimension of underlying Euclidean space have to be imposed in order to obtain the compactness of certain families of open sets with respect to the Hausdorff distance.

Let us first state a shape optimization problem of an elliptic equation in an open sets class $\mathcal{C}$ in $\mathbb{R}^N$. We denote by $U(x, r)$ the open ball and by $B(x, r)$ the closed ball of $\mathbb{R}^N$, both centered at $x$ with radius $r$ throughout the paper. Suppose that $B^* \subset \mathbb{R}^N$ is a bounded domain such that $B^* \supset U(0, 2R_0)$ for a given positive constant $R_0$. In the Sobolev space $H^1_0(B^*)$, the inner product induced norm is given by $\|u\|^2_{H^1_0(B^*)} = \int_{B^*} |\nabla u|^2 dx$. Let $A \in M_{N \times N}(C^1(B^*))$ be a smooth symmetric matrix function, and suppose that

$$\alpha |\xi|^2 \leq \langle A\xi, \xi \rangle \quad \forall \xi \in \mathbb{R}^N,$$

where $\alpha > 0$ is a constant. Define the operator $A : H^1_0(B^*) \to H^{-1}(B^*)$ associated with $A$ as follows:

$$A = \text{div}(A\nabla).$$

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For any open set \( \Omega \in \mathcal{C} \), where \( \mathcal{C} \) is some class of open sets contained in \( B^* \), consider the Dirichlet problem in \( \Omega \):

\[
-Au_\Omega = f, \quad u_\Omega \in H^1_0(\Omega),
\]

where \( f \in H^{-1}(B^*) \) is a given function. The (weak) solution of (1.3) is considered to satisfy (1.3) in the variational sense that

\[
\int_\Omega (A\nabla u_\Omega, \nabla \phi)_{L^2(\Omega)} \, dx = \langle f|\Omega, \phi \rangle_{H^{-1}(\Omega) \times H^1_0(\Omega)} \quad \forall \phi \in C^\infty_0(\Omega),
\]

where \( f|\Omega \) denotes the restriction of the distribution \( f \) in \( \Omega \).

Equation (1.4) admits a unique solution \( u_\Omega \in H^1_0(\Omega) \), which is considered to be defined in whole \( B^* \) by assigning zero on \( B^* \setminus \Omega \), the complementary set of \( \Omega \) in \( B^* \). In this way, we have \( u_\Omega \in H^1_0(B^*) \) and \( \|u_\Omega\|_{H^1_0(B^*)} = \|u_\Omega\|_{H^1_0(\Omega)} \). In what follows, when we mention the solution of (1.3), we implicitly take this zero extension without explanation.

Set

\[
J(\Omega) = \frac{1}{2} \int_{B^*} |u_\Omega - g|^2 \, dx,
\]

where \( g \in L^2(B^*) \) is a given function. A shape optimization in the class \( \mathcal{C} \) is to seek an \( \Omega^* \in \mathcal{C} \) such that

\[
J(\Omega^*) = \inf \{ J(\Omega); \Omega \in \mathcal{C} \}.
\]

\( \Omega^* \) is then called the optimal solution to the problem (1.5).

It is seen that in order to study problem (1.5), we need to endow with topology for the open sets class \( \mathcal{C} \). This is realized by the Hausdorff distance between their complementary sets for any two given open sets. That is, for any \( \Omega_1, \Omega_2 \in \mathcal{C} \), the Hausdorff distance \( \rho(\Omega_1, \Omega_2) \) is defined as [1]

\[
\rho(\Omega_1, \Omega_2) = \max \left\{ \sup_{x \in B^* \setminus \Omega_1} \text{dist}(x, B^* \setminus \Omega_2), \sup_{y \in B^* \setminus \Omega_2} \text{dist}(B^* \setminus \Omega_1, y) \right\},
\]

where \( \text{dist}(\cdot, \cdot) \) denotes the Euclidean metric of \( \mathbb{R}^N \). In this way, \( (\mathcal{C}, \rho) \) becomes a metric space [4, 11]. A sequence \( \{\Omega_n\} \subset \mathcal{C} \) is said to be convergent to \( \Omega \in \mathcal{C} \), which is denoted by \( \Omega_n \rightarrow^\rho \Omega \), if \( \rho(\Omega_n, \Omega) \rightarrow 0 \) as \( n \rightarrow \infty \).

In this work, we introduce three new classes of open sets \( \mathcal{C}_i, i = 1, 2, 3 \), in a general Euclidean space \( \mathbb{R}^N (N \geq 1) \). We show that each class is compact under the Hausdorff distance (1.6). The regularity of class \( \mathcal{C}_1 \cap \mathcal{C}_2 \) is specified, by which we show the existence of optimal solution in the shape optimization problem mentioned for the class \( \mathcal{C}_1 \cap \mathcal{C}_2 \).

We proceed as follows. In section 2, we introduce some preliminary notation and define the classes of open sets \( \mathcal{C}_i (i = 1, 2, 3) \). The main results are then stated. Section 3 is devoted to the proof of main results.

2. Main results. Define

\[
\delta(K_1, K_2) = \max \left\{ \sup_{x \in K_1} \text{dist}(x, K_2), \sup_{y \in K_2} \text{dist}(y, K_1) \right\},
\]

which is also called the Hausdorff distance between two compact subsets \( K_1 \) and \( K_2 \) of \( \mathbb{R}^N \). It is seen from (1.6) that \( \rho(\Omega_1, \Omega_2) = \delta(B^* \setminus \Omega_1, B^* \setminus \Omega_2) \) for any open sets \( \Omega_1, \Omega_2 \subset B^* \). Hence

\[
\Omega_n \rightarrow^\rho \Omega \iff B^* \setminus \Omega_n \delta \rightarrow B^* \setminus \Omega.
\]
Lemmas 2.1 through 2.4 are brought from [4, 11], which will be used in what follows.

**Lemma 2.1.** Let $K, K_n, n \in \mathbb{N}$, be compact subsets of $\mathbb{R}^N$ such that $K_n \overset{\mathcal{L}}{} K$. Then $K$ is the set of all accumulation points of the sequences $\{x_n\}_{n \in \mathbb{N}}$ such that $x_n \in K_n$ for every $n \in \mathbb{N}$.

**Remark 2.1.** It follows from Lemma 2.1 and the definition of $\mathcal{L}$ that for any given $\varepsilon > 0$, there exists an integer $M(\varepsilon) > 0$ such that for all $m \geq M(\varepsilon)$, it has $K \subseteq \bigcup_{x \in K_m} U(x, \varepsilon)$ and $K_m \subseteq \bigcup_{x \in K} U(x, \varepsilon)$.

**Lemma 2.2.** Let $K, \tilde{K}, K_n, n \in \mathbb{N}$, be compact subsets of $\mathbb{R}^N$ such that $K_n \overset{\mathcal{L}}{} K$ and $K_n \overset{\mathcal{L}}{} \tilde{K}$. If $K_n \subseteq \tilde{K}_n$ for every $n$, then $K \subseteq \tilde{K}$.

**Lemma 2.3 (Γ-property for open set class).** For any given class of open sets $\mathcal{C}$, if $\{\Omega_n\}_{n=1}^{\infty} \subseteq \mathcal{C}, \Omega \in \mathcal{C}$, and $\Omega_n \overset{\mathcal{L}}{} \Omega$, then for each open subset $\Lambda$ with $\overline{\Lambda} \subseteq \Omega$, there exists a positive integer $n_{\Lambda}$ depending on $\Lambda$ such that $\overline{\Lambda} \subseteq \Omega_n$ for all $n \geq n_{\Lambda}$.

**Lemma 2.4.** Suppose that $\Omega_n \subseteq B^*, n \in \mathbb{N}$, are bounded open sets of $\mathbb{R}^N$. Then there exist an open set $\Omega \subseteq B^*$ and a subsequence $\{\Omega_{n_k}\}_{k=1}^{\infty}$ of $\{\Omega_n\}_{n=1}^{\infty}$ such that $\Omega_{n_k} \overset{\mathcal{L}}{} \Omega$. In particular, $(\mathcal{O}, \delta)$ is a compact metric space, where $\mathcal{O} = \{K \subseteq B^* | K$ is compact.$\}$

Before stating the main results, we need the following definitions.

**Definition 2.5.** Let $\Omega$ be a bounded set in $\mathbb{R}^N$, $x_0 \in \partial \Omega$. We say that $\Omega$ satisfies the interior ball condition [8] at $x_0$ if there exists an open ball $U(y(x_0), r(x_0)) \subseteq \Omega$ with $x_0 \in \partial U(y(x_0), r(x_0))$, where $y(x_0) \in \mathbb{R}^N$ is a point and $r(x_0)$ is a positive number, both depending on $x_0$.

$\Omega$ is said to satisfy the interior ball condition if for every point $x_0 \in \partial \Omega$, $\Omega$ satisfies the interior ball condition at $x_0$.

$\Omega$ is said to satisfy the uniformly interior ball condition if there exists a $r_\Omega > 0$ such that for every point $x_0 \in \partial \Omega$, $\Omega$ satisfies the interior ball condition at $x_0$ and $r(x_0) \geq r_\Omega$.

**Definition 2.6.** Let $\Omega$ be a bounded set in $\mathbb{R}^N$, $x_0 \in \partial \Omega$. $\Omega$ is said to satisfy the exterior ball condition at $x_0$ if there exists a closed ball $B(y(x_0), r(x_0)) \subseteq B^* \setminus \Omega$ with $x_0 \in \partial B(y(x_0), r(x_0))$, where $y(x_0) \in \mathbb{R}^N$ is a point and $r(x_0)$ is a positive number, both depending on $x_0$.

$\Omega$ is said to satisfy the exterior ball condition if for every point $x_0 \in \partial \Omega$, $\Omega$ satisfies the exterior ball condition at $x_0$.

$\Omega$ is said to satisfy the uniformly exterior ball condition if there exists a $r_\Omega > 0$ such that for every point $x_0 \in \partial \Omega$, $\Omega$ satisfies the exterior ball condition at $x_0$ and $r(x_0) \geq r_\Omega$.

**Definition 2.7.** An open set $\Omega \subseteq \mathbb{R}^N$ is said to have property $(C_M)$ if for any $x, y \in \Omega$, there exists a connected compact set $K$ with $x, y \in K$, such that $K \subseteq \Omega$ and $\bigcup_{z \in K} U(z, \frac{d^*}{2}) \subseteq \Omega$; here $d^* := \min\{\text{dist}(x, \partial \Omega), \text{dist}(y, \partial \Omega)\}$, and $M > 1$ is a given constant.

Let $R_0 > 0, r_0 > 0, R > 0$. We introduce three classes of open sets $\mathcal{C}_i, i = 1, 2, 3$:

\[
\begin{align*}
\mathcal{C}_1 &= \{\Omega \subseteq B(0, R_0) \subseteq B^* | \\
& \quad \text{$\Omega$ satisfies the uniformly interior ball condition and $r_\Omega \geq r_0$}\}.
\end{align*}
\]

\[
\begin{align*}
\mathcal{C}_2 &= \{\Omega \subseteq B(0, R_0) \subseteq B^* | \text{there exists an $x_\Omega \in \Omega$ such that} \\
& \quad \text{$U(x_\Omega, R) \subseteq \Omega$, $\Omega$ satisfies the uniformly exterior ball condition} \\
& \quad \text{and $r_\Omega \geq r_0$}\}.
\end{align*}
\]

\[
\begin{align*}
\mathcal{C}_3 &= \{\Omega \subseteq B(0, R_0) \subseteq B^* | \text{there exists an $x_\Omega \in \Omega$ such that} \\
& \quad \text{$U(x_\Omega, R) \subseteq \Omega$, $\Omega$ has the property $(C_M)$}\}.
\end{align*}
\]
Example 2.1. In this example, we give some open sets that either possess \((C_M)\) property or not.

- All open balls of \(\mathbb{R}^N\) have \((C_M)\) property.
- Let \(\Omega \subset \mathbb{R}^2\) be the interior domain surrounded by the following three curves \(\Gamma_i, i = 1, 2, 3:\)
  \[ \Gamma_1: \{(x,y) \in \mathbb{R}^2 | y = x^2, x \in [0,1]\}; \]
  \[ \Gamma_2: \{(x,y) \in \mathbb{R}^2 | y = -x^2, x \in [0,1]\}; \]
  \[ \Gamma_3: \{(x,y) \in \mathbb{R}^2 | x = 1 + \sqrt{1-y^2}, y \in [-1,1]\}. \]

It is obvious that \(\Omega\) has property \((C_M)\). However, \(\partial\Omega\) is not smooth since \((0,0) \in \partial\Omega\) is a cusp point of \(\Omega\).

- Let \(\Omega \subset \mathbb{R}^2\) be the interior domain surrounded by the following four curves \(\Gamma_i, i = 1, 2, 3, 4:\)
  \[ \Gamma_1: \{(x,y) \in \mathbb{R}^2 | (x+2)^2 + y^2 = 1, x \in [-3,\frac{-3}{2}]\}; \]
  \[ \Gamma_2: \{(x,y) \in \mathbb{R}^2 | (x-2)^2 + y^2 = 1, x \in [\frac{3}{2}, 3]\}; \]
  \[ \Gamma_3: \{(x,y) \in \mathbb{R}^2 | y = \frac{\sqrt{2}}{x}, x \in [-\frac{3}{2}, \frac{3}{2}]\}; \]
  \[ \Gamma_3: \{(x,y) \in \mathbb{R}^2 | y = -\frac{\sqrt{2}}{x}, x \in [-\frac{3}{2}, \frac{3}{2}]\}. \]

Since \((-2,0), (2,0) \in \Omega\) and \(U((-2,0),1) \cup (-2,0) \subset \Omega\), there is no connected compact set \(K\) with \((-2,0), (2,0) \in K\) such that \(K \subset \Omega\) and \(\bigcup_{z \in K} U(z,1) \subset \Omega\). So this defined \(\Omega\) has no \((C_M)\) property with \(M \in (1, \frac{2\sqrt{2}}{3})\).

We are now in a position to state the main results of this paper.

**Theorem 2.8.** For every given \(i \in \{1, 2, 3\}\), if \(\{\Omega_m\}_{m=1}^{\infty} \subset C_i\), then there exist a subsequence \(\{\Omega_{m_k}\}_{k=1}^{\infty}\) of \(\{\Omega_m\}_{m=1}^{\infty}\) and \(\Omega \subset C_i\) such that

\[ \Omega_{m_k} \overset{\rho}{\to} \Omega \text{ as } k \to \infty. \]

In other words, each \((C_i, \rho)\) is a compact metric space. Moreover, for any \(i, j = 1, 2, 3\), \((C_i \cap C_j, \rho)\) is also a compact metric space.

**Remark 2.2.** In [12], the open sets class of

\[ C_l = \{ \Omega \in C_l | \Omega \subset B(0,R_0) \text{ is open, and the number of connected components of } B(0,R_0) \setminus \Omega \text{ is } \leq l \} \]

is discussed. However, our class \(C_3\) is different with \(C_l\) because for any \(l\) and \(M > 1\) large enough, we can find an \(\Omega \in C_3\) so that the number of connected components of \(\overline{B^2} \setminus \Omega\) is just \(l\). The advantage of \(C_3\) is that any open set in \(C_3\) is connected.

One of the main results of Theorem 2.9 says that the class \(C_1 \cap C_2\) possesses a special geometric property that each open set in \(C_1 \cap C_2\) is of \(C^{1,1}\) [9, p. 94]. This is, to some extent, the inverse statement about the well-known fact that for any surface of \(\mathbb{R}^n\) with smooth boundary, there is always an interior ball at any point of the boundary [8, pp. 330–331].

**Theorem 2.9.** Let \(\Omega \subset C_1 \cap C_2\). Then \(\Omega \subset C^{1,1}\).

Theorem 2.9 cannot be improved by Example 2.2.

**Example 2.2.** Let \(\Omega \subset \mathbb{R}^2\) be the interior domain surrounded by the following four curves \(\Gamma_i, i = 1, 2, 3, 4:\)

\[ \Gamma_1: \{(x,y) \in \mathbb{R}^2 | x \in [-1,1], y = 1\}; \]
\[ \Gamma_2: \{(x,y) \in \mathbb{R}^2 | x \in [-1,1], y = -1\}; \]
\[ \Gamma_3: \{(x,y) \in \mathbb{R}^2 | x = 1 + \sqrt{1-y^2}, y \in [-1,1]\}; \]
\[ \Gamma_4: \{(x,y) \in \mathbb{R}^2 | x = -1 - \sqrt{1-y^2}, y \in [-1,1]\}. \]

Then \(\Omega \in C^{1,1}\) but \(\Omega \notin C^2\).
This contradiction implies that (3.2) is valid. So one can find an 
\( \varepsilon \) such that

\[
\{ x_n \} \subset \mathbb{R}^N \text{ and } \{ r_n \} \subset \mathbb{R} \text{ be such that } r_n \geq r_0 > 0 \text{ and } \]

\[
U(x_n, r_n) \supset D \subset \mathbb{R}^N. \]

Then there exist an \( x \in \mathbb{R}^N \) and a subsequence of \( \{ x_n \} \), denoted by \( \{ x_{nk} \} \), such that \( U(x, r_0) \subset D \) and \( x_{nk} \to x \) as \( k \to \infty \). Furthermore, if \( r_n \to r_1 \) as \( n \to \infty \), then \( D = U(x, r_1) \).

Proof. Since \( U(x_n, r_n) \subset B^* \) and \( B^* \) is a bounded set in \( \mathbb{R}^N \), there exists a subsequence of \( \{ x_n \} \), still denoted by itself without confusion, such that

\[
(3.1) \quad x_n \to x
\]

for some \( x \in \overline{B^*} \).

We first prove that

\[
U(x, r_0) \subset D.
\]

In fact, if (3.2) fails, then there exists a \( y \in U(x, r_0) \), \( y \notin D \). Since \( U(x_n, r_n) \supset D \), we have \( \overline{B^*} \setminus U(x_n, r_n) \xrightarrow{\delta} \overline{B^*} \setminus D \). It follows from Lemma 2.1 that there exists a sequence \( \{ y_n \} \subset \overline{B^*} \setminus U(x_n, r_n) \) so that

\[
(3.3) \quad y_n \to y \text{ as } n \to \infty.
\]

Hence

\[
d(y_n, x_n) \geq r_n \geq r_0.
\]

Passing to the limit on the left side of the inequality above as \( n \to \infty \) gives

\[
d(y, x) \geq r_0.
\]

This contradiction implies that (3.2) is valid.

Next, we show that if \( x_n \to x \) and \( r_n \to r_1 \) as \( n \to \infty \), then \( D = U(x, r_1) \).

If \( U(x, r_1) \subset D \) is not true, then there exists a \( y \in U(x, r_1) \), \( y \notin D \), and hence \( \varepsilon_0 = r_1 - |y - x| > 0 \). Since \( x_n \to x \) and \( r_n \to r_1 \) as \( n \to \infty \), one can find an \( n_1 > 0 \) such that \( |x_{n_1} - x| \leq \frac{\varepsilon_0}{4} \), \( |r_n - r_1| \leq \frac{\varepsilon_0}{4} \) for all \( n \geq n_1 \). Since \( y \notin D \), as in the same arguments to get (3.3), there exists a sequence \( \{ z_n \} \) satisfying

\[
z_n \in \overline{B^*} \setminus U(x_n, r_n) \quad \text{and} \quad z_n \to y.
\]

So one can find an \( n_2 > 0 \) such that \( |z_n - y| \leq \frac{\varepsilon_0}{4} \) for all \( n \geq n_2 \). This means that for all \( n \geq n_0 \), \( n_0 = \max\{n_1, n_2\} \), we have \( r_1 - \frac{\varepsilon_0}{4} \leq r_n \leq |x_n - z_n| \leq |x_n - x| + |x - y| + |y - z_n| \leq \frac{\varepsilon_0}{4} + (r_1 - \varepsilon_0) + \frac{\varepsilon_0}{4} = r_1 - \frac{\varepsilon_0}{4} \). This is a contradiction. So \( U(x, r_1) \subset D \).

If \( D \subset U(x, r_1) \) is not true, then there exists a \( z \in D \), \( z \notin B(x, r_0) \). Since \( D \) is open, one can find an \( \varepsilon_1 > 0 \) so that \( B(z, \varepsilon_1) \subset D \) and \( B(z, \varepsilon_1) \cap B(x, r_0) = \emptyset \). By Lemma 2.3, there exists an \( n_0 > 0 \) such that \( B(z, \varepsilon_1) \subset U(x_n, r_0) \) for all \( n \geq n_0 \). Hence \( r_0 + \varepsilon_1 \leq |x - z| \leq |x - x_n| + |x_n - z| \leq |x - x_n| + r_0 \). Passing to the limit
as \( n \to \infty \) leads to a contradiction that \( r_0 + \varepsilon_1 \leq r_0 \). This shows \( D \subseteq U(x, r_1) \), or \( D = U(x, r_1) \). The proof is complete. □

In a similar way, we can prove Lemma 3.2.

**Lemma 3.2.** Let \( \{x_n\} \subseteq \mathbb{R}^N \) and \( \{r_n\} \subseteq \mathbb{R} \) be such that \( r_n \geq r_0 > 0 \) and \( B(x_n, r_n) \xrightarrow{\delta} D \subseteq \mathbb{R}^N \). Then there exist an \( x \in \mathbb{R}^N \) and a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( B(x, r_0) \subseteq D \), and \( x_{n_k} \to x \) as \( k \to \infty \). Moreover, if \( r_n = r_0 \) for all \( n \in \mathbb{N} \), then \( D = B(x, r_0) \).

**Lemma 3.3.** Assume \( \Omega_n \xrightarrow{\delta} \Omega \) and \( x \in \partial \Omega \). Then there exist \( x_{n_l} \in \partial \Omega_{n_l} \) for all \( l \in \mathbb{N} \) such that \( x_{n_l} \to x \) as \( l \to \infty \).

**Proof.** For any given \( n \in \mathbb{N} \), since \( x \in \partial \Omega \), it has \( U(x, \frac{1}{n}) \cap \Omega \neq \emptyset \). Hence one can find \( y_n \in U(x, \frac{1}{n}) \cap \Omega \), and \( r_n > 0 \) so that \( B(y_n, r_n) \subseteq U(x, \frac{1}{n}) \cap \Omega \). By the \( \Gamma \)-property claimed in Lemma 2.3, there exists an \( m_n > 0 \) such that \( B(y_n, r_n) \subseteq \Omega_m \) for all \( m \geq m_n \).

On the other hand, since \( x \in \partial \Omega \subseteq B^* \setminus \Omega \), by Lemma 2.2 there exist \( z_k \in B^* \setminus \Omega_k \) such that

\[ z_k \to x \quad \text{as} \quad k \to \infty. \]

So one can find a subsequence of \( \{z_k\}_{k=1}^{\infty} \), still denoted by itself without confusion, such that \( |z_{k+1} - x| \leq |z_k - x| \) for all \( k \in \mathbb{N} \). Now take \( k_n > 0 \) sufficiently large so that \( |z_k - x| < \frac{1}{n} \) for all \( k \geq k_n \). Hence for all \( p \geq N_n \), \( N_n = \max \{m_n, k_n\} \), it has \( B(y_n, r_n) \subseteq \Omega_p \) and \( z_p \in U(x, \frac{1}{n}) \). In particular, \( y_n, z_n \in U(x, \frac{1}{n}) \), \( y_n \in \Omega_{N_n} \), and \( z_n, \notin \Omega_{N_n} \). So there exists an \( x_{N_n} \in \partial \Omega_{N_n} \) such that \( x_{N_n} \in U(x, \frac{1}{n}) \). This shows that the sequence \( \{x_{N_n}\}_{n=1}^{\infty} \) is the designed sequence by the arbitrary choice of \( n \in \mathbb{N} \). □

**Lemma 3.4.** Let \( \{K_n\}_{n=1}^{\infty} \subseteq B^* \) be a sequence of connected compact sets, and \( K_n \xrightarrow{\delta} K \). Then \( K \) is also a connected compact set.

**Proof.** By Lemma 2.4, \( K \) is compact. We need only show that \( K \) is connected as well. Otherwise, there exist at least two subsets of \( K \) that are disconnected. Denote one of them as \( K_1 \), and \( K_2 = K \setminus K_1 \). Then both \( K_1 \) and \( K_2 \) are compact and nonempty, and \( l_0 = \text{dist}(K_1, K_2) > 0 \).

By Remark 2.1, there exists an \( n_1 > 0 \) such that for all \( n \geq n_1 \), we have

\[ K_n \subseteq \bigcup_{z \in K} U \left( z, \frac{l_0}{4} \right) = \bigcup_{z \in K_1} U \left( z, \frac{l_0}{4} \right) \bigcup \bigcup_{z \in K_2} U \left( z, \frac{l_0}{4} \right). \]

Since \( K_n \) is connected, it must have

\[ K_n \subseteq \bigcup_{z \in K_1} U \left( z, \frac{l_0}{4} \right) \quad \text{or} \quad K_n \subseteq \bigcup_{z \in K_2} U \left( z, \frac{l_0}{4} \right). \]

On the other hand, since \( K_n \xrightarrow{\delta} K \), there exists an \( n_2 > 0 \) such that for all \( m, n \geq n_2 \),

\[ \delta(K_m, K_n) < \frac{l_0}{4}, \]

i.e.,

\[ K_m \subseteq \bigcup_{z \in K_n} U \left( z, \frac{l_0}{4} \right). \]

Take \( n_0 = \max \{n_1, n_2\} \). From (3.4) we may suppose \( K_0 \subseteq \bigcup_{z \in K_1} U \left( z, \frac{l_0}{4} \right) \), and by
(3.5), for all \( n \geq n_0 \), we have

\[
K_n \subset \bigcup_{z \in K_{n_0}} U \left( z; \frac{\delta}{4} \right) \subset \bigcup_{z \in K_1} U \left( z; \frac{\delta}{2} \right).
\]

Since \( \bigcup_{z \in K_1} U \left( z; \frac{\delta}{4} \right) \cap \bigcup_{z \in K_2} U \left( z; \frac{\delta}{4} \right) = \emptyset \), we have for all \( n \geq n_0 \) that \( K_n \subset \bigcup_{z \in K_1} U \left( z; \frac{\delta}{4} \right) \), i.e., \( \delta(K_n, K_2) \geq \frac{\delta}{4} \) for all \( n \geq n_0 \). By Lemma 2.1, \( K_2 = \emptyset \). This contradicts the assumption. The proof is complete.

**Lemma 3.5.** Let \( \Omega \) be a bounded open set and \( d(K) = \text{dist}(K, \partial \Omega) \), where \( K \subset \Omega \) is a compact set. Then for all compact sets \( K_1, K_2 \subset \Omega \), it has

\[
|d(K_1) - d(K_2)| \leq \delta(K_1, K_2).
\]

**Proof.** We may assume without loss of generality that \( d(K_1) \leq d(K_2) \). So we only need to show that \( d(K_2) \leq d(K_1) + \delta(K_1, K_2) \). Since \( K_1, \partial \Omega \) are compact, there exist \( x_1 \in K_1, y_1 \in \partial \Omega \) such that \( d(K_1) = |x_1 - y_1| \). Note that \( \text{dist}(x_1, K_2) \leq \text{dist}(x_1, K_1, K_2) \leq \delta(K_1, K_2) \). So \( d(K_2) = \text{dist}(K_2, \partial \Omega) \leq \text{dist}(y_1, K_2) \leq \text{dist}(x_1, K_2) + |x_1 - y_1| \leq \delta(K_1, K_2) + d(K_1) \). \( \square \)

**Lemma 3.6.** Let \( K, K_n, n \in \mathbb{N} \) be compact sets in \( B^* \), and \( K \cup K_n \xrightarrow{\delta} C \). If \( \text{dist}(K, K_n) \geq \varepsilon_0 > 0 \) for some \( \varepsilon_0 \), then \( C \setminus K \) is a compact set, dist\((K, C \setminus K) > 0 \), and \( K_n \xrightarrow{\delta} C \setminus K \), where dist\((K, D) = \inf_{x \in K, y \in D} |x - y| \) for any given set \( D \).

**Proof.** Since \( K \subset K \cup K_n \), by Lemma 2.2, it has \( K \subset C \).

If dist\((K, C \setminus K) = 0 \), then there exist an \( x \in K \) and \( \{x_n\} \subset C \setminus K \) such that \( x_n \to x \) as \( n \to \infty \). For every \( x_n \in C \setminus K \), since dist\((x_n, K) > 0 \), it follows from Lemma 2.1 that there exists a sequence \( \{x_{nm}\} \) such that \( x_{nm} \in K_m \) and \( x_{nm} \to x_n \) as \( m \to \infty \). For every \( n \in \mathbb{N} \), we can take \( x_{nm_n} \in K_{m_n} \) with \( m_n \geq m_n - 1 \) such that \( |x_{nm_n} - x_n| \leq \frac{\delta}{2} \), and hence we can find a subsequence \( \{x_{nm_n}\}_{n=1}^{\infty} \) of \( \{x_{nm}\}_{n=1}^{\infty} \) such that \( x_{nm_n} \in K_{m_n} \), and \( x_{nm_n} \to x \) as \( n \to \infty \), which contradicts the assumption that dist\((K, K_{m_n}) \geq \varepsilon_0 > 0 \). Hence dist\((K, C \setminus K) > 0 \). Since \( C \) is compact, by Lemma 2.4, \( C \setminus K \) is compact as well. Finally, by Lemma 2.1, we have \( K_n \xrightarrow{\delta} C \setminus K \). \( \square \)

**Proof of Theorem 2.8.** For every given \( i \in \{1, 2, 3\} \), by Lemma 2.4 there exist an \( \Omega \subset B^* \) and a subsequence of \( \{\Omega_m\}_{m=1}^{\infty} \), still denoted by itself without confusion, such that \( \Omega_m \xrightarrow{\delta} \Omega \). So we need only show that \( \Omega \in \mathcal{C}_i \) for each \( i \).

(i) \( i = 1 \). Let \( x \in \partial \Omega \). It follows from Lemma 3.3 that there exists a sequence \( \{x_{nm}\}_{n=1}^{\infty} \) with \( x_{nm} \in \partial \Omega_m \) for all \( n \in \mathbb{N} \) such that \( x_{nm} \to x \) as \( n \to \infty \). We may denote this \( \{x_{nm}\}_{n=1}^{\infty} \).\( \{x_{n}\}_{n=1}^{\infty} \) without loss of generality. Hence \( x_n \in \partial \Omega_n \) for every \( n \in \mathbb{N} \) and \( x_n \to x \) as \( n \to \infty \). Since \( \Omega_n \) satisfies the uniformly interior ball condition for every \( n \in \mathbb{N} \), we can find \( y_n \in \Omega_n \) with \( U(y_n, r_0) \subset \Omega_n \) and \( x_n \in \partial U(y_n, r_0) \). By Lemma 3.1, there exists a subsequence of \( \{U(y_n, r_0)\}_{n=1}^{\infty} \), still denoted by itself without confusion, and a \( y \in \mathbb{R}^N \) such that \( y_n \to y \), and \( U(y_n, r_0) \xrightarrow{\delta} U(y, r_0) \). Since \( B^* \setminus \Omega_n \subset B^* \setminus U(y_n, r_0) \), it follows from Lemma 2.2 that \( U(y, r_0) \subset \Omega \). This shows that \( \Omega \) is not empty. Now it suffices to show that \( x \in \partial U(y, r_0) \).

If \( x \notin B(y, r_0) \), then \( \varepsilon_0 = \text{dist}(x, B(y, r_0)) > 0 \). So \( r_0 < \varepsilon_0 + r_0 = |x - y| \leq |x - x_n| + |x_n - y_n| + |y_n - y| = |x - x_n| + r_0 + |y_n - y| \to r_0 \) as \( n \to \infty \). This contradiction completes the proof of (i).

(ii) \( i = 2 \). By the definition of \( \mathcal{C}_2 \), there exists a \( U(x_n, R) \subset \Omega_n \) for every \( n \in \mathbb{N} \). It then follows from Lemmas 2.4 and 3.1 that there exists an \( x \in \Omega \) and a subsequence of \( \{U(x_n, R)\} \), still denoted by itself, such that \( U(x_n, R) \xrightarrow{\delta} U(x, R) \) and \( x_n \to x \) as
$n \to \infty$. Let $\tilde{K}_n = \overline{B}^c \setminus U(x_n, R), \bar{K} = \overline{B}^c \setminus U(x, R); K_n = \overline{B}^c \setminus \Omega_n, K = \overline{B}^c \setminus \Omega$. Then $\delta(K_n, K) \to 0$ and $\delta(\bar{K}_n, \bar{K}) \to 0$. By Lemma 2.2, $U(x, R) \subset \Omega$, which shows that $\Omega$ is not empty.

Using Lemma 3.2 and the same argument in the proof of case (i), we can prove that $(C_2, \rho)$ is a compact metric space.

(iii) $i = 3$. The proof of $\Omega \in C_3$ will be split into several steps.

Step 1. $\Omega$ is not empty. This is the same as that in the proof of case (ii).

Step 2. $\Omega$ is connected.

Suppose that this is false. Then there exist at least two nonempty open subsets $\Omega_1, \Omega_2$ such that $\Omega = \Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$. Since $\Omega_1, \Omega_2$ are open sets, there exist two points $x_1 \in \Omega_1, x_2 \in \Omega_2$ and a $d > 0$ such that $U(x_1, d) \subset \Omega_1$ and $U(x_2, d) \subset \Omega_2$, here $d^* = \min\{\text{dist}(x_1, \partial \Omega_1), \text{dist}(x_2, \partial \Omega_2)\}$. For any $0 < r_0 < d^*$, by Lemma 2.3, there exists a $N_0 > 0$ such that for all $n \geq N_0$ it has $B(x_1, r_0) \cup B(x_2, r_0) \subset \Omega_n$, in particular, $U(x_1, r_0) \cup U(x_2, r_0) \subset \Omega_n$. By the definition of $C_3$, for every $n \geq N_0$ there exists a connected compact set $K_n$ such that $x_1, x_2 \in K_n$ and $\cup_{z \in K_n} U(z, \frac{d^*}{2}) \subset \Omega_n$ for some $M > 0$. Because $\Omega_1 \cap \Omega_2 = \emptyset$, $K_n \not\subset \Omega_1 \cup \Omega_2$. So there exist some $y_n \in K_n \setminus \Omega$ for all $n \geq N_0$. We thus get a sequence of points $\{y_n\}_{n=N_0}^\infty, \{y_n\}_{n=N_0}^\infty \cap \Omega = \emptyset$. Since $\{y_n\}_{n=N_0}^\infty \subset B^*$ and $B^*$ is bounded, there exists a subsequence of $\{y_n\}_{n=N_0}^\infty$, still denoted by itself, and a $y \not\in \Omega$ such that $y_n \to y$ as $n \to \infty$.

Now consider the ball $U(y, \frac{d^*}{2})$. Since $y_n \to y$ as $n \to \infty$, there exists a $N_1 > 0$ such that for all $n \geq N_1$, it has $|y_n - y| < \frac{r_0}{2M}$. On the other hand, by $\cup_{z \in K_n} B(z, \frac{d^*}{2}) \subset \Omega_n$, it has $U(y_n, \frac{d^*}{2}) \subset \Omega_n$. Hence $U(y, \frac{d^*}{2}) \subset \Omega_n$ for all $n \geq N_2 = \max\{N_0, N_1\}$. With the same argument as that in Step 1 or by Lemma 2.2 directly, we have $U(y, \frac{d^*}{2}) \subset \Omega$, and hence $U(y, \frac{d^*}{2}) \subset \Omega$ by letting $r_0 \to d^*$, which contradicts the fact $y \not\in \Omega$. This shows $\Omega$ is connected.

Step 3. $\Omega \in C_3$. To do this we only need to prove that $\Omega$ has the property $(C_M)$.

For any $x, y \in \Omega$, set

$$d_0 = d_0(x, y) = \sup\{d(K) | K \text{ is connected compact set and } x, y \in K, \cup_{z \in K} U(z, d) \subset \Omega\}.$$ 

Since $\Omega$ is an open set, $\Omega$ is locally path connected. Furthermore, since $\Omega$ is connected claimed by Step 2, it follows that $\Omega$ is a path connected set by the point topological theory. Hence for any $x, y \in \Omega$, there exists at least one continuous path $f : [0, 1] \to \Omega$ such that $f(0) = x, f(1) = y$. This shows that $d_0 = d_0(x, y)$ does exist.

Choose a connected compact sequence $\{K_n\}_{n=1}^\infty$ such that $d(K_n) \to d_0$ as $n \to \infty$. Since $K_n \subset B^*, n \in \mathbb{N}$, by Lemma 2.4, there exist a set $K$ and a subsequence of $\{K_n\}_{n=1}^\infty$, still denoted by itself, such that $\delta(K_n, K) \to 0$. By Lemma 3.5, we know that $d(K)$ is a continuous function of $K$ under the Hausdorff distance. Hence $d(K) = d_0$.

It follows from Lemma 3.4 that $K$ is a connected compact set, and by Lemma 2.1 $x, y \in K$. We claim that $\cup_{z \in K} U(z, d_0) \subset \Omega$.

Actually, if this is false, then there exists a point $x_0 \in [\cup_{z \in K} U(z, d_0)] \setminus \Omega$. We assume $\text{dist}(x_0, K) = |x_0 - z_0|$ with $z_0 \in K$ and denote $\varepsilon_0 = d_0 - |x_0 - z_0| > 0$. Then $U(x_0, \frac{d^*}{2}) \subset \cup_{z \in K} U(z, d_0 - \frac{d^*}{2})$. By Remark 2.1 $K \subset \cup_{z \in K_n} U(z, \frac{d^*}{2})$ for all sufficiently large $n$, we have $U(x_0, \frac{d^*}{2}) \subset \cup_{z \in K_n} U(z, d_0 - \frac{d^*}{2}) \subset \cup_{z \in K_n} U(z, d_0) \subset \Omega_n$ for $d_0 - d(K_n) < \frac{d^*}{2}$. With the same argument as that in Step 1, we obtain $U(x_0, \frac{d^*}{2}) \subset \Omega$. This contradicts the assumption that $x_0 \not\in \Omega$. So $\cup_{z \in K} U(z, d_0) \subset \Omega$.

Now we show that $d_0 \geq \frac{d^*}{2}$, where $d^* = \min\{\text{dist}(x, \partial \Omega), \text{dist}(y, \partial \Omega)\}$.

Since $B(x, r), B(y, r) \subset \Omega$ for any given $0 < r < d^*$, there exists a $N(r) > 0$ such that for all $n \geq N(r)$ it has $B(x, r), B(y, r) \subset \Omega_n$. Since $\Omega_n \in C_3$, there exists
$Q_n \subset \Omega_n$, $Q_n$ is a connected compact set such that $\cup_{z \in Q_n} U(z, \frac{r}{M}) \subset \Omega_n$ for all $n \geq N(r)$. By Lemma 2.4, there exist a set $Q$ and a subsequence of $\{Q_n\}_{n=N(r)}^\infty$ still denoted by itself, such that $\delta(Q_n, Q) \to 0$ as $n \to \infty$. By Lemma 3.4, we know that $Q$ is also a connected compact set. We claim that $\cup_{z \in Q_n} U(z, \frac{r}{M}) \subset \Omega$.

If this is false, then there exists an $x^* \in \left[\cup_{z \in Q} U(z, \frac{r}{M})\right] \setminus \Omega$. So there exists a $z^* \in Q$ such that $|x^* - z^*| = \text{dist}(x^*, Q)$. Denote $z^* = \frac{r}{M} - |x^* - z^*|$.

Then $U(x^*, \frac{r}{2M}) \subset \cup_{z \in Q} U(z, \frac{r}{M})$. By Remark 2.1, we know that $Q \subset \cup_{z \in Q_n} U(z, \frac{r}{M})$ for all $n \geq N^* = \max\{N(r), N(z^*)\}$, and hence $U(x^*, \frac{r}{M}) \subset \cup_{z \in Q_n} U(z, \frac{r}{M}) \subset \Omega_n$ for all $n \geq N^*$. With the same argument in Step 1, we have $U(x^*, \frac{r}{M}) \subset \Omega$. This contradicts the assumption that $x^* \notin \Omega$. So $\cup_{z \in Q} U(z, \frac{r}{M}) \subset \Omega$.

Since $0 < r < d^*$ is arbitrary, passing to the limit as $r \to d^*$ gives $\cup_{z \in Q} U(z, \frac{r}{M}) \subset \Omega$. By the definition of $d_0$, we have $d_0 \geq \frac{d^*}{M}$. This shows $\Omega \in \mathcal{C}_3$.  

**Lemma 3.7.** Let $\Omega \in \mathcal{C}_1 \cap \mathcal{C}_2$, where $\mathcal{C}_1, \mathcal{C}_2$ are defined by (2.1). Then for every $x_0 \in \partial \Omega$, there exists a straight line $L(x_0)$ passing through the point $x_0$ such that all centers of the exterior and interior balls at $x_0$ are lying in this line. Moreover, $L(x_0)$ is unique.

**Proof.** Since $x_0 \in \partial \Omega$, and $\Omega \in \mathcal{C}_1 \cap \mathcal{C}_2$, there exist interior ball $U(y_0, r_0) \subset \Omega$ with $x_0 \in U(y_0, r_0)$ and exterior ball $B(z_0, r_0) \subset B^* \setminus \Omega$ with $x_0 \in \partial B(z_0, r_0)$. So $B(y_0, r_0) \cap B(z_0, r_0) = \{x_0\}$. This shows that $x_0, y_0, z_0$ lie in a straight line. Denote this line as $L(x_0)$. For any interior ball $U(y_1, r) \subset \Omega$ with $x_0 \in \partial U(y_1, r)$, we must have $B(y_1, r) \cap B(z_0, r_0) = \{x_0\}$, and hence $y_1 \in L(x_0)$. In the same way, for any exterior ball $B(z_1, r)$, we have $z_1 \in L(x_0)$. We have thus proved the uniqueness of $L(x_0)$.

For any $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, in what follows we write $x' = (x_1, \ldots, x_{N-1})$.

Take $x_0 = (x_0, 0, \ldots, 0) \in \partial \Omega$, where $\Omega \in \mathcal{C}_1 \cap \mathcal{C}_2$. By the definitions of $\mathcal{C}_1$ and $\mathcal{C}_2$, there exist $y_0, z_0 \in \mathbb{R}^N$ such that $U(y_0, r_0) \subset \Omega$, $B(z_0, r_0) \subset B^* \setminus \Omega$, and $x_0 \in \partial U(y_0, r_0), x_0 \in \partial B(z_0, r_0)$. By Lemma 3.7, we may assume that $x_0 = 0$ and $L(x_0)$ is the $x_N$-axis upon rotating and relabeling the coordinate axes if necessary, such that $x_0 = (0, 0), y_0 = (0, r_0)$, and $z_0 = (0, -r_0)$. We get the following lemma.

**Lemma 3.8.** For any $x' \in U(0, \frac{r_0}{2}) \subset \mathbb{R}^{N-1}$, the line $L(x') = \{(x', s); s \in \mathbb{R}\}$ intersects with the set $\left[U(0, \frac{r_0}{2}) \times (-\frac{r_0}{2}, \frac{r_0}{2})\right] \cap \partial \Omega$ at only one point.

**Proof.** Suppose the conclusion is false. Then there exists a $\xi' \in U(0, \frac{r_0}{10}) \subset \mathbb{R}^{N-1}$ such that $(\xi', \xi_1' \xi_N') \in \mathcal{L}_{C} \cap U(0, \frac{r_0}{2}) \times (-\frac{r_0}{2}, \frac{r_0}{2}) \cap \partial \Omega$. We assume without loss of generality that $\xi_1' > 2r_0$. Since $(\xi', \xi_1') \in \partial \Omega$, by Lemma 3.7 there exists a unique line $L((\xi', \xi_1'))$ such that $L(\xi', \xi_1') \subset \Omega$ is an interior ball with $\zeta = (\xi', \xi_N') \in L((\xi', \xi_1'))$ and $(\xi', \xi_1') \in \partial U(\xi, r_0)$. Thus we can take the center of the exterior ball to be $(2\xi' - \xi_1', 2\xi_N' - \xi_N)$. Since $(\xi', \xi_1') \notin U(y_0, r_0)$, it has $|\xi'|^2 + |\xi_1' - r_0|^2 \geq r_0^2$, and so $(\xi_1')^2 - 2r_0 \xi_N + |\xi'|^2 \geq 0$. This implies

$$\xi_N \geq r_0 + \sqrt{r_0^2 - |\xi'|^2} \quad \text{or} \quad \xi_N \leq r_0 - \sqrt{r_0^2 - |\xi'|^2}.$$  

By $-\frac{r_0}{2} < \xi_N < \frac{r_0}{2}$, we get

$$\xi_N \leq r_0 - \sqrt{r_0^2 - |\xi'|^2}.$$  

On the other hand, by $(\xi', \xi_1') \notin U(z_0, r_0)$, it has $|\xi'|^2 + |\xi_1' + r_0|^2 \geq r_0^2$. Similar to (3.6), we have

$$\xi_1' \geq -r_0 + \sqrt{r_0^2 - |\xi'|^2}.$$
This together with (3.6) shows that

\begin{equation}
- r_0 + \sqrt{r_0^2 - |\xi'|^2} \leq \xi_N^1 \leq r_0 - \sqrt{r_0^2 - |\xi'|^2}.
\end{equation}

Similarly, we also get

\begin{equation}
- r_0 + \sqrt{r_0^2 - |\xi'|^2} \leq \xi_N^2 \leq r_0 - \sqrt{r_0^2 - |\xi'|^2}.
\end{equation}

Combining (3.8) and (3.9) gives

\begin{equation}
\frac{\xi_N^1 - \xi_N^2}{2} \leq r_0 - \sqrt{r_0^2 - |\xi'|^2}.
\end{equation}

Since \((\xi', \xi_N^1) \in \partial U(\zeta, r_0)\), it has

\begin{equation}
|\xi' - \zeta'|^2 + (\xi_N^1 - \zeta_N)^2 = r_0^2.
\end{equation}

By \((\xi', \xi_N^2) \not\in U(\zeta, r_0)\), we get

\begin{align*}
r_0^2 & \leq |\xi' - \zeta'|^2 + |\xi_N^2 - \zeta_N|^2 \\
& = |\xi' - \zeta'|^2 + |\xi_N^2 - \xi_N^1 + \xi_N^1 - \zeta_N|^2 \\
& = |\xi' - \zeta'|^2 + (\xi_N^2 - \xi_N^1)^2 + 2(\xi_N^2 - \xi_N^1)(\xi_N^1 - \zeta_N) + (\xi_N^1 - \zeta_N)^2 \\
& \text{by (3.11)} \\
& \leq r_0^2 + (\xi_N^2 - \xi_N^1)^2 + 2(\xi_N^2 - \xi_N^1)(\xi_N^1 - \zeta_N),
\end{align*}

which shows that

\begin{equation}
\xi_N^1 - \zeta_N \leq \frac{\xi_N^1 - \xi_N^2}{2}.
\end{equation}

On the other hand, by \((\xi', \xi_N^2) \not\in U((2\xi' - \zeta', 2\xi_N^1 - \zeta_N), r_0)\), it has \(|\xi' - \zeta'|^2 + |\xi_N^1 - \zeta_N + \xi_N^1 - \xi_N^2|^2 \geq r_0^2\), which implies

\begin{equation}
-(\xi_N^1 - \zeta_N) \leq \frac{\xi_N^1 - \xi_N^2}{2}.
\end{equation}

This together with (3.12) shows that

\begin{equation}
|\xi_N^1 - \zeta_N| \leq \frac{\xi_N^1 - \xi_N^2}{2}.
\end{equation}

Since \(0 \not\in U(\zeta, r_0)\), it has

\begin{align*}
r_0^2 & \leq |\xi'|^2 + |\zeta_N|^2 \\
& = |\xi' - \xi' + \zeta|'^2 + |\zeta_N - \xi_N^1 + \xi_N^1|^2 \\
& = |\xi' - \zeta'|^2 + 2\xi' \cdot (\xi' - \zeta) + |\zeta'|^2 + (\zeta_N - \xi_N^1)^2 + 2\xi_N^1(\zeta_N - \xi_N^1) + (\xi_N^1)^2 \\
& \text{by (3.11)} \\
& = r_0^2 + 2\xi' \cdot (\xi' - \zeta) + |\zeta'|^2 + 2\xi_N^1(\zeta_N - \xi_N^1) + (\xi_N^1)^2,
\end{align*}

where \(\xi' \cdot (\xi' - \zeta') = \sum_{i=1}^{N-1} \xi_i (\zeta_i - \xi_i)\). Therefore

\begin{equation}
2\xi' \cdot (\xi' - \zeta') \leq |\xi'|^2 + 2\xi_N^1(\zeta_N - \xi_N^1) + (\xi_N^1)^2.
\end{equation}
On the other hand, by $0 \notin U((2\xi' - \zeta', 2\xi^1_N - \zeta_N), r_0)$, it has

$$r_0^2 \leq |\xi' - \zeta' + \xi| + |\xi_N - \zeta_N + \xi_N^1|^2$$

$$= |\xi' - \zeta' + 2\xi' \cdot (\zeta' - \zeta') + |\xi'|^2 + (\xi_N^1 - \zeta_N)^2 + 2\xi_N^1(\xi_N^1 - \zeta_N) + (\xi_N^1)^2$$

by (3.11)

$$= r_0^2 + 2\xi' \cdot (\zeta' - \zeta') + |\xi'|^2 + 2\xi_N^1(\xi_N^1 - \zeta_N) + (\xi_N^1)^2,$$

which implies

(3.16) 

$$-2\xi' \cdot (\zeta' - \zeta') \leq |\xi'|^2 + 2\xi_N^1(\xi_N^1 - \zeta_N) + (\xi_N^1)^2.$$ 

The distance between the points $z_0 = (0, -r_0)$ and $\zeta = (\zeta', \zeta_N)$ can be estimated as follows:

$$|z_0 - \zeta|^2 = |\zeta'|^2 + |\zeta_N + r_0|^2$$

$$= |\zeta' - \zeta'|^2 + |\zeta_N - \xi_N^1 + r_0|^2$$

$$= |\zeta' - \zeta'|^2 + 2\xi' \cdot (\zeta' - \zeta') + |\xi'|^2 + |\zeta_N - \xi_N^1|^2$$

$$+ 2(\zeta_N - \xi_N^1)(\xi_N^1 + r_0) + |\xi_N + r_0|^2$$

by (3.11)

$$= r_0^2 + 2\xi' \cdot (\zeta' - \zeta') + |\xi'|^2 + 2(\zeta_N - \xi_N^1)(\xi_N^1 + r_0) + |\xi_N + r_0|^2$$

by (3.16)

$$\leq r_0^2 + |\xi'|^2 + 2\xi_N^1(\zeta_N - \zeta_N) + |\xi_N^1|^2 + |\xi'|^2$$

$$+ 2(\zeta_N - \xi_N^1)(\xi_N^1 + r_0) + |\xi_N + r_0|^2$$

$$= 2r_0^2 + 2|\xi'|^2 + 2|\xi_N^1|^2 + 2r_0\xi_N^1 + 2r_0(\zeta_N - \xi_N^1)$$

by (3.8), (3.10), and (3.14)

$$\leq 2r_0^2 + 2|\xi'|^2$$

$$+ 2(r_0 - \sqrt{r_0^2 - |\xi'|^2})^2 + 4r_0(r_0 - \sqrt{r_0^2 - |\xi'|^2})$$

(17)

$$< (2r_0)^2.$$ 

This lead to a contradiction that $U(\zeta, r_0) \cap U(z_0, r_0) \neq \emptyset$. The proof is complete.

Remark. 3.1. From Lemma 3.8, we see that there exists a function $f : U(0, \frac{r_0}{16}) \subset \mathbb{R}^N \rightarrow \mathbb{R}$, $x' \mapsto f(x')$ such that $f(0) = 0$ and $\{(x', f(x')) | x' \in U(0, \frac{r_0}{16})\} = [U(0, \frac{r_0}{16}] \times (-\frac{r_0}{16}, \frac{r_0}{16})] \cap \partial \Omega$.

Corollary 3.9. Let $f$ be given in Remark 3.1 and $U(\zeta, r_0) = U((\zeta', \zeta_N), r_0)$ be the interior ball at $(x', f(x')) \in \partial \Omega$ with $x' \in U(0, \frac{r_0}{16}) \subset \mathbb{R}^N$. Then

$$\zeta_N - f(x') > \frac{r_0}{2}.$$

Proof. Suppose it is false. Then there exists a $\zeta' \in U(0, \frac{r_0}{16})$ such that $(\zeta', f(\zeta')) \in \partial \Omega$, and

$$\zeta_N - f(x') \leq \frac{r_0}{2}.$$

In the same way as the proof of (3.17), we also get the contradiction that $U(\zeta, r_0) \cap U(z_0, r_0) \neq \emptyset$. This completes the proof.

Proof of Theorem 2.9. Take $x_0 = (x_0', x_0N) \in \partial \Omega$. By the definitions of $C_1$ and $C_2$, there exist $y_0, z_0 \in \mathbb{R}^N$ such that $U(y_0, r_0) \subset \Omega, B(z_0, r_0) \subset B^* \setminus \Omega$, and $x_0 \in \partial U(y_0, r_0), x_0 \in \partial B(z_0, r_0)$. By Lemma 3.7, we can assume that $x_0 = 0$, and $L(x_0)$ is the $x_0N$-axis upon rotating and relabeling the coordinate axes if necessary, such that we have $x_0 = (0, 0), y_0 = (0, r_0)$, and $z_0 = (0, -r_0)$. 

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By Remark 3.1, there exists a function $f : U(0, \frac{r_0}{16}) \subset \mathbb{R}^{N-1} \to \mathbb{R}$, $\xi' \to f(\xi')$ such that $f(0) = 0$ and $\{\xi', f(\xi') \mid \xi' \in U(0, \frac{r_0}{16})\} = \{U(0, \frac{r_0}{16}) \times (-\frac{r_0}{16}, \frac{r_0}{16}) \} \cap \partial \Omega$.

The proof of Theorem 2.9 will be split into several steps.

Step 1. $f$ is a continuous function. In fact, for any $\xi', \eta' \in U(0, \frac{r_0}{16})$, by Remark 3.1, $\xi', f(\xi'))$, $(\eta', f(\eta')) \in \partial \Omega$. So there exists an interior ball $U(\xi, \eta, r_0)$ at $(\xi', f(\xi'))$, and we can take the exterior ball to be $U((2\xi' - \xi, 2f(\xi') - \xi_N), r_0)$.

Thus
\begin{equation}
(3.18) \quad |\xi' - \eta'|^2 + |\xi_N - f(\eta')|^2 \geq r_0^2, \quad |\xi' - \xi_N|^2 - |2f(\xi') - \xi_N - f(\eta')|^2 \geq r_0^2.
\end{equation}

This together with the equality $|\xi' - \xi_N|^2 + |\xi_N - f(\xi')|^2 = r_0^2$ and the first inequality of (3.18) gives

$$
r_0^2 \leq |\xi' - \xi_0 + \xi' - \eta'|^2 + |\xi_N - f(\xi') + f(\xi') - f(\eta')|^2
$$
\begin{equation}
= |\xi' - \xi_N|^2 + 2(\xi' - \xi_N) \cdot (\xi' - \eta') + |\xi' - \eta'|^2
+ (\xi_N - f(\xi'))^2 + 2(\xi_N - f(\xi'))(f(\xi') - f(\eta')) + (f(\xi') - f(\eta'))^2
\end{equation}
\begin{equation}
= r_0^2 + (f(\xi') - f(\eta'))^2 + 2(\xi_N - f(\xi'))(f(\xi') - f(\eta')) + (f(\xi') - f(\eta'))^2.
\end{equation}

Since $|f(\xi') - f(\eta')| < \frac{r_0}{16}$, by Corollary 3.9, it must have
\begin{equation}
f(\xi') - f(\eta') \geq -\left((\xi_N - f(\xi')) + \sqrt{(\xi_N - f(\xi'))^2 - 2(\xi' - \xi_N) \cdot (\xi' - \eta') + |\xi' - \eta'|^2}\right)
- \frac{2(\xi' - \xi_N) \cdot (\xi' - \eta') + |\xi' - \eta'|^2}{(\xi_N - f(\xi')) + \sqrt{(\xi_N - f(\xi'))^2 - 2(\xi' - \xi_N) \cdot (\xi' - \eta') + |\xi' - \eta'|^2}}.
\end{equation}

Similarly, it has
\begin{equation}
f(\xi') - f(\eta') \leq (\xi_N - f(\xi')) - \sqrt{(\xi_N - f(\xi'))^2 - 2(\xi' - \xi_N) \cdot (\xi' - \eta') + |\xi' - \eta'|^2}
- \frac{-2(\xi' - \xi_N) \cdot (\xi' - \eta') + |\xi' - \eta'|^2}{(\xi_N - f(\xi')) + \sqrt{(\xi_N - f(\xi'))^2 - 2(\xi' - \xi_N) \cdot (\xi' - \eta') + |\xi' - \eta'|^2}}.
\end{equation}

So,
\begin{equation}
(3.19) \quad |f(\xi') - f(\eta')| \leq C|\xi' - \eta'|
\end{equation}
for some constant $C > 0$. This shows that $f$ is Lipschitz continuous in $U(0, \frac{r_0}{16})$.

In what follows, we use $C$ to denote constant depending on $r_0$ and $N$ only although it may vary in different context.

Step 2. $\frac{\partial f}{\partial x_i}(0)$ = 0 for every $i \in \{1, \ldots, N - 1\}$. Actually, by (3.8) we have
\begin{equation}
\left| \frac{f(0, \ldots, 0, x_i, \ldots, 0) - f(0)}{x_i - 0} \right| \leq \frac{|x_i|}{r_0 + \sqrt{r_0^2 - |x_i|^2}}.
\end{equation}

This shows that $\frac{\partial f}{\partial x_i}(0)$ exists and $\frac{\partial f}{\partial x_i}(0) = 0$ for every $i \in \{1, \ldots, N - 1\}$.

Step 3. For any $\xi' \in U(0, \frac{r_0}{16})$, we have
\begin{equation}
(3.20) \quad |f(\xi')| \leq C|\xi'|^2.
\end{equation}

Indeed, by (3.8) it has
\begin{equation}
|f(\xi')| \leq r_0 - \sqrt{r_0^2 - |\xi'|^2} = \frac{|\xi'|^2}{r_0 + \sqrt{r_0^2 - |\xi'|^2}} \leq C|\xi'|^2.
\end{equation}

Step 4. For any $\omega' \in U(0, \frac{r_0}{16})$, we have
\begin{equation}
(3.21) \quad \left| \frac{\partial f}{\partial \xi_j}(\omega') \right| \leq C, \quad \forall \ j \in \{1, \ldots, N - 1\}.
\end{equation}
Suppose that \( U(\zeta, r_0) \) is the interior ball at \( (\omega', f(\omega')) \). Then we may take the exterior ball as \( U((2\omega'-\zeta', 2f(\omega')-\zeta_N), r_0) \).

Since \( Q = (\omega', f(\omega')) \in \partial \Omega \), by a rotation and a translation, we can make, in a new coordinate, \( Q = (0, \ldots, 0, 0) \), the interior ball \( U((0, r_0), r_0) \), and the exterior ball \( U((0, r_0), r_0) \) and \( \{(\eta'; g(\eta')) \in \mathbb{R}^N | \eta' \in U(0, r_0) \cap \mathbb{R}^N \} = [U(0, r_0) \times (-\frac{3}{4}, \frac{3}{4})] \cap \partial \Omega \), where \( g \) is a function in the new coordinate defined as \( f \) in Corollary 3.9. After doing so, we have three facts: (i) \( Q \) becomes the origin; (ii) the vector \( (\zeta' - \omega', \zeta_N - f(\omega')) \) becomes \( (0, r_0) \); (iii) the function \( g \) has the same properties as \( f \).

Now we show that there exists an \( N \times N \) orthogonal matrix \( B_{N \times N} \) such that

\[
B_{N \times N}(\zeta' - \omega', \zeta_N - f(\omega')) = (0, r_0).
\]

Actually, since \( (\zeta' - \omega', \zeta_N - f(\omega')) \) is a vector of length \( r_0 \) in \( \mathbb{R}^N \), there exist \( N-1 \) number of orthogonal unit vectors \( \alpha_1, \ldots, \alpha_{N-1} \) in \( \mathbb{R}^N \) such that \( \alpha_1, \ldots, \alpha_{N-1} \) are orthogonal with \( (\zeta' - \omega', \zeta_N - f(\omega')) \). Take \( B_{N \times N} = (\alpha_1^\top, \ldots, \alpha_{N-1}^\top, (\zeta' - \omega', \zeta_N - f(\omega'))^\top) \); here \( \alpha_i^\top \) represents the transpose of the column vector \( \alpha_i \). In this way, \( B_{N \times N} \) is an orthogonal matrix: \( B_{N \times N}^{-1} = B_{N \times N}^\top \). Write \( B_{N \times N} = (b_{ij})_{i,j=1,\ldots,N} \). Then \( b_{NN} = \frac{\zeta_N - f(\omega')}{r_0} \). By Corollary 3.9, \( b_{NN} \neq 0 \).

By the definition of \( B_{N \times N} \) and (i), (ii), (iii), it is easy to see that

\[
B_{N \times N} ((\xi', f(\xi')) - (\omega', f(\omega'))) = (\eta', g(\eta')),
\]

where \( \xi' \) is in some neighborhood \( O(\omega') \) of \( \omega' \). This implies that

\[
\begin{align*}
\eta_1 &= \sum_{j=1}^{N-1} b_{1j}(\xi_j - \omega_j) + b_{1N}(f(\xi') - f(\omega')) , \\
\eta_2 &= \sum_{j=1}^{N-1} b_{2j}(\xi_j - \omega_j) + b_{2N}(f(\xi') - f(\omega')) , \\
\ldots \ldots . & \\
\eta_{N-1} &= \sum_{j=1}^{N-1} b_{N-1,j}(\xi_j - \omega_j) + b_{N-1,N}(f(\xi') - f(\omega')) , \\
g(\eta') &= \sum_{j=1}^{N-1} b_{N,j}(\xi_j - \omega_j) + b_{NN}(f(\xi') - f(\omega')).
\end{align*}
\]

Hence

\[
F(\xi') := \sum_{j=1}^{N-1} b_{N,j}(\xi_j - \omega_j) + b_{NN}(f(\xi') - f(\omega')) - g \left( \sum_{j=1}^{N-1} b_{1j}(\xi_j - \omega_j) + b_{1N}(f(\xi') - f(\omega')) , \ldots , \right. \\
\left. \sum_{j=1}^{N-1} b_{N-1,j}(\xi_j - \omega_j) + b_{N-1,N}(f(\xi') - f(\omega')) \right) = 0.
\]
Let \( e_j' = (0, \ldots, 0, t_j, 0, \ldots, 0) \in \mathbb{R}^{N-1}, \) \( j \in \{1, \ldots, N-1\}, \) be a vector with \( j \)th component being \( t_j, \) the others being \( 0, \) and \( \omega' + e_j' \in O(\omega'). \) Since \( F(\omega' + e_j') = 0, \) we get

\[
\begin{align*}
|b_{Nj}t_j + b_{NN}(f(\omega' + e_j') - f(\omega'))| &= \left| g(b_{1j}t_j + b_{1N}(f(\omega' + e_j') - f(\omega'))), \ldots, \\
& \quad b_{N-1,j}t_j + b_{N-1,N}(f(\omega' + e_j') - f(\omega'))) \right| \\
& \leq C \sum_{i=1}^{N-1} |b_{ij}t_j + b_{iN}(f(\omega' + e_j') - f(\omega'))|^2 \\
& \leq C \sum_{i=1}^{N-1} (|b_{ij}|^2 + |b_{iN}|^2)t_j^2 \\
& \leq Ct_j^2.
\end{align*}
\]

This shows

\[
\left| b_{Nj} + b_{NN} \frac{f(\omega' + e_j') - f(\omega')}{t_j} \right| \leq C|t_j|.
\]

Hence \( \frac{\partial f}{\partial \xi_j}(\omega') \) exists, and

\begin{equation}
\left(3.23\right)
\frac{\partial f}{\partial \xi_j}(\omega') = -\frac{b_{Nj}}{b_{NN}} \forall j = 1, \ldots, N - 1.
\end{equation}

Moreover, since \( b_{NN} = \frac{\omega_{N-1} - f(\omega')}{\omega_{N-1}} > \frac{1}{2} \) by virtue of Corollary 3.9, and \( \sum_{j=1}^{N} b_{Nj}^2 = 1, \) we finally arrive at (3.21).

**Step 5.** \( \frac{\partial f}{\partial \xi_j} : U(0, \frac{\pi}{4N})(\subset \mathbb{R}^{N-1}) \to \mathbb{R} \) is Lipschitz continuous for every \( j \in \{1, \ldots, N-1\}. \) Let the surface \( S : \xi_N - f(\xi') = 0, \) \( \xi' \in U(0, \frac{\pi}{4N}). \) For every \( j \in \{1, \ldots, N-1\}, \) take \( \varepsilon_j = \sqrt{(\frac{\omega_j}{2})^2 - \sum_{i \neq j} \omega_i^2} + \omega_j \) and \( \varepsilon_j' = \sqrt{(\frac{\omega_j}{2})^2 - \sum_{i \neq j} \omega_i^2} - \omega_j. \) Then \( \{\omega', e_j' | t_j \in (\varepsilon_j, \varepsilon_j')\} = \{x' \in U(0, \frac{\pi}{4N}) | x_i = \omega_i, i = 1, \ldots, N-1, i \neq j\}. \) Note that the curves \( \Gamma_j : \xi_N - f(\omega' + e_j') = 0, \) \( t_j \in (\varepsilon_j, \varepsilon_j') \) are in \( S \) passing through the point \( (\omega', f(\omega')). \) So the vector \( (0, \ldots, 0, 1, 0, \ldots, 0, \frac{\partial f}{\partial \xi_j}(\omega')) \in \mathbb{R}^N \) with \( j \)th component being 1 is the tangent vector of the curve \( \Gamma_j \) at \( (\omega', f(\omega')). \) Furthermore, the vector \( (0, \ldots, 0, 1, 0, \ldots, 0, \frac{\partial f}{\partial \xi_j}(\omega')) \) is orthogonal to the vector \( \left(0, r_0 \right) = B_{N \times N}(\zeta - (\omega', f(\omega'))). \) Hence

\[
\begin{align*}
0 &= \left[ B_{N \times N}(\zeta - (\omega', f(\omega'))) \right] \left[ B_{N \times N} \left(0, \ldots, 0, 1, 0, \ldots, 0, \frac{\partial f}{\partial \xi_j}(\omega')\right) \right] \\
&= (\zeta - (\omega', f(\omega'))) \cdot \left[ B_{N \times N}^\top B_{N \times N} \left(0, \ldots, 0, 1, 0, \ldots, 0, \frac{\partial f}{\partial \xi_j}(\omega')\right) \right] \\
&= (\zeta - (\omega', f(\omega'))) \cdot \left(0, \ldots, 0, 1, 0, \ldots, 0, \frac{\partial f}{\partial \xi_j}(\omega')\right).
\end{align*}
\]
That is, the vector \((\zeta - (\omega', f(\omega')))\) is orthogonal with the vectors \((0, \ldots, 0, 1, 0, \ldots, 0,\ \frac{\partial f}{\partial \xi_j}(\omega'))\), \(j = 1, \ldots, N - 1\). Thus

\[
(3.24) \quad \zeta - (\omega', f(\omega')) = \gamma \left( - \frac{\partial f}{\partial \xi_1}(\omega'), \ldots, - \frac{\partial f}{\partial \xi_{N-1}}(\omega'), 1 \right),
\]

where \(\gamma\) is a constant with \(|\gamma| = \frac{r_0}{\sqrt{\sum_{j=1}^{N-1} (\frac{\partial f}{\partial \xi_j}(\omega'))^2 + 1}} = r_0 b_{NN}\).

For any \(t_j \in (0, \varepsilon_j^2)\), since

\[
|\zeta' - \omega' - e_j'|^2 + |\zeta_N - f(\omega' + e_j')|^2 \geq r_0^2,
\]

by \(\zeta' - \omega'|^2 + |\zeta_N - f(\omega')|^2 = r_0^2\) and (3.24), it has

\[
0 \leq -2e_j' \cdot (\zeta' - \omega') + t_j^2 + 2(\zeta_N - f(\omega'))(f(\omega') - f(\omega' + e_j')) + (f(\omega') - f(\omega' + e_j'))^2
= 2\gamma t_j \frac{\partial f}{\partial \xi_j}(\omega') + t_j^2 + 2\gamma(f(\omega') - f(\omega' + e_j')) + (f(\omega') - f(\omega' + e_j'))^2,
\]

and hence

\[
(3.25) \quad -2\gamma \left[ \frac{f(\omega' + e_j') - f(\omega')}{t_j} - \frac{\partial f}{\partial \xi_j}(\omega') \right] \leq t_j + \frac{(f(\omega' + e_j') - f(\omega'))^2}{t_j}.
\]

On the other hand, since \(U((2\omega' - \zeta', 2f(\omega') - \zeta_N), r_0)\) is the exterior ball, it has

\[
|2\omega' - \zeta' - \omega' - e_j'|^2 + |2f(\omega') - \zeta_N - f(\omega' + e_j')|^2 \geq r_0^2.
\]

In the same way, we can also get

\[
(3.26) \quad -2\gamma \left[ \frac{f(\omega' + e_j') - f(\omega')}{t_j} - \frac{\partial f}{\partial \xi_j}(\omega') \right] \leq t_j + \frac{(f(\omega' + e_j') - f(\omega'))^2}{t_j}.
\]

By virtue of (3.25) and (3.26), we have

\[
(3.27) \quad \left| \frac{f(\omega' + e_j') - f(\omega')}{t_j} - \frac{\partial f}{\partial \xi_j}(\omega') \right| \leq \frac{1}{2\gamma} \left[ t_j + \left| \frac{f(\omega' + e_j') - f(\omega')}{t_j} \right|^2 \right] - \frac{1}{2\gamma} \left[ t_j + \left| \frac{\partial f}{\partial \xi_j}(\omega' + \beta) \right|^2 t_j \right],
\]

By the mean value theorem, there is a \(\beta = (0, \ldots, 0, s_j, 0, \ldots, 0)\), \(s_j \in (0, t_j)\) such that

\[
|f(\omega' + e_j') - f(\omega')| = \left| \frac{\partial f}{\partial \xi_j}(\omega' + \beta) \right| t_j.
\]

This together with (3.27) gives

\[
\left| \frac{f(\omega' + e_j') - f(\omega')}{t_j} - \frac{\partial f}{\partial \xi_j}(\omega') \right| \leq \frac{1}{2\gamma} \left[ t_j + \left| \frac{f(\omega' + e_j') - f(\omega')}{t_j} \right|^2 \right] - \frac{1}{2\gamma} \left[ t_j + \left| \frac{\partial f}{\partial \xi_j}(\omega' + \beta) \right|^2 t_j \right]
\leq \frac{1}{2\gamma} \left[ t_j + \left| \frac{\partial f}{\partial \xi_j}(\omega' + \beta) \right|^2 t_j \right]
\leq C|t_j|.
\]
The same result holds for any $t_j \in (-\varepsilon_j^1, 0)$. So, for any $t_j \in (-\varepsilon_j^1, \varepsilon_j^2)$,

$$\left| \frac{f(\omega' + \varepsilon_j') - f(\omega')}{t_j} - \frac{\partial f}{\partial \xi_j}(\omega') \right| \leq C|t_j|.$$ 

Thus

$$\left| \frac{\partial f}{\partial \xi_j}(\omega' + \varepsilon_j') - \frac{\partial f}{\partial \xi_j}(\omega') \right| \leq \left| \frac{\partial f}{\partial \xi_j}(\omega' + \varepsilon_j') - \frac{f(\omega' + \varepsilon_j') - f(\omega')}{t_j} \right|$$

$$+ \left| \frac{f(\omega' + \varepsilon_j') - f(\omega')}{t_j} - \frac{\partial f}{\partial \xi_j}(\omega') \right| \leq C|t_j|.$$

(3.28)

For any $\omega', \eta' \in U(0, \frac{r_0}{64N})$, and $k \in \{1, \ldots, N-1\}$, it must have

$$||(\eta_1, \ldots, \eta_k, \omega_k, \omega_{k+1}, \ldots, \omega_{N-1}) - (\eta_1, \ldots, \eta_k, \omega_k, \omega_{k+1}, \ldots, \omega_{N-1})||$$

$$= \sqrt{\sum_{i=1}^{k+1} \eta_i^2 + \eta_k^2} + \sum_{i=k+1}^{N-1} \omega_i^2 \leq \sqrt{\left(\frac{r_0}{64N}\right)^2 + \left(\frac{r_0}{32N}\right)^2 + \left(\frac{r_0}{64N}\right)^2}$$

$$= \frac{\sqrt{6}r_0}{64N}.$$

This implies

$$||(\eta_1, \ldots, \eta_k, \omega_k, \omega_{k+1}, \ldots, \omega_{N-1})||$$

$$\leq ||(\eta_1, \ldots, \eta_k, \omega_k, \omega_{k+1}, \ldots, \omega_{N-1}) - (\eta_1, \ldots, \eta_k, \omega_k, \omega_{k+1}, \ldots, \omega_{N-1})||$$

$$+ ||(\eta_1, \ldots, \eta_{k-1}, \omega_{k-1}, \omega_{k+1}, \ldots, \omega_{N-1}) - (\eta_1, \ldots, \eta_{k-1}, \omega_{k-1}, \omega_{k+1}, \ldots, \omega_{N-1})||$$

$$+ \cdots + ||(\eta_1, \omega_2, \ldots, \omega_{N-1}) - \omega' + |\omega'||$$

$$\leq k \frac{\sqrt{6}r_0}{64N} + \frac{r_0}{64N} \leq \frac{r_0}{16}.$$ 

Thus $(\eta_1, \ldots, \eta_k, \omega_k, \omega_{k+1}, \ldots, \omega_{N-1}) \in U(0, \frac{r_0}{16})$. This together with (3.28) gives

$$\left| \frac{\partial f}{\partial \xi_j}(\omega') - \frac{\partial f}{\partial \xi_j}(\eta') \right| \leq \left| \frac{\partial f}{\partial \xi_j}(\omega') - \frac{\partial f}{\partial \xi_j}(\eta_1, \omega_2, \ldots, \omega_{N-1}) \right|$$

$$+ \left| \frac{\partial f}{\partial \xi_j}(\eta_1, \omega_2, \ldots, \omega_{N-1}) - \frac{\partial f}{\partial \xi_j}(\eta_1, \eta_2, \omega_3, \ldots, \omega_{N-1}) \right|$$

$$+ \cdots + \left| \frac{\partial f}{\partial \xi_j}(\eta_1, \ldots, \eta_{N-2}, \omega_{N-1}) - \frac{\partial f}{\partial \xi_j}(\eta') \right|$$

$$\leq C|\omega_1 - \eta_1| + \cdots + C|\omega_{N-1} - \eta_{N-1}|$$

$$\leq C|\omega' - \eta'|,$$

which shows that $\frac{\partial f}{\partial \xi_j} : U(0, \frac{r_0}{64N}) \subset \mathbb{R}^N \to \mathbb{R}$ is Lipschitz continuous for every $j \in \{1, \ldots, N-1\}$.

**Step 6.** $\Omega \cap [U(0, \frac{r_0}{64N}) \times (-\frac{r_0}{4}, \frac{r_0}{4})] = \{ \xi \in \mathbb{R}^N \mid f(\xi') < \xi_N \} \cap [U(0, \frac{r_0}{64N}) \times (-\frac{r_0}{4}, \frac{r_0}{4})]$. If the claim is not true, then there exists a $\xi = (\xi', \xi_N) \in \mathbb{R}^N$ such that
\[ f(\xi') < \xi_N \text{ and } \xi \notin \Omega. \] Let \( U(\xi, r_0) \) be an interior ball at \( (\xi', f(\xi')) \in \partial \Omega. \) By Corollary 3.9, there exists an \( \varepsilon_0 > 0 \) such that \( \{ (\xi', t) | f(\xi') < t < f(\xi') + \varepsilon \} \subset U(\xi, r_0) \subset \Omega. \) We thus have a point \( (\xi', \xi_N) \in \partial \Omega \) with \( f(\xi') < \xi_N. \) This contradicts Lemma 3.8 since \( (\xi', f(\xi')) \in \partial \Omega \) and \( (\xi', \xi_N) \in \partial \Omega. \)

Finally, define

\[
\Psi : U \left( 0, \frac{r_0}{64N} \right) (\subset \mathbb{R}^N) \rightarrow D = \Psi \left( U \left( 0, \frac{r_0}{64N} \right) \right) (\subset \mathbb{R}^N),
\]

\[
\xi = (\xi', \xi_N) \mapsto y = (\xi', \xi_N - f(\xi')).
\]

By the definition of \( C^{1,1} \) \( [9, \text{p. 94}] \), the proof is ended if we can show that \( \Psi \) is one-to-one, and

\[
\begin{align*}
(i) & \quad \Psi \left( U \left( 0, \frac{r_0}{64N} \right) \cap \Omega \right) \subset \mathbb{R}^N; \\
(ii) & \quad \Psi \left( U \left( 0, \frac{r_0}{64N} \right) \cap \partial \Omega \right) \subset \partial \mathbb{R}^N; \\
(iii) & \quad \Psi \in C^{1,1} \left( \left( U \left( 0, \frac{r_0}{64N} \right) \right) \cap \Omega \right), \quad \Psi^{-1} \in C^{1,1} (D).
\end{align*}
\]

Obviously, \( \Psi \) is one-to-one. By Steps 1 through 6, we see that \( \Psi \) satisfies (i), (ii), and the first part of (iii). Since \( \Psi^{-1}(y) = (y', y_N + f(y')) \), the second part of (iii) is also true by Steps 1 through 6. This completes the proof. \( \square \)

\textbf{Proof for Example 2.2.} Note that for any \( r_0 \leq 1 \) and any \( x \in \partial \Omega \), there exist the interior ball \( U(y, r_0) \subset \Omega \) with \( x \in \partial U(y, r_0) \) and the exterior ball \( B(y, r_0) \subset B^* \setminus \Omega \) with \( x \in \partial B(y, r_0) \). \( \Omega \in C^{1,1} \) has been proved in Theorem 2.9. So we only need to show \( \Omega \notin C^2 \). Since \( (1, 1) \in \partial \Omega \), define the function \( f : (0, 2) \rightarrow \mathbb{R} \) by

\[
f(x) = \begin{cases} 1, & x \in (0, 1]; \\ \sqrt{1 - (x-1)^2}, & x \in [1, 2). \end{cases}
\]

Then \( f''(1) = 0 \), but \( f''''(1) = -1 \). This shows that \( f''''(1) \) does not exist. \( \square \)

\textbf{Lemma 3.10.} Let \( \Omega \subset \mathbb{R}^N \) and \( \Omega \in \mathcal{C}_1 \cap \mathcal{C}_2 \). Then for any \( x \in \Omega \), there exists a ball \( U(y, \frac{r_0}{2}) \) such that \( x \in U(y, \frac{r_0}{2}) \subset \Omega \), and for any \( x \in B^* \setminus \Omega \), there exists a ball \( B(y, r_0) \) such that \( x \in B(y, r_0) \subset B^* \setminus \Omega \). Moreover, for any \( x \in (B^* \setminus \Omega)^\circ \), there exists a ball \( U(y_1, \frac{r_0}{2}) \) such that \( x \in U(y_1, \frac{r_0}{2}) \subset (B^* \setminus \Omega)^\circ \), where \( (B^* \setminus \Omega)^\circ \) is the interior of the set \( B^* \setminus \Omega \).

\textbf{Proof.} Let

\[ \Lambda = \{ r > 0 | \text{ there exists a ball } U(\tilde{y}, r) \text{ such that } x \in U(\tilde{y}, r) \subset \Omega \}. \]

Obviously \( \Lambda \neq \emptyset \). Take \( b_0 = \sup \Lambda \), and \( \{ r_n \} \subset \Lambda \) with \( r_n \rightarrow b_0 \) as \( n \rightarrow \infty \). Then there exists a sequence \( \{ y_n \} \subset \Omega \) with \( x \in U(y_n, r_n) \subset \Omega \) for all \( n \in \mathbb{N} \). Since \( \Omega \) is bounded, there exists a subsequence of \( \{ y_n \} \), still denoted by itself, and a \( \tilde{y} \in \mathbb{R}^N \) such that \( y_n \rightarrow \tilde{y} \) as \( n \rightarrow \infty \). By Lemmas 2.4 and 3.1, there exists a subsequence of \( \{ U(y_n, r_n) \} \), still denoted by itself, such that \( U(y_n, r_n) \xrightarrow{n\to\infty} U(\tilde{y}, b_0) \). Furthermore, Lemma 2.2 tells us that \( U(y_n, r_n) \subset \Omega \). On the other hand, since \( x \in U(y_n, r_n) \), it has \( |x - y_n| < r_n \), and hence \( |x - \tilde{y}| \leq b_0 \) by letting \( n \rightarrow \infty \). In other words, \( x \in B(\tilde{y}, b_0) \).

If \( \partial U(\tilde{y}, b_0) \cap \partial \Omega = \emptyset \), then there exists a \( b_1 > b_0 \) such that \( x \in B(\tilde{y}, b_1) \subset U(\tilde{y}, b_1) \subset \Omega \) because \( U(\tilde{y}, b_0) \subset \Omega \). This contradicts the definition of \( b_0 \). In other words, \( \partial U(\tilde{y}, b_0) \cap \partial \Omega \neq \emptyset \).

Take \( z_0 \in \partial U(\tilde{y}, b_0) \cap \partial \Omega \). Then \( U(\tilde{y}, b_0) \) is an interior ball at \( z_0 \in \partial \Omega \). Since \( \Omega \in \mathcal{C}_1 \) satisfies the uniformly exterior ball condition, we take \( r_0 \) as that in Definition
2.6. Now we need only prove $b_0 \geq r_0$. Actually, if this is false, then there exists a $y^* \in \Omega$ such that $U(y^*, r_0) \subseteq \Omega$ and $z_0 \in \partial U(y^*, r_0)$. By Lemma 3.7, $z_0, y, y^*$ lie in the unique line $L(z_0)$. Hence $x \in U(y^*, r_0)$ since $x \in \Omega$ and $r_0 > b_0$. This contradicts the definition of $b_0$.

Since $U(y, b_0) \subseteq \Omega$, $x \in B(y, b_0)$ and $b_0 \geq r_0$, there exists a $y \in \Omega$ such that $x \in U(y, \frac{b_0}{2})$ by the definition of $b_0$. This proves the first part of Lemma 3.10. Now we prove the second part.

Indeed, if $x \in \partial \Omega$, then there exists a $y \in B^* \setminus \Omega$ such that $x \in B(y, r_0)$ and $B(y, r_0) \subseteq B^* \setminus \Omega$ by $\Omega \in C_2$. If $x \in (B^* \setminus \Omega)^\circ$, along the same lines as the proof of the first part, we can get a ball $B(y, r_0)$ such that $x \in B(y, r_0) \subseteq B^* \setminus \Omega$.

Last, along the same lines as the proof of the first part, we can also prove the last part. This completes the proof of the results.

Finally, for the sake of readability and completeness, we give short proofs for Theorems 2.10 and 2.11, but it should be pointed out that with the smoothness of the open sets in $C_1 \cap C_2$ claimed by Theorem 2.9, the approaches used in what follows are practically available in [6, 11, 13] for other optimization problems.

**Proof of Theorem 2.10.** Since $(C_1 \cap C_2, \rho)$ is a compact metric space and $\{\Omega_n\}_{n=1}^\infty \subseteq C_1 \cap C_2$, we have $\Omega \in C_1 \cap C_2$. Consider the sequence $\{B^* \setminus \Omega_n\}_{n=1}^\infty$. By Lemma 2.4, there exists an open set $\Omega^* \subset B^*$ such that $\Omega^* \setminus \Omega_n \overset{\delta}{\to} \Omega^*$. By Lemma 2.3 we only need to prove $B^* \setminus \Omega = \Omega^*$.

First, we show $\Omega^* \subset B^* \setminus \Omega$. For every $x \in \Omega^*$, there exists a $r_1 > 0$ such that $B(x, r_1) \subset \Omega^*$. By Lemma 2.3, there exists a $n(x, r_1) > 0$ such that for all $n \geq n(x, r_1)$, $B(x, r_1) \subset B^* \setminus \Omega_n$. Moreover, $U(x, r_1) \subset B^* \setminus \Omega_n$ for all $n \geq n(x, r_1)$.

If $U(x, r_1) \not\subseteq B^* \setminus \Omega$, i.e., $U(x, r_1) \cap \Omega \neq \emptyset$, then there exists a $y \in U(x, r_1)$ and $y \in \Omega$. If $y \in \Omega$, then there exist $r_2 > 0$, $r_2 < r_1$ such that $U(y, r_2) \subset U(x, r_1) \cap \Omega$; if $y \in \partial \Omega$, since $\Omega \in C_2$, there exists a $z \in \Omega$ such that $U(z, r_0) \subset \Omega$ and $y \in \partial U(z, r_0)$. Hence we can take $U(x, r_2) \subset U(x, r_1) \cap U(z, r_0) \subset U(x, r_1) \cap \Omega$. In any case, there exists a ball $U(x_1, r_2) \subset U(x, r_1) \cap \Omega$. Furthermore, $B(x_1, \frac{r_2}{2}) \subset U(x_1, r_1) \cap \Omega$. By Lemma 2.3, there exists a $n(x_1, r_2) > 0$ such that for all $n \geq n(x_1, r_2)$, $B(x_1, \frac{r_2}{2}) \subset \Omega_n$ due to $\Omega_n \overset{\delta}{\to} \Omega$. Take $J = \max\{n(x, r_1), n(x_1, r_2)\}$; we have $U(x_1, r_2) \subset \Omega_J$ and $U(x_1, r_2) \subset U(x_1, r_1) \subset B^* \setminus \Omega$, which is a contradiction. Hence $U(x_1, r_1) \subset \Omega^*$.

This shows $\Omega^* \subset B^* \setminus \Omega$.

Second, we show $B^* \setminus \Omega \subset \Omega^*$. Otherwise, there exists an $x \in B^* \setminus \Omega$ but $x \not\in \Omega^*$. That is, $x \not\in \Omega$. Since $B^* \setminus \Omega_n \overset{\delta}{\to} \Omega^*$, it follows that $B^* \setminus \Omega_n \overset{\delta}{\to} \Omega^*$ in particular, $\partial B^* \cup \Omega_n \overset{\delta}{\to} \partial B^* \setminus \Omega^*$. By Lemma 3.6, $B^* \setminus \Omega^*$ is a closed set, and $\Omega_n \overset{\delta}{\to} B^* \setminus \Omega$ for the fact that $\inf_{x \in \partial B^*, y \in \Omega_n} |x - y| \geq R_0 > 0$. Since $x \in B^* \setminus \Omega$, there exists a sequence $\{x_n\}$ such that $x_n \in \Omega_n$, and $x_n \to x$ as $n \to \infty$. By Lemma 3.10 and $\Omega_n \in C_1$, there exist $y_n \in \Omega_n$ such that $x_n \in B(y_n, \frac{r_n}{2})$ and $U(y_n, \frac{r_n}{2}) \subset \Omega_n$. By Lemmas 2.4 and 3.1, there exists a subsequence of $\{y_n\}$, still denoted by itself, such that $y_n \to y$ and $U(y_n, \frac{r_n}{2}) \overset{\delta}{\to} U(y, \frac{r_0}{2}), U(y, \frac{r_0}{2}) \subset \Omega$. Since $|x - y| = \lim_{n \to \infty} |x_n - y_n| \leq \frac{r_n}{2}$, it has $x \in B(y, \frac{r_0}{2}) \subset \Omega$. This contradiction shows that $B^* \setminus \Omega \subset \Omega^*$. The proof is complete.

In order to prove Theorem 2.11, we first prove the following lemmas.

**Lemma 3.11.** Let $\{F_n\} \subseteq \overline{B^*}$ be a sequence of compact sets and $F_n \overset{\delta}{\to} F$. Then for any $w \in H^1_0(\Omega)$, there exists a sequence $w_n \in H^1_0(\Omega_n)$, $\Omega_n = B^* \setminus F_n$, $\Omega = B^* \setminus F$, such that

$$
\lim_{n \to \infty} w_n = w \text{ in } H^1(B^*).
$$
Assume the following:

and from (3.32)

Extending $u$ in (3.32) gives

This is the required result.

**Lemma 3.12.** For a sequence of bounded open sets $\{\Omega_n\}_{n=1}^{\infty} \subset B^*$, let $u_n$ be the solution of equation (3.30) following

\begin{equation}
- Au_n = f \text{ in } \Omega_n, \quad u_n \in H^1_0(\Omega_n).
\end{equation}

Assume the following:

(i) $\Omega_n \overset{\text{d}}{\rightarrow} \Omega$, a.e., $B^* \setminus \Omega_n \overset{\delta}{\rightarrow} B^* \setminus \Omega$.

(ii) $\chi_{B^* \setminus \Omega_n} \to l$ in $L^\infty$ weak star topology, $l > 0$ a.e. in $B^* \setminus \Omega$.

(iii) If $w \in H^1(B^*)$, then $w|_{\Omega} \in H^1_0(\Omega)$, where $\chi_{\Omega}$ denotes the characteristic function of $\Omega$.

Then $u_n \to u$ in $H^1(\Omega)$, where $u$ is the solution of (3.31):

\begin{equation}
- Au = f \text{ in } \Omega, \quad u \in H^1_0(\Omega).
\end{equation}

**Proof.** The solution of (3.30) satisfies

\begin{equation}
\int_{\Omega_n} \langle A\nabla u_n, \nabla \phi \rangle dx = \langle f|_{\Omega_n}, \phi \rangle_{H^{-1}(\Omega_n)} \forall \phi \in H^1_0(\Omega_n).
\end{equation}

Extending $u_n$ by zero in $B^* \setminus \Omega_n$ yields

\begin{equation}
\int_{B^*} \langle A\nabla u_n, \nabla \tilde{\phi} \rangle dx = \langle f, \tilde{\phi} \rangle_{H^{-1}(B^*)} \forall \tilde{\phi} \in H^1_0(B^*), \quad \tilde{\phi}|_{\Omega_n} \in H^1_0(\Omega_n).
\end{equation}

Taking $\tilde{\phi} = u_n$ in (3.32) gives

$$\alpha \|\nabla u_n\|_{H^1_0(B^*)}^2 \leq \|f\|_{H^{-1}(B^*)}\|u_n\|_{H^1_0(B^*)}.$$ 

This together with the Poincaré inequality shows that $\{u_n\}$ is bounded in $H^1_0(B^*)$. Therefore, there exists an accumulation point $u^*$ of $\{u_n\}$ in weak star topology of $H^1(B^*)$.

From Lemma 3.11, for a fixed $\phi \in H^1_0(\Omega)$, it has

$$\tilde{\phi}_n \to \tilde{\phi} \text{ in } H^1_0(B^*), \quad \tilde{\phi}_n \in H^1_0(B^*),$$

and from (3.32)

$$\int_{B^*} \langle A\nabla u_n, \nabla \tilde{\phi}_n \rangle dx = \langle f, \tilde{\phi}_n \rangle_{H^{-1}(B^*)} \times H^1_0(B^*).$$

Passing to the limit as $n \to \infty$ gives

$$\int_{B^*} \langle A\nabla u, \nabla \tilde{\phi} \rangle dx = \langle f, \tilde{\phi} \rangle_{H^{-1}(B^*)} \times H^1_0(B^*).$$
So \( u \) is the weak solution of (3.31) in \( B^* \). Now we show that
\[
u \in H^1_0(\Omega).
\]
Actually, since
\[
\chi_{\overline{B^*}\setminus \Omega_n} u_n = 0,
\]
passing to the limit in the above equality as \( n \to \infty \), and taking assumptions (i) through (iii) into account, we arrive at \( lu = 0 \), and so \( u = 0 \) in \( \overline{B^*}\setminus \Omega \). This completes the proof. \( \square \)

**Lemma 3.13.** If \( \Omega, \Omega_n \in \mathcal{C}_1 \cap \mathcal{C}_2 \) and \( \Omega_n \xrightarrow{L^p} \Omega \), then
\[
\chi_{\Omega_n} \to \chi_{\Omega} \text{ in } L^\infty.
\]

**Proof.** Let \( l \) be such that
\[
\chi_{\Omega_n} \to l \text{ in } L^\infty(B^*) \text{ weak star topology.}
\]
We claim that for any \( x \in \Omega \), there exists a \( y \in \Omega \) such that \( x \in U(y, r_0) \subset \Omega \).
Answer: Actually, for \( x \in \Omega \), by Lemma 3.10, there exists a \( y \in \Omega \) such that \( x \in U(y, \frac{r_0}{2}) \subset \Omega \). So, owing to Lemma 2.3, there exists an \( n_1 > 0 \) such that \( U(y, \frac{r_0}{2}) \subset \Omega_n \) for all \( n > n_1 \). Hence
\[
\int_{U(y, \frac{r_0}{2})} dx = \lim_{n \to \infty} \int_{U(y, \frac{r_0}{2})} dx = \lim_{n \to \infty} \int_{U(y, \frac{r_0}{2})} \chi_{\Omega_n} dx = \int_{U(y, \frac{r_0}{2})} l dx.
\]
This shows that \( l = 1 \) in \( U(y, \frac{r_0}{2}) \) and \( l = 1 \) in \( \Omega \).
For any \( x \in B^* \setminus \overline{\Omega} \), by Lemma 2.10, there exists a ball \( U(y, \frac{r_0}{2}) \) such that \( x \in U(y, \frac{r_0}{2}) \subset B(y, \frac{r_0}{2}) \subset B^* \setminus \overline{\Omega} \). By Theorem 2.10, there exists a \( n_x > 0 \) such that for all \( n \geq n_x \), it has \( U(y, \frac{r_0}{2}) \subset B(y, \frac{r_0}{2}) \subset B^* \setminus \overline{\Omega_n} \). Hence
\[
\int_{B(y, \frac{r_0}{2})} l dx = \lim_{n \to \infty} \int_{B(y, \frac{r_0}{2})} \chi_{\Omega_n} dx = 0.
\]
This implies \( l = 0 \) in \( B(y, r_0) \), and \( l = 0 \) in \( B^* \setminus \overline{\Omega} \).
Finally, by Theorem 2.9, \( L^N(\partial \Omega) = 0 \), where \( L^N \) is the \( N \)-dimensional Lebesgue measure; hence we can take \( l = 0 \) on \( \partial \Omega \). The result then follows. \( \square \)

**Proof of Theorem 2.11.** Let \( \{ \Omega_n \} \) be a minimizing sequence for the problem (1.5). From Theorem 2.8, there exist an \( \Omega \) and a subsequence of \( \{ \Omega_n \} \), still denoted by itself, such that \( \Omega_n \xrightarrow{L^p} \Omega \). Obviously, assumption (i) of Lemma 3.12 is valid. Assumption (ii) of Lemma 3.12 is shown in Lemma 3.13. Assumption (iii) of Lemma 3.12 is a direct result of \( \partial \Omega \in C^{1,1} \) claimed by Theorem 2.9. So all assumptions of Lemma 3.12 are satisfied. The result then follows from Lemma 3.12.
Finally, since \( \Omega \in C^{1,1} \) is the optimal solution, by Corollary 8.36 of [9] we have the solution \( u_{\Omega} \) of (1.3) satisfies \( u_{\Omega} \in C^{1,1}(\Omega) \). This completes the proof. \( \square \)

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