



Optimal actuator location of minimum norm controls for heat equation with general controlled domain [☆]

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Abstract

In this paper, we study optimal actuator location of the minimum norm controls for a multi-dimensional heat equation with control defined in the space $L^2(\Omega \times (0, T))$. The actuator domain is time-varying in the sense that it is only required to have a prescribed Lebesgue measure for any moment. We select an optimal actuator location so that the optimal control takes its minimal norm over all possible actuator domains. We build a framework of finding the Nash equilibrium so that we can develop a sufficient and necessary condition to characterize the optimal relaxed solutions for both actuator location and corresponding optimal control of the open-loop system. The existence and uniqueness of the optimal classical solutions are therefore concluded. As a result, we synthesize both optimal actuator location and corresponding optimal control into a time-varying feedbacks.

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1. Introduction and main results

Different to lumped parameter systems, the location of actuator where optimal control optimizes the performance in systems governed by partial differential equations (PDEs) can often be chosen [14]. Using a simple duct model, it is shown in [13] that the noise reduction performance depends strongly on actuator location. An approximation scheme is developed in [14] to find optimal location of the optimal controls for abstract infinite-dimensional systems to minimize cost functional with the worst choice of initial condition. In fact, the actuator location problem has been attracted widely by many researchers in different contexts but most of them are for one-dimensional PDEs, as previously studied elsewhere [3,4,9,10,20,21,25], to name just a few. Numerical research is one of most important perspectives [3,15,18,19,24], among many others.

However, there are few results available in literature for multi-dimensional PDEs. In [16], a problem of optimizing the shape and position of the damping set for internal stabilization of a linear wave equation in \mathbb{R}^N , $N = 1, 2$ is considered. The paper [17] considers a numerical approximation of null controls of the minimal L^∞ -norm for a linear heat equation with a bounded potential. An interesting study is presented in [20] where the problem of determining a measurable subset of maximizing the L^2 norm of the restriction of the corresponding solution to a homogeneous wave equation on a bounded open connected subset over a finite time interval is addressed. In [8], the shape optimal design problems related to norm optimal and time optimal of null controlled heat equation have been considered. However, the controlled domains in [8] are limited to some special class of open subsets measured by the Hausdorff metric. The same limitations can also be found in shape optimization problems discussed in [6,7]. Very recently, some optimal shape and location problems of sensors for parabolic equations with random initial data have been considered in [22].

In this paper, we consider optimal actuator location of the minimal norm controls for a multi-dimensional internal null controllable heat equation over an open bounded domain Ω in \mathbb{R}^d space and the duration $[0, T]$. Our internal actuator domains are quite general: they are varying over all possible measurable subsets $\omega(t)$ of Ω where $\omega(t)$ is only required to have a prescribed measure for any decision making moment. This work is different from [22] yet one result (**Theorem 1.1**) can be considered as a refined multi-dimensional generalization of paper [19] where one-dimensional problem is considered, as well as a solution to a similar but open problem for parabolic equation mentioned in paper [21].

Let us first state our problem. Suppose that $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) is a non-empty bounded domain with analytic boundary $\partial\Omega$. Let $T > 0$, $\alpha \in (0, 1)$, and let $m(\cdot)$ be the Lebesgue measure on \mathbb{R}^d . Denote by

$$\mathcal{W} = \{ \omega \subset \Omega \mid \omega \text{ is measurable with } m(\omega) = \alpha \cdot m(\Omega) \}, \tag{1.1}$$

and

$$\mathcal{W}_{s,T} = \{ w \in L^\infty(\Omega \times (s, T); \{0, 1\}) \mid m(\{x \mid w(x, t) = 1\}) \equiv \alpha \cdot m(\Omega) \text{ a.e. } t \in (s, T) \}. \tag{1.2}$$

It is assumed that $a(x, t)$ is analytic in $\Omega \times (0, T)$. For any $s \in [0, T)$ and $\xi \in L^2(\Omega)$, consider the following controlled heat equation

$$\begin{cases} y_t(x, t) - \Delta y(x, t) + a(x, t)y(x, t) = w(x, t)u(x, t) & \text{in } \Omega \times (s, T), \\ y(x, t) = 0 & \text{on } \partial\Omega \times (s, T), \\ y(x, s) = \xi(x) & \text{in } \Omega, \end{cases} \quad (1.3)$$

where $w \in \mathcal{W}_{s,T}$ is said to be, by abuse of notation, the actuator location, and $u \in L^2(\Omega \times (s, T))$ is the control. It is well known that Equation (1.3) admits a unique mild solution which is denoted by $y(\cdot; w, u; s, \xi)$ or $y((x, t); w, u; s, \xi)$ when it is necessary.

The minimal norm control problem can be stated as follows. For any given $s \in [0, T)$, $\xi \in L^2(\Omega)$, and $w \in \mathcal{W}_{s,T}$, find a minimal norm control to solve the following optimal control problem:

Problem (NP) $_{w}^{s,\xi}$: $N(w; s, \xi) \triangleq \inf \{ \|u\|_{L^2(\Omega \times (s, T))} \mid y((x, T); w, u; s, \xi) = 0 \}$.

We want to find an optimal actuator location determined by state and design the corresponding optimal time-varying feedback control. More precisely, we want to find two time-varying feedbacks: $\mathcal{F} : [0, T) \times L^2(\Omega) \mapsto \mathcal{W}$ and $\mathcal{G} : [0, T) \times L^2(\Omega) \mapsto L^2(\Omega)$ so that for any $s \in [0, T)$ and $\xi \in L^2(\Omega)$, the evolution equation

$$\begin{cases} y'(t) - \Delta y(t) + a(\cdot, t)y(t) = \mathcal{F}(t, y(t))\mathcal{G}(t, y(t)) & t \in (s, T), \\ y(s) = \xi, \end{cases} \quad (1.4)$$

admits a unique mild solution $y^{\mathcal{F},\mathcal{G}}(\cdot; s, \xi)$ satisfying $y^{\mathcal{F},\mathcal{G}}(T; s, \xi) = 0$ and

$$\|u^{\mathcal{F},\mathcal{G}}(s, \xi)\|_{L^2(\Omega \times (s, T))} = N(w^{\mathcal{F},\mathcal{G}}(s, \xi); s, \xi) = \inf_{w \in \mathcal{W}_{s,T}} N(w; s, \xi), \quad (1.5)$$

where

$$w^{\mathcal{F},\mathcal{G}}(s, \xi) \triangleq \mathcal{F}(\cdot, y^{\mathcal{F},\mathcal{G}}(\cdot; s, \xi)) \in \mathcal{W}_{s,T}, \quad (1.6)$$

and

$$u^{\mathcal{F},\mathcal{G}}(s, \xi) \triangleq \mathcal{G}(\cdot, y^{\mathcal{F},\mathcal{G}}(\cdot; s, \xi)) \in L^2(\Omega \times (s, T)). \quad (1.7)$$

Notice that in (1.4), the linear operator $-\Delta$ is of Dirichlet boundary conditions.

To solve this problem, we need to discuss the following open-loop problem with $s \in [0, T)$ and $\xi \in L^2(\Omega)$ being fixed. In particular, we need the existence and uniqueness for optimal classical solutions to open-loop problem. A classical optimal actuator location of the minimal norm control problem with respect to (s, ξ) is to seek $w^{s,\xi} \in \mathcal{W}_{s,T}$ to minimize $N(w; s, \xi)$:

Problem (CP) $^{s,\xi}$:
$$\begin{aligned} \bar{N}(s, \xi) &\triangleq \inf_{w \in \mathcal{W}_{s,T}} N(w; s, \xi) = N(w^{s,\xi}; s, \xi) \\ &= \inf_{w \in \mathcal{W}_{s,T}} \inf_{u \in L^2(\Omega \times (s, T))} \{ \|u\|_{L^2(\Omega \times (s, T))} \mid y((x, T); w, u; \xi) = 0 \}. \end{aligned}$$

If such $w^{s,\xi}$ exists, we say that $w^{s,\xi}$ is an optimal actuator location of the optimal minimal norm controls with respect to (s, ξ) . For Problem $(NP)_w^{s,\xi}$, we will apply the duality theory in the sense of Fenchel (see, e.g., [5,23,11]), namely, we will solve the following dual problem of $(NP)_w^{s,\xi}$ [12,26]:

$$\text{Problem (DP)}_w^{s,\xi} : V(w; s, \xi) \triangleq \inf_{z \in L^2(\Omega)} J^{s,\xi}(z; w) \triangleq \frac{1}{2} \|w\varphi(\cdot; z)\|_{L^2(\Omega \times (s,T))}^2 + \langle \xi, \varphi(s; z) \rangle,$$

where $\varphi(\cdot, z)$ is the solution to the following equation

$$\begin{cases} \varphi_t(x, t) + \Delta\varphi(x, t) - a(x, t)\varphi(x, t) = 0 & \text{in } \Omega \times (s, T), \\ \varphi(x, t) = 0 & \text{on } \partial\Omega \times (s, T), \\ \varphi(x, T) = z(x) & \text{in } \Omega. \end{cases} \tag{1.8}$$

Furthermore, it is derived (see Lemma 2.5 later) that

$$V(w; s, \xi) = -\frac{1}{2}N(w; s, \xi)^2, \quad \forall (s, \xi) \in [0, T] \times L^2(\Omega), \quad w \in \mathcal{W}_{s,T}. \tag{1.9}$$

Thus

$$\begin{aligned} \frac{1}{2}\bar{N}(s, \xi)^2 &= \inf_{w \in \mathcal{W}_{s,T}} \frac{1}{2}N(w; s, \xi)^2 \\ &= \inf_{w \in \mathcal{W}_{s,T}} \left[- \inf_{z \in L^2(\Omega)} \left(\frac{1}{2} \|w\varphi(\cdot; z)\|_{L^2(\Omega \times (s,T))}^2 + \langle \xi, \varphi(s; z) \rangle \right) \right] \\ &= - \sup_{w \in \mathcal{W}_{s,T}} \inf_{z \in L^2(\Omega)} \left[\frac{1}{2} \|w\varphi(\cdot; z)\|_{L^2(\Omega \times (s,T))}^2 + \langle \xi, \varphi(s; z) \rangle \right]. \end{aligned} \tag{1.10}$$

Therefore Problem $(CP)^{s,\xi}$ can be transformed into the following Stackelberg Problem

$$\text{Problem (SP)}^{s,\xi} : \sup_{w \in \mathcal{W}_{s,T}} \inf_{z \in L^2(\Omega)} \left[\frac{1}{2} \|w\varphi(\cdot; z)\|_{L^2(\Omega \times (s,T))}^2 + \langle \xi, \varphi(s; z) \rangle \right].$$

Since $\mathcal{W}_{s,T}$ lacks of compactness, it is nature to extend the feasible set $\mathcal{W}_{s,T}$ to a relaxed set $\mathcal{B}_{s,T}$ (see (1.14)) to ensure the existence. But it is well known that usually

$$\sup_{w \in \mathcal{W}_{s,T}} \inf_{z \in L^2(\Omega)} \neq \sup_{\beta \in \mathcal{B}_{s,T}} \inf_{z \in L^2(\Omega)} \tag{1.11}$$

in the framework of game theory.

One novelty of present work is that the results derived from the relaxed case can be returned back to the classical case. It is difficult to verify directly if (1.11) is true or not. Our way is to prove that any relaxed solution is also classical by using a sufficient and necessary condition for relaxed solutions. As for two-level optimization Problem $(SP)^{s,\xi}$, it is still not easy to obtain a sufficient and necessary condition. It is especially critical that Problem $(DP)_w^{s,\xi}$ may have no solution in $L^2(\Omega)$ though Problem $(NP)_w^{s,\xi}$ always admits its solution, which is another difficulty. We observe keenly that in these cases, the Stackelberg game problem can be transformed into a

Nash equilibrium problem in a zero-sum game framework, for which a sufficient and necessary condition for the optimal solutions (actuator location and control) can be derived.

Define

$$Z = \left\{ z \in H^{-1/2}(\Omega) \mid \varphi(\cdot; z) \in L^2(\Omega \times (0, T)) \right\}, \tag{1.12}$$

where $\varphi(\cdot, z)$ is the solution to Equation (1.8) with $s = 0$. One of the main results of this paper is [Theorem 1.1](#).

Theorem 1.1. *Let $T > 0$, $\alpha \in (0, 1)$, and let $a(x, t)$ be analytic in $\Omega \times (0, T)$. Problem $(CP)^{s,\xi}$ admits a unique solution for any $(s, \xi) \in [0, T) \times L^2(\Omega) \setminus \{0\}$. In addition, \bar{w} is a solution to Problem $(CP)^{s,\xi}$ if and only if there is $\bar{z} \in Z$ such that (\bar{w}, \bar{z}) is a Nash equilibrium of the following two-person zero-sum game problem: Find $(\bar{w}, \bar{z}) \in \mathcal{W}_{s,T} \times Z$ such that*

$$\begin{aligned} \frac{1}{2} \|\bar{w}\varphi(\cdot; \bar{z})\|_{L^2(\Omega \times (s,T))}^2 + \langle \xi, \varphi(s; \bar{z}) \rangle &= \sup_{w \in \mathcal{W}_{s,T}} \left[\frac{1}{2} \|w\varphi(\cdot; \bar{z})\|_{L^2(\Omega \times (s,T))}^2 + \langle \xi, \varphi(s; \bar{z}) \rangle \right], \\ \frac{1}{2} \|\bar{w}\varphi(\cdot; \bar{z})\|_{L^2(\Omega \times (s,T))}^2 + \langle \xi, \varphi(s; \bar{z}) \rangle &= \inf_{z \in Z} \left[\frac{1}{2} \|\bar{w}\varphi(\cdot; z)\|_{L^2(\Omega \times (s,T))}^2 + \langle \xi, \varphi(s; z) \rangle \right]. \end{aligned} \tag{1.13}$$

As a corollary of the above theorem, we have the following result.

Corollary 1.2. *Let $T > 0$, $\alpha \in (0, 1)$, and let $a(x, t)$ be analytic in $\Omega \times (0, T)$. There are two maps: $\mathcal{F} : [0, T) \times L^2(\Omega) \mapsto \mathcal{W}$ and $\mathcal{G} : [0, T) \times L^2(\Omega) \mapsto L^2(\Omega)$ so that for any $s \in [0, T)$ and $\xi \in L^2(\Omega) \setminus \{0\}$, Equation (1.4) admits a unique mild solution $y^{\mathcal{F},\mathcal{G}}(\cdot; s, \xi)$ satisfying $y^{\mathcal{F},\mathcal{G}}((x, T); s, \xi) = 0$ and (1.5)–(1.7).*

We proceed as follows. Define

$$\mathcal{B}_{s,T} = \left\{ \beta \in L^\infty(\Omega \times (s, T); [0, 1]) \mid \int_{\Omega} \beta^2(x, t) dx \equiv \alpha \cdot m(\Omega) \text{ a.e. } t \in (s, T) \right\} \tag{1.14}$$

as a relaxed set of $\mathcal{W}_{s,T}$. In section 2, we discuss the minimum norm control Problem $(NP)_\beta^{s,\xi}$ in the relaxed case by replacing $w \in \mathcal{W}_{s,T}$ with $\beta \in \mathcal{B}_{s,T}$, and present Problem $(SP)^{s,\xi}$ in the relaxed case. Section 3 will be devoted to discussing properties of solutions to Problem $(SP)^{s,\xi}$ in the relaxed case. Finally, we prove [Theorem 1.1](#), and [Corollary 1.2](#) is proved by the synthetic method.

It should be pointed out that [Corollary 1.2](#) is of time-varying feedback which does not give any concrete algorithm to synthesize these maps. Essentially speaking, the main difficulty lies in solving the problem with fixed initial data. For completely initial value free problem, we refer to recent important works [\[20–22\]](#).

2. Relaxed minimum norm control problem $(NP)_\beta^{s,\xi}$

Let $(s, \xi) \in [0, T) \times L^2(\Omega)$ be fixed. For any $\beta \in \mathcal{B}_{s,T}$. Consider the following system:

$$\begin{cases} y_t(x, t) - \Delta y(x, t) + a(x, t)y(x, t) = \beta(x, t)u(x, t) & \text{in } \Omega \times (s, T), \\ y(x, t) = 0 & \text{on } \partial\Omega \times (s, T), \\ y(x, s) = \xi(x) & \text{in } \Omega, \end{cases} \quad (2.1)$$

where once again the control $u \in L^2(\Omega \times (s, T))$, and the solution of (2.1) is denoted by $y(\cdot; \beta, u)$. Accordingly, Problem $(NP)_w^{s,\xi}$ is changed into a relaxation problem of the following:

Problem $(NP)_\beta^{s,\xi}$: $N(\beta; s, \xi) \triangleq \inf \{ \|u\|_{L^2(\Omega \times (s, T))} \mid y(\cdot, T); \beta, u = 0 \}$.

Let us first show the null controllability for controlled system (2.1), which is deduced by building the “observability inequality” (2.2) for system (1.8).

Lemma 2.1. *For any $\beta \in \mathcal{B}_{s,T}$, there exists positive constant C_β such that the solution of (1.8) satisfies*

$$\|\varphi(s; z)\|_{L^2(\Omega)} \leq C_\beta \|\beta\varphi(\cdot; z)\|_{L^2(\Omega \times (s, T))}, \quad \forall z \in L^2(\Omega), \quad (2.2)$$

where C_β is independent of $z \in L^2(\Omega)$.

Proof. It is well known that system (2.1) is null controllable if and only if the “observability inequality” (2.2) holds for the dual system (1.8). Let $w \in \mathcal{W}_{s,T}$. An observability inequality on the measurable set ω :

$$\|\varphi(s; z)\|_{L^2(\Omega)} \leq \hat{C}_w \|\omega\varphi(\cdot; z)\|_{L^2(\Omega \times (s, T))}, \quad \forall z \in L^2(\Omega), \quad (2.3)$$

has been derived in [2] for some $\hat{C}_w > 0$. Now for any $\beta \in \mathcal{B}_{s,T}$, let

$$E = \left\{ (x, t) \in \Omega \times (s, T) \mid \beta(x, t) \geq \sqrt{\alpha/2} \right\}, \quad \lambda = \frac{m(E)}{m(\Omega \times (s, T))}.$$

By

$$\begin{aligned} & 1 \cdot m(\{\beta \geq \sqrt{\alpha/2}\}) + \alpha/2 \cdot m(\{\beta < \sqrt{\alpha/2}\}) \\ & \geq \iint_{\{\beta \geq \sqrt{\alpha/2}\}} \beta^2(x, t) dx dt + \iint_{\{\beta < \sqrt{\alpha/2}\}} \beta^2(x, t) dx dt \\ & = \iint_{\Omega \times (s, T)} \beta^2(x, t) dx dt = \alpha(T - s)m(\Omega), \end{aligned}$$

here and in what follows, we denote $\{\beta \geq \sqrt{\alpha/2}\}$ by $\{(x, t) \in \Omega \times (s, T) \mid \beta(x, t) \geq \sqrt{\alpha/2}\}$. It follows that

$$\lambda + (1 - \lambda)\alpha/2 \geq \alpha.$$

Consequently, $\lambda \geq \frac{\alpha}{2 - \alpha}$. This means that E is not a zero-measure set. It then follows from (2.3) with $w = \chi_E$ and $\beta \geq \sqrt{\alpha/2} \chi_E$ that

$$\|\varphi(s; z)\|_{L^2(\Omega)} \leq \hat{C}_w \|w\varphi(\cdot; z)\|_{L^2(\Omega \times (s, T))} \leq \frac{\sqrt{2}\hat{C}_w}{\sqrt{\alpha}} \|\beta\varphi(\cdot; z)\|_{L^2(\Omega \times (s, T))}.$$

This is (2.2) by taking $C_\beta = \sqrt{2}\hat{C}_w/\sqrt{\alpha}$. \square

2.1. Relaxed dual problem $(DP)_\beta^{s, \xi}$

Now we present the relaxed dual problem

Problem $(DP)_\beta^{s, \xi}$: $V(\beta; s, \xi) \triangleq \inf_{z \in L^2(\Omega)} J^{s, \xi}(z; \beta) \triangleq \frac{1}{2} \|\beta\varphi(\cdot; z)\|_{L^2(\Omega \times (s, T))}^2 + \langle \xi, \varphi(s, z) \rangle.$

Since there may have no solution in $L^2(\Omega)$ for Problem $(DP)_\beta^{s, \xi}$, we need to introduce a class of spaces $\{\bar{Y}_\beta, \beta \in \mathcal{B}_{s, T}\}$. Let

$$Y = \{\varphi(\cdot; z) \mid z \in L^2(\Omega)\} \subset L^2(\Omega \times (s, T)), \tag{2.4}$$

where $\varphi(\cdot; z)$ is the solution of (1.8) with the initial data $z \in L^2(\Omega)$. Obviously, Y is a linear space from the linearity of PDE (1.8).

Lemma 2.2. *Let Y be defined by (2.4). For each $\beta \in \mathcal{B}_{s, T}$, define a functional in Y by*

$$F_0(\varphi) = \|\beta\varphi\|_{L^2(\Omega \times (s, T))}, \quad \forall \varphi \in Y.$$

Then (Y, F_0) is a linear normed space. We denote this normed space by Y_β .

Proof. It suffices to show that $F_0(\psi) = \|\beta\psi\|_{L^2(\Omega \times (s, T))} = 0$ implies $\psi = 0$. Actually, by $F_0(\psi) = 0$, it follows that

$$\sqrt{\alpha/2} \|\chi_{\{\beta \geq \sqrt{\alpha/2}\}} \psi\|_{L^2(\Omega \times (s, T))} \leq \|\beta\psi\|_{L^2(\Omega \times (s, T))} = 0.$$

By the unique continuation (see, e.g., [21]) for heat equation, we arrive at $\psi(x, t) = 0$. \square

Denote by

$$\bar{Y}_\beta = \text{the completion of the space } Y_\beta. \tag{2.5}$$

It is usually hard to characterize \bar{Y}_β . However, we have the following description for \bar{Y}_β .

Lemma 2.3. Let $\beta \in \mathcal{B}_{s,T}$, and let \bar{Y}_β be defined by (2.5). Then under an isometric isomorphism, any element of \bar{Y}_β can be expressed as a function $\hat{\varphi} \in C([s, T]; L^2(\Omega))$ which satisfies (in the sense of weak solution)

$$\begin{cases} \hat{\varphi}_t(x, t) + \Delta \hat{\varphi}(x, t) - a(x, t)\hat{\varphi}(x, t) = 0 & \text{in } \Omega \times (s, T), \\ \hat{\varphi}(x, t) = 0 & \text{on } \partial\Omega \times (s, T), \end{cases} \tag{2.6}$$

and $\beta\hat{\varphi} = \lim_{n \rightarrow \infty} \beta\varphi(\cdot; z_n)$ in $L^2(\Omega \times (s, T))$ for some sequence $\{z_n\} \subset L^2(\Omega)$, where $\varphi(\cdot; z_n)$ is the solution of (1.8) with initial value $z = z_n$.

Proof. Let $\bar{\psi} \in (\bar{Y}_\beta, \bar{F}_0)$, where $(\bar{Y}_\beta, \bar{F}_0)$ is the completion of (Y_β, F_0) . By definition, there is a sequence $\{z_n\} \subset L^2(\Omega)$ such that

$$\bar{F}_0(\varphi(\cdot; z_n) - \bar{\psi}) \rightarrow 0,$$

from which, one has

$$F_0(\varphi(\cdot; z_n) - \varphi(\cdot; z_m)) = \bar{F}_0(\varphi(\cdot; z_n) - \varphi(\cdot; z_m)) \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

In other words,

$$\|\beta\varphi(\cdot; z_n) - \beta\varphi(\cdot; z_m)\|_{L^2(\Omega \times (s, T))} \rightarrow 0 \text{ as } n, m \rightarrow \infty. \tag{2.7}$$

Hence, there exists $\hat{\psi} \in L^2(\Omega \times (s, T))$ such that

$$\beta\varphi(\cdot; z_n) \rightarrow \hat{\psi} \text{ strongly in } L^2(\Omega \times (s, T)). \tag{2.8}$$

Let $\{T_k\} \subset (s, T)$ be such that $T_k \nearrow T$. i.e. T_k is strictly monotone increasing and converges to T . Denote $\varphi_n \equiv \varphi(\cdot; z_n)$.

(a). For T_1 , by the observability inequality (2.2), and (2.7),

$$\begin{aligned} \|\varphi(T_2; z_n)\|_{L^2(\Omega)} &\leq C(1)\|\beta\varphi(\cdot; z_n)\|_{L^2(T_2, T; L^2(\Omega))} \\ &\leq C(1)\|\beta\varphi(\cdot; z_n)\|_{L^2(\Omega \times (s, T))} \leq C(1) \sup_m \|\beta\varphi(\cdot; z_m)\|_{L^2(\Omega \times (s, T))}, \quad \forall n \in \mathbb{N}. \end{aligned}$$

Hence, there exists a subsequence $\{\varphi_{1n}\}$ of $\{\varphi_n\}$ and $\varphi_{01} \in L^2(\Omega)$ such that

$$\varphi_{1n}(T_2) = \varphi(T_2; z_{1n}) \rightarrow \varphi_{01} \text{ weakly in } L^2(\Omega).$$

This together with the fact:

$$\begin{cases} (\varphi_{1n})_t(x, t) + \Delta \varphi_{1n}(x, t) - a(x, t)\varphi_{1n}(x, t) = 0 & \text{in } \Omega \times (s, T_2), \\ \varphi_{1n}(x, t) = 0 & \text{on } \partial\Omega \times (s, T_2), \\ \varphi_{1n}(x, T_2) = \varphi(T_2; z_{1n}) & \text{in } \Omega, \end{cases}$$

shows that there exists $\psi_1 \in L^2(s, T_2; L^2(\Omega)) \cap C([s, T_2 - \delta]; L^2(\Omega))$ for all $\delta > 0$, satisfies

$$\begin{cases} (\psi_1)_t(x, t) + \Delta\psi_1(x, t) - a(x, t)\psi_1(x, t) = 0 & \text{in } \Omega \times (s, T_2), \\ \psi_1(x, t) = 0 & \text{on } \partial\Omega \times (s, T_2), \\ \psi_1(x, T_2) = \varphi_{01}(x) & \text{in } \Omega, \end{cases}$$

and for all $\delta \in (0, T_2)$,

$$\varphi_{1n} \rightarrow \psi_1 \text{ strongly in } L^2([s, T_2]; L^2(\Omega)) \cap C([s, T_2 - \delta]; L^2(\Omega)).$$

In particular,

$$\varphi_{1n} \rightarrow \psi_1 \text{ strongly in } L^2([s, T_2]; L^2(\Omega)) \cap C([s, T_1]; L^2(\Omega)), \tag{2.9}$$

and

$$\beta\varphi_{1n} \rightarrow \beta\psi_1 \text{ strongly in } L^2([s, T_2]; L^2(\Omega)). \tag{2.10}$$

These together with (2.8) and (2.10) yield

$$\beta\psi_1 = \hat{\psi} \text{ in } L^2([s, T_1]; L^2(\Omega)).$$

(b). Along the same way as (a), we can find a subsequence $\{\varphi_{2n}\}$ of $\{\varphi_{1n}\}$, and $\psi_2 \in L^2([s, T_3]; L^2(\Omega)) \cap C([s, T_3 - \delta]; L^2(\Omega))$ for all $\delta > 0$ satisfying

$$\begin{cases} (\psi_2)_t(x, t) + \Delta\psi_2(x, t) - a(x, t)\psi_2(x, t) = 0 & \text{in } \Omega \times (s, T_3), \\ \psi_2(x, t) = 0 & \text{on } \partial\Omega \times (s, T_3), \end{cases}$$

and

$$\varphi_{2n} \rightarrow \psi_2 \text{ strongly in } L^2([s, T_3]; L^2(\Omega)) \cap C([s, T_2]; L^2(\Omega)).$$

This, together with (2.9), leads to

$$\psi_2|_{[s, T_1]} = \psi_1,$$

and

$$\beta\psi_2 = \hat{\psi} \text{ in } L^2([s, T_2]; L^2(\Omega)).$$

(c). Similarly to (a) and (b), we can find a sequence $\{\psi_k\}$ which satisfies, for each $k \in \mathbb{N}^+$, that

- $\psi_k \in L^2([s, T_{k+1}]; L^2(\Omega)) \cap C([s, T_k]; L^2(\Omega))$;
- $\psi_{k+1}|_{[s, T_k]} = \psi_k$;
- ψ_k satisfies

$$\begin{cases} (\psi_k)_t(x, t) + \Delta\psi_k(x, t) - a(x, t)\psi_k(x, t) = 0 & \text{in } \Omega \times (s, T_{k+1}), \\ \psi_k(x, t) = 0 & \text{on } \partial\Omega \times (s, T_{k+1}). \end{cases}$$

- $\beta\psi_k = \hat{\psi}$ in $L^2([s, T_k]; L^2(\Omega))$.

Define

$$\psi(\cdot, t) = \psi_k(\cdot, t), \quad t \in [s, T_k].$$

Then, $\psi(x, t)$ is well defined on $[s, T]$, which satisfies $\psi \in L^2([s, T]; L^2(\Omega)) \cap C([s, T]; L^2(\Omega))$,

$$\begin{cases} \psi_t(x, t) + \Delta\psi(x, t) - a(x, t)\psi(x, t) = 0 & \text{in } \Omega \times (s, T), \\ \psi(x, t) = 0 & \text{on } \partial\Omega \times (s, T), \end{cases}$$

and

$$\beta\psi = \hat{\psi} = \lim_{n \rightarrow \infty} \beta\varphi(\cdot; z_n).$$

Under an isometric isomorphism, we can say $\bar{\psi} = \psi$. This complete the proof of the lemma. \square

We define an operator $\mathcal{T} : Y \rightarrow L^2(\Omega)$ by

$$\mathcal{T}(\varphi(\cdot; z)) = \varphi(s; z), \quad \forall z \in L^2(\Omega), \tag{2.11}$$

which is well-defined because $Y \subset C([0, T]; L^2(\Omega))$. Define an operator $\mathcal{T}_\beta : \beta\bar{Y}_\beta \rightarrow L^2(\Omega)$ by

$$\mathcal{T}_\beta(\beta\psi) = \psi(s), \quad \forall \psi \in \bar{Y}_\beta. \tag{2.12}$$

By Lemma 2.3, the operator \mathcal{T}_β is also well-defined. In addition, it follows from the observability inequality claimed by Lemma 2.1 that the linear operator \mathcal{T}_β is bounded.

Lemma 2.4. *If $\beta \in \mathcal{B}_{s,T}$, then the operator \mathcal{T}_β defined by (2.12) is compact.*

Proof. By the observability inequality claimed by Lemma 2.1, it follows that the operator $\beta\bar{Y}_\beta \rightarrow L^2(\Omega)$ defined by

$$\beta\psi(\cdot, \cdot) \rightarrow \psi(\cdot, (T + s)/2), \quad \forall \psi \in \bar{Y}_\beta$$

is bounded. Also by the property of heat equation, the operator defined by

$$\varphi(\cdot, (T + s)/2) \rightarrow \varphi(\cdot, s), \quad \forall \varphi \in \bar{Y}_\beta$$

is compact. As a composition operator from the above two operators, \mathcal{T}_β is compact as well. \square

Now we tune to discuss the solution to Problem $(DP)_\beta^{s,\xi}$ with extended domain. From the notation of \mathcal{T}_β , we could rewrite the functional $J^{s,\xi}(\cdot; \beta)$ in Problem $(DP)_\beta^{s,\xi}$ as follows:

$$J^{s,\xi}(\zeta; \beta) = \frac{1}{2} \|\zeta(\cdot)\|_{L^2(\Omega \times (s,T))}^2 + \langle \xi, \mathcal{T}_\beta(\zeta) \rangle, \quad \forall \psi \in Y.$$

Let us expand the domain of $J^{s,\xi}(\cdot; \beta)$ as follows:

$$\hat{J}_\beta^{s,\xi}(\zeta) = \frac{1}{2} \|\zeta(\cdot)\|_{L^2(\Omega \times (s,T))}^2 + \langle \xi, \mathcal{T}_\beta(\zeta) \rangle \quad \text{for any } \zeta \in \beta \bar{Y}_\beta$$

and denote

Problem $(\widehat{DP})_\beta^{s,\xi} : V(\beta; s, \xi) = \inf_{\zeta \in \beta \bar{Y}_\beta} \hat{J}_\beta^{s,\xi}(\zeta) \triangleq \frac{1}{2} \|\zeta(\cdot)\|_{L^2(\Omega \times (s,T))}^2 + \langle \xi, \mathcal{T}_\beta(\zeta) \rangle$

In the above, the first equation holds from Lemma 2.3 and the continuity of τ_β .

2.2. Relationship between Problems $(NP)_\beta^{s,\xi}$ and $(\widehat{DP})_\beta^{s,\xi}$

In this subsection, we present two properties on the relationship between Problems $(NP)_{p,\beta}$ and $(\widehat{DP})_\beta^{s,\xi}$.

Lemma 2.5. *Let $s \in (0, T)$, $\xi \in L^2(\Omega) \setminus \{0\}$, and $\beta \in \mathcal{B}_{s,T}$. Then Problem $(\widehat{DP})_\beta^{s,\xi}$ admits a unique nonzero solution in $\beta \bar{Y}_\beta$, denoted by $\bar{\zeta}$, and the control defined by*

$$\bar{u}(x, t) = \bar{\zeta}(x, t), \quad (x, t) \in \Omega \times (s, T) \text{ a.e.} \tag{2.13}$$

is an optimal control to Problem $(NP)_\beta^{s,\xi}$. Moreover,

$$V(\beta; s, \xi) = -\frac{1}{2} N(\beta; s, \xi)^2. \tag{2.14}$$

Proof. Since $L^2(\Omega \times (s, T))$ is reflexive. Thus, $\beta \bar{Y}_\beta$, as a closed subspace of $L^2(\Omega \times (s, T))$, is also reflexive. Meanwhile, one can directly check that $\hat{J}^{s,\xi}(\cdot, \beta)$ is strictly convex and coercive in $\beta \bar{Y}_\beta$. Hence, $\hat{J}^{s,\xi}(\cdot, \beta)$ has a unique minimizer $\bar{\zeta}$. It follows from the unique continuity of heat equations, the map from \bar{Y}_β to $\beta \bar{Y}_\beta$ is one-to-one. Thus there is unique $\bar{\psi} \in \bar{Y}_\beta$ such that $\bar{\zeta} = \beta \bar{\psi}$.

We prove

$$\bar{\zeta} \neq 0 \text{ and } \bar{\psi} \neq 0 \text{ in } L^2(\Omega \times (s, T)). \tag{2.15}$$

Indeed, if this is not true, then it must hold that $V(\beta; s, \xi) = 0$. We claim that $\{\varphi(s, z) | z \in L^2(\Omega)\}$ is dense in $L^2(\Omega)$. Once the claim holds, there is $z \in L^2(\Omega)$ such that $\langle \xi, \varphi(s, z) \rangle < 0$ because $\xi \neq \{0\}$. But,

$$0 = V(\beta; s, \xi) \leq J^{s,\xi}(\varepsilon z, \beta) = \frac{1}{2} \varepsilon^2 \|\beta \varphi(\cdot; z)\|_{L^2(\Omega \times (s,T))}^2 + \varepsilon \langle \xi, \varphi(s, z) \rangle < 0,$$

where the last inequality holds as $\varepsilon > 0$ is small enough.

Now we show that $\{\varphi(s, z) | z \in L^2(\Omega)\}$ is dense in $L^2(\Omega)$. Recalling the dual system (1.8), we define the operator L in $L^2(\Omega)$ by

$$Lz = \varphi(s, z) \text{ for any } z \in L^2(\Omega).$$

Notice that

$$\{\varphi(s, z) | z \in L^2(\Omega)\} \text{ is dense in } L^2(\Omega) \Leftrightarrow \overline{\mathcal{R}(L)} = L^2(\Omega) \Leftrightarrow \mathcal{N}(L^*) = \{0\},$$

where the last equivalence holds because of $\overline{\mathcal{R}(L)} = \mathcal{N}(L^*)^\perp$. For any $\hat{z} \in L^2(\Omega)$, consider the following equation:

$$\begin{cases} \hat{\varphi}_t(t) - \Delta \hat{\varphi}(t) + a(T - t)\hat{\varphi}(t) = 0, \\ \hat{\varphi}(s) = \hat{z}. \end{cases}$$

A direct verification shows that

$$L^*(\hat{z}) = \hat{\varphi}(T).$$

By the backward uniqueness for heat equation, we have $\mathcal{N}(L^*) = \{0\}$, and this leads to (2.15).

Now, we show that the control defined by (2.13) is optimal to Problem $(NP)_\beta^{s,\xi}$. Since $\bar{\zeta}(x, t)$ is optimal, we have

$$\langle \bar{u}, \zeta \rangle_{L^2(\Omega \times (s,T))} + \langle \xi, \mathcal{T}_\beta(\zeta)(s) \rangle = 0, \quad \forall \zeta \in \beta \bar{Y}_\beta. \tag{2.16}$$

Taking $\zeta = \beta\varphi(\cdot; z)$ for any $z \in L^2(\Omega)$ in (2.16), a straightforward calculation shows that

$$y(T; \beta, \bar{u}) = 0.$$

If $\hat{u}(\cdot, \cdot)$ satisfies

$$y(T; \beta, \hat{u}) = 0, \tag{2.17}$$

we will show that

$$\|\bar{u}\|_{L^2(\Omega \times (s,T))} \leq \|\hat{u}\|_{L^2(\Omega \times (s,T))}, \tag{2.18}$$

from which we see that $\bar{u}(\cdot, \cdot)$ is an optimal solution to Problem $(NP)_\beta^{s,\xi}$.

Now, we prove (2.18). By (2.17),

$$-\langle \xi, \varphi(s, z) \rangle = \langle y(T; \beta, \hat{u}), z \rangle - \langle \xi, \varphi(s; z) \rangle = \langle \varphi(\cdot; z), \beta \hat{u}(\cdot) \rangle_{L^2(\Omega \times (s,T))}, \quad \forall z \in L^2(\Omega),$$

which is rewritten as

$$-\langle \xi, \mathcal{T}_\beta(\zeta)(s) \rangle = \langle \hat{u}, \zeta \rangle_{L^2(\Omega \times (s,T))}, \quad \forall \zeta \in \beta Y_\beta.$$

By the density argument, the above still holds for any $\zeta \in \beta \bar{Y}_\beta$. It then follows from (2.16) that

$$\langle \bar{u}, \zeta \rangle_{L^2(\Omega \times (s,T))} = \langle \hat{u}, \zeta \rangle_{L^2(\Omega \times (s,T))}, \quad \forall \zeta \in \beta \bar{Y}_\beta.$$

Taking $\zeta = \bar{\zeta}$ in above equality, we have

$$\langle \bar{u}, \bar{\zeta} \rangle_{L^2(\Omega \times (s, T))} = \langle \hat{u}, \bar{\zeta} \rangle_{L^2(\Omega \times (s, T))}. \tag{2.19}$$

On the other hand, it follows from (2.13) that

$$\|\bar{u}\|_{L^2(\Omega \times (s, T))}^2 = \|\bar{\zeta}\|_{L^2(\Omega \times (s, T))}^2 = \langle \bar{\zeta}, \bar{u} \rangle_{L^2(\Omega \times (s, T))}. \tag{2.20}$$

By (2.20) and (2.19),

$$\begin{aligned} \|\bar{u}\|_{L^2(\Omega \times (s, T))}^2 &= \langle \bar{u}, \bar{\zeta} \rangle_{L^2(\Omega \times (s, T))} = \langle \hat{u}, \bar{\zeta} \rangle_{L^2(\Omega \times (s, T))} \\ &\leq \|\hat{u}\|_{L^2(\Omega \times (s, T))} \cdot \|\bar{\zeta}\|_{L^2(\Omega \times (s, T))} = \|\hat{u}\|_{L^2(\Omega \times (s, T))} \cdot \|\bar{u}\|_{L^2(\Omega \times (s, T))}. \end{aligned}$$

The inequality $\|\bar{u}\|_{L^2(\Omega \times (s, T))} \leq \|\hat{u}\|_{L^2(\Omega \times (s, T))}$ then follows immediately because $\bar{u} \neq 0$.

With a straightforward calculation, we can obtain

$$V(\beta; s, \xi) = \frac{1}{2} \|\bar{\zeta}\|_{L^2(\Omega \times (s, T))}^2 + \langle \xi, \mathcal{T}_\beta(\bar{\zeta})(s) \rangle.$$

This, together with (2.16), (2.13) and the optimality of \bar{u} , implies (2.14). \square

Now we present relaxed optimal actuator location of the minimal norm control problem with respect to (s, ξ) :

Problem (RP)^{s,ξ} : $\inf_{\beta \in \mathcal{B}_{s, T}} \inf_{u \in L^2(\Omega \times (s, T))} \{ \|u\|_{L^2(\Omega \times (s, T))} \mid y(T; \omega, u; y_0) = 0 \}. \tag{2.21}$

By the same argument in Section 1, Problem (RP)^{s,ξ} is equivalent to the following Problem (SP)^{s,ξ} in the relaxed case.

Problem (RSP)^{s,ξ} : $\sup_{\beta \in \mathcal{B}_{s, T}} \inf_{z \in L^2(\Omega)} \left[\frac{1}{2} \|\beta \varphi(\cdot; z)\|_{L^2(\Omega \times (s, T))}^2 + \langle \xi, \varphi(s; z) \rangle \right]. \tag{2.22}$

3. Relaxed Stackelberg game problem

Let us recall some basic facts of the two-person zero-sum game problem. There are two players: Emil and Frances. Emil takes his strategy x from his strategy set E and Frances takes his strategy y from his strategy set F . Let $f : E \times F$ be the index cost function. Emil wants to minimize the function F while Frances wants to maximize F . In the framework of two-person zero-sum game, any solution to $\inf_{x \in E} \sup_{y \in F} f(x, y)$ is called a conservative strategy of Emil while any solution to $\sup_{y \in F} \inf_{x \in E} f(x, y)$ is called a conservative strategy of Frances [28]. For a game problem, the Nash equilibrium is the most important concept.

Definition 3.1. Suppose that E and F are strategy sets of Emil and Frances, respectively. Let $f : E \times F \mapsto \mathbb{R}$ be an index cost functional. We call $(\bar{x}, \bar{y}) \in E \times F$ a Nash equilibrium if,

$$f(\bar{x}, y) \leq f(\bar{x}, \bar{y}) \leq f(x, \bar{y}), \quad \forall x \in E, y \in F.$$

The following result is well known, see, for instance, Proposition 8.1 of [1, p. 121]. It connects the Stackelberg equilibrium with the Nash equilibrium.

Proposition 3.2. *The following conditions are equivalent.*

- (i) (\bar{x}, \bar{y}) is a Nash equilibrium;
- (ii) $V^+ = V^-$ and \bar{x} is a conservative strategy of Emil (equivalently, the following equation holds):

$$V^+ \triangleq \inf_{x \in E} \sup_{y \in F} f(x, y) = \sup_{y \in F} f(\bar{x}, y),$$

and \bar{y} is a conservative strategy of Frances (equivalently, the following equation holds):

$$V^- \triangleq \sup_{y \in F} \inf_{x \in E} f(x, y) = \inf_{x \in E} f(x, \bar{y}).$$

When $V^+ = V^-$, we say that the game problem attains its value at V^+ .

Notice that Problem (RSP)^{s,ξ} is a typical Stackelberg game problem and we will discuss it in the framework of two-person zero-sum game theory. Let

$$\mathcal{B}_{s,T}^2 = \left\{ b \in L^\infty(\Omega \times (s, T); [0, 1]) \mid \int_{\Omega} b(x, t) dx \equiv \alpha \cdot m(\Omega) \text{ a.e. } t \in (s, T) \right\} \tag{3.1}$$

and define an index cost functional by

$$F(b, \psi) = -\frac{1}{2} \iint_{\Omega \times (s, T)} b(x, t) \psi^2(x, t) dx dt - \langle \xi, \psi(s) \rangle, \quad \forall \psi \in Y, b \in \mathcal{B}_{s,T}^2. \tag{3.2}$$

We assume that Emil who controls the function $b \in \mathcal{B}_{s,T}^2$ wants to minimize F and likewise, Frances who controls the function $\psi \in Y$ wants to maximize F . Then Problem (RSP) has the following equivalent form:

Problem (RSP1) : $V^+ \triangleq \inf_{b \in \mathcal{B}_{s,T}^2} \sup_{\psi \in Y} F(b, \psi) = \inf_{b \in \mathcal{B}_{s,T}^2} \sup_{\psi \in \bar{Y}_\beta} F(b, \psi) \tag{3.3}$

with $\beta = \sqrt{b}$.

Theorem 3.3. *Problem (RSP1) admits a solution in $\mathcal{B}_{s,T}^2$.*

Proof. For any $\psi \in Y$, it is clear that the functional $F(\cdot, \psi)$ is linear and hence it is weakly* lower semi-continuous. Let $X = L^\infty(\Omega)$ be equipped with the weak* topology. Then $F(\cdot, \psi)$ is lower semi-continuous under the topology of X . If we denote

$$\hat{F}(b) = \sup_{\psi \in Y} F(b, \psi), \forall b \in \mathcal{B}_{s,T}^2,$$

then $\hat{F}(b)$ is also lower semi-continuous. In addition, since $\mathcal{B}_{s,T}^2$ is compact under the topology of X , there exists at least one solution solving $\inf_{b \in \mathcal{B}_{s,T}^2} \hat{F}(b)$. Therefore, the game Problem (RSP1)

admits a solution in $\mathcal{B}_{s,T}^2$. \square

3.1. Value attainability for zero-sum game

In this subsection, we will make use of the game theory to discuss value attainability for above two-person zero-sum game. More precisely, denote by

$$\textbf{Problem (RSP2): } V^- \triangleq \sup_{\psi \in Y} \inf_{b \in \mathcal{B}_{s,T}^2} F(b, \psi). \tag{3.4}$$

Once $V^+ = V^-$, we say that the above two-person zero-sum game attains its value. Furthermore, it is possible to characterize the conservative strategy of Frances (solutions to Problem (RSP1)) by using Proposition 3.2. To this end, we introduce an intermediate value \hat{V} and prove successively that $V^- = \hat{V}$ under topological assumptions, and that $\hat{V} = V^+$ under convexity assumptions.

We denote by \mathcal{K} all the finite subsets of Y . For any $K \in \mathcal{K}$, set

$$V_K = \inf_{b \in \mathcal{B}_{s,T}^2} \sup_{\psi \in K} F(b, \psi), \hat{V} \triangleq \inf_{K \in \mathcal{K}} V_K = \sup_{K \in \mathcal{K}} \inf_{b \in \mathcal{B}_{s,T}^2} \sup_{\psi \in K} F(b, \psi). \tag{3.5}$$

Then

$$V^- \leq \hat{V} \leq V^+. \tag{3.6}$$

Lemma 3.4. Let V^+ and \hat{V} be defined by (RSP1) and (RSP2) respectively. Then

$$V^+ = \hat{V}. \tag{3.7}$$

Proof. For any $\psi \in Y$, it is clear that the functional $F(\cdot, \psi)$ is sequentially weakly* lower semi-continuous. Furthermore, for any $K = \{\psi_1, \psi_2, \dots, \psi_n\} \in \mathcal{K}$, functional $\sup_{\psi \in K} F(\cdot, \psi)$ is

also sequentially weakly* lower semi-continuous. This, together with the compactness of $\mathcal{B}_{s,T}^2$, implies that there is $b_K \in \mathcal{B}_{s,T}^2$ such that

$$\sup_{\psi \in K} F(b_K, \psi) = \inf_{b \in \mathcal{B}_{s,T}^2} \sup_{\psi \in K} F(b, \psi).$$

It then follows from the definition of \hat{V} that

$$F(b_K, \psi) \leq \sup_{\hat{\psi} \in K} F(b_K, \hat{\psi}) = \inf_{b \in \mathcal{B}_{s,T}^2} \sup_{\hat{\psi} \in K} F(b, \hat{\psi}) \leq \sup_{\hat{K} \in \mathcal{K}} \inf_{b \in \mathcal{B}_{s,T}^2} \sup_{\hat{\psi} \in \hat{K}} F(b, \hat{\psi}) = \hat{V}, \quad \forall \psi \in K. \tag{3.8}$$

If we denote by

$$S_\psi \triangleq \left\{ b \in \mathcal{B}_{s,T}^2 \mid F(b, \psi) \leq \hat{V} \right\}$$

for any $\psi \in Y$, then it follows from (3.8) that $b_K \in \bigcap_{\psi \in K} S_\psi$ and hence

$$\bigcap_{\psi \in K} S_\psi \neq \emptyset \text{ for any } K \in \mathcal{K}. \tag{3.9}$$

In addition, since $F(\cdot, \psi)$ is weakly* lower semi-continuous, S_ψ is weakly* closed in $L^\infty(\Omega \times (s, T))$ as well. In other words, S_ψ is closed under the weak* topology of $L^\infty(\Omega \times (s, T))$. We claim that

$$\bigcap_{\psi \in Y} S_\psi \neq \emptyset. \tag{3.10}$$

Indeed, if the above condition fails, then $\bigcup_{\psi \in Y} \mathcal{B}_{s,T}^2 \setminus S_\psi = \mathcal{B}_{s,T}^2$. It follows from the compactness of $\mathcal{B}_{s,T}^2$ that there is $\hat{K} \in \mathcal{K}$ such that

$$\bigcup_{\psi \in \hat{K}} \mathcal{B}_{s,T}^2 \setminus S_\psi = \mathcal{B}_{s,T}^2.$$

This contradicts to (3.9). Select \bar{b} in the set $\bigcap_{\psi \in Y} S_\psi$. Then

$$\sup_{\psi \in Y} F(\bar{b}, \psi) \leq \hat{V},$$

and so

$$\inf_{b \in \mathcal{B}_{s,T}^2} \sup_{\psi \in Y} F(b, \psi) \leq \hat{V}.$$

This, together with (3.6), completes the proof of the lemma. \square

The following Proposition 3.5 is Proposition 8.3 of [1, p. 132].

Proposition 3.5. *Let \hat{E} and \hat{F} be two convex sets and let the function $f(\cdot, \cdot)$ be defined in $\hat{E} \times \hat{F}$. Let \mathcal{F} be the set of all finite subsets of \hat{F} and*

$$\hat{V} = \sup_{K \in \mathcal{F}} \inf_{x \in \hat{E}} \sup_{\psi \in K} f(x, y), \quad V^- = \sup_{y \in \hat{F}} \inf_{x \in \hat{E}} f(x, y).$$

Suppose that a) for any $y \in \hat{F}$, $x \rightarrow f(x, y)$ is convex; and b) for any $x \in \hat{E}$, $x \rightarrow f(x, y)$ is concave. Then $\hat{V} = V^-$.

Theorem 3.6. Let V^+ and \hat{V} be defined by (RSP1) and (RSP2), respectively. Then

$$V^- = V^+. \tag{3.11}$$

Proof. It is clear that both $\mathcal{B}_{s,T}^2$ and Y are convex. We can verify directly that the functional $F(\cdot, \psi)$ is linear and hence is convex for any and $\psi \in Y$. In addition, the functional $F(b, \cdot)$ is concave for any $b \in \mathcal{B}_{s,T}^2$. Thus $\hat{V} = V^-$ in terms of Proposition 3.5. The equality (3.11) is then derived by applying Lemma 3.4. This completes the proof of the lemma. \square

3.2. Nash equilibrium

The value attainability for a given two-person zero-sum game is a necessary condition to the existence of the Nash equilibrium. To discuss further about the solution to the Stackleberg game Problem (RSP1) or equivalently Problem (RSP)^{s,ξ}, we need to discuss another Stackleberg game Problem (RSP2), in other words, we should discuss the following problem:

$$\inf_{\psi \in Y} \sup_{b \in \mathcal{B}_{s,T}^2} \left[\frac{1}{2} \int_s^T \int_{\Omega} b(x, t) \psi(x, t)^2 dx dt + \langle \xi, \psi(s) \rangle \right]. \tag{3.12}$$

Define a non-negative nonlinear functional on Y by

$$NF(\psi) = \sup_{b \in \mathcal{B}_{s,T}^2} \left(\int_s^T \int_{\Omega} b(x, t) \psi(x, t)^2 dx dt \right)^{\frac{1}{2}}, \quad \forall \psi \in Y. \tag{3.13}$$

Lemma 3.7. Let $NF(\cdot)$ be the functional defined by (3.13). Then $NF(\cdot)$ is a norm for the space Y defined by (2.4).

Proof. It is clear that

$$NF(\psi) \geq 0, \quad \forall \psi \in Y \text{ and } \psi = 0 \Rightarrow NF(\psi) = 0.$$

By the relation between $\mathcal{B}_{s,T}$ and $\mathcal{B}_{s,T}^2$,

$$NF(\psi) = \sup_{\beta \in \mathcal{B}_{s,T}} \|\beta \psi\|_{L^2(\Omega \times (s, T))}.$$

Furthermore, if $NF(\psi) = 0$, then $\beta \psi = 0$ for any $\beta \in \mathcal{B}_{s,T}$. Take

$$\hat{\beta}(x, t) \equiv \chi_{\omega_1}(x) \text{ with } m(\omega_1) = \alpha \cdot m(\Omega).$$

It then follows from the unique continuation for heat equation [2] that $\psi(x, t) = 0$. Therefore, $NF(\psi) = 0$ if and only if $\psi(x, t) = 0$. Finally, a direct computation shows that

$$NF(c\psi) = |c|NF(\psi), \forall \psi \in Y, c \in \mathbb{R}.$$

By

$$\|\beta(\psi_1 + \psi_2)\|_{L^2(\Omega \times (s, T))} \leq \|\beta\psi_1\|_{L^2(\Omega \times (s, T))} + \|\beta\psi_2\|_{L^2(\Omega \times (s, T))}, \quad \forall \beta \in \mathcal{B}_{s, T},$$

we have

$$\begin{aligned} & \left(\int_s^T \int_{\Omega} b(x, t)(\psi_1(x, t) + \psi_2(x, t))^2 dx dt \right)^{\frac{1}{2}} \\ & \leq \left(\int_s^T \int_{\Omega} b(x, t)\psi_1(x, t)^2 dx dt \right)^{\frac{1}{2}} + \left(\int_s^T \int_{\Omega} b(x, t)\psi_2(x, t)^2 dx dt \right)^{\frac{1}{2}}. \end{aligned}$$

So,

$$NF(\psi_1 + \psi_2) \leq NF(\psi_1) + NF(\psi_2).$$

This shows that NF is a norm of the space Y . \square

Definition 3.8. Owing to Lemma 3.7, we can denote the norm given by the functional $NF(\cdot)$ as $\|\cdot\|_{NF}$. It is clear that the space $(Y, \|\cdot\|_{NF})$ is a normed linear space. We set $(\bar{Y}, \|\cdot\|_{NF})$ as the completion space of $(Y, \|\cdot\|_{NF})$.

Along the same line in the proof of Lemma 2.3, we have Lemma 3.9.

Lemma 3.9. Under an isometric isomorphism, any element of \bar{Y} can be expressed as a function $\hat{\varphi} \in C([0, T]; L^2(\Omega))$ which satisfies (in the sense of weak solution)

$$\begin{cases} \hat{\varphi}_t(x, t) + \Delta \hat{\varphi}(x, t) - a(x, t)\hat{\varphi}(x, t) = 0 & \text{in } \Omega \times (s, T), \\ \hat{\varphi}(x, t) = 0 & \text{on } \partial\Omega \times (s, T), \end{cases}$$

and $NF(\hat{\varphi}) = \lim_{n \rightarrow \infty} NF(\varphi(\cdot; z_n))$ for some sequence $\{z_n\} \subset L^2(\Omega)$, where $\varphi(\cdot; z_n)$ is the solution of (1.8) with initial value $z = z_n$.

We present a further characterization of \bar{Y} .

Lemma 3.10. Let Z be defined as (1.12). Then

$$\bar{Y} = \{\varphi(\cdot; z) \mid z \in Z\}, \tag{3.14}$$

where $\varphi(\cdot, z)$ is the solution to (1.8). Moreover,

$$\sup_{\psi \in Y} F(b, \psi) = \sup_{\psi \in \bar{Y}} F(b, \psi) = \sup_{\psi \in \bar{Y}_\beta} F(b, \psi). \tag{3.15}$$

Proof. We claim by virtue of [Lemma 3.9](#) that

$$\overline{Y} \subseteq L^2(\Omega \times (s, T)). \tag{3.16}$$

Indeed, suppose that $n_0 \in \mathbb{N}$ so that $n_0 \geq 1/\alpha$. There are n_0 measurable subsets $\omega_1, \omega_2, \dots, \omega_{n_0}$ of Ω such that

$$m(\omega_j) = \alpha \cdot m(\Omega), \quad \forall j \in \{1, 2, \dots, n_0\}, \quad \bigcup_{j=1}^{n_0} \omega_j = \Omega.$$

The inclusion (3.16) then follows from

$$\begin{aligned} \int_s^T \int_{\Omega} \psi(x, t)^2 dx dt &\leq \int_s^T \left(\sum_{j=1}^{n_0} \int_{\Omega} \chi_{\omega_j}(x) \psi(x, t)^2 dx \right) dt \\ &\leq \sum_{j=1}^{n_0} \int_s^T \int_{\Omega} \chi_{\omega_j}(x) \psi(x, t)^2 dx dt \leq n_0 \|\psi\|_{NF}^2. \end{aligned} \tag{3.17}$$

Since $\psi(x, t)$ is a generalized function defined on $\Omega \times (s, T)$ and belongs to $L^2(\Omega \times (s, T))$, and $\Omega \times T$ is the boundary of $\Omega \times (s, T)$, the inclusion (3.16), together with the trace theorem, implies (3.14). Furthermore, for any $\beta \in \mathcal{B}_{s,T}$, by

$$\|\beta\psi\|_{L^2(s,T;L^2(\Omega))} \leq NF(\psi), \quad \forall \psi \in Y,$$

it follows that

$$\overline{Y} \subseteq \overline{Y_\beta}, \quad \forall \beta \in \mathcal{B}_{s,T}. \tag{3.18}$$

Since Y is dense in $\overline{Y_\beta}$ and $\sup_{\psi \in Y} F(b, \psi) = \sup_{\psi \in \overline{Y_\beta}} F(b, \psi)$ with $b = \beta^2$, we obtain (3.15). \square

Now, we discuss the following game problem (with the extend domain of Problem (RSP2) or Problem (3.12)).

$$\begin{aligned} \textbf{Problem (RSP2')} : \quad &\inf_{\psi \in \overline{Y}} \sup_{b \in \mathcal{B}_{s,T}^2} \left[\frac{1}{2} \int_s^T \int_{\Omega} b(x, t) \psi(x, t)^2 dx dt + \langle \xi, \psi(s) \rangle \right] \\ &= \inf_{\psi \in \overline{Y}} \left[\frac{1}{2} \|\psi\|_{NF}^2 + \langle \xi, \psi(s) \rangle \right]. \end{aligned} \tag{3.19}$$

Notice that the functional in Problem (RSP2') is strictly convex, coercive, and continuous. Besides, \overline{Y} , as a closed subspace of $L^2(\Omega \times (s, T))$, is also reflexive. Similarly to [Lemma 2.5](#), we have [Lemma 3.11](#).

Lemma 3.11. For any $s \in [0, T)$ and $\xi \in L^2(\Omega) \setminus \{0\}$, Problem (RSP2') admits a unique nonzero solution.

Now we present the Nash equilibrium problem of two-person zero-sum game:

Problem (NEGP) : To find $\bar{b} \in \mathcal{B}_{s,T}^2, \bar{\psi} \in \bar{Y}$ such that

$$\begin{aligned} F(\bar{b}, \bar{\psi}) &= \sup_{\psi \in \bar{Y}} F(\bar{b}, \psi) \\ &= \inf_{b \in \mathcal{B}_{s,T}^2} F(b, \bar{\psi}), \end{aligned} \tag{3.20}$$

where $F(b, \psi)$ is defined by (3.2). The following **Theorem 3.12** is about existence of Nash equilibrium to the two-person zero-sum game Problem (NEGP).

Theorem 3.12. *Let $\bar{\psi}$ be a solution to Problem (RSP2'). Then Problem (NEGP) admits at least one Nash equilibrium. Furthermore, if $\bar{\beta}$ is a relaxed optimal actuator location to Problem (RP)^{s,ξ}, then $(\bar{b} = \bar{\beta}^2, \bar{\psi})$ is a Nash equilibrium to Problem (NEGP). Conversely, if $(\hat{b}, \hat{\psi})$ is a Nash equilibrium of Problem (NEGP), then $\hat{\psi} = \bar{\psi}$, and $\hat{\beta} = \hat{b}^{1/2}$ is a relaxed optimal actuator location to Problem (RP)^{s,ξ}.*

Proof. In terms of (3.15),

$$V^+ = \inf_{b \in \mathcal{B}_{s,T}^2} \sup_{\psi \in Y} F(b, \psi) = \inf_{b \in \mathcal{B}_{s,T}^2} \sup_{\psi \in \bar{Y}} F(b, \psi). \tag{3.21}$$

Notice that

$$V^- = \sup_{\psi \in Y} \inf_{b \in \mathcal{B}_{s,T}^2} F(b, \psi) \leq \sup_{\psi \in \bar{Y}} \inf_{b \in \mathcal{B}_{s,T}^2} F(b, \psi) \leq \inf_{b \in \mathcal{B}_{s,T}^2} \sup_{\psi \in \bar{Y}} F(b, \psi).$$

It follows from **Theorem 3.6** that

$$\inf_{b \in \mathcal{B}_{s,T}^2} \sup_{\psi \in \bar{Y}} F(b, \psi) = \sup_{\psi \in \bar{Y}} \inf_{b \in \mathcal{B}_{s,T}^2} F(b, \psi). \tag{3.22}$$

Furthermore, by (3.21) and the relation between $\mathcal{B}_{s,T}$ and $\mathcal{B}_{s,T}^2$,

$$\begin{aligned} \text{if } \bar{\beta} \text{ is a solution to Problem (RSP)}^{s,\xi}, \text{ then } \bar{b} \text{ is a solution to } \inf_{b \in \mathcal{B}_{s,T}^2} \sup_{\psi \in \bar{Y}} F(b, \psi); \\ \text{if } \bar{b} \text{ is a solution to } \inf_{b \in \mathcal{B}_{s,T}^2} \sup_{\psi \in \bar{Y}} F(b, \psi), \text{ then } \bar{\beta} \text{ is a solution to Problem (RSP)}^{s,\xi}, \end{aligned} \tag{3.23}$$

where $\bar{b} = \bar{\beta}^2$. Recalling **Proposition 3.2**, we have the following results:

- Equation (3.22) ensures that Problem (NEGP) attains its value;
- Problem (RSP2') admits a unique solution $\bar{\psi}$ by **Lemma 3.11**;
- Problem (RSP1) admits a solution by **Theorem 3.3** and (3.23).

It follows from **Proposition 3.2** that Problem (NEGP) admits at least one Nash equilibrium. Furthermore, if \bar{b} is a solution to $\inf_{b \in \mathcal{B}_{s,T}^2} \sup_{\psi \in \bar{Y}} F(b, \psi)$, then $(\bar{b}, \bar{\psi})$ is a Nash equilibrium to Problem

(NEGP). Conversely, if $(\hat{b}, \hat{\psi})$ is a Nash equilibrium of Problem (NEGP), then \hat{b} is a solution to problem $\inf_{b \in \mathcal{B}_{s,T}^2} \sup_{\psi \in \bar{\mathcal{Y}}} F(b, \psi)$ and $\hat{\psi}$ solves $\sup_{\psi \in \bar{\mathcal{Y}}} \inf_{b \in \mathcal{B}_{s,T}^2} F(b, \psi)$. By the uniqueness from

Lemma 3.11, it holds that $\hat{\psi} = \bar{\psi}$. This, together with (3.23) and the equivalence between Problem (RSP)^{s,ξ} and Problem (RP)^{s,ξ}, proves **Theorem 3.12** directly. □

4. Proof of the main results

In this section, we present proofs for **Theorem 1.1** and a time-varying feedback.

4.1. Existence and uniqueness of relaxed optimal actuator location

Though we have derived the existence for the relaxation problem (RP)^{s,ξ}, existence for the optimal actuator location to the classical problem (CP)^{s,ξ} is still not known. To this purpose, we need to learn more about the optimal relaxed actuator location $\bar{\beta}$. Recall **Theorem 3.12** that if $\bar{\beta}$ is a relaxed actuator location, then $\bar{b} = \bar{\beta}^2$ solves Problem $\sup_{\psi \in \bar{\mathcal{Y}}} F(\bar{b}, \psi)$. That is to say, \bar{b} solves

$$\sup_{b \in \mathcal{B}_{s,T}^2} \int_s^T \int_{\Omega} b(x, t) \bar{\psi}(x, t)^2 dx dt. \tag{4.1}$$

Further, if we denote

$$\Gamma = \left\{ \gamma \in L^\infty(\Omega; [0, 1]) \mid \int_{\Omega} \gamma(x) dx = \alpha \cdot m(\Omega) \right\},$$

then

$$\int_{\Omega} \bar{b}(x, t) \bar{\psi}(x, t)^2 dx = \sup_{\gamma \in \Gamma} \int_{\Omega} \gamma(x) \bar{\psi}(x, t)^2 dx, \quad t \in (s, T) \text{ a.e.} \tag{4.2}$$

and

$$\bar{b}(\cdot, t) \in \operatorname{argmax}_{\gamma \in \Gamma} \int_{\Omega} \gamma(x) \bar{\psi}(x, t)^2 dx, \quad t \in (s, T) \text{ a.e.} \tag{4.3}$$

Therefore, we need to discuss the following problem

$$\sup_{\gamma \in \Gamma} \int_{\Omega} \gamma(x) \phi(x) dx, \tag{4.4}$$

where $\phi \in L^1(\Omega)$. Similar problem has been discussed in [20] where Γ is replaced by \mathcal{W} . But for the sake of completeness, we present here a short proof.

Let us define, for any $\phi \in L^1(\Omega)$ and $c \in \mathbb{R}$, that

$$\begin{aligned} \{\phi \geq c\} &= \{x \in \Omega \mid \phi(x) \geq c\}, & \{\phi = c\} &= \{x \in \Omega \mid \phi(x) = c\}, \\ \{\phi > c\} &= \{x \in \Omega \mid \phi(x) > c\}, & \{\phi < c\} &= \{x \in \Omega \mid \phi(x) < c\}. \end{aligned} \tag{4.5}$$

Let

$$M_\phi(c) = m(\{\phi \geq c\}) \text{ for any } \phi \in L^1(\Omega) \text{ and } c \in \mathbb{R}. \tag{4.6}$$

It is clear that the function $M_\phi(c)$ is monotone decreasing with respect to c . By

$$\lim_{\varepsilon \rightarrow 0^+} \{\phi \geq c - \varepsilon\} = \bigcap_{\varepsilon > 0} \{\phi \geq c - \varepsilon\} = \{\phi \geq c\},$$

we have

$$\lim_{\varepsilon \rightarrow 0^+} M_\phi(c - \varepsilon) = M_\phi(c), \quad \forall c \in \mathbb{R}. \tag{4.7}$$

This shows that $M_\phi(c)$ is continuous from the left for any given $\phi \in L^1(\Omega)$. Since

$$\lim_{c \rightarrow +\infty} M_\phi(c) = 0, \quad \lim_{c \rightarrow -\infty} M_\phi(c) = m(\Omega),$$

the real c_ϕ given by

$$c_\phi = \max \{c \in \mathbb{R} \mid M_\phi(c) \geq \alpha \cdot m(\Omega)\}, \tag{4.8}$$

is well-defined. Hence

$$M_\phi(c_\phi) \geq \alpha \cdot m(\Omega) \geq M_\phi(c_\phi +) \stackrel{\Delta}{=} \lim_{\varepsilon \rightarrow 0^+} M_\phi(c_\phi + \varepsilon), \tag{4.9}$$

and

$$M_\phi(c_\phi + \varepsilon) < \alpha m(\Omega), \quad \forall \varepsilon > 0. \tag{4.10}$$

Let

$$\bar{\alpha}_\phi \stackrel{\Delta}{=} \frac{M_\phi(c_\phi)}{m(\Omega)}, \quad \underline{\alpha}_\phi \stackrel{\Delta}{=} \frac{M_\phi(c_\phi +)}{m(\Omega)}. \tag{4.11}$$

It follows from (4.9) that

$$\bar{\alpha}_\phi \geq \alpha \geq \underline{\alpha}_\phi. \tag{4.12}$$

Since

$$\lim_{\varepsilon \rightarrow 0^+} \{\phi \geq c + \varepsilon\} = \bigcup_{\varepsilon > 0} \{\phi \geq c + \varepsilon\} = \{\phi > c\},$$

it follows that

$$M_\phi(c_\phi+) = m(\{\phi > c_\phi\}).$$

By the definition of $\underline{\alpha}_\phi$ in (4.11),

$$m(\{\phi > c_\phi\}) = \underline{\alpha}_\phi \cdot m(\Omega). \tag{4.13}$$

This, together with (4.11) and (4.12), implies that

$$m(\{\phi = c_\phi\}) = (\bar{\alpha}_\phi - \underline{\alpha}_\phi)m(\Omega) \geq (\alpha - \underline{\alpha}_\phi)m(\Omega). \tag{4.14}$$

The following result is about problem (4.4).

Lemma 4.1. *Let $\phi \in W_0^{1,1}(\Omega)$. If $\phi(x) \neq 0$ is analytic in Ω , then Problem (4.4) admits a unique solution $\bar{\gamma}$. Furthermore, it holds that*

$$\bar{\gamma} \in \mathcal{W}. \tag{4.15}$$

Proof. Because $\phi(x)$ is analytic, it is clear that

$$m(\{\phi = c\}) = 0 \text{ or } m(\{\phi = c\}) = m(\Omega) \text{ for any } c \in \mathbb{R}. \tag{4.16}$$

Furthermore, we claim that

$$m(\{\phi = c\}) = 0 \text{ for any } c \in \mathbb{R}. \tag{4.17}$$

Indeed, it follows from $\phi \neq 0$ and (4.16) that

$$m(\{\phi = 0\}) = 0.$$

On the other hand, suppose there is $c \neq 0$ such that

$$m(\{\phi = c\}) = m(\Omega).$$

That is to say, $\phi(x) = c$ in Ω almost everywhere. Then the trace of $\phi(x)$ is just c . This contradicts $\phi \in W_0^{1,1}(\Omega)$. The claim is then proved.

Let $c_\phi, \bar{\alpha}_\phi, \underline{\alpha}_\phi$ defined in (4.8) and (4.11). It follows from (4.14) and (4.17) that

$$\bar{\alpha}_\phi = \underline{\alpha}_\phi = \alpha.$$

It follows from (4.13) and (4.14) that

$$m(\{\phi \geq c_\phi\}) = \alpha \cdot m(\Omega).$$

That is, $\{\phi \geq c_\phi\} \in \mathcal{W}$. Since Γ is the convex hull of $\{\chi_\omega | \omega \in \mathcal{W}\}$, it holds

$$\sup_{\gamma \in \Gamma} \int \gamma \phi dx = \sup_{\omega \in \mathcal{W}} \int \chi_\omega \phi dx.$$

If we can show that

$$\int \chi_{\{\phi \geq c_\phi\}} \phi dx > \int \chi_\omega \phi dx, \quad \forall \omega \in \mathcal{W}, \chi_\omega \neq \chi_{\{\phi \geq c_\phi\}},$$

then $\chi_{\{\phi \geq c_\phi\}}$ is the unique solution to problem (4.4) and belongs to \mathcal{W} . To this purpose, let $\omega_1 = \omega \setminus \{\phi \geq c_\phi\}$, $\omega_2 = \{\phi \geq c_\phi\} \setminus \omega$, and $\omega_3 = \omega \cap \{\phi \geq c_\phi\}$. Since ω and $\{\phi \geq c_\phi\}$ belong to \mathcal{W} , it holds

$$m(\omega_1) = m(\omega_2) \neq 0.$$

On the other hand, since

$$\phi(x) \geq c_\phi > \phi(y) \quad \forall x \in \omega_2, y \in \omega_1,$$

we thus have

$$\int \chi_{\{\phi \geq c_\phi\}} \phi dx = \int_{\omega_2} \phi dx + \int_{\omega_3} \phi dx > \int_{\omega_1} \phi dx + \int_{\omega_3} \phi dx = \int \chi_\omega \phi dx.$$

Therefore, $\chi_{\{\phi \geq c_\phi\}}$ is the unique solution to problem (4.4) and belongs to \mathcal{W} . \square

Proof of Theorem 1.1. Recall that the coefficient $a(x, t)$ is analytic. Thus the solution to Equation (1.8) with the initial condition $z \in L^2(\Omega)$ is also analytic in $\Omega \times (s, T)$ [2]. As the solution to Problem (RSP2'),

$$\bar{\psi}(\cdot, T - \varepsilon) \in L^2(\Omega) \quad \text{for any } \varepsilon > 0.$$

Thus $\bar{\psi}$ is analytic in $\Omega \times (s, T - \varepsilon)$. By the arbitrariness of ε , $\bar{\psi}$ is analytic in $\Omega \times (s, T)$. On the other hand, it follows from the smooth effect of the heat equation that

$$\bar{\psi}(\cdot, t) \in H_0^1(\Omega) \quad \text{for any } t \in (s, T).$$

Those, together with the non-singularity of $\bar{\psi}$, imply that

$$\bar{\psi}(\cdot, t)^2 \text{ is nonzero analytic in } \Omega \text{ and } \bar{\psi}(\cdot, t)^2 \in W_0^{1,1}(\Omega) \text{ for any } t \in (s, T).$$

By Lemma 4.1 and (4.2), \bar{b} is unique and belongs to $\mathcal{W}_{s,T}$. Therefore, it follows from Theorem 3.12 that any relaxed optimal actuator location must be classical and unique. We thus complete the proof of the theorem. \square

Proof of Corollary 1.2. We use the synthetic method, to obtain the time-varying feedback and prove the corresponding result by the dynamic programming approach. The synthetic method is a method to be used to construct a time-varying feedback control through open-loop control reflected mathematically by (4.18) and (4.19) later (see, e.g., [27]).

Now, for any $(s, \xi) \in [0, T) \times L^2(\Omega) \setminus \{0\}$, denote the optimal actuator location by $w^{s,\xi} \in \mathcal{W}_{s,T}$ and the corresponding optimal control of Problem (NP) $_{w^{s,\xi}}^{s,\xi}$ by $u^{s,\xi} \in L^2(\Omega \times (s, T))$. Write the corresponding optimal trajectory by $y^{s,\xi} \in C([s, T]; L^2(\Omega))$. Based on these notations, we

begin to define $\mathcal{F} : [0, T) \times L^2(\Omega) \mapsto \mathcal{W}$ by

$$\mathcal{F}(s, \xi) = w^{s,\xi}(s) \text{ for any } (s, \xi) \in [0, T) \times L^2(\Omega), \tag{4.18}$$

and define $\mathcal{G} : [0, T) \times L^2(\Omega) \mapsto L^2(\Omega)$ by

$$\mathcal{G}(s, \xi) = u^{s,\xi}(s) \text{ for any } (s, \xi) \in [0, T) \times L^2(\Omega). \tag{4.19}$$

The above definition is well-defined. Indeed, as the solution of Problem (RSP2'), $\bar{\psi} \in C([s, T); L^2(\Omega))$. It follows from (4.3) that $w^{s,\xi} \in C([s, T); L^2(\Omega))$. By Lemma 2.5, $u^{s,\xi} \in w^{s,\xi} \bar{Y}_{w^{s,\xi}}$. This, together with the continuity of $w^{s,\xi}$, implies that $u^{s,\xi} \in C([s, T); L^2(\Omega))$.

Fix $(s, \xi) \in [0, T) \times L^2(\Omega)$. We will show that $y^{s,\xi}$ defined as above is just the unique solution of Equation (1.4) satisfying $y^{\mathcal{F},\mathcal{G}}((x, T); s, \xi) = 0$ and (1.5)–(1.7). The proof will be carried out by the following several steps.

Step 1: $u^{s,\xi}|_{[t,T)}$ is the solution to Problem (NP) $_{w^{s,\xi}|_{[t,T)}, t, y^{s,\xi}(t)}$.

Notice that

$$y\left(T; w^{s,\xi}|_{[t,T)}, u^{s,\xi}|_{[t,T)}; t, y^{s,\xi}(t)\right) = y\left(T; w^{s,\xi}, u^{s,\xi}; s, \xi\right) = 0.$$

If there is $v \in L^2(\Omega \times (t, T))$ such that

$$y\left(T; w^{s,\xi}|_{[t,T)}, v; t, y^{s,\xi}(t)\right) = 0 \text{ with } \|v\|_{L^2(\Omega \times (t,T))} < \|u^{s,\xi}|_{[t,T)}\|_{L^2(\Omega \times (t,T))},$$

by setting

$$\hat{v}(r) = \begin{cases} u^{s,\xi}(r), & \text{when } r \in [s, t) \\ v(r) & \text{when } r \in [t, T), \end{cases}$$

we find that $\hat{v} \in L^2(\Omega \times (s, T))$ satisfies $y(T; w^{s,\xi}, \hat{v}; s, \xi) = 0$ and $\|\hat{v}\|_{L^2(\Omega \times [s,T))} < \|u^{s,\xi}\|_{L^2(\Omega \times [s,T))}$. This means that \hat{v} solves Problem (NP) $_{w^{s,\xi}, s, \xi}^{s,\xi}$, which contradicts with the optimality of $u^{s,\xi}$ and thus leads to claim of step 1.

Step 2: $w^{s,\xi}|_{[t,T)}$ is the solution to Problem (CP) $_{w^{s,\xi}|_{[t,T)}, t, y^{s,\xi}(t)}$.

Assume the above claim is false. Then there is $\hat{w} \in \mathcal{W}_{t,T}$ solving Problem (CP) $_{w^{s,\xi}|_{[t,T)}, t, y^{s,\xi}(t)}$. Denote by $\tilde{v} \in L^2(\Omega \times (t, T))$ the solution to Problem (NP) $_{\hat{w}, t, y^{s,\xi}(t)}$. By setting

$$\tilde{w}(r) = \begin{cases} w^{s,\xi}(r), & \text{when } r \in [s, t) \\ \hat{w}(r) & \text{when } r \in [t, T), \end{cases} \quad \hat{v}(r) = \begin{cases} u^{s,\xi}(r), & \text{when } r \in [s, t) \\ \tilde{v}(r) & \text{when } r \in [t, T), \end{cases}$$

we find that $y(T; \tilde{w}, \hat{v}; s, \xi) = 0$. Now we claim

$$\|\hat{v}\|_{L^2(\Omega \times [s,T))} < \|u^{s,\xi}\|_{L^2(\Omega \times [s,T))}. \tag{4.20}$$

Indeed, by Step 1,

$$N \left(w^{s,\xi} \Big|_{[t,T)}; t, y^{s,\xi}(t) \right) = \left\| u^{s,\xi} \Big|_{[t,T)} \right\|_{L^2(\Omega \times (t,T))}.$$

Because $N(\hat{w}; t, y^{s,\xi}(t)) = \|\tilde{v}\|_{L^2(\Omega \times (t,T))}$, it follows from the unique optimality of \hat{w} that

$$\left\| u^{s,\xi} \Big|_{[t,T)} \right\|_{L^2(\Omega \times (t,T))} > \|\tilde{v}\|_{L^2(\Omega \times (t,T))}.$$

This implies (4.20) and hence \tilde{w} solves Problem (CP)^{s,ξ} which is impossible. Thus we conclude the claim of Step 2.

Step 3: $y^{s,\xi}$ is the unique solution to (1.4) satisfying $y^{s,\xi}(T) = 0$ and (1.5)–(1.7).

It is clear that $y^{s,\xi}(T) = 0$. From Step 2, we have

$$\mathcal{F}(t, y^{s,\xi}(t)) = w^{s,\xi}(t) \quad \text{for any } t \in [s, T). \tag{4.21}$$

By Step 1, we have

$$\mathcal{G}(t, y^{s,\xi}(t)) = u^{s,\xi}(t) \quad \text{for any } t \in [s, T). \tag{4.22}$$

Thus $y^{s,\xi}$ is a solution to (1.4). In addition, it follows from (4.21)–(4.22) and the definition of $w^{\mathcal{F},\mathcal{G}}(s, \xi)$ and $u^{\mathcal{F},\mathcal{G}}(s, \xi)$ that

$$w^{\mathcal{F},\mathcal{G}}(s, \xi) = w^{s,\xi} \in \mathcal{W}_{s,T}, \quad u^{\mathcal{F},\mathcal{G}}(s, \xi) = u^{s,\xi} \in L^2(\Omega \times (s, T)). \tag{4.23}$$

This gives (1.6)–(1.7). The identities (1.5) follow straightforwardly from the optimality of $u^{s,\xi}$. Therefore, $y^{s,\xi}$ is a solution to (1.4) satisfying (1.5)–(1.7).

Finally, we come up uniqueness. From (1.4), we find that $w^{\mathcal{F},\mathcal{G}}(s, \xi)$ is a solution to Problem (CP)^{s,ξ}. The identities (4.23) follow from the uniqueness. In addition, as the solution to (1.4), $y^{s,\xi}$ must satisfy the following equation

$$\begin{cases} y_t(x, t) - \Delta y(x, t) + a(x, t)y(x, t) = (w^{s,\xi} u^{s,\xi})(x, t) & \text{in } \Omega \times (s, T), \\ y(x, t) = 0 & \text{on } \partial\Omega \times (s, T), \\ y(x, s) = \xi(x) & \text{in } \Omega. \end{cases}$$

It is clear that $y^{s,\xi}$ is the unique solution. We thus complete the proof. \square

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