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REVIEW

Review and new theoretical perspectives on active disturbance rejection control for uncertain finite-dimensional and infinite-dimensional systems

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Abstract The active disturbance rejection control (AD RC), first proposed by Jingqing Han in late 1980s, is a powerful control technology being able to deal with external disturbances and internal uncertainties in large scale for control systems in engineering applications. This survey paper will articulate, from a theoretical perspective, the origin, ideology and progress of ADRC for not only uncertain finite-dimensional systems but also uncertain infinite-dimensional ones. Some recent theoretical developments, general framework and unsolved problems of ADRC for finite-dimensional systems with mismatched disturbances and uncertainties by output feedback, uncertain finite-dimensional stochastic systems, uncertain infinite-dimensional systems described by both the wave equation and the fractional-order partial differential equation are successively addressed, from which we see the challenges and opportunities for this remarkable emerging control technology to various types of control systems.

Keywords Active disturbance rejection control · Extended state observer · Boundary control · Disturbance · Stochastic systems · Infinite-dimensional systems · Fractional-order PDE

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1 Introduction

Copying with disturbances and uncertainties is the eternal theme in control theory due to the ubiquitousness of the uncertainties and disturbances in most of the industrial control systems, which most often cause negative effects on performance and even stability of control systems [1–4]. There many control approaches have been developed since 1970s to cope with disturbances or uncertainties through disturbance attenuation and disturbance rejection. Among many others, stochastic control [5–9] and robust control [10–14] are two major disturbance attenuation methods, where the former is often applicable for attenuating disturbances in the form of noises with known statistical characteristics and the latter is to deal with more general norm bounded disturbances and uncertainties without concerning their statistical characteristics. For robust control approaches, the H_∞ and H_2 control approaches have been developed to attenuate the disturbances so that its influence to controlled output is kept into a desired level [15–18]. However, most of the robust controller designs are based on the worst case scenario that make control relatively conservative. On the other hand, the ideology of disturbance rejection could be found in adaptive control [19,20], internal model principle [21,22], disturbance observer-based control (DOBC) [23–26] and active disturbance rejection control (ADRC) [27–32], where the disturbances or uncertainties (or both of them) are estimated in real time and are eliminated in feedback loop so that these disturbance rejection control approaches have good anti-disturbance performance. However, the disturbances or uncertainties estimated and cancelled in the adaptive control framework and internal model principle are only in the form of internal unknown parameters and are some "almost known" ones generated from a dynamic exosystem, respectively.

Based on the error-driven rather than model-based thought, the powerful proportional–integral–derivative (PID) control law proposed during the period of the 1920s–1940s has dominated control practice for one century. This contributes largely to its model-free nature, while most other control theories are relying on the mathematical models. However, there are some limitations in existing PID framework in practical applications (see, e.g.,[31]) that (1) The setpoint is often chosen from some nonsmooth functions such as the step function, not applicable to most dynamics systems since the controlled output and the control signal will have a sudden jump in this sense. (2) The derivative part in PID may be not practically feasible because the classical differentiation is quite sensitive to noise and may amplify the noise. (3) The weight sum of the three terms in PID may not be the best choice based on the current and the past of the error and its rate of change. (4) The integral term in PID could lead to other limitations like saturation and reduced stability margin because of phase lag. The ADRC framework, as an alternative of PID, provides some corresponding technical and conceptual solutions to these limitations.

Motivated by the practical demands from industry to surmount available PID framework and the new challenges in control designs for systems subject to more general disturbances and uncertainties and improving performances of disturbance rejection and robustness, the idea of estimation/cancellation strategy has been fostered and enhanced by well-known ADRC, an almost model-free control technology proposed by Jingqing Han in the late 1980s [27–31]. ADRC is composed of three parts which include the tracking differentiator (TD), the extended state observer (ESO) and the ESO-based feedback control. The first part TD is a relatively independent part that carries forward PID directly, which not only acts as the derivative extraction, but also provides a transient process that the output of plant can reasonably track to avoid sudden jump in PID. The estimation/cancellation characteristic of ADRC is embodied in the configurations of ESO and ESO-based feedback control, which are capable of dealing with large-scale "total disturbance" representing the total effects of all potential unmodeled system dynamics, external disturbances, unknown control gain coefficient or even the part difficultly coped with by engineers, as long as they influence the performance of the controlled output. The concept of "disturbance" is significantly refined in this framework because all uncertainties affecting the performance of controlled output are regarded as "internal disturbance" of the plant, combined with the external disturbances to form the "total disturbance." The total disturbance is regarded as a signal of time from the "timescale" no matter they are state variables, inputs, outputs ones and the disturbances, which is reflected in observable measured output and then can be estimated. ESO is the most important part of ADRC, designed for the real-time estimation of not only the unmeasured state but also the total disturbance in large scale by the measured output. Once the total disturbance is estimated, an ESO-based feedback control for the stabilization or the output tracking of the nominal systems (without disturbances and uncertainties) and feed-forward compensation link, can be designed to compensate the total disturbance in real time and guarantee satisfactory performance and robustness of the resulting closed loop. It can be seen that ADRC is a systematic estimation/cancellation strategy to deal with disturbances and uncertainties in large scale compared with the adaptive control and internal model principle aforementioned. In addition, some nonlinear feedback combinations are explored in ADRC, not just the weight sum of the three terms in PID, have been proved to be very effective in improving performance and practicality from practices in the beginning and theory till recently. Finally, the conventional PID control problem could be transformed to real-time estimation and rejection of the total disturbance with a simplified controller without the integral term in PID, i.e., a PD controller. That is, the limitations of PID caused by the integral term mentioned above could be overcame in ADRC.

In the past two decades, the effectiveness and practicality of ADRC have been demonstrated in many engineering applications such as flywheel energy storage systems [33], robot control [34], predictive control for quadrotor helicopter [35], hydraulic servo systems [36], power plants [37,38] and many other ones in [39–61], to name just a few. Specially, the ADRC control technology has been applied in the general purpose control chips produced by Texas Instruments [62] and Freescale Semiconductor [63] and has been experimented in Parker Hannifin Parflex hose extrusion plant and across multiple production lines for over 8 months with the improvement of product performance capability index (Cpk) by 30% and the reduction of the energy consumption over 50% [64].

Although the theoretical research of ADRC lags behind its practical applications generally, some progresses, however, have been made in recent years. This survey paper will demonstrate a comprehensive review of ADRC for controlled plants from finite-dimensional systems to infinite-dimensional ones from a theoretical perspective. In particular, the recent theoretical developments, essentials and unsolved problems of ADRC for finite-dimensional systems with mismatched disturbances and uncertainties by output feedback, uncertain finite-dimensional stochastic systems, uncertain infinite-dimensional systems described by both the wave equation and the fractional-order partial differential equation are introduced, which summarize some latest theoretical developments and were not presented in available survey papers like our previous ones [65–67].

We proceed as follows. In the next section, Sect. 2, the configuration and basic idea of ADRC are articulated. In Sect. 3, overall review of theoretical progresses of ADRC for uncertain finite-dimensional and infinite-dimensional systems is presented. The recent theoretical developments and essentials of ADRC for finite-dimensional systems with mismatched disturbances and uncertainties by output feedback, uncertain finite-dimensional stochastic systems, uncertain infinite-dimensional systems described by the wave equation and uncertain infinite-dimensional systems described by the fractional-order partial differential equation are addressed in Sects. 4, 5, 6 and 7, respectively. Some further theoretical problems to be considered are summarized in Sect. 8, followed up concluding remarks in Sect. 9.

We use the following notations throughout this paper. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $\mathbb{E}$ denotes the mathematical expectation; for a vector or matrix K, $K^\top$ represents its transpose; for a scalar K, $|K|$ denotes its absolute value; $\|K\|$ represents the Euclidean norm of a vector K, and the corresponding induced norm when K is a matrix; $\lambda_{\max}(K)$ denotes the maximal eigenvalue of the symmetric real matrix K; $C(\Omega)$ denotes the set of all continuous functions from Ω to concerning Euclidean space; $C^s(\Omega)$ denotes the set of all continuous differentiable functions from Ω to concerning Euclidean space up to s-order; $L^2(0,1)$ is a Hilbert space whose elements are those square integrable measurable functions on $(0,1)$; $H^1(0,1) \triangleq \{\phi : \phi \in L^2(0,1), \phi' \in L^2(0,1)\}$; $L^\infty(0,\infty)$ is a function space whose elements are the essentially bounded measurable functions; $L^2_{\mathrm{loc}}(0,\infty)$ is locally summable function space whose elements are square integrable on every compact subset of $(0,\infty)$; the space $H^1_{\mathrm{loc}}(0,\infty)$ consists of all the functions ϕ satisfying $\phi, \phi' \in L^2_{\mathrm{loc}}(0,\infty)$.

2 Basic idea of ADRC and its framework

As mentioned in the last section, the ADRC is composed of three parts which include tracking differentiator (TD), extended state observer (ESO) and ESO-based feedback control.

Let us begin with the introduction of TD, which is the first and relatively independent component of ADRC. One of the main functions of TD is to recover the derivatives $r^{(i)}(t)$ $(i = 1, \ldots, n)$ of a given reference signal $r(t)$ through the $r(t)$ itself. Mathematically, TD is described by

$$\begin{cases} \dot{z}_{1R}(t) = z_{2R}(t), \\ \quad \vdots \\ \dot{z}_{nR}(t) = z_{(n+1)R}(t), \\ \dot{z}_{(n+1)R}(t) = R^n \psi\left(z_{1R}(t) - r(t), \dfrac{z_{2R}(t)}{R}, \ldots, \right. \\ \left. \dfrac{z_{(n+1)R}(t)}{R^n}\right), \quad \psi(0,0,\ldots,0) = 0, \end{cases} \tag{1}$$

where R is the tuning parameter, and $\psi : \mathbb{R}^{n+1} \to \mathbb{R}$ is a locally Lipschitz continuous function chosen so that the zero equilibrium state of the following reference-free system is globally asymptotically stable:

$$\begin{cases} \dot{z}_1(t) = z_2(t), \\ \dot{z}_2(t) = z_3(t), \\ \quad \vdots \\ \dot{z}_{n+1}(t) = \psi(z_1(t), \ldots, z_{n+1}(t)). \end{cases} \tag{2}$$

A second-order TD which is the special case of (1) with $n = 1$, was first proposed in [27] where a first convergence proof was presented. It is, however, proved afterward that it is true only for constant signal $r(t)$ in [68]. A rigorous theoretical proof was given in [68] showing that if system (2) is globally asymptotically stable and the reference signal $r(t)$ is differentiable and $\sup_{t\in[0,\infty)} |\dot{r}(t)| < \infty$, then, the solution of designed TD (1) is convergent in the sense that for any given initial value and any $T > 0$,

$$\lim_{R\to\infty} |z_{1R} - r(t)| = 0 \ \text{ uniformly in } [T, \infty). \tag{3}$$

This convergence result reveals that $z_{iR}(t)$ can be regarded as an approximation of the derivative $r^{(i-1)}(t)$ for each $i = 2, \ldots, n+1$, as long as the latter exists in the classical sense or is considered as the generalized derivative by considering $r(t)$ as a generalized function [68].

The linear TD, which is the special case of (1) when $\psi(\cdot)$ is a linear function, is as follows [68]:

$$\begin{cases} \dot{z}_{1R}(t) = z_{2R}(t), \\ \quad \vdots \\ \dot{z}_{nR}(t) = z_{(n+1)R}(t), \\ \dot{z}_{(n+1)R}(t) = R^n \left(k_1(z_{1R}(t) - r(t)) + \dfrac{k_2 z_{2R}(t)}{R} + \right. \\ \qquad \left. \cdots + \dfrac{k_{n+1} z_{(n+1)R}(t)}{R^n} \right), \end{cases} \tag{4}$$

where k_i $(i = 1, 2, \ldots, n+1)$ are the parameters such that the following matrix

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ddots & 1 \\ k_1 & k_2 & k_3 & \cdots & k_{n+1} \end{pmatrix}_{(n+1)\times(n+1)} \tag{5}$$

is Hurwitz. It was proved in [68] that if $\sup_{t\in[0,\infty), 1\le k\le n+1} |r^{(k)}(t)| < \infty$, then, linear TD (4) is convergent in the sense that for any given initial value and any $T > 0$,

$$\lim_{R\to\infty} |z_{1R}(t) - r(t)| = 0 \ \text{ uniformly in } [T, \infty), \tag{6}$$

$$\lim_{R\to\infty} |z_{iR}(t) - r^{(i-1)}(t)| = 0 \ \text{ uniformly in } [T, \infty), \quad i = 2, \ldots, n+1. \tag{7}$$

This convergence reveals that $z_{iR}(t)$ can be well recognized as the derivative $r^{(i-1)}(t)$ for each $i = 2, \ldots, n+1$ provided that the latter exists in the classical sense.

Another function of TD is that it can play a role as a transient profile so that the controlled output of plant can effectively track a relatively smooth signal to avoid sudden jump in PID. Namely, the trajectory to be tracked by controlled output of the plant in engineering applications is $z_{1R}(t)$ instead of $r(t)$ which could be jumping like step function, which makes the reference signal smooth and then the control signal could also be made smooth. The transient profile function of TD was indicated by Han in [31].

It is worth noting that for general nonlinear TD presented by (1) and linear TD (4), there are still not explicit estimation errors given in existing literature. However, for some special nonlinear TD, explicit estimation errors can be given [69].

We take the single-input single-output (SISO) system as an example to address another two components of ADRC as follows:

$$\begin{cases} x^{(n)}(t) = f(t, x(t), \dot{x}(t), \ldots, x^{(n-1)}(t), w(t)) + bu(t), \\ y(t) = x(t) \end{cases}$$

that can be rewritten as

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = x_3(t), \\ \quad \vdots \\ \dot{x}_n(t) = f(t, x_1(t), \ldots, x_n(t), w(t)) + bu(t), \\ y(t) = x_1(t), \end{cases} \tag{8}$$

where $u(t)$ is the input, $y(t)$ is the measured output, $f(\cdot) : [0, \infty) \times \mathbb{R}^{n+1} \to \mathbb{R}$ is an unknown system function, b is the control coefficient which is not exactly known but has a nominal value b_0 sufficiently closed to b, and $w(t)$ is the external disturbance. $x_{n+1}(t) \triangleq f(t, x_1(t), \ldots, x_n(t), w(t)) + (b - b_0)u(t)$ is regarded as the total disturbance (extended state) of system (8) including the nonlinear coupling effects of both internal unmodeled dynamics and external disturbance.

The second and also key component of ADRC is the ESO which is an extension of the state observer by adding an augmented state variable designed to estimate the total disturbance. In general, ESO is a systematic scheme for real-time estimation of not only the unmeasured state but also the total disturbance that could contain uncertainties coming from internal structure of system and external disturbance, where the former is regarded as the "internal disturbance" of system. The total effects of this "internal disturbance" and external disturbance are refined into the total disturbance or "extended state" to be estimated by ESO.

A first ESO with multiple tuning parameters was proposed by Han in late 1980s [28] as follows:

$$\begin{cases} \dot{\hat{x}}_1(t) = \hat{x}_2(t) - \alpha_1 g_1(\hat{x}_1(t) - y(t)), \\ \dot{\hat{x}}_2(t) = \hat{x}_3(t) - \alpha_2 g_2(\hat{x}_1(t) - y(t)), \\ \quad \vdots \\ \dot{\hat{x}}_n(t) = \hat{x}_{n+1}(t) - \alpha_n g_n(\hat{x}_1(t) - y(t)) + b_0 u(t), \\ \dot{\hat{x}}_{n+1}(t) = -\alpha_{n+1} g_{n+1}(\hat{x}_1(t) - y(t)). \end{cases} \tag{9}$$

The core idea of ESO (9) is that for some properly selected functions $g_i(\cdot)$ $(i = 1, 2, \ldots, n+1)$ and parameters α_i $(i = 1, 2, \ldots, n+1)$, the unmeasured states $x_i(t)$ $(i = 1, 2, \ldots, n)$ and the total disturbance $x_{n+1}(t)$ of system (8) can be estimated in real time by the states $\hat{x}_i(t)$ $(i = 1, 2, \ldots, n)$ and $\hat{x}_{n+1}(t)$ of ESO (9) designed by making use of the input $u(t)$ and the output $y(t)$ of system (8), respectively. The "fal" function is the nonlinear gain one commonly used in ESO (9) in practice defined as follows:

$$g_i(e) = \text{fal}(e, \alpha_i, \delta) = \begin{cases} \frac{e}{\delta^{1-\alpha_i}}, & |e| \le \delta, \\ |e|^{\alpha_i} \text{sign}(e), & |e| > \delta, \end{cases} \tag{10}$$

where $0 < \alpha_i < 1$, $\delta > 0$ are tuning parameters. Many computer simulations and engineering practices have confirmed that ESO (9) with nonlinear functions $g_i(\cdot)$ $(i = 1, 2, \ldots, n+1)$ of form (10) is very effective in the real-time estimation of unmeasured state and total disturbance with good performances including small peaking value and better measurement noise tolerance. The convergence analysis of estimation error of ESO (9) with "fal" gain functions is not available due to its special nonlinear structure until recently made in [70].

For the purpose of easy use in practice, Zhiqiang Gao introduces simplified one-parameter tuning linear ESO (11) in terms of bandwidth [71] as follows:

$$\begin{cases} \dot{\hat{x}}_1(t) = \hat{x}_2(t) + \frac{a_1}{\varepsilon}(y(t) - \hat{x}_1(t)), \\ \dot{\hat{x}}_2(t) = \hat{x}_3(t) + \frac{a_2}{\varepsilon^2}(y(t) - \hat{x}_1(t)), \\ \quad \vdots \\ \dot{\hat{x}}_n(t) = \hat{x}_{n+1}(t) + \frac{a_n}{\varepsilon^n}(y(t) - \hat{x}_1(t)) + b_0 u(t), \\ \dot{\hat{x}}_{n+1}(t) = \frac{a_{n+1}}{\varepsilon^{n+1}}(y(t) - \hat{x}_1(t)), \end{cases} \tag{11}$$

where a_i $(i = 1, 2, \ldots, n+1)$ are designed parameters such that the following matrix is Hurwitz:

$$\begin{pmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_n & 0 & 0 & \ddots & 1 \\ -a_{n+1} & 0 & 0 & \cdots & \end{pmatrix}_{(n+1)\times(n+1)}, \tag{12}$$

$\varepsilon > 0$ is the tuning parameter, and $\frac{1}{\varepsilon}$ is the observer bandwidth. Theoretically, the faster the total disturbance varies, the smaller the tuning parameter ε should be tuned correspondingly. The convergence analysis of one-parameter tuning linear ESO (11) was presented in [72] showing that the estimation errors of linear ESO (11) are bounded and their bounds are monotonously decreasing with their respective bandwidths.

As a special case of (9) and a nonlinear generalization of linear ESO (11), a one-parameter tuning nonlinear ESO was proposed in [73] as follows:

$$\begin{cases} \dot{\hat{x}}_1(t) = \hat{x}_2(t) + \varepsilon^{n-1} g_1\left(\frac{y(t) - \hat{x}_1(t)}{\varepsilon^n}\right), \\ \dot{\hat{x}}_2(t) = \hat{x}_3(t) + \varepsilon^{n-2} g_2\left(\frac{y(t) - \hat{x}_1(t)}{\varepsilon^n}\right), \quad \vdots \\ \dot{\hat{x}}_n(t) = \hat{x}_{n+1}(t) + g_n\left(\frac{y(t) - \hat{x}_1(t)}{\varepsilon^n}\right) + b_0 u(t), \\ \dot{\hat{x}}_{n+1}(t) = \frac{1}{\varepsilon} g_{n+1}\left(\frac{y(t) - \hat{x}_1(t)}{\varepsilon^n}\right), \end{cases} \tag{13}$$

where $g_i(\cdot)$ $(i = 1, 2, \ldots, n+1)$ are some appropriate chosen functions. The convergence analysis of the one-parameter tuning nonlinear ESO was first presented in [73] with sufficient conditions being given for the selection of $g_i(\cdot)$ $(i = 1, 2, \ldots, n+1)$.

The third and also the last component of ADRC is the ESO-based feedback control. The control objective of ADRC is to design a state and disturbance observer-based, i.e., an ESO-based feedback control so that the closed-loop output $y(t)$ tracks a given reference signal $r(t)$, and keeps all $x_i(t)$ to be bounded. The better case for the latter is certainly that $x_i(t)$ tracks $r^{(i-1)}(t)$ for each $i = 2, 3, \ldots, n$ where the stabilization problem at the origin is a special case by letting $r(t) \equiv 0$. The ESO-based feedback control can be designed as follows:

$$u(t) = \frac{1}{b_0}\Big[\phi(\hat{x}_1(t) - r(t), \ldots, \hat{x}_n(t) - r^{(n-1)}(t)) + r^n(t) - \hat{x}_{n+1}(t)\Big], \tag{14}$$

where "$-\hat{x}_{n+1}(t)$" plays a role in cancelling the total disturbance $x_{n+1}(t)$, and $\phi(\cdot)$ is a (linear or nonlinear) feedback control law chosen such that the zero equilibrium state of the following target error system is asymptotically stable:

$$\begin{cases} \dot{e}_1(t) = e_2(t), \\ \dot{e}_2(t) = e_3(t), \\ \quad \vdots \\ \dot{e}_n(t) = \phi(e_1(t), \ldots, e_n(t)), \phi(0, 0, \ldots, 0) = 0. \end{cases} \tag{15}$$

As pointed out above, TD can also serve as a transient profile for output tracking where $y(t)$ tracks $z_{1R}(t)$ instead of $r(t)$ to avoid setpoint jump. Thus, the last component of ADRC can be designed as the TD and ESO-based feedback control as follows:

$$u(t) = \frac{1}{b_0}\big[\phi(\hat{x}_1(t) - z_{1R}(t), \ldots, \hat{x}_n(t) - z_{nR}(t)) + z_{(n+1)R}(t) - \hat{x}_{n+1}(t)\big]. \tag{16}$$

Under some assumptions, see, for example the main results of [32,65], the ADRC closed loop is practically convergent in the sense that for any given initial value, it holds

$$\begin{aligned} &\lim_{\substack{t\to\infty \\ \varepsilon\to 0}} [x_i(t) - \hat{x}_i(t)] = 0,\ 1 \le i \le n+1, \\ &\lim_{\substack{t\to\infty \\ \varepsilon\to 0}} [y(t) - r(t)] = 0, \\ &\lim_{\substack{t\to\infty \\ \varepsilon\to 0}} [x_i(t) - r^{(i-1)}(t)] = 0,\ 2 \le i \le n, \end{aligned} \tag{17}$$

without using the relatively independent TD component; or there exists a constant $R_0 > 0$ such that for all $R > R_0$ and $T > 0$, it holds

$$\begin{aligned} &\lim_{\substack{t\to\infty \\ \varepsilon\to 0}} [x_i(t) - \hat{x}_i(t)] = 0,\ 1 \le i \le n+1, \\ &\lim_{\substack{t\to\infty \\ \varepsilon\to 0}} [x_i(t) - z_{iR}(t)] = 0,\ 1 \le i \le n, \\ &\lim_{R\to\infty} |z_{1R}(t) - r(t)| = 0 \text{ uniformly in } t \in [T, \infty), \end{aligned} \tag{18}$$

using the relatively independent TD component. Specially, when $r(t) \equiv 0$, then, $z_{iR}(t) \equiv 0$ and the ADRC closed loop deduces the practical stability.

3 Overview of theoretical progresses of ADRC

If either the ESO or the ESO-based feedback control is nonlinear, the ADRC is referred commonly as a nonlinear ADRC and to be a linear one otherwise. The stability characteristics of linear ADRC for nonlinear time-varying plants subject to vast dynamic uncertainties were first addressed in [72] revealing that both estimation and tracking errors are bounded with their bounds proportion to the bandwidths of linear ESO. The global and semi-global convergence of the nonlinear ADRC for a class of multi-input multi-output (MIMO) nonlinear systems with large uncertainty was investigated in [74]. On the one hand, a series of progresses concerning ADRC designs and convergence for various kinds of uncertain finite-dimensional systems have been made up to date. An adaptive ESO with time-varying observer gains was proposed for nonlinear disturbed systems, and the convergence proof of estimation errors was presented in [75]; The backstepping ADRC design was proposed for uncertain nonlinear systems, and the corresponding closed-loop convergence was established in [76–78]. The analysis of linear ADRC was carried out via the well-known internal model control (IMC) framework [79]. The controller based on both the ESO and the projected gradient estimator was designed for a class of uncertain nonlinear dynamical systems with zero dynamics without a "good" prior estimate for the uncertainties in the input channel required by the conventional ADRC, and the corresponding closed-loop convergence was investigated in [80]. A switching control scheme composed of linear ADRC and nonlinear ADRC was proposed, and its stability was analyzed in [81]. The stability of ADRC was addressed for uncertain nonlinear systems by singular perturbations analysis in [82], a flatness-based approach [83] and some novel Lyapunov approaches [84,85], respectively; the filtering problem for a class of uncertain MIMO systems with measurement noise was addressed by ESO in [86], the augment observer in view of ESO and its convergence analysis for a large class of uncertain nonlinear systems was investigated in [87], and some improved ADRC designs and corresponding closed-loop convergence analysis were presented in [88,89]. The nonlinear ESO using "fal" functions and the corresponding ESO-based output feedback control were designed for nonlinear systems with mismatched uncertainties, and the convergence of the closed-loop systems was proved in [90]. The convergence analysis and designs of ADRC for uncertain nonminimum phase systems [91], uncertain nonlinear fractional-order systems [92,93], uncertain nonaffine-in-control nonlinear systems [94], nonlinear systems with mismatched disturbances and uncertainties [90,95–100], uncertain time-delay systems [101–105], uncertain networked control systems [106,107], uncertain nonlinear systems with measurement uncertainty [108] and uncertain stochastic systems [109–113] have been investigated.

On the other hand, both ADRC designs and convergence for various uncertain infinite-dimensional systems described by partial differential equations (PDEs), have also attracted more attentions. Some primary theoretical researches of ADRC for uncertain PDEs can be found in boundary feedback stabilization for one-dimensional Euler–Bernoulli beam equations with control matched external disturbance [114–116], a one-dimensional anti-stable wave equation with control matched external disturbance [117], a one-dimensional Schrödinger equation with control matched external disturbance [118], a one-dimensional rotating disk-beam system with boundary input disturbances [119] and a one-dimensional unstable heat equation by a dynamic boundary ADRC compensator [120]. The ADRC was also developed on boundary state feedback stabilization for multi-dimensional wave equation with control matched external disturbance [121] and multi-dimensional Kirchhoff equation with control matched external disturbance [122]. The idea in these literatures is that the control matched external disturbance is refined into associated disturbed ordinary differential equations (ODEs) by test functions, and then the conventional ESO designs for uncertain finite-dimensional systems can be adopted for real-time estimation of the disturbance. Based on the estimate of the control matched external disturbance, the disturbance can be approximatively cancelled in the closed loop of the uncertain PDEs by feedforward compensation so that the conventional boundary feedback control for the PDEs without disturbance can be designed to obtain closed-loop's stability. A first result on output feedback stabilization for a one-dimensional anti-stable wave equation subject to boundary control matched disturbance by the ADRC approach was presented in [123] where a variable structured unknown input state observer was designed by the output of the PDE system. The boundary output feedback stabilization by ADRC approach was first addressed for uncertain multi-dimensional Kirchhoff plate with boundary observation subject to external disturbance, where the real-time estimation of disturbance is conducted by use of infinitely many time-dependent test functions [124]. An infinite-dimensional disturbance estimator, also served as a TD, was designed to extract real signal from disturbed velocity signal by the ADRC approach, and the design strategy was adopted in the boundary output feedback stabilization for a multi-dimensional wave equation with position and disturbed velocity measurements [125]. A series of boundary output feedback stabilization problems for uncertain PDEs by designing an infinite-dimensional disturbance estimator to estimate the disturbance can be found in [126–128]. The output tracking for a one-dimensional wave equation and a multi-dimensional heat equation subject to unmatched general disturbance and noncollocated control was, respectively, investigated in [129,130], where the mismatched disturbance is only supposed to be in $L^{\infty}(0, \infty)$, which is not necessarily smooth. The ADRC for uncertain fractional PDEs is just started from the boundary Mittag–Leffler stabilization for a unstable time fractional anomalous diffusion equation with boundary control subject to the control matched external disturbance [131] and a unstable time fractional hyperbolic PDE by boundary control and boundary measurement [132].

4 ADRC for finite-dimensional nonlinear systems with mismatched disturbances and uncertainties

Most of literature address mainly ADRC for essentially integral chain systems with control matched disturbances and uncertainties satisfying some matching conditions. However, the control mismatched disturbances and uncertainties, that is, those in different channels from control inputs, are more general and widely exist in practical applications, see, for example, some practical systems in [24, p. 22]. Generally speaking, the mismatched disturbances and uncertainties cannot be attenuated completely from the state equation no matter what controller is designed [133]. One of the most feasible objectives in this case is to eliminate the disturbances and uncertainties from the output channel in steady state, i.e., the output tracking control objective. Related theoretical progresses have been made recently. The output tracking for nonlinear systems with mismatched disturbances and uncertainties was investigated in [76–78] by combining the ADRC approach and a constructive backstepping control strategy, with state feedback controller was designed. An output tracking problem for multiple-input multiple-output (MIMO) lower-triangular nonlinear systems with mismatched disturbances and uncertainties by the ADRC strategy by state feedback was investigated in [95]. The output tracking for SISO and MIMO lower-triangular nonlinear systems with mismatched disturbances and uncertainties via the ADRC strategy by output feedback was developed in [96] and [97], respectively. The output tracking for MIMO lower-triangular nonlinear systems via the ADRC approach based on both nonlinear ESO constructed by "fal" functions and output feedback was addressed in [90].

To make the ideology of ADRC for finite-dimensional nonlinear systems with mismatched disturbances and uncertainties more clearly, in this section, we only use a second-order system with mismatched disturbance and uncertainties for demonstration. As for the ADRC on more general n-th-order nonlinear systems with mismatched disturbances and uncertainties, we refer to [96]. The following results come from [96] or can be easily concluded from [96].

The second-order system with mismatched disturbance and uncertainties considered here is as follows:

$$\begin{cases} \dot{x}_1(t) = x_2(t) + h_1(x_1(t), w(t)), \\ \dot{x}_2(t) = h_2(x_1(t), x_2(t), w(t)) + u(t), \\ y(t) = x_1(t), \end{cases} \tag{19}$$

where $x(t) = (x_1(t), x_2(t))^\top \in \mathbb{R}^2$ is the system state, $y(t) \in \mathbb{R}$ is the measured controlled output, $u(t) \in \mathbb{R}$ is the control input, and $w(t) \in \mathbb{R}$ is the unknown exogenous signal or external disturbance. The functions $h_i \in C^{3-i}(\mathbb{R}^{i+1})$ $(i = 1, 2)$ represent unknown systems dynamics. It can be seen that nonlinear system uncertainties and unknown external disturbance are in all channels of system (19), not only in the control channel.

The control objective here is to design an output feedback control by the ADRC approach, such that for all initial state in given compact set, the closed-loop state $x(t)$ is bounded and the closed-loop output $y(t)$ tracks practically a given, bounded, reference signal $r(t)$ whose derivatives $\dot{r}(t), \ddot{r}(t), r^{(3)}(t)$ are assumed to be bounded. Set

$$(r_1(t), r_2(t), r_3(t), r_4(t)) = (r(t), \dot{r}(t), \ddot{r}(t), r^{(3)}(t)). \tag{20}$$

As announced in [30,67,96], a key point to apply ADRC is to lump various kinds of systems dynamics and external disturbances affecting performance of the controlled systems into the total disturbance, which is a vital procedure in making the control problem become simple whatever the plant is complicated or not. Thus, the total disturbance should be observed by use of some measurable states so that the estimation/cancellation strategy of ADRC can be implemented. Thus, the first step is to refine the total disturbance that affects the system performance. Let us make the following state transformation keeping the same measured controlled output $y(t)$:

$$\begin{cases} \bar{x}_1(t) = x_1(t), \\ \bar{x}_2(t) = x_2(t) + h_1(x_1(t), w(t)). \end{cases} \tag{21}$$

It follows from (19), (21) that system (19) is equivalently transformed into an essentially integral-chain system with control matched total disturbance as follows:

$$\begin{cases} \dot{\bar{x}}_1(t) = \bar{x}_2(t), \\ \dot{\bar{x}}_2(t) = \bar{x}_3(t) + u(t), \\ y(t) = \bar{x}_1(t) = x_1(t), \end{cases} \tag{22}$$

where $\bar{x}_3(t)$ is the actual total disturbance influencing the controlled output $y(t)$ of system (19) given by

$$\bar{x}_3(t) = h_2(x_1(t), x_2(t), w(t)) + \frac{\partial h_1(x_1(t), w(t))}{\partial x_1}(x_2(t) + h_1(x_1(t), w(t))) + \frac{\partial h_1(x_1(t), w(t))}{\partial w}\dot{w}(t). \tag{23}$$

System (22) is exactly observable because it is easily concluded that for any $L > 0$, $(y(t), u(t)) \equiv 0, t \in [0, L] \Rightarrow \bar{x}_3(t) \equiv 0, t \in [0, L]; (\bar{x}_1(0), \bar{x}_2(0)) = 0$ (see, e.g., [134, p.5, Definition 1.2]). This indicates that $y(t)$ contains all information of $\bar{x}_3(t)$ and then a natural thought is to use $y(t)$ to estimate the actual total disturbance $\bar{x}_3(t)$. If this is feasible, that is, $y(t) \Rightarrow \hat{\bar{x}}_3(t) \approx \bar{x}_3(t)$, then the actual total disturbance $\bar{x}_3(t)$ can be approximately cancelled by designing $u(t) = u_0(t) - \hat{\bar{x}}_3(t)$ and system (22) is approximately equivalent to the following linear time-invariant system

$$\begin{cases} \dot{\bar{x}}_1(t) = \bar{x}_2(t), \\ \dot{\bar{x}}_2(t) = u_0(t), \\ y(t) = \bar{x}_1(t) = x_1(t), \end{cases} \tag{24}$$

where the control law $u_0(t)$ can be easily designed for the output tracking of simplified system (24).

As indicated above, the key point in the ADRC design is how to estimate the actual total disturbance $\bar{x}_3(t)$ by the measured output $y(t)$. By taking these points into account, a third-order one-parameter tuning linear ESO is designed for system (22) as follows:

$$\begin{cases} \dot{\hat{\bar{x}}}_1(t) = \hat{\bar{x}}_2(t) + \frac{a_1(y(t)-\hat{\bar{x}}_1(t))}{\varepsilon}, \\ \dot{\hat{\bar{x}}}_2(t) = \hat{\bar{x}}_3(t) + \frac{a_2(y(t)-\hat{\bar{x}}_1(t))}{\varepsilon^2}, \\ \dot{\hat{\bar{x}}}_3(t) = \dfrac{a_3(y(t) - \hat{\bar{x}}_1(t))}{\varepsilon^3}, \end{cases} \tag{25}$$

where $\varepsilon > 0$ is the tuning gain parameter and a_i $(i = 1, 2, 3)$ are the parameters such that the matrix

$$M_1 = \begin{pmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{pmatrix} \tag{26}$$

is Hurwitz. For example, a_i $(i = 1, 2, 3)$ are often chosen in practice as $a_1 = 3, a_2 = 3, a_3 = 1$, in which case all eigenvalues of M_1 are -1 and then M_1 is Hurwitz. The main idea of linear ESO (25) is to choose some appropriate parameters a_i $(i = 1, 2, 3)$, such that the $\hat{\bar{x}}_i(t)$ approaches $\bar{x}_i(t)$ for each $i = 1, 2, 3$ by tuning the gain parameter ε, where the absolute values of the estimation errors $|\bar{x}_i(t) - \hat{\bar{x}}_i(t)|$ $(i = 1, 2, 3)$ are inversely proportional to ε.

Motivated by (14), ESO (25)-based output feedback control is designed as

$$u(t) = k_1 \text{sat}_{Q_1}(\hat{\bar{x}}_1(t) - r_1(t)) + k_2 \text{sat}_{Q_2}(\hat{\bar{x}}_2(t) - r_2(t)) - \text{sat}_{Q_3}(\hat{\bar{x}}_3(t)) + r_3(t), \tag{27}$$

where the output feedback control gain parameters k_i $(i = 1, 2)$ are chosen such that the following target error system is asymptotically stable:

$$\begin{cases} \dot{e}_1(t) = e_2(t), \\ \dot{e}_2(t) = k_1 e_1(t) + k_2 e_2(t), \end{cases} \tag{28}$$

that is, the following matrix is Hurwitz:

$$M_2 = \begin{pmatrix} 0 & 1 \\ k_1 & k_2 \end{pmatrix}_{n \times n}, \tag{29}$$

and $\text{sat}_{Q_i}(\cdot)$ $(i = 1, 2, 3)$ are the continuous differentiable saturation odd functions to limit the peaking value in control signal defined by (the counterpart for $t \in (-\infty, 0]$ is obtained by symmetry)

$$\text{sat}_{Q_i}(z) = \begin{cases} z, & 0 \le z \le Q_i, \\ -\frac{1}{2}z^2 + (Q_i + 1)z - \frac{1}{2}Q_i^2, & Q_i < z \le Q_i + 1, \\ Q_i + \frac{1}{2}, & z > Q_i + 1, \end{cases} \tag{30}$$

with Q_i $(1 \le i \le 3)$ are constants to be specified.

We notice that ESO (25)-based output feedback control (27) is essentially an error-driven PD feedback control combined with a feedforward term "$-\text{sat}_{Q_3}(\hat{\bar{x}}_3(t)) + r_3(t)$," where $-\text{sat}_{Q_3}(\hat{\bar{x}}_3(t))$ is designed to cancel in real time the actual total disturbance $\bar{x}_3(t)$ and $r_3(t)$ is for the compensation of the second derivative of the reference signal $r(t)$.

The convergence of the closed-loop system composed of (19), (25), (27) is a special case of Theorem 2.1 of [96], which is summarized in following Theorem 1.

Theorem 1 *Suppose that there exists a positive constant C such that $\|x(0)\| \le C$, $\sup_{t \ge 0} \|(w(t), \dot{w}(t), \ddot{w}(t))\|$*

$\leq C$ and $\sup_{t\geq 0}|r_i(t)| \leq C$ for all $i = 1, 2, 3$. Then, the closed-loop system composed of (19), (25), (27) has the following convergence:

(i) *The closed-loop state $x(t)$ is bounded: $\|x(t)\| \leq \Gamma$ for all $t \geq 0$, where Γ is an ε-independent positive constant;*
(ii) *The output $y(t)$ of system (19) tracks practically the reference signal $r(t)$ in the sense that: For any $\sigma > 0$, there exists a constant $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*)$,*

$$|y(t) - r(t)| \leq \sigma \text{ uniformly in } t \in [t_\varepsilon, \infty),$$

where $t_\varepsilon > 0$ is an ε-dependent constant. In particular,

$$\limsup_{t\to\infty} |y(t) - r(t)| \leq \sigma.$$

Remark 1 The ADRC designs like (27) and the main results like Theorem 1 and others in literature [90,95–97] indicate that the ADRC approach can be effective in the output tracking for finite-dimensional nonlinear systems with mismatched disturbances and uncertainties, which pushes forward potentially practical applications and further theoretical research. It is also worth noting that the unknown dynamics functions $h_i \in C^{3-i}(\mathbb{R}^{i+1})$ $(i = 1, 2)$ could be fast-varying even be nonlinear growth such as the functions $h_1(x_1, w) = e^{x_1+w}$ and $h_2(x_1, x_2, w) = e^{x_1+x_2+w}$. This is because the closed-loop states under the ADRC controller are bounded so that the total disturbance is bounded in the closed loop and can thus be estimated by ESO and cancelled by the feedback.

Remark 2 It should be pointed out that the backstepping ADRC approach proposed in [76–78] can also be applied in output tracking control problem without using the state transformation adopted in this section, and the smooth assumptions about the unknown system functions and external disturbances can therefore be relaxed to a large extent. However, the ADRC approach proposed in this section is by output feedback instead of state feedback via backstepping approach, and the ADRC controller structure in this section is much simpler than the backstepping ADRC one that it avoids the "explosion of complexity" inevitable in the backstepping ADRC approach.

5 ADRC for uncertain finite-dimensional stochastic systems

A commonly known fact is that in engineering applications, stochastic disturbances are much more common. A series of theoretical researches concerning the ADRC approach to output feedback stabilization for uncertain finite-dimensional stochastic systems with control matched bounded stochastic noises of unknown statistical characteristics and unmodeled dynamics can be found in [109–113], where the considered bounded noises exist widely in practical systems [135–137]. Although the ADRC approach is applicable to n-th-order SISO uncertain stochastic systems [109,110], lower triangular uncertain stochastic systems [111] and MIMO uncertain stochastic systems [112,113], we use a second-order SISO example for the sake of simplicity and clarity.

Now, we consider a second-order SISO uncertain stochastic system with control matched bounded stochastic noises of unknown statistical characteristics and unmodeled dynamics as follows:

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = f(t, x(t), w(t)) + bu(t), \\ y(t) = x_1(t), \end{cases} \tag{31}$$

where $x(t) = (x_1(t), x_2(t))^\top \in \mathbb{R}^2$ is the state, $u(t) \in \mathbb{R}$ is the control input, and $y(t) \in \mathbb{R}$ is the measured output. The functions $f : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}$ are unknown; $b \neq 0$ is the control coefficient which is not exactly known yet has a nominal value b_0 that is sufficiently closed to b. The $w(t) \triangleq \psi(t, B(t)) \in \mathbb{R}$ for some bounded unknown function $\psi(\cdot) : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is the external stochastic disturbance, where $B(t)$ is a one-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with Ω being a sample space, $\mathcal{F}$ being a σ field, $\{\mathcal{F}_t\}_{t\geq 0}$ being a filtration and P being the probability measure.

As pointed out in papers [109–113] that the external stochastic noise $w(t)$ is quite general from the perspectives of theory and practice. Firstly, the external stochastic noise $w(t)$ has large stochastic uncertainty since the function $\psi(\cdot)$ is unknown. Secondly, the disturbance without stochastic characteristics investigated via ADRC in aforementioned literature is just a special case of the $w(t)$ where the defining function $\psi(\cdot)$ is only with respect to the time variable t: $w(t) \triangleq \psi(t)$.

Finally, for the stochastic case, the $w(t)$ covers bounded stochastic noises considered in many practical systems, see, for instance [135–137].

Since $f(\cdot)$ is an unknown function, the equilibrium states of uncertain stochastic system (31) cannot be determined or even the existence cannot be guaranteed, the stabilization of uncertain stochastic system (31) is the stabilization at the equilibrium state of its nominal system (the part without disturbances and uncertainty), i.e., the stabilization at the origin. Thus, the control objective is to design an output feedback control such that for any initial state, the closed-loop system is mean square practically stable as stated in succeeding Theorem 2.

The stochastic total disturbance affecting system performance is refined as follows:

$$x_3(t) \triangleq f(t, x(t), w(t)) + (b - b_0)u(t), \tag{32}$$

which represents the total coupling effects of unknown system dynamics, external stochastic disturbance with unknown defining function and uncertainty caused by the deviation of control parameter b from its nominal value b_0.

System (31) is exactly observable because we can easily obtain that for any $L > 0$, $(y(t), u(t)) \equiv 0, t \in [0, L] \Rightarrow x_3(t) \equiv 0, t \in [0, L]$; $(x_1(0), x_2(0)) = 0$ (see, e.g., [134, p.5, Definition 1.2]), which indicates that the real-time information of stochastic total disturbance $x_3(t)$ and the state $x(t) = (x_1(t), x_2(t))^\top$ could be identified via the measured output $y(t)$.

Thus, it is quite reasonable to design an observer for estimation of both the stochastic total disturbance and unmeasured state by use of $y(t)$. Motivated by (11), a third-order one-parameter tuning linear ESO for system (31) is designed as follows:

$$\begin{cases} \dot{\hat{x}}_1(t) = \hat{x}_2(t) + \frac{a_1(y(t) - \hat{x}_1(t))}{\varepsilon}, \\ \dot{\hat{x}}_2(t) = \hat{x}_3(t) + \frac{a_2(y(t) - \hat{x}_1(t))}{\varepsilon^2} + b_0 u(t), \\ \dot{\hat{x}}_3(t) = \dfrac{a_3(y(t) - \hat{x}_1(t))}{\varepsilon^3}, \end{cases} \tag{33}$$

where $\varepsilon > 0$ is the tuning gain parameter and a_i $(i = 1, 2, 3)$ are the parameters such that the matrix M_1 defined in (26) is Hurwitz.

Motivated by (14) with the reference signal satisfying $r(t) \equiv 0$, ESO (33)-based output feedback control of very simple control structure is designed as follows:

$$u(t) = \frac{1}{b_0}\left[k_1 \hat{x}_1(t) + k_2 \hat{x}_2(t) - \hat{x}_3(t)\right], \tag{34}$$

where $-\hat{x}_3(t)$ is used for the real-time approximate cancellation of the stochastic total disturbance $x_3(t)$ defined in (32) and the output feedback control law $u_0(t) \triangleq k_1 \hat{x}_1(t) + k_2 \hat{x}_2(t)$ is designed to stabilize the nominal part of system (31):

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = u_0(t), \end{cases} \tag{35}$$

that is, the output feedback control gain parameters k_i $(i = 1, 2)$ are chosen such that the matrix M_2 defined in (29) is Hurwitz.

To address the resulting ADRC's closed-loop stability, following Assumptions ((A1)) and ((A2)) are required, where the former is about the unknown function defining the external stochastic disturbance and the latter is a prior assumption about the unknown function $f(\cdot)$.

Assumption (A1) The $\psi(t, \vartheta) : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable with respect to their arguments, and there exists a (known) constant $C > 0$ such that for all $\vartheta \in \mathbb{R}$,

$$|\psi(t, \vartheta)| + \left|\frac{\partial \psi(t, \vartheta)}{\partial t}\right| + \left|\frac{\partial \psi(t, \vartheta)}{\partial \vartheta}\right| + \left|\frac{\partial^2 \psi(t, \vartheta)}{\partial \vartheta^2}\right| \le C. \tag{36}$$

Assumption (A2) The $f(\cdot)$ is twice continuously differentiable with respect to their arguments. There exist (known) constants $D_i > 0$ $(i = 1, 2, 3)$ and a non-negative continuous function $\varsigma \in C(\mathbb{R}; \mathbb{R})$ such that for all $t \ge 0$, $x = (x_1, x_2)^\top \in \mathbb{R}^2$ and $w \in \mathbb{R}$,

$$\left|\frac{\partial f(t, x, w)}{\partial t}\right| \le D_1 + D_2\|x\| + \varsigma(w); \tag{37}$$

$$\left\|\frac{\partial f(t, x, w)}{\partial x}\right\| + \left|\frac{\partial f(t, x, w)}{\partial w}\right| + \left|\frac{\partial^2 f(t, x, w)}{\partial w^2}\right| \le D_3 + \varsigma(w); \tag{38}$$

The conditions of Assumption ((A1)) and (37), (38) in Assumption ((A2)) are about the Itô differential (or "variation") of the stochastic total disturbance to make sure that the "variation" is bounded or can be "absorbed" by decaying parts in the closed loop. These assumptions are reasonable because the ADRC controller is based on feedforward compensation by use of the estimate of the stochastic total disturbance and the control energy must be limited in practice.

Let Q be the positive definite matrix solution satisfying the Lyapunov equations $QM_1 + M_1^\top Q = -I_{3\times 3}$ with M_1 defined in (26) and $I_{3\times 3}$ being the 3×3 unit matrix.

The main result on practical mean square convergence of the closed loop comprised of (31), (33) and (34) which includes practical mean square stability of the closed loop and ESO's practical mean square estimation of unmeasured state and stochastic total disturbance, can be concluded directly from Theorem 2.1 of [110].

Theorem 2 *Suppose that Assumptions (A1)–(A2) hold and* $|b - b_0| < \frac{|b_0|}{2\lambda_{\max}(Q)a_3}$. *Then, for any initial value, the closed loop composed of* (31), (33) *and* (34) *has the practical mean square convergence in the sense that there are a constant* $\varepsilon^* > 0$ *and an* ε*-dependent constant* $t_\varepsilon^* > 0$ *with* $\varepsilon \in (0, \varepsilon^*)$ *such that*

$$\mathbb{E}|x_i(t) - \hat{x}_i(t)|^2 \le \Gamma \varepsilon^{2n+3-2i}, \quad \forall t \ge t_\varepsilon^*, \ i = 1, 2, 3,$$

and

$$\mathbb{E}|x_i(t)|^2 \le \Gamma\varepsilon, \quad \forall t \ge t_\varepsilon^*, \ i = 1, 2,$$

where $\Gamma > 0$ *is an* ε*-independent constant. As a result,*

$$\limsup_{t\to\infty} \mathbb{E}|x_i(t)|^2 \le \Gamma\varepsilon, \quad i = 1, 2.$$

Remark 3 From a theoretical perspective, it can be shown from Assumption (A1) and Theorem 2 that the bounded stochastic noise like "$\sin(\alpha_1 t + \alpha_2 B(t))$" or "$\cos(\alpha_1 t + \alpha_2 B(t))$" with $B(t)$ being a Brownian motion and α_i $(i = 1, 2)$ being unknown constants can be coped with by ADRC whatever it is the high-frequency stochastic noise or not.

6 ADRC for an uncertain infinite-dimensional system: wave equation

In this section, we introduce ADRC for an uncertain infinite-dimensional system by considering the stabilization of the wave equation using two kinds of disturbance estimators to cope with the external disturbance and interior uncertainty. The first estimator is based on the conventional ESO by using a test function, and the second one relies on an infinite-dimensional system which is related to the original system.

Although the ADRC approach can be applied to both the anti-stable wave equation and unstable wave equation [123,126,138], to make the ideology of ADRC for infinite-dimensional systems more clearly, we consider a one-dimensional Lyapunov stable wave equation whose solution does not decay to zero provided that there is no control input as follows:

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \ \ x \in (0,1), \ t > 0, \\ w(0,t) = 0, \ \ t \ge 0, \\ w_x(1,t) = F(t) + u(t), \ \ t \ge 0, \end{cases} \tag{39}$$

where (w, w_t) is the state, $u(t)$ is the control (or input), and $F(t) \triangleq f(w(\cdot,t), w_t(\cdot,t)) + \mathrm{d}(t)$ is the total disturbance with the function $f(\cdot)$ being unknown. Here $f(w(\cdot,t), w_t(\cdot,t))$ represents the boundary unknown interior uncertainty and $\mathrm{d}(t)$ is the external disturbance. The control objective is to design a feedback control law $u(t)$ so that the closed-loop state of system (39) converges to zero by rejecting the total disturbance $F(t)$. The state space is taken as $\mathcal{H} = H_L^1(0,1) \times L^2(0,1)$, where $H_L^1(0,1) = \{\phi \in H^1(0,1) | \phi(0) = 0\}$. The usual inner product of $\mathcal{H}$ is given by

$$\langle(\phi_1,\psi_1),(\phi_2,\psi_2)\rangle_{\mathcal{H}} = \int_0^1 [\phi_1'(x)\overline{\phi_2'(x)} + \psi_1(x)\overline{\psi_2(x)}]dx.$$

for any $(\phi_1,\psi_1) \in \mathcal{H}$, $(\phi_2,\psi_2) \in \mathcal{H}$.

It is well known that $u(t) = -kw_t(1,t)$ with $k > 0$ can exponentially stabilize system (39) provided that $F(t) \equiv 0$. However, this control law is not robust with respect to the boundary disturbance. To see this, let $F(t)$ be a constant, i.e., $F(t) \equiv C$. Then, $(w, w_t) = (Cx, 0)$ is a nonzero solution of system (39) even if we design the control law $u(t) = -kw_t(1,t)$.

We define the operator A as follows:

$$\begin{cases} A(\phi,\psi) = (\psi, \phi''), \quad \forall(\phi,\psi) \in D(A), \\ D(A) = \Big\{(\phi,\psi) \in \mathcal{H} \cap (H^2(0,1) \times H^1(0,1)) | \\ \psi(0) = 0, \ \phi'(1) = 0\Big\}. \end{cases} \tag{40}$$

Then, system (39) can be written as

$$\frac{d}{dt}(w(\cdot,t), w_t(\cdot,t)) = A(w(\cdot,t), w_t(\cdot,t)) + B(F(t) + u(t)), \tag{41}$$

where $B = (0, \delta(x-1))$. It is easy to verify that A generates a C_0-group e^{At} on $\mathcal{H}$ and B is admissible for e^{At} [139]. Moreover, from Lemma A.2 of [128], system (39) has a unique solution provided that $u, d \in L^2_{\text{loc}}(0,\infty)$ and $f(\cdot)$ satisfies the local Lipschitz condition on $\mathcal{H}$.

To adopt the conventional ESO to deal with the total disturbance $F(t)$, we first assume that $f(\cdot) \equiv 0$, $d \in H^1_{\text{loc}}(0,\infty) \cap L^\infty(0,\infty)$. In this case, system (39) has only the external disturbance $\mathrm{d}(t)$. The measurement is the full state (w, w_t).

Define $Y(t) = \int_0^1 x w_t(x,t)dx$, where x is regarded as a test function. Finding the derivative of Y with respect to t and using the boundary condition of system (39), it is easy to see that $Y(t)$ satisfies

$$\dot{Y}(t) = -w(1,t) + u(t) + \mathrm{d}(t). \tag{42}$$

It can be seen that (42) is a one-dimensional ODE subject to external disturbance $\mathrm{d}(t)$. That is, the external disturbance $\mathrm{d}(t)$ is refined into uncertain finite-dimensional system (42).

Motivated by one-parameter tuning linear ESO design (11), we design a one-parameter tuning linear ESO to estimate the external disturbance $\mathrm{d}(t)$:

$$\begin{cases} \dot{\widehat{Y}}_\varepsilon(t) = -w(1,t) + \widehat{d}_\varepsilon(t) + \dfrac{2}{\varepsilon}\left[Y(t) - \widehat{Y}_\varepsilon(t)\right] + u(t), \\ \dot{\widehat{d}}_\varepsilon(t) = \dfrac{1}{\varepsilon^2}\left[Y(t) - \widehat{Y}_\varepsilon(t)\right], \end{cases} \tag{43}$$

where ε is the tuning parameter, and $\hat{d}_\varepsilon(t)$ is regarded as an estimate of $\mathrm{d}(t)$. The convergence of the estimation errors is summarized as following Lemma 1 that comes from [140].

Lemma 1 *Assume that the disturbance $\mathrm{d}(t)$ and its derivative $\dot{d}(t)$ are uniformly bounded with an upper bound M. Let $Y(t) = \int_0^1 x w_t(x,t)dx$. Then, the estimation errors of linear ESO (43) satisfy*

$$\lim_{t\to\infty} |\widehat{Y}_\varepsilon(t) - Y(t)| = \lim_{t\to\infty} |\widehat{d}_\varepsilon(t) - \mathrm{d}(t)| = \mathcal{O}(\varepsilon) \ \ as\ \varepsilon \to 0.$$

Moreover, for any fixed $T > 0$,

$$\int_0^T |\widehat{Y}_\varepsilon(t) - Y(t)|\mathrm{d}t = \int_0^T |\widehat{d}_\varepsilon(t) - \mathrm{d}(t)|\mathrm{d}t = \mathcal{O}(\varepsilon) \quad as\ \varepsilon \to 0,$$

$$\int_0^T |\widehat{Y}_\varepsilon(t) - Y(t)|^2\mathrm{d}t = \int_0^T |\widehat{d}_\varepsilon(t) - \mathrm{d}(t)|^2\mathrm{d}t = \mathcal{O}(\varepsilon^{-1}) \quad as\ \varepsilon \to 0. \tag{44}$$

By (44), $\int_0^T |\hat{d}_\varepsilon(t) - \mathrm{d}(t)|\mathrm{d}t$ is uniformly bounded in ε for any fixed $T > 0$, and $\int_0^T |\hat{d}_\varepsilon(t) - \mathrm{d}(t)|^2\mathrm{d}t$ is unbounded in ε. By [139, Theorem 4.8], we only have admissibility with $L^2_{loc}(0,\infty)$ control yet not the admissibility with $L^1_{loc}(0,\infty)$ control. To overcome this difficulty, the control law is proposed by

$$u(t) = -kw_t(1,t) - \mathrm{sat}(\widehat{d}(t))$$

where $\mathrm{sat}(x) = \min\{M+1, \max\{x, -M-1\}\}$ is a saturate function.

The resulting closed-loop system is governed by

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \quad x \in (0,1),\ t > 0, \\ w(0,t) = 0, \quad t \geq 0, \\ w_x(1,t) = -kw_t(1,t) - \mathrm{sat}(\widehat{d}(t)) + \mathrm{d}(t), \\ \dot{\widehat{Y}}_\varepsilon(t) = \widehat{d}_\varepsilon(t) + \frac{2}{\varepsilon}[Y(t) - \widehat{Y}_\varepsilon(t)] \\ \quad -w(1,t) - kw_t(1,t) - \mathrm{sat}(\widehat{d}(t)), \\ \dot{\widehat{d}}_\varepsilon(t) = \dfrac{1}{\varepsilon^2}[Y(t) - \widehat{Y}_\varepsilon(t)]. \end{cases} \tag{45}$$

The following result can be easily concluded from [123, 140].

Theorem 3 *Suppose that $d \in H^1_{\text{loc}}(0,\infty)$ and there exists a positive constant M such that $|\mathrm{d}(t)| \leq M$ for all $t \geq 0$. Let $Y(t) = \int_0^1 x w_t(x,t)dx$. Then, system (45) is practically stable in the sense that*

$$\limsup_{t\to\infty} \left\{ \int_0^1 [w_t^2(x,t) + w_x^2(x,t)]\mathrm{d}x + |\widehat{Y}_\varepsilon(t)| + |\widehat{d}_\varepsilon(t) - \mathrm{d}(t)| \right\} \leq C\varepsilon, \tag{46}$$

where $C > 0$ is a constant independent of ε.

In closed-loop system (45), the external disturbance is estimated by a conventional ESO that may be high gain because the gain $1/\varepsilon$ may be large due to the fact that the energy of the state is in inverse proportion to the tuning parameter ε. In addition, the control law in (45) is based on the full state feedback.

Next we will consider the output feedback exponential stabilization of system (39). The output measurement is supposed to be

$$y_{\mathrm{m}}(t) = \{w_x(0,t), w(1,t)\}, \quad t \geq 0.$$

Compared with the full state feedback, the output measurement is only two boundary signals. We first use the output signal $w_x(0,t)$ to design an auxiliary system as follows:

$$\begin{cases} v_{tt}(x,t) = v_{xx}(x,t), \quad x \in (0,1), \ t > 0, \\ v_x(0,t) = c_0 v(0,t) + c_1 v_t(0,t) + w_x(0,t), \\ v_x(1,t) = u(t), \quad t \geq 0, \end{cases} \tag{47}$$

which is used to bring the total disturbance from original system (39) into an exponentially stable system. Here $c_0, c_1 > 0$. To understand this idea, we introduce a new variable $p(x,t)$ given by

$$p(x,t) = w(x,t) - v(x,t). \tag{48}$$

Combining (39) with (47), we can conclude that

$$\begin{cases} p_{tt}(x,t) = p_{xx}(x,t), \quad x \in (0,1), \ t > 0, \\ p_x(0,t) = c_0 p(0,t) + c_1 p_t(0,t), \quad t \geq 0, \\ p_x(1,t) = F(t), \quad t \geq 0. \end{cases} \tag{49}$$

We consider (49) in the energy space $\mathbf{H} = H^1(0,1) \times L^2(0,1)$ with the norm

$$\langle(\phi_1,\psi_1),(\phi_2,\psi_2)\rangle_{\mathbf{H}} = \int_0^1 [\phi_1'(x)\overline{\phi_2'(x)} + \psi_1(x)\overline{\psi_2(x)}]dx + \phi_1(0)\overline{\phi_2(0)}.$$

In system (49), the total disturbance $F(t)$ can be regarded as an input. Indeed, system (49) can be written as

$$\frac{d}{dt}(p(\cdot,t), p_t(\cdot,t)) = A_1(p(\cdot,t), p_t(\cdot,t)) + B_1 F(t),$$

where the operator A_1 is defined by

$$\begin{cases} A_1(\phi,\psi) = (\psi,\phi''), \quad \forall(\phi,\psi) \in D(A_1), \\ D(A_1) = \Big\{(\phi,\psi) \in H^2(0,1) \times H^1(0,1) | \\ \phi'(0) = c_0\phi(0) + c_1\psi(0), \ \phi'(1) = 0\Big\}, \end{cases} \tag{50}$$

and the operator B_1 is defined by $B_1 = (0, \delta(x-1))$. It is well known that A_1 generates an exponentially stable C_0-semigroup $e^{A_1 t}$ and B_1 is admissible for $e^{A_1 t}$. The well-posedness and the boundedness of the solution of system (49) are a special case of Lemma 2.2 of [138], which is summarized in following lemma 2.

Lemma 2 *For any initial state $(p(\cdot,0), p_t(\cdot,0)) \in \mathbf{H}$, suppose that $F \in L^\infty(0,\infty)$. Then, there exists a unique solution to (49) such that $(p, p_t) \in C(0,\infty;\mathbf{H})$ and for some $M > 0$, it holds*

$$\sup_{t\geq 0} \|(p(\cdot,t), p_t(\cdot,t))\|_{\mathbf{H}} \leq M. \tag{51}$$

In system (49), we regard $y_o(t) \triangleq p(1,t)$ as an output of system (49). Next, by making use of system (49) and the output $p(1,t)$, we propose the second auxiliary system as follows:

$$\begin{cases} z_{tt}(x,t) = z_{xx}(x,t), \quad x \in (0,1), \ t > 0, \\ z_x(0,t) = c_0 z(0,t) + c_1 z_t(0,t), \quad t \geq 0, \\ z(1,t) = p(1,t), \quad t \geq 0, \end{cases} \tag{52}$$

which is used to estimate the total disturbance $F(t)$. To understand this idea, we regard $z_x(1,t)$ as an output of system (52) and denote

$$q(x,t) = z(x,t) - p(x,t). \tag{53}$$

It is seen that $q(x,t)$ satisfies

$$\begin{cases} q_{tt}(x,t) = q_{xx}(x,t), \\ q_x(0,t) = c_0 q(0,t) + c_1 q_t(0,t), \quad q(1,t) = 0. \end{cases} \tag{54}$$

We consider system (54) in the energy Hilbert state space $\mathbb{H} = H_R^1(0,1) \times L^2(0,1)$, where $H_R^1(0,1) = \{\phi \in H^1(0,1) | \ \phi(1) = 0\}$. System (54) can be rewritten as

$$\frac{d}{dt}(q(\cdot,t), q_t(\cdot,t)) = A_2(q(\cdot,t), q_t(\cdot,t)),$$

where

$$\begin{cases} A_2(\phi,\psi) = (\psi,\phi''), \ \forall(\phi,\psi) \in D(A_2), \\ D(A_2) = \Big\{(\phi,\psi) \in H^2(0,1) \times H^1(0,1) | \\ \phi'(0) = c_0\phi(0) + c_1\psi(0), \ \phi(1) = 0\Big\}. \end{cases} \tag{55}$$

It is well known [141, Theorem 3] that $e^{A_2 t}$ is an exponentially stable operator semigroup on $\mathbb{H}$. Thus, for any initial state $(q_0, q_1) \in \mathbb{H}$, system (54) has a unique solution $(q(\cdot,t), q_t(\cdot,t)) = e^{A_2 t}(q_0, q_1) \in C(0,\infty;\mathbb{H})$, which decays exponentially. Moreover, for some $\alpha > 0$, we have (see [138, Remark 2.3])

$$e^{\alpha t} q_x(1,t) \in L^2(0,\infty).$$

Denote $q_x(1,t) = z_x(1,t) - p_x(1,t) = z_x(1,t) - F(t)$. Hence, $z_x(1,t)$ can be regarded as an approximation of the total disturbance $F(t)$. Using this approximation, we propose an state observer for system (39) as follows:

$$\begin{cases} \widehat{w}_{tt}(x,t) = \widehat{w}_{xx}(x,t), \quad x \in (0,1),\ t > 0, \\ \widehat{w}_x(0,t) = c_0\widehat{w}(0,t) + c_1\widehat{w}_t(0,t) + w_x(0,t), \\ \widehat{w}_x(1,t) = u(t) + z_x(1,t), \quad t \geq 0, \end{cases} \tag{56}$$

where $z_x(1,t)$ is the estimate of the total disturbance $F(t)$. To confirm that (56) is an appropriate observer for system (39), we set

$$\widetilde{w}(x,t) = \widehat{w}(x,t) - w(x,t).$$

It is easy to verify that $\widetilde{w}(x,t)$ satisfies

$$\begin{cases} \widetilde{w}_{tt}(x,t) = \widetilde{w}_{xx}(x,t), \quad x \in (0,1),\ t > 0, \\ \widetilde{w}_x(0,t) = c_0\widetilde{w}(0,t) + c_1\widetilde{w}_t(0,t), \quad t \geq 0, \\ \widetilde{w}_x(1,t) = q_x(1,t), \quad t \geq 0. \end{cases} \tag{57}$$

To demonstrate the exponential stability of system (57), we introduce the variable $\beta(x,t)$ given by

$$\beta(x,t) = \widetilde{w}(x,t) - q(x,t).$$

From (54) and (57), it follows that

$$\begin{cases} \beta_{tt}(x,t) = \beta_{xx}(x,t), \quad x \in (0,1),\ t > 0, \\ \beta_x(0,t) = c_0\beta(0,t) + c_1\beta_t(0,t), \quad t \geq 0, \\ \beta_x(1,t) = 0, \quad t \geq 0. \end{cases} \tag{58}$$

Consider system (58) in the state space $\mathbf{H}$. It is well known that for any $(\beta(\cdot,0), \beta_t(\cdot,0)) \in \mathbf{H}$, system (58) has a unique solution $(\beta, \beta_t) \in C(0,\infty; \mathbf{H})$. By this fact and the exponential stability of system (54), we can immediately obtain the following conclusion.

Lemma 3 *For any initial state* $(\widetilde{w}(\cdot,0), \widetilde{w}_t(\cdot,0)) \in \mathbf{H}$, $q_x(1,t)$ *is generated by (54), there exists a unique solution to* (57) *such that* $(\widetilde{w}, \widetilde{w}_t) \in C(0,\infty; \mathbf{H})$, *and for some* $\alpha, M, \mu > 0$, *it holds* $e^{\alpha t}\widetilde{w}_t(1,t) \in L^2(0,\infty)$ *and*

$$\|(\widetilde{w}(\cdot,t), \widetilde{w}_t(\cdot,t))\|_{H^1(0,1)\times L^2(0,1)} \leq Me^{-\mu t}. \tag{59}$$

Since the state observer and the estimate of the total disturbance are obtained, an observer-based feedback control law can be designed naturally as follows:

$$u(t) = -c_2\widehat{w}_t(1,t) - z_x(1,t).$$

By this control, since $q_x(1,t) = z_x(1,t) - F(t)$ and $\widehat{w}_t(1,t) = \widetilde{w}_t(1,t) + w_t(1,t)$, system (39) becomes

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \quad x \in (0,1),\ t > 0, \\ w(0,t) = 0, \quad t \geq 0, \\ w_x(1,t) = F(t) - c_2\widehat{w}_t(1,t) - z_x(1,t) \\ = -c_2 w_t(1,t) + \theta(t), \end{cases} \tag{60}$$

where $\theta(t) = -c_2\widetilde{w}_t(1,t) - q_x(1,t)$ satisfies $e^{\alpha t}\theta(t) \in L^2(0,\infty)$. Define the operator A_3 given by

$$\begin{cases} A_2(\phi,\psi) = (\psi,\phi''), \quad \forall(\phi,\psi) \in D(A_3), \\ D(A_3) = \Big\{(\phi,\psi) \in \mathcal{H} \cap (H^2(0,1) \times H^1(0,1)) | \\ \psi(0) = 0,\ \phi'(1) = -c_2\psi(1)\Big\}. \end{cases} \tag{61}$$

Then, system (60) can be written as

$$\frac{d}{dt}(w(\cdot,t), w_t(\cdot,t)) = A_3(w(\cdot,t), w_t(\cdot,t)) + B\theta(t),$$

where $B = (0, \delta(x-1))$. It is well known that A_3 generates an exponentially stable C_0-semigroup $e^{A_3 t}$ on $\mathcal{H}$ and B is admissible for $e^{A_3 t}$ [139]. The exponential stability of the solution of (60) follows from Lemma 2.1 in [138].

Collecting (47), (52) and (56), we obtain the closed-loop system of (39) described by

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \\ w(0,t) = 0, \quad t \geq 0, \\ w_x(1,t) = f(w(\cdot,t), w_t(\cdot,t)) + \mathrm{d}(t) \\ \quad -c_2\widehat{w}_t(1,t) - z_x(1,t), \\ v_{tt}(x,t) = v_{xx}(x,t), \quad x \in (0,1), \\ v_x(0,t) = c_0 v(0,t) + c_1 v_t(0,t) + w_x(0,t), \\ v_x(1,t) = -c_2\widehat{w}_t(1,t) - z_x(1,t), \\ z_{tt}(x,t) = z_{xx}(x,t), \\ z_x(0,t) = c_0 z(0,t) + c_1 z_t(0,t), \\ z(1,t) = w(1,t) - v(1,t), \\ \widehat{w}_{tt}(x,t) = \widehat{w}_{xx}(x,t), \\ \widehat{w}_x(0,t) = c_0\widehat{w}(0,t) + c_1\widehat{w}_t(0,t) + w_x(0,t), \\ \widehat{w}_x(1,t) = -c_2\widehat{w}_t(1,t). \end{cases} \tag{62}$$

We consider system (62) in the state space $\mathcal{X} = \mathcal{H} \times \mathbf{H}^3$. The main result on the stability and the well-posedness of system (62) can be proved similarly as Theorem 4.3 of [138].

Theorem 4 *Suppose that the parameters $c_0, c_1, c_2 > 0$, $f : \mathcal{H} \to \mathbb{R}$ are continuous, and $d \in L^\infty(0,\infty)$ or $d \in L^2(0,\infty)$. For any initial state $(w_0, w_1, v_0, v_1, z_0, z_1, \widehat{w}_0, \widehat{w}_1) \in \mathcal{X}$ with the compatibility conditions*

$$z_0(1) + v_0(1) - w_0(1) = 0, \tag{63}$$

there exists a unique solution to (62) such that $(w, w_t, v, v_t, z, z_t, \widehat{w}, \widehat{w}_t) \in C(0,\infty;\mathcal{X})$,

$$\|(w(\cdot,t), w_t(\cdot,t), \widehat{w}(\cdot,t), \widehat{w}_t(\cdot,t))\|_{\mathcal{H}\times\mathbf{H}} \le Me^{-\mu t}\|(w_0, w_1, v_0, v_1, \widehat{w}_0, \widehat{w}_1)\|_{\mathcal{X}}, \forall\, t \ge 0, \tag{64}$$

for some $M, \mu > 0$, and

$$\sup_{t\ge 0}\|(v(\cdot,t), v_t(\cdot,t), z(\cdot,t), z_t(\cdot,t))\|_{\mathbf{H}^2} < \infty.$$

7 ADRC for uncertain infinite-dimensional fractional-order systems

In this section, we introduce the ADRC approach to stabilization for uncertain infinite-dimensional fractional-order systems, which is just initiated in 2019 [131]. The following main results come from [131]. The controlled plant is the following one-dimensional time fractional-order anomalous diffusion equation (TFADE) with Neumann boundary control and boundary disturbance:

$$\begin{cases} {}_0^C D_t^\alpha w(x,t) = w_{xx}(x,t) + \lambda(x)w(x,t), \\ w_x(0,t) = -qw(0,t), \\ w_x(1,t) = u(t) + \mathrm{d}(t), \\ w(x,0) = w_0(x), \end{cases} \tag{65}$$

where $x \in (0,1)$, $t \ge 0$, $w(x,t)$ is the state, $u(t)$ is the control input, $\lambda \in C[0,1]$, $\mathrm{d}(t)$ represents an unknown external disturbance which is only supposed to satisfy d, ${}_0^C D_t^\alpha d \in L^\infty(0,\infty)$. ${}_0^C D_t^\alpha w(x,t)$ stands for the Caputo derivative which is a regularized fractional derivative of $w(x,t)$ with respect to the time variable t, that is,

$$ {}_0^C D_t^\alpha w(x,t) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{\partial}{\partial t}\int_0^t (t-s)^{-\alpha} w(x,s)ds - t^{-\alpha}w(x,0)\right].$$

It is well known that

$$\lim_{\alpha\to 1^-} {}_0^C D_t^\alpha w(x,t) = \frac{\partial w(x,t)}{\partial t}.$$

Noting that system (65) is unstable without control and disturbance. When the external disturbance flows in the control end, the stabilization problem for (65) becomes much more complicated. The control objective here is to design a state feedback control law $u(t)$ so that the close-loop state of system (65) converges to zero in the Mittag–Leffler sense by rejecting the external disturbance $\mathrm{d}(t)$.

For the reader's convenience, we present the usual definitions of the Mittag–Leffler function and the Mittag–Leffler stability, which can be founded in [131].

Definition 1 The one-parameter Mittag–Leffler function and two-parameter Mittag–Leffler function are defined by

$$E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + 1)}$$

$$\text{and}\quad E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)},$$

respectively, where $\alpha > 0, \beta > 0$. In particular, $E_{\alpha,1}(z) = E_\alpha(z)$ and $E_1(z) = E_{1,1}(z) = e^z$.

Definition 2 (Mittag–Leffler Stability). The solution of (65) is said to be Mittag–Leffler stable if

$$\|w(\cdot,t)\|_{L^2(0,1)} \le \{m(\|w(\cdot,0)\|_{L^2(0,1)})E_\alpha(-\lambda t^\alpha)\}^b,$$

where $\alpha \in (0,1)$, $\lambda > 0$, $b > 0$, $m(0) = 0$, $m(s) \ge 0$, and $m(s)$ is locally Lipschitz in $s \in \mathbb{R}$ with Lipschitz constant m_0.

Since $E_\alpha(-\lambda t^\alpha) \le \frac{M}{1+\lambda t^\alpha}$ for some $M > 0$ and all $t \ge 0$, it is seen that the Mittag–Leffler stability implies the Lyapunov asymptotic stability, that is, $\lim_{t\to\infty}\|w(\cdot,t)\|_{L^2(0,1)} = 0$.

To obtain the estimation of the external disturbance, we propose two auxiliary systems, one is to bring the disturbance from original system (65) into a Mittag–Leffler stable system, and the other one is to estimate the external disturbance. The following steps were presented in [131].

Step 1: The first auxiliary system is given by :

$$\begin{cases} {}_0^C D_t^\alpha v(x,t) = v_{xx}(x,t) + \lambda(x)w(x,t) \\ \quad -c[v(x,t) - w(x,t)], \\ v_x(0,t) = -qw(0,t), \qquad v_x(1,t) = u(t), \\ v(x,0) = v_0(x), \end{cases} \tag{66}$$

where the gain c is a positive designed parameter used to regulate the convergence speed. Let $\widehat{v}(x,t)=v(x,t)-w(x,t)$. It is easy to check that $\widehat{v}(x,t)$ satisfies

$$\begin{cases} {}_0^C D_t^{\alpha}\widehat{v}(x,t)=\widehat{v}_{xx}(x,t)-c\widehat{v}(x,t),\\ \widehat{v}_x(0,t)=0, \qquad \widehat{v}_x(1,t)=-\mathrm{d}(t),\\ \widehat{v}(x,0)=\widehat{v}_0(x)=v_0(x)-w_0(x), \end{cases} \tag{67}$$

and the external disturbance is coming into system (67) to be Mittag–Leffler stable shown in the following lemma.

Lemma 4 ([131]) *Suppose that* $c>0$, *and* $d, {}_0^C D_t^{\alpha} d\in L^{\infty}(0,\infty)$. *For any initial value* $\widehat{v}(\cdot,0)\in L^2(0,1)$, *there exists a unique solution to* (67) *such that* $\widehat{v}\in C(0,\infty;L^2(0,1))$ *satisfying* $\sup_{t\geq 0}\|\widehat{v}(\cdot,t)\|_{L^2(0,1)}<+\infty$. *Moreover, if* $d\equiv 0$, *then* $\|\widehat{v}(\cdot,t)\|_{L^2(0,1)}\leq ME_{\alpha}(-\mu t^{\alpha})$ *with* $M,\mu>0$.

Step 2: For system (66), we design a second auxiliary system to estimate the external disturbance:

$$\begin{cases} {}_0^C D_t^{\alpha} z(x,t)=z_{xx}(x,t)-cz(x,t),\\ z_x(0,t)=0, \qquad z(1,t)=w(1,t)-v(1,t),\\ z(x,0)=z_0(x), \end{cases} \tag{68}$$

where c is a positive designed parameter which is exactly the same as that in (66). Let $p(x,t)=-z(x,t)-\widehat{v}(x,t)$. It is easy to verify that $p(x,t)$ satisfies

$$\begin{cases} {}_0^C D_t^{\alpha} p(x,t)=p_{xx}(x,t)-cp(x,t),\\ p_x(0,t)=0, \qquad p(1,t)=0,\\ p(x,0)=p_0(x), \end{cases} \tag{69}$$

which is a Mittag–Leffler stable system and serves as a target system for the design of disturbance estimator.

System (69) can be rewritten as

$${}_0^C D_t^{\alpha} p(\cdot,t)=Ap(\cdot,t), \quad p(x,0)=p_0(x),$$

where the operator $A: D(A)(\subset L^2(0,1))\to L^2(0,1)$ is given by

$$\begin{cases} [Af](x)=f''(x)-cf(x),\\ D(A)=\{f\in H^2(0,1)|\ f'(0)=0,\ f(1)=0\}. \end{cases} \tag{70}$$

The well-posedness and stability of (69) can be found in [131]. Moreover, we have the following result.

Lemma 5 ([131]) *Suppose that* $c>0$. *For any initial value* $p(\cdot,0)\in D(A)$, *the solution of* (69) *satisfies* $|p_x(1,t)|\leq ME_{\alpha}(-\mu t^{\alpha})$ *with some* $M,\mu>0$.

Clearly, we have

$$p_x(1,t)=\mathrm{d}(t)-z_x(1,t),$$

which, together with Lemma 5, implies that $z_x(1,t)$ could be an approximation of the external disturbance $\mathrm{d}(t)$.

Finally, let us put system (66) and system (68) together. We then obtain a disturbance estimator for system (65) as follows:

$$\begin{cases} {}_0^C D_t^{\alpha} v(x,t)=v_{xx}(x,t)+\lambda(x)w(x,t)\\ \quad -c[v(x,t)-w(x,t)],\\ v_x(0,t)=-qw(0,t), \qquad v_x(1,t)=u(t),\\ {}_0^C D_t^{\alpha} z(x,t)=z_{xx}(x,t)-cz(x,t),\\ z_x(0,t)=0, \qquad z(1,t)=w(1,t)-v(1,t), \end{cases} \tag{71}$$

where the external disturbance is estimated in the way of $\mathrm{d}(t)\approx z_x(1,t)$ because of $p_x(1,t)=z_x(1,t)-\mathrm{d}(t)$ and Lemma 5.

With disturbance estimator (71), we next present a stabilizing control for system (65). For this purpose, we introduce an invertible transformation $w\to\widehat{w}$ [142]:

$$\widehat{w}(x,t)=w(x,t)-\int_0^x k(x,y)w(y,t)dy, \tag{72}$$

where the kernel function $k(x,y)$ is the solution of the following partial differential equation:

$$\begin{cases} k_{xx}(x,y)-k_{yy}(x,y)=(\lambda(y)+c)k(x,y),\\ k_y(x,0)+qk(x,0)=0,\\ k(x,x)=-q-\dfrac{1}{2}\displaystyle\int_0^x(\lambda(y)+c)dy. \end{cases} \tag{73}$$

Under transformation (72), system (65) is equivalent to:

$$\begin{cases} {}_0^C D_t^{\alpha}\widehat{w}(x,t)=\widehat{w}_{xx}(x,t)-c\widehat{w}(x,t),\\ \widehat{w}_x(0,t)=0,\\ \widehat{w}_x(1,t)=u(t)+d(t)-k(1,1)w(1,t)\\ \quad -\displaystyle\int_0^1 k_x(1,y)w(y,t)dy. \end{cases} \tag{74}$$

If the disturbance $d(t)$ vanishes, the stabilizing control law was designed in [142] as

$$u(t) = k(1,1)w(1,t) + \int_0^1 k_x(1,y)w(y,t)dy. \quad (75)$$

However, when the external disturbance $d(t)$ is nonzero, control law (75) cannot stabilize system (65).

Since we have concluded that the external disturbance $d(t)$ could be approximated by $z_x(1,t)$, it is natural to propose the following disturbance estimator-based feedback control:

$$u(t) = -z_x(1,t) + k(1,1)w(1,t) + \int_0^1 k_x(1,y)w(y,t)dy. \quad (76)$$

It is seen that the "$-z_x(1,t)$" term in (76) is used to compensate for the external disturbance $d(t)$, and the other terms are the feedback control designed to stabilize system (74) without the external disturbance $d(t)$ suggested by (75).

Closed-loop system (65) under disturbance estimator-based feedback control (76) is:

$$\begin{cases} {}_0^C D_t^\alpha w(x,t) = w_{xx}(x,t) + \lambda(x)w(x,t), \\ w_x(0,t) = -qw(0,t), \\ w_x(1,t) = -z_x(1,t) + k(1,1)w(1,t) \\ \quad + \int_0^1 k_x(1,y)w(y,t)y + d(t), \\ {}_0^C D_t^\alpha v(x,t) = v_{xx}(x,t) + \lambda(x)w(x,t) \\ \quad -c[v(x,t) - w(x,t)], \\ v_x(0,t) = -qw(0,t), \\ v_x(1,t) = -z_x(1,t) + k(1,1)w(1,t) \\ \quad + \int_0^1 k_x(1,y)w(y,t)y, \\ {}_0^C D_t^\alpha z(x,t) = z_{xx}(x,t) - cz(x,t), \\ z_x(0,t) = 0, \quad z(1,t) = v(1,t) - w(1,t). \end{cases} \quad (77)$$

We consider closed-loop system (77) in $\mathcal{H} = [L^2(0,1)]^3$, and its convergence can be summarized in following Theorem 5 obtained in [131].

Theorem 5 ([131]) *Let $k(x,y)$ be the solution of (73). Suppose that $c > 0$, and $d, {}_0^C D_t^\alpha d \in L^\infty(0,\infty)$. For any initial value $(w(\cdot,0), v(\cdot,0), z(\cdot,0)) \in \mathcal{H}$, there exists a unique solution to (77) such that $(w,v,z) \in C(0,\infty;\mathcal{H})$ satisfying $\|w(\cdot,t)\|_{L^2(0,1)} \le ME_\alpha(-\mu t^\alpha)$ with some $M, \mu > 0$, and $\sup_{t\ge0}\|(v(\cdot,t), z(\cdot,t))\|_{[L^2(0,1)]^2} < +\infty$. If we assume further that $d(t) \equiv 0$, then, there exist two constants $M', \mu' > 0$ such that $\|(v(\cdot,t), z(\cdot,t))\|_{\mathbb{H}^2} \le M'E_\alpha(-\mu' t^\alpha),\ \forall t \ge 0$.*

To end this section, we emphasize that although the conventional one-parameter tuning linear ESO is extended to fractional ESO (see [92,93]), the fractional ESO seems not be applied to fractional infinite-dimensional system, which is remarkably different from the stabilization problem of the wave equation with boundary disturbance and uncertainty by conventional ESO to estimate the boundary total disturbance presented in Sect. 6. This difference leads to the fact that we are not able to obtain corresponding practical stability for the fractional infinite-dimensional system. In order to explain it clearly, we use the test function to refine the disturbance and the control in the boundary into an ODE, and we denote two new variables $Y(t)$ and $Z(t)$ as follows:

$$\begin{aligned} Y(t) &= \int_0^1 h(x)w(x,t)dx, \\ Z(t) &= \int_0^1 [h(x)\lambda(x) + h''(x)]w(x,t)dx, \end{aligned} \quad (78)$$

where $h(x)$ is any test function satisfying $h \in C^2[0,1]$ with $h(0) = h'(0) = h'(1) = 0$ and $h(1) = 1$. Obviously, we can take a simple example as $h(x) = x^2(3-2x)$. A simple exercise shows $Y(t)$ and $Z(t)$ are governed by

$${}_0^C D_t^\alpha Y(t) = u(t) + d(t) + Z(t). \quad (79)$$

Motivated by the fractional ESO design in [92,93], the corresponding fractional ESO for ODE (79) subject to external disturbance $d(t)$ can be designed as follows:

$$\begin{cases} {}_0^C D_t^\alpha \widehat{Y}(t) = u(t) + \widehat{d}(t) + Z(t) - \beta_1[\widehat{Y}(t) - Y(t)], \\ {}_0^C D_t^\alpha \widehat{d}(t) = -\beta_2[\widehat{Y}(t) - Y(t)], \end{cases}$$

where $\beta_1 = 2\omega_o$ and $\beta_2 = \omega_o^2$ with ω_o being the linear bandwidth parameterization [92]. It follows from [93, Lemma 2] that

$$\begin{aligned} \lim_{t\to\infty}\sup|\widehat{Y}(t) - Y(t)| &\le \frac{M}{\omega_o^2}, \\ \lim_{t\to\infty}\sup|\widehat{d}(t) - d(t)| &\le \frac{2M}{\omega_o}, \end{aligned} \quad (80)$$

where $M = \sup_{t\ge0}|{}_0^C D_t^\alpha d(t)|$. It is clearly seen from (80) that the larger the bandwidth ω_o is , the smaller the estimation errors of the fractional-order ESO become. However, a drawback of this design is that the output noise will be amplified when the bandwidth ω_o is large.

From (80), we obtain an estimate $\widehat{d}(t)$ of the external disturbance $d(t)$. However, the ADRC based on this ESO design seems unable to reject the external disturbance in fractional PDEs like the one satisfying d, ${}_0^C D_t^\alpha d \in L^\infty(0,\infty)$. Actually, by the ADRC strategy, the control law should be designed by

$$u(t) = -\widehat{d}(t) + k(1,1)w(1,t) + \int_0^1 k_x(1,y)w(y,t)dy. \tag{81}$$

With this control law, the closed loop becomes

$$\begin{cases} {}_0^C D_t^\alpha w(x,t) = w_{xx}(x,t) + \lambda(x)w(x,t), \\ w_x(0,t) = -qw(0,t), \\ w_x(1,t) = -\widehat{d}(t) + k(1,1)w(1,t) + d(t) \\ \quad + \int_0^1 k_x(1,y)w(y,t)dy, \\ {}_0^C D_t^\alpha \widehat{Y}(t) = k(1,1)w(1,t) - \beta_1[\widehat{Y}(t) - Y(t)] \\ \quad + Z(t) + \int_0^1 k_x(1,y)w(y,t)dy, \\ {}_0^C D_t^\alpha \widehat{d}(t) = -\beta_2[\widehat{Y}(t) - Y(t)], \end{cases} \tag{82}$$

where $Y(t)$ and $Z(t)$ are given by (78), $\beta_1 = 2\omega_o$ and $\beta_2 = \omega_o^2$. Using transformation (72) and the error variables $\widetilde{Y}(t) = \widehat{Y}(t) - Y(t)$, $\widetilde{d}(t) = \widehat{d}(t) - d(t)$, system (82) is equivalent to:

$$\begin{cases} {}_0^C D_t^\alpha \widehat{w}(x,t) = \widehat{w}_{xx}(x,t) - c\widehat{w}(x,t), \\ \widehat{w}_x(0,t) = 0, \quad \widehat{w}_x(1,t) = -\widetilde{d}(t), \\ {}_0^C D_t^\alpha \widetilde{Y}(t) = \widetilde{d}(t) - \beta_1 \widetilde{Y}(t), \\ {}_0^C D_t^\alpha \widetilde{d}(t) = -\beta_2 \widetilde{Y}(t) - {}_0^C D_t^\alpha d(t). \end{cases} \tag{83}$$

If $\alpha = 1$, by linear system theory [139], with the similar estimation techniques used for the wave equation, we can conclude that

$$\lim_{t\to\infty,\omega_o\to\infty} \sup \|\widehat{w}(\cdot,t)\|_{L^2(0,1)} = 0. \tag{84}$$

However, when $\alpha \in (0,1)$, the admissibility theory for fractional system is not available. By (80), we have

$$\lim_{t\to\infty} \sup |\widetilde{d}(t)| \le \frac{2M}{\omega_o}, \quad \lim_{t\to\infty} \sup |{}_0^C D_t^\alpha \widetilde{d}(t)| \le 2M,$$

where $M = \sup_{t\ge 0} |{}_0^C D_t^\alpha d(t)|$. By Comparing (83) with (67), we can only obtain that $\sup_{t\ge 0} \|\widehat{w}(\cdot,t)\|_{L^2(0,1)} < +\infty$ by Lemma 4 but not the practical stability concluded as that in (84).

8 Some further theoretical problems

In this section, we summarize some open theoretical problems on active disturbance rejection control (AD RC) to be further considered specially according to previous four sections.

Firstly, with regard to ADRC for finite-dimensional nonlinear systems with mismatched disturbances and uncertainties in Sect. 4, we point out an unresolved and interesting problem of the ADRC design and convergence analysis for uncertain systems without satisfying the matching condition. The problem is again output tracking for the following lower triangular nonlinear system subject to mismatched stochastic disturbances and uncertainties by the ADRC approach:

$$\begin{cases} \dot{x}_1(t) = x_2(t) + h_1(x_1(t), w_1(t)), \\ \dot{x}_2(t) = x_3(t) + h_2(x_1(t), x_2(t), w_2(t)), \\ \quad \vdots \\ \dot{x}_n(t) = h_n(x(t), w_n(t)) + bu(t), \\ y(t) = x_1(t), \end{cases} \tag{85}$$

where the mathematical symbols are defined as those in system (8), $h_i : \mathbb{R}^{i+1} \to \mathbb{R}$ $(i = 1, 2, \ldots, n)$ are unknown functions representing mismatched unmodeled dynamics, and $w_i(t)$ $(i = 1, 2, \ldots, n)$ are bounded stochastic noises existing widely in practical systems like the ones in [135–137] and defined in Sect. 5. A major obstacle of the ADRC design and convergence analysis here is that the paths of the bounded stochastic noises are nowhere differentiable almost surely so that the mismatched bounded stochastic noises cannot be refined into the control input channel by the straightforward state transformation method proposed in Sect. 4.

Secondly, with regard to ADRC for uncertain finite-dimensional stochastic systems in Sect. 5, we point out an unresolved and important problem of the ADRC design and convergence analysis for uncertain stochastic systems. It is about the ADRC design and convergence analysis for widely existing Itô-type stochastic nonlinear system with large uncertainties. Take the following simple first-order stochastic system driven by

Brownian motion as an example:

$$\begin{cases} dx_1(t) = [w(t) + u(t)]\,dt + \sigma\,dB(t), \\ y(t) = x_1(t), \end{cases} \tag{86}$$

where $B(t)$ is a one-dimensional standard Brownian motion, σ is a constant representing the additive noise intensity, and $w(t)$ is the bounded external disturbance with unknown stochastic characteristics whose variation with respect to t is also bounded. The conventional ADRC design for this simple system (86) is even not feasible because the conventional ESO design cannot be used for the real-time estimation of the external disturbance $w(t)$, which is demonstrated in detail in section 2.2 of [110]. The essential difficulty is caused by the diffusion term of system (86) [110]. A possibly feasible solution to the ESO design for Itô-type stochastic nonlinear system with large uncertainties is to introduce dynamic time-varying gain making full use of the real-time information of the control input $u(t)$ and measured output $y(t)$, such that there exists a well trade-off between the estimation performance and the noise sensitivity.

Thirdly, with regard to ADRC for both finite-dimensional nonlinear systems with mismatched disturbances and uncertainties in Sect. 4 and uncertain finite-dimensional stochastic systems in Sect. 5, we point out an interesting open problem. That is, since measurement noise exists widely in practice, the design and performance analysis of ADRC for uncertain systems with measurement noise becomes an open problem theoretically. Similar to the trade-off existing between the speed of state reconstruction and the immunity to measurement noise caused by high-gain observers [143], a natural trade-off between fast reconstruction of both the states and the total disturbance and the tolerance to measurement noise caused by ESO is inevitable and should be explored. Recently, the ADRC has been addressed for a class of uncertain systems with measurement uncertainty by seizing the equivalent total effect of multiple uncertainties in both dynamics and measurement, where the measurement uncertainty is without stochastic characteristic and the existence of its higher derivative is required [108]. However, for the measurement noise of stochastic characteristic, the approach proposed in [108] is infeasible since the paths of the measurement noise are nowhere differentiable almost surely. In this scheme, the switched-gain approach [143] may be one of the feasible approaches.

Finally, with regard to ADRC for uncertain infinite-dimensional systems in Sect. 6 and fractional-order systems in Sect. 7, we point out two unresolved and interesting problems of the ADRC design and convergence analysis for fractional systems. The first one is the output feedback stabilization for system (65). Section 7 presents the full state feedback for system (65); however, the in-domain term $\lambda(x)w(x,t)$ leads to the difficulty of the design in terms of the boundary output measurement. The second problem is the stabilization of uncertain fractional wave systems when the fractional-order $\alpha \in (1,2)$. Consider the system described by

$$\begin{cases} {}_0^C D_t^\alpha y(x,t) = y_{xx}(x,t), \quad x \in (0,1), \ t \geq 0, \\ y_x(0,t) = p\,{}_0^C D_t^r y(0,t), \quad y_x(1,t) = u(t) + \psi(t), \\ y(x,0) = y_0(x), \ y_t(x,0) = y_1(x), \ 0 \leq x \leq 1, \end{cases} \tag{87}$$

where $\alpha \in (1,2)$, $r \in (0,1)$, $y(x,t)$ is the state, p is a constant, $u(t)$ is the control input, $\psi(t)$ is an unknown external disturbance, and $(y_0(x), y_1(x))$ is the initial state. The case where $(\alpha, r) = (2,1)$, $p < 0$ or $(\alpha, r) = (2,0)$, $p < 0$ and the case where $\alpha \in (0,1)$, $p < 0$ and $r = 0$ were studied in [138] and [131], respectively. Unlike the case $\alpha \in (0,1)$ and $\alpha = 2$, the main difficulties are that the fractional Lyapunov method is not applicable for the order $\alpha \in (1,2)$ (the key inequality ${}_0^C D_t^\alpha x^2(t) \leq 2x(t){}_0^C D_t^\alpha x(t)$ fails when $\alpha \in (1,2)$) and that the eigenfunctions of the associated system operator do not form Riesz basis due to the fractional boundary condition, which lead to the obstacle in proving the stability.

9 Concluding remarks

In this survey paper, the origin, general framework, ideology and theoretical progresses of active disturbance rejection control (ADRC) have been articulated at length. The plant addressed by the ADRC approach, from a theoretical perspective, is comprehensively concerning not only uncertain finite-dimensional systems but also uncertain infinite-dimensional systems. Some very recent theoretical developments have been specially highlighted by including finite-dimensional systems with mismatched disturbances and uncertainties by output feedback, uncertain finite-dimensional stochastic systems, uncertain infinite-dimensional systems described by wave equation and fractional-order partial differential equation, where the essences of ADRC for these kinds of controlled systems and some potential further developments have been specially introduced.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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